

A SOBOLEV ROUGH PATH EXTENSION THEOREM VIA REGULARITY STRUCTURES

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ABSTRACT. We show that every \mathbb{R}^d -valued Sobolev path with regularity α and integrability p can be lifted to a Sobolev rough path provided $1/2 > \alpha > 1/p \vee 1/3$. The novelty of our approach is its use of ideas underlying Hairer’s reconstruction theorem generalized to a framework allowing for Sobolev models and Sobolev modelled distributions. Moreover, we show that the corresponding lifting map is locally Lipschitz continuous with respect to the inhomogeneous Sobolev metric.

Key words: fractional Sobolev space, Lyons–Victoir extension theorem, reconstruction theorem, regularity structures, rough path.

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1. INTRODUCTION

The cornerstone of rough path theory is the concept of a rough path in the sense of Lyons [Lyo98]. Unlike a classical path X from an interval $[0, T]$ to the Euclidean space \mathbb{R}^d , a rough path $\mathbf{X} = (X, \mathbb{X})$ contains the additional information \mathbb{X} representing, loosely speaking, the iterated integrals of the path X against itself. However, at least in general, there is no simple canonical way to ensure the existence of these iterated integrals. This has led to the fundamental question whether every \mathbb{R}^d -valued path X can be lifted to a rough path \mathbf{X} in the sense that the projection of \mathbf{X} onto the path-level¹ is X .

A first affirmative and non-trivial² answer was given by Lyons and Victoir [LV07], proving, in particular, that an \mathbb{R}^d -valued Hölder continuous path can always be lifted to a Hölder continuous weakly geometric rough path. Using a re-parameterization argument, this directly reveals an extension theorem in terms of p -variation. While proof of Lyons and Victoir is considered to be non-constructive, an explicit approach, based on so-called Fourier normal ordering, was obtained by Unterberger [Unt10]. More recently, further constructive proofs of the Lyons–Victoir extension theorem were derived by Broux and Zambotti [BZ21], using a sewing lemma for low regularity, and by Tapia and Zambotti [TZ20], using an explicit form of the Baker–Campbell–Hausdorff formula. Notice that in [TZ20] a Lyons–Victoir extension theorem is provided allowing even for anisotropic Hölder continuous paths, i.e., allowing each component of the underlying path to have a different Hölder regularity. Moreover, in [LPT18] a Lyons–Victoir extension theorem for Sobolev paths is proven, using a discrete characterization of (non-linear) fractional Sobolev spaces.

The present work is a companion paper of [LPT18] and continues the above line of research. We explore an approach based on Hairer’s theory of regularity structures [Hai14], which goes back to [FH14], and show that every path with Sobolev regularity $\alpha \in (1/3, 1/2)$ and

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¹That is, the first coordinate projection $\mathbf{X} = (X, \mathbb{X}) \mapsto X$.

²Of course, for a smooth path a rough path lift can be constructed by, e.g., Riemann–Stieltjes integration.

integrability $p > 1/\alpha$ can be lifted to a weakly geometric rough path possessing exactly the same Sobolev regularity. While the rough path lift of a Hölder continuous path is a known and fairly simple application of Hairer’s reconstruction theorem ([Hai14, Theorem 3.10]), see [FH14, Proposition 13.23] or [Bra19], lifting a Sobolev path lies outside the current framework of regularity structures and thus requires some serious additional effort. Indeed, we need to use a Sobolev topology on the space of modelled distributions, as introduced in [HL17] and [LPT21b] (see also [HR20]) and additionally to generalize the definition of models from the originally required Hölder bounds to some more general Sobolev bounds. In other words, we cannot apply Hairer’s reconstruction theorem directly and instead need to generalize the essential features of Hairer’s reconstruction operator to our setting allowing for Sobolev models and Sobolev modelled distributions, see Remark 3.11 for a more detailed discussion. The rough path lift essentially relying on Hairer’s reconstruction operator constitutes an explicit construction of a rough path above a given \mathbb{R}^d -valued path. In contrast to the rough path lift obtained in the spirit of Lyons and Victoir, the corresponding lifting map turns out to be continuous, cf. [FH14, Proposition 13.23] or [Bra19]. Indeed, we prove that the corresponding lifting map is locally Lipschitz continuous with respect to the inhomogeneous Sobolev metric.

Organization of the paper: In Section 2 we introduce the notion of a rough path and some basic definitions. In Section 3 we construct the rough path lifts of Sobolev paths and show that the corresponding lifting map is locally Lipschitz continuous.

2. SOBOLEV ROUGH PATH AND BASIC NOTATION

We start by introducing the notion of Sobolev rough paths in Subsection 2.1 and some basic definitions in Subsection 2.2.

2.1. Sobolev rough path. Since we focus throughout the entire work on the Sobolev regularity $\alpha \in (1/3, 1/2)$, we only present the definitions below in the necessary generality to deal with this regularity. A more general treatment of Sobolev rough paths can be found in [LPT18, LPT21a] and for more comprehensive introduction to rough path theory, see e.g. [LCL07, FV10, FH14].

We first recall the underlying algebra structure of a rough path, which can be conveniently described by the free nilpotent Lie group $G^2(\mathbb{R}^d)$. Let \mathbb{R}^d be the Euclidean space with norm $|\cdot|$ for $d \in \mathbb{N}$ and let $C^{1\text{-var}}([0, T]; \mathbb{R}^d)$ be the space of all continuous functions $Z: [0, T] \rightarrow \mathbb{R}^d$ of finite variation. For a path $Z \in C^{1\text{-var}}([0, T]; \mathbb{R}^d)$, its step-2 signature is defined by

$$S_2(Z)_{s,t} := \left(1, \int_{s < u < t} dZ_u, \int_{s < u_1 < u_2 < t} dZ_{u_1} \otimes dZ_{u_2} \right) \in T^2(\mathbb{R}^d) := \bigoplus_{k=0}^2 (\mathbb{R}^d)^{\otimes k},$$

where $(\mathbb{R}^d)^{\otimes n}$ denotes the n -tensor space of \mathbb{R}^d with the convention $(\mathbb{R}^d)^{\otimes 0} := \mathbb{R}$, cf. [FV10, Definition 7.2]. We equip $T^2(\mathbb{R}^d)$ with the standard addition $+$, tensor multiplication \otimes and scalar product, and denote by π_i the projection from $T^2(\mathbb{R}^d)$ onto the i -th level, for $i = 0, 1, 2$. The corresponding space of all these lifted paths is the step-2 free nilpotent group (w.r.t. \otimes)

$$G^2(\mathbb{R}^d) := \{S_2(Z)_{0,1} : Z \in C^{1\text{-var}}([0, T]; \mathbb{R}^d)\} \subset T^2(\mathbb{R}^d).$$

On $G^2(\mathbb{R}^d)$ we work with the Carnot–Carathéodory metric d_{cc} , which is given by

$$d_{cc}(g, h) := \|g^{-1} \otimes h\|_{cc} \quad \text{for } g, h \in G^2(\mathbb{R}^d),$$

where $\|\cdot\|_{cc}$ is the Carnot–Caratheodory norm defined via [FV10, Theorem 7.32], cf. [FV10, Definition 7.41]. The metric d_{cc} turns $G^2(\mathbb{R}^d)$ into a complete geodesic metric space. For a path $\mathbf{X}: [0, T] \rightarrow G^2(\mathbb{R}^d)$, we set $\mathbf{X}_{s,t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_t$ for any subinterval $[s, t] \subset [0, T]$. We refer to [FV10, Chapter 7] for a more comprehensive introduction to $G^N(\mathbb{R}^d)$.

Analogously to [LPT18, LPT21a], we want to consider rough paths with fractional Sobolev regularity. For this purpose, let us recall the definition of Sobolev regularity for functions mapping into a metric space (E, \mathbf{d}) . For $\alpha \in (0, 1)$, $p \in (1, +\infty)$ and a (continuous) function $f: [0, T] \rightarrow E$ we define the *fractional Sobolev regularity* by

$$(2.1) \quad \|f\|_{W_p^\alpha} := \|f\|_{W_p^\alpha; [0, T]} := \left(\iint_{[0, T]^2} \frac{\mathbf{d}(f(u), f(v))^p}{|v - u|^{\alpha p + 1}} du dv \right)^{1/p} + \left(\int_{[0, T]} \mathbf{d}(x_0, f(u))^p du \right)^{1/p}$$

and in the case of $p = +\infty$ we set

$$\|f\|_{W_p^\alpha} := \|f\|_{W_p^\alpha; [0, T]} := \sup_{u, v \in [0, T]} \frac{\mathbf{d}(f(u), f(v))}{|v - u|^\alpha} + \sup_{u \in [0, T]} \mathbf{d}(x_0, f(u))$$

for an arbitrary $x_0 \in E$. The latter case is also known as Hölder regularity. The space $W_p^\alpha([0, T]; E)$ consists of all continuous functions $f: [0, T] \rightarrow E$ such that $\|f\|_{W_p^\alpha} < +\infty$. Notice that the fractional Sobolev space $W_p^\alpha([0, T]; E)$ is independent of the reference point x_0 . The Sobolev regularity leads naturally to the notion of (fractional) Sobolev rough paths.

Definition 2.1 (Sobolev rough path). Let $\alpha \in (1/3, 1/2)$ and $p \in (1, +\infty]$ be such that $\alpha > 1/p$. The space $W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ consists of all paths $\mathbf{X}: [0, T] \rightarrow G^2(\mathbb{R}^d)$ such that

$$\|\mathbf{X}\|_{W_p^\alpha} := \left(\iint_{[0, T]^2} \frac{d_{cc}(\mathbf{X}_s, \mathbf{X}_t)^p}{|t - s|^{\alpha p + 1}} ds dt \right)^{1/p} < +\infty.$$

The space $W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ is called the *weakly geometric Sobolev rough path space* and $\mathbf{X} \in W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ is called a *weakly geometric rough path of Sobolev regularity* (α, p) or in short *Sobolev rough path*.

Remark 2.2. Suppose that $\alpha \in (1/3, 1/2)$ and $p \in (1, +\infty]$ be such that $\alpha > 1/p$, as done in Definition 2.1. By [FV06, Theorem 2], every weakly geometric rough path of Sobolev regularity (α, p) is a continuous weakly geometric rough path of finite $1/\alpha$ -variation. Therefore, the definition of the Sobolev regularity $\|\mathbf{X}\|_{W_p^\alpha}$ for a (continuous) rough path \mathbf{X} does not require the additional L^p -regularity and, thus, we did drop it in definition of $\|\mathbf{X}\|_{W_p^\alpha}$, following the previous definitions of Sobolev rough paths, see e.g. [LPT21a, Definition 2.2].

2.2. Basic notation and function spaces. As usual, \mathbb{Z} denotes the set of integers, $\mathbb{N} := \{1, 2, \dots\}$ are the natural numbers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $z \in \mathbb{R}$ we write $\lfloor z \rfloor := \max\{y \in \mathbb{Z} : y \leq z\}$. The ball in \mathbb{R}^k , around $x \in \mathbb{R}^k$ with radius $R > 0$ is denoted by $B(x, R)$. For two real functions a, b depending on variables x one writes $a \lesssim b$ or $a \lesssim_z b$ if there exists a constant $C(z) > 0$ such that $a(x) \leq C(z) \cdot b(x)$ for all x , and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

The space $L^p := L^p(\mathbb{R}^k, dx)$, $p \geq 1$, is the Lebesgue space, that is, the space of all functions f such that $\int_{\mathbb{R}^k} |f(x)|^p dx < +\infty$. We also set $L_\lambda^q := L^q((0, 1), \lambda^{-1} d\lambda)$ for $q \geq 1$ and write $L^p(\mathbb{R}^k; B)$ for the L^p -space of functions $f: \mathbb{R}^k \rightarrow B$ where B is a Banach space. The notation $\langle f, g \rangle$ is used for the L^2 -inner product of f and g as well as the evaluation of the distribution f against the test function g .

The space ℓ^p is the Banach space of all sequences $(x_n)_{n \in \mathbb{N}}$ of real numbers such that $\sum_{n \in \mathbb{N}} |x_n|^p < +\infty$ and the corresponding norm is denoted by $\|\cdot\|_{\ell^p}$. The space ℓ_n^p , for $n \in \mathbb{N}$, is the Banach space of all sequences $u(x) \in \mathbb{R}$, $x \in \Lambda_n := \{2^{-n}k : k \in \mathbb{Z}\}$, such that

$$\|u(x)\|_{\ell_n^p} := \left(\sum_{x \in \Lambda_n} 2^{-nd} |u(x)|^p \right)^{1/p} < +\infty.$$

The space $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^d)$ is the space of Schwartz distributions, that is, the topological dual of the space of compactly supported infinitely differentiable functions.

The space of Hölder continuous functions $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}$ of order $r \geq 0$ is denoted by \mathcal{C}^r , that is, φ is bounded (not necessarily continuous) if $r = 0$, Hölder continuous for $0 < r \leq 1$ (which amounts precisely to Lipschitz continuous for $r = 1$, the derivative does not necessarily exist everywhere). For $r > 1$ not an integer the function φ is $[r]$ -times continuously differentiable and the derivatives of order $[r]$ are Hölder continuous of order $r - [r]$. The space \mathcal{C}^r is equipped with the norm

$$\|\varphi\|_{\mathcal{C}^r} := \sum_{j=0}^{[r]} \|D^j \varphi\|_{\infty} + \mathbf{1}_{r > [r]} \|D^{[r]} \varphi\|_{r - [r]},$$

where $\|\cdot\|_{\beta}$ denotes the β -Hölder norm for $\beta \in (0, 1]$, and $\|\cdot\|_{\infty}$ denotes the supremum norm. If a function $\varphi \in \mathcal{C}^r$ has compact support, we say $\varphi \in \mathcal{C}_0^r$. Additionally, we use $\varphi \in \mathcal{B}^r$ if $\varphi \in \mathcal{C}_0^r$ is such that $\|\varphi\|_{\mathcal{C}^r} \leq 1$ and $\text{supp } \varphi \subset B(0, 1)$, and $\varphi \in \mathcal{B}_n^r$ for $n \in \mathbb{N}$ if $\varphi \in \mathcal{B}^r$ and φ annihilates all polynomials of degree at most n . We set $\mathcal{B}_{-n}^r(\mathbb{R}^k) := \mathcal{B}^r(\mathbb{R}^k)$ for all $n \in \mathbb{N}$.

3. LIFTING SOBOLEV PATHS TO SOBOLEV ROUGH PATHS

This section is devoted to show that every path of suitable Sobolev regularity can be lifted to a weakly geometric rough path *possessing exactly the same Sobolev regularity*. To prove this statement, we proceed via an approach based on Hairer's reconstruction theorem appearing in the theory of regularity structures [Hai14], which requires not only to use a Sobolev topology on the space of modelled distributions, as introduced in [HL17] and [LPT21b] (see also [HR20]), but additionally to generalize the definition of models from the originally assumed Hölder bounds to more general Sobolev bounds. For a further discussion on this point we refer the end of Subsection 3.1.

3.1. Elements of regularity structures in a Sobolev setting. In order to construct a Sobolev rough path lift of a Sobolev path relying on Hairer's theory of regularity structures, we introduce the essential ingredients of the theory in the following. For more detailed introductions we refer to [Hai15, CW17]. Let us start by recalling the definition of a regularity structure as given in [Hai14, Definition 2.1].

Definition 3.1. A triplet $\mathcal{T} = (A, T, G)$ is called *regularity structure* if it consists of the following three objects:

- An *index set* $A \subset \mathbb{R}$, which is locally finite³ and bounded from below, with $0 \in A$.
- A *model space* $T = \bigoplus_{\alpha \in A} T_{\alpha}$, which is a graded vector space with each T_{α} a Banach space and $T_0 \approx \mathbb{R}$. Its unit vector is denoted by $\mathbf{1}$.

³That is, A does not contain any cluster point.

- A *structure group* G consisting of linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $a \in T_\alpha$ it holds

$$\Gamma a - a \in \bigoplus_{\beta \in A; \beta < \alpha} T_\beta.$$

Moreover, $\Gamma \mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

For any $\tau \in T$ and $\alpha \in A$ we denote by $\mathcal{Q}_\alpha \tau$ the projection of τ onto T_α and set $|\tau|_\alpha := \|\mathcal{Q}_\alpha \tau\|$. Furthermore, for $\gamma > \min A$ we set $T_\gamma^- := \bigoplus_{\alpha \in A_\gamma} T_\alpha$ where $A_\gamma := \{\alpha \in A : \alpha < \gamma\}$.

In view of the definition of (real-valued) Sobolev spaces (see e.g. [HL17, Definition 2.1]) and of models with global bounds (see [HL17, Definition 2.8]), we introduce a Sobolev version of models with global bounds.

Definition 3.2 (Sobolev model). Let $\mathcal{T} = (A, T, G)$ be a regularity structure. For $p \in [1, +\infty]$ and $\gamma > \min A$ a *Sobolev model* is a pair (Π, Γ) that satisfies the following conditions:

- $\Pi = (\Pi_x)_{x \in \mathbb{R}^d}$ is a collection of linear maps $\Pi_x: \mathcal{T}_\gamma^- \rightarrow \mathcal{D}'(\mathbb{R}^d)$ such that

$$\|\Pi\|_p := \sup_{\zeta \in A_\gamma} \sup_{\tau \in T_\zeta^-} |\tau|_\zeta^{-1} \left\| \left\| \sup_{\eta \in \mathcal{B}_{[\zeta]}^r(\mathbb{R}^d)} \frac{|\langle \Pi_x \tau, \eta_x^\lambda \rangle|}{\lambda^\zeta} \right\|_{L^p(dx)} \right\|_{L_\lambda^p} < +\infty,$$

where η_x^λ is defined below in Definition 3.6 and $r = \lfloor \min A \rfloor + 1$.

- $\Gamma = (\Gamma_{x,y})_{x,y \in \mathbb{R}^d}$ fulfills $\Gamma_{x,y} \in G$ for all $x, y \in \mathbb{R}^d$ and

$$\|\Gamma\|_p := \sup_{\beta < \zeta \in A_\gamma} \sup_{\tau \in T_\zeta^-} |\tau|_\zeta^{-1} \left\| \left\| \frac{|\Gamma_{x,x+h} \tau|_\beta}{\|h\|^{\zeta-\beta}} \right\|_{L^p(dx)} \right\|_{L_h^p} < +\infty,$$

where $\|g\|_{L_h^p} = (\int_{B(0,1)} |g(h)|^p \frac{dh}{\|h\|})^{1/p}$.

Remark 3.3. *The Sobolev model could also be defined locally in the sense that the L^p -norm with respect to x is taken on compact subsets of \mathbb{R}^d , which is closer to the original definition of models given in [Hai14, Definition 2.17]. However, for our purpose the global bounds are the more convenient ones. Moreover, a non-Euclidean scaling can be included in Definition 3.2 and the extension to more general Besov bounds can be achieved by replacing the L_λ^p -norm by an L_λ^q -norm for $q \in [1, +\infty]$.*

Remark 3.4. *While the definition of Sobolev models seems to be the canonical one for our later choice of a regularity structure, cf. Example 3.7 below, in general different regularity structures might lead to other natural choices of models with Sobolev type bounds.*

Following [HL17] and [LPT21b], we introduce the Sobolev space of modelled distributions. Notice that the definition of modelled distributions depends on the definition of models and thus the generalized definition of models in Definition 3.2 also leads to more general notion of modelled distributions.

Definition 3.5. Let $\mathcal{T} = (A, T, G)$ be a regularity structure with a model (Π, Γ) , $\gamma \in \mathbb{R}$ and $p, q \in [1, +\infty)$. The *Besov space* $\mathcal{D}_{p,q}^\gamma$ consists of all measurable functions $f: \mathbb{R}^d \rightarrow T_\gamma^-$ such

that

$$\begin{aligned} \|f\|_{\gamma,p,q} &:= \sum_{\alpha \in A_\gamma} \| |f(x)|_\alpha \|_{L^p(dx)} \\ &+ \sum_{\alpha \in A_\gamma} \left(\int_{h \in B(0,1)} \left\| \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_\alpha}{\|h\|^{\gamma-\alpha}} \right\|_{L^p(dx)}^q \frac{dh}{\|h\|^d} \right)^{\frac{1}{q}} < +\infty. \end{aligned}$$

We refer to $\mathcal{D}_{p,p}^\gamma$ as *Sobolev space of modelled distributions*.

The corresponding Sobolev and Besov spaces consisting of real-valued distributions are introduced in the next definition following [HL17, Definition 2.1]. For a more comprehensive treatment of these function spaces we refer to [Tri10].

Definition 3.6. Let $\alpha < 0$, $p, q \in [1, +\infty)$ and $r \in \mathbb{N}$ such that $r > |\alpha|$, and define

$$\eta_x^\lambda(y) := \lambda^{-k} \eta(\lambda^{-1}(y_1 - x_1), \dots, \lambda^{-1}(y_k - x_k))$$

for $\lambda \in (0, 1]$, $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \dots, y_k) \in \mathbb{R}^k$.

For $\alpha < 0$, the *Besov space* $\mathcal{B}_{p,q}^\alpha := \mathcal{B}_{p,q}^\alpha(\mathbb{R}^k)$ is the space of all distributions ξ on \mathbb{R}^k such that

$$\left\| \left\| \sup_{\eta \in \mathcal{B}^r(\mathbb{R}^k)} \frac{|\langle \xi, \eta_x^\lambda \rangle|}{\lambda^\alpha} \right\|_{L^p(dx)} \right\|_{L_\lambda^q} < +\infty.$$

For $\alpha \geq 0$, the *Besov space* $\mathcal{B}_{p,q}^\alpha := \mathcal{B}_{p,q}^\alpha(\mathbb{R}^k)$ is the space of all distributions ξ on \mathbb{R}^k such that

$$\left\| \sup_{\eta \in \mathcal{B}^r(\mathbb{R}^k)} |\langle \xi, \eta_x^\lambda \rangle| \right\|_{L^p(dx)} < +\infty \quad \text{and} \quad \left\| \left\| \sup_{\eta \in \mathcal{B}_{[\alpha]}^r(\mathbb{R}^k)} \frac{|\langle \xi, \eta_x^\lambda \rangle|}{\lambda^\alpha} \right\|_{L^p(dx)} \right\|_{L_\lambda^q} < +\infty.$$

The *Sobolev space* W_p^α is defined as $W_p^\alpha := W_p^\alpha(\mathbb{R}^k) := \mathcal{B}_{p,p}^\alpha(\mathbb{R}^k)$.

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ has the fractional Sobolev regularity defined in (2.1) (with $[0, T]$ replaced by \mathbb{R} and $\mathbf{d} = |\cdot|$ induced by the Euclidean norm on \mathbb{R}) if and only if it is an element in $W_p^\alpha(\mathbb{R})$ in the sense of Definition 3.6 and the two norms are equivalent, see e.g. [Tri10] and [Sim90]. Note that these two norms remain equivalent when they are restricted to bounded interval $[0, T]$, see e.g. [Sch11, Theorem 1.11]. In fact, thanks to [Sch11, Theorem 1.9] for any fractional Sobolev function f defined on $[0, T]$ satisfying (2.1), there exists a bounded extension operator Ext such that $g = \text{Ext}(f)$ is a fractional Sobolev function defined on \mathbb{R} and $g|_{[0,1]} = f$. This allows us to work globally with g instead of f . In view of this observation, subsequently we will not distinguish the “localized function” f and its extension g defined on \mathbb{R} .

The construction of a rough path lift for a \mathbb{R}^d -valued path with suitable Sobolev regularity is based on the following regularity structure.

Example 3.7. Let $\alpha \in (1/3, 1/2)$ and $p \in (1, +\infty)$ such that $\alpha > 1/p$ and suppose that $W \in W_p^\alpha(\mathbb{R})$. The path W induces a regularity structure (A, T, G) via

$$A = \{\alpha - 1, 0\}, \quad T = T_{\alpha-1} \oplus T_0 = \langle \dot{W} \rangle \oplus \langle \mathbf{1} \rangle, \quad G = \{\text{Id}_T\}$$

and an associated Sobolev model (Π, Γ) via

$$\Pi_x(\dot{W}) := \dot{W}, \quad \Pi_x(\mathbf{1}) := 1 \in \mathbb{R} \quad \text{and} \quad \Gamma_{x,y} := \text{Id}_T, \quad \text{for all } x, y \in \mathbb{R},$$

where \dot{W} stands for the distributional derivative of W . Indeed, for $\tau = \tau_0 \mathbf{1} \in T_0$ with $\tau_0 \in \mathbb{R}$ we have

$$\left\| \left\| \sup_{\eta \in \mathcal{B}_0^r(\mathbb{R}^d)} \frac{|\langle \Pi_x \tau, \eta_x^\lambda \rangle|}{\lambda^0} \right\|_{L^p(dx)} \right\|_{L_\lambda^p} = 0,$$

since in this case any test function $\eta \in \mathcal{B}_0^r(\mathbb{R})$ annihilating constants has a vanishing mean. For $\tau = \tau_{\alpha-1} \dot{W} \in \mathcal{T}_{\alpha-1}$ with $\tau_{\alpha-1} \in \mathbb{R}$ we have

$$\begin{aligned} \left\| \left\| \sup_{\eta \in \mathcal{B}^r(\mathbb{R}^d)} \frac{|\langle \Pi_x \tau, \eta_x^\lambda \rangle|}{\lambda^{\alpha-1}} \right\|_{L^p(dx)} \right\|_{L_\lambda^p} &= |\tau_{\alpha-1}| \left\| \left\| \sup_{\eta \in \mathcal{B}^r(\mathbb{R}^d)} \frac{|\langle \dot{W}, \eta_x^\lambda \rangle|}{\lambda^{\alpha-1}} \right\|_{L^p(dx)} \right\|_{L_\lambda^p} \\ &= |\tau_{\alpha-1}| \|\dot{W}\|_{W_p^{\alpha-1}} \lesssim |\tau_{\alpha-1}|, \end{aligned}$$

since $\dot{W} \in W_p^{\alpha-1}(\mathbb{R})$ for $W \in W_p^\alpha(\mathbb{R})$ by [Tri10, Theorem 2.3.8]. This shows that $\|\Pi\|_p = \|\dot{W}\|_{W_p^{\alpha-1}}$. The estimate for $\|\Gamma\|_p$ holds since $|\Gamma_{x,y}\tau|_\beta = |\tau|_\beta = 0$ for any $\tau \in \mathcal{T}_\zeta$, $\beta < \zeta$ and $x, y \in \mathbb{R}$.

Given a two-dimensional path $(Y, W) \in W_p^\alpha$, in order to construct a rough path lift via Hairer's theory of regularity structure, the key idea goes as follows, cf. [FH14, Proposition 13.23]: W induces a regularity $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a model (Π, Γ) as defined in Example 3.7 and Y induces a modelled distribution with negative regularity in the sense of Definition 3.5. Then, in the case of a Hölder continuous path (Y, W) an application of Hairer's reconstruction operator ([Hai14, Theorem 3.10]) leads to a rough path lift with the same Hölder regularity.

While the reconstruction theorem for modelled distributions with negative Sobolev regularity (but for the original Hölder type models) was recently established in [LPT21b, Theorem 2.11], it is not sufficient to lift a Sobolev path to a rough path with the same Sobolev regularity. First, one loses already regularity when constructing a Hölder type model starting with a Sobolev path. Second, the classical bounds (relying on Hölder type models) obtained for the reconstruction operator, see [LPT21b, Theorem 2.11], are not sufficient and would lead again to a loss of regularity.

As a consequence, we have to derive sharper bounds for the reconstruction operator for lifting Sobolev paths to rough path in the case of Sobolev models, see (3.1) and Remark 3.11 below.

3.2. Sobolev rough path lift via the reconstruction operator. In this subsection we construct a Sobolev rough path lift of a \mathbb{R}^d -valued Sobolev path with regularity $\alpha \in (1/3, 1/2)$.

Theorem 3.8. *Let $\alpha \in (1/3, 1/2)$ and $p \in (1, +\infty]$ be such that $\alpha > 1/p$. For every Sobolev path $X \in W_p^\alpha([0, T]; \mathbb{R}^d)$ there exists a rough path lift $\mathbf{X} := (X, \mathbb{X}) \in W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ of X .*

Let us first observe that it is sufficient to prove Theorem 3.8 for a \mathbb{R}^2 -valued path $X = (Y, W) \in W_p^\alpha([0, T]; \mathbb{R}^2)$. The d -dimensional case immediately follows from successively applying the 2-dimensional case. Secondly, we extend X continuously from $[0, T]$ to \mathbb{R} such that $X = (Y, W) \in W_p^\alpha(\mathbb{R}; \mathbb{R}^2)$ for $\alpha \in (1/3, 1/2)$ and $\alpha > 1/p$. By classical Besov embeddings (see e.g. [Tri10]), we note that $X \in \mathcal{B}_{\infty, \infty}^{\alpha-1/p}$ and thus $\sup_{x \in \mathbb{R}} |X(x)| < +\infty$.

Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be the regularity structure induced by the second component W with the corresponding model (Π, Γ) as defined in Example 3.7. The first component Y induces a modelled distribution $\dot{Z}: \mathbb{R} \rightarrow \mathcal{T}$ by setting $\dot{Z}(x) := Y_x \dot{W}$ for $x \in \mathbb{R}$. Then, we have $\dot{Z} \in \mathcal{D}_{p,p}^\gamma$

with $\gamma := 2\alpha - 1$ in the sense of Definition 3.5. Indeed, note that with $\zeta = \alpha - 1$ the translation bound of \dot{Z} is equal to

$$\begin{aligned} & \left(\int_{h \in [-1, 1]} \left\| \frac{|\dot{Z}(x+h) - \Gamma_{x+h, x} \dot{Z}(x)|_\zeta}{|h|^{\gamma-\zeta}} \right\|_{L^p(dx)}^p \frac{dh}{|h|} \right)^{\frac{1}{p}} \\ &= \left(\int_{h \in [-1, 1]} \left\| \frac{|Y(x+h) - Y(x)|}{|h|^\alpha} \right\|_{L^p(dx)}^p \frac{dh}{|h|} \right)^{\frac{1}{p}} \lesssim \|Y\|_{W_p^\alpha} < +\infty, \end{aligned}$$

where the inequality follows from the equivalence of Sobolev norms, see e.g. [Tri10] and [Sim90]. Similarly one can easily show that \dot{Z} also has a bounded L^p -norm as $Y \in W_p^\alpha$ satisfies this property by definition.

Before coming to the actual proof of Theorem 3.8, we need to establish the analog of the reconstruction theorem similar to [LPT21b, Theorem 2.11], that is, we need to show the existence of the reconstruction operator \mathcal{R} , which is required for lifting Sobolev paths to Sobolev rough paths (Lemma 3.9), mapping modelled distributions into a Sobolev space and the required bound (3.1) below (Lemma 3.10).

Namely, for the regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ and the Sobolev model (Π, Γ) as defined Example 3.7, there exists a distribution $\mathcal{R}\dot{Z} \in W_p^{\alpha-1}$ satisfying that

$$(3.1) \quad \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \mathcal{R}\dot{Z} - \Pi_x \dot{Z}(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^{\frac{p}{\lambda}}} \lesssim \|\Pi\|_p (1 + \|\Gamma\|_p) \left\| \dot{Z} \right\|_{\gamma, p, p}.$$

Note that $L^{p/2}$ -norms are used in the Estimate (3.1) instead of L^p -norms, as usually obtained for the reconstruction operator, see Remark 3.11 below.

In order to define $\mathcal{R}\dot{Z}$, let $r \in \mathbb{N}$ be such that $r > |\alpha - 1 - \frac{1}{p}|$ (we will see later why such special r is needed). We fix $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ and $\psi: \mathbb{R} \rightarrow \mathbb{R}$ both in \mathcal{C}_0^r as the father wavelet and mother wavelet, respectively, of a wavelet analysis on \mathbb{R} which has the following properties:

- (1) For every polynomial P of degree at most r there exists a polynomial \hat{P} such that

$$\sum_{y \in \mathbb{Z}} \hat{P}(y) \varphi(x-y) = P(x), \quad x \in \mathbb{R}.$$

- (2) For every $y \in \mathbb{Z}$ one has $\int_{\mathbb{R}} \varphi(x) \varphi(x-y) dx = \delta_{y,0}$.
(3) There exist coefficients $(a_k)_{k \in \mathbb{Z}}$ with only finitely many non-zero values such that

$$\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k), \quad x \in \mathbb{R}.$$

- (4) The function ψ annihilates all polynomials of degree at most r .
(5) For any $n \geq 0$, the set

$$\{\varphi_x^n : x \in \Lambda_n\} \cup \{\psi_x^m : x \in \Lambda_m, m \geq n\}$$

constitutes an orthonormal basis of L^2 .

Here we used the notation

$$\varphi_x^n(y) := 2^{\frac{n}{2}} \varphi(2^n(y-x)) \quad \text{and} \quad \psi_x^n(y) := 2^{\frac{n}{2}} \psi(2^n(y-x)),$$

for $x, y \in \mathbb{R}$ and $\Lambda_n := \{2^{-n}k : k \in \mathbb{Z}\}$. For more details on wavelet analysis we refer the reader to [Mey92] and [Dau88] or in our particular setting to [HL17, Section 2.1].

As in the proof of [LPT21b, Theorem 2.11], we define

$$(3.2) \quad \mathcal{R}f := \sum_{n \in \mathbb{N}_0} \sum_{x \in \Lambda_n} \langle \Pi_x \bar{f}^n(x), \psi_x^n \rangle \psi_x^n + \sum_{x \in \Lambda_0} \langle \Pi_x \bar{f}^0(x), \varphi_x^0 \rangle \varphi_x^0,$$

where $f := \dot{Z}$ and $\bar{f}^n(x) := \int_{B(x, 2^{-n})} 2^n \Gamma_{x,y} f(y) dy$ for $x \in \Lambda_n$, cf. [HL17, (2.8)].

Lemma 3.9. *The distribution $\mathcal{R}f$ defined in (3.2) is well-defined and belongs to $W_p^{\alpha-1}$.*

Proof. We set for every $n \geq 0$, $x \in \Lambda_n$ a real number

$$a_x^{n,\psi} := \langle \mathcal{R}f, \psi_x^n \rangle = \langle \Pi_x \bar{f}^n(x), \psi_x^n \rangle,$$

and for $x \in \Lambda_0$, $b_x^0 := \langle \mathcal{R}f, \varphi_x \rangle = \langle \Pi_x \bar{f}^0(x), \varphi_x \rangle$. Invoking [HL17, (2.2)], it suffices to show that

$$\left\| \left\| \frac{a_x^{n,\psi}}{2^{-\frac{n}{2}-n(\alpha-1)}} \right\|_{\ell_n^p} \right\|_{\ell^p} < +\infty \quad \text{and} \quad \|b_x^0\|_{\ell_0^p} < +\infty.$$

To this end, we remark that by the definition of \bar{f}^n and the fact that in our setting $\Gamma_{x,y} = \text{Id}_{\mathcal{T}}$, $\Pi_x f(y) = Y_y \dot{W}$, it holds that

$$|a_x^{n,\psi}| \leq \int_{B(x, 2^{-n})} 2^n |\langle \Pi_x \Gamma_{x,y} f(y), \psi_x^n \rangle| dy = \int_{B(x, 2^{-n})} 2^n |Y_y| |\langle \dot{W}, \psi_x^n \rangle| dy.$$

It follows that

$$\begin{aligned} \left\| \left\| \frac{a_x^{n,\psi}}{2^{-\frac{n}{2}-n(\alpha-1)}} \right\|_{\ell_n^p} \right\|_{\ell^p} &\leq \left(\sum_{x \in \Lambda_n} 2^{-n} \left(\int_{B(x, 2^{-n})} 2^n |Y_y| \frac{|\langle \dot{W}, \psi_x^n \rangle|}{2^{-\frac{n}{2}-n(\alpha-1)}} dy \right)^p \right)^{\frac{1}{p}} \\ &\lesssim |Y|_\infty \left(\sum_{x \in \Lambda_n} 2^{-n} \left(\frac{|\langle \dot{W}, \psi_x^n \rangle|}{2^{-\frac{n}{2}-n(\alpha-1)}} \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

and therefore

$$\left\| \left\| \frac{a_x^{n,\psi}}{2^{-\frac{n}{2}-n(\alpha-1)}} \right\|_{\ell_n^p} \right\|_{\ell^p} \lesssim \left\| \left\| \frac{|\langle \dot{W}, \psi_x^n \rangle|}{2^{-\frac{n}{2}-n(\alpha-1)}} \right\|_{\ell_n^p} \right\|_{\ell^p} < +\infty,$$

since $W \in W_p^\alpha$ by assumption. The same argument gives us $\|b_x^0\|_{\ell_0^p} < +\infty$. Hence, we conclude that $\mathcal{R}f \in W_p^{\alpha-1}$ by using [HL17, Proposition 2.4]. \square

As a next step we show Bound (3.1) for our Sobolev model.

Lemma 3.10. *The distribution $\mathcal{R}f$ defined in (3.2) satisfies Bound (3.1).*

Proof. For fixed $x \in \mathbb{R}$, $\lambda \in (0, 1]$ and $\eta \in \mathcal{B}^r$, we have

$$\langle \mathcal{R}f - \Pi_x f(x), \eta_x^\lambda \rangle = \sum_{n \geq 0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle + \sum_{y \in \Lambda_0} \langle \mathcal{R}f - \Pi_x f(x), \varphi_y \rangle \langle \varphi_y, \eta_x^\lambda \rangle,$$

where in our case

$$\begin{aligned} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle &= \langle \Pi_y \bar{f}^n(y) - \Pi_x f(x), \psi_y^n \rangle \\ &= \int_{B(y, 2^{-n})} 2^n \langle \Pi_y (\Gamma_{y,z} f(z) - \Gamma_{y,x} f(x)), \psi_y^n \rangle dz \\ &= \int_{B(y, 2^{-n})} 2^n \langle (Y_z - Y_x) \dot{W}, \psi_y^n \rangle dz \end{aligned}$$

and the same expression holds for $\langle \mathcal{R}f - \Pi_x f(x), \varphi_y \rangle$. It follows that

$$(3.3) \quad |\langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle| \leq \int_{B(y, 2^{-n})} 2^n |Y_z - Y_x| |\langle \dot{W}, \psi_y^n \rangle| dz.$$

As in the proof of [LPT21b, Theorem 3.1] we use $\|\cdot\|_{L_{n_0}^q(d\lambda)}$ to denote the L^q -norm with respect to the finite measure (with the total mass $\ln 2$) $\lambda^{-1} \mathbf{1}_{(2^{-n_0-1}, 2^{-n_0}]}$ $d\lambda$, and we consider two quantities

$$(3.4) \quad \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n \leq n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L_{n_0}^{\frac{p}{2}}(d\lambda)}$$

and

$$(3.5) \quad \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n > n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L_{n_0}^{\frac{p}{2}}(d\lambda)}.$$

Since $\left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L_\lambda^{\frac{p}{2}}}$ is bounded by the sum of the $\ell^{\frac{p}{2}}(n_0 \in \mathbb{N}_0)$ -norms of (3.4) and (3.5), it suffices to establish the bound (3.1) for the $\ell^{\frac{p}{2}}(n_0 \in \mathbb{N}_0)$ -norm of each term. (One can easily bound the terms in the expansion of $\mathcal{R}f - \Pi_x f(x)$ involved with φ_y^0 by using a similar argument.)

Step 1: We first give an estimate for the Term (3.4). As in the proof of [HL17, Theorem 3.1], we note that for $\lambda \in (2^{-n_0-1}, 2^{-n_0})$ and $n \leq n_0$ one has $|\langle \psi_y^n, \eta_x^\lambda \rangle| \lesssim 2^{n/2}$ uniformly over all $y \in \Lambda_n$, $\eta \in \mathcal{B}^r$, $x \in \mathbb{R}$ and $n \leq n_0$. Moreover, this inner product vanishes as soon as $|x - y| > C2^{-n}$ for some constant C . Hence, inserting Inequality (3.3) we obtain that

$$\begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n \leq n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L_{n_0}^{\frac{p}{2}}(d\lambda)} \\ & \lesssim \sum_{n \leq n_0} \left\| \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \int_{B(y, 2^{-n})} 2^n \frac{|Y_z - Y_x| |\langle \dot{W}, \psi_y^n \rangle|}{2^{-n_0\gamma - \frac{n}{2}}} dz \right\|_{L^{\frac{p}{2}}(dx)} \\ & \lesssim \sum_{n \leq n_0} 2^{(n_0-n)\gamma} \left\| \int_{B(x, C'2^{-n})} 2^n \frac{|Y_z - Y_x| |\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \langle \dot{W}, \psi_y^n \rangle|}{2^{-n\gamma - \frac{n}{2}}} dz \right\|_{L^{\frac{p}{2}}(dx)} \\ & \lesssim \sum_{n \leq n_0} 2^{(n_0-n)\gamma} \left\| \int_{B(0, C'2^{-n})} 2^n \frac{|Y_{x+h} - Y_x|}{2^{-n\alpha}} dh \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{L^{\frac{p}{2}}(dx)}, \end{aligned}$$

where we used $\gamma = 2\alpha - 1 = \alpha + (\alpha - 1)$ in the last line.

For each $n \leq n_0$, by the above observation we can further deduce that

$$\begin{aligned} & \left\| \int_{B(0, C'2^{-n})} 2^n \frac{|Y_{x+h} - Y_x|}{2^{-n\alpha}} dh \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{L^{\frac{p}{2}}(dx)} \\ & \lesssim \int_{B(0, C'2^{-n})} 2^n \left(\int_{\mathbb{R}} \left(\frac{|Y_{x+h} - Y_x|}{2^{-n\alpha}} \right)^{\frac{p}{2}} \left(\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} dh \\ & \lesssim \int_{B(0, C'2^{-n})} 2^n \left(\int_{\mathbb{R}} \left(\frac{|Y_{x+h} - Y_x|}{2^{-n\alpha}} \right)^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left(\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p dx \right)^{\frac{1}{p}} dh, \end{aligned}$$

where we used Minkowski's integral inequality for the measures dx and $2^n dh$ on $B(0, C'2^{-n})$, and the Hölder inequality in the last inequality.

Now we look at the term $\int_{\mathbb{R}} \left(\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p dx$. It can be written as

$$\sum_{z \in \Lambda_n} \int_{x \in B(z, 2^{-n-1})} \left(\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p dx,$$

which can be bounded by $\sum_{z \in \Lambda_n} \int_{x \in B(z, 2^{-n-1})} \left(\sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p dx$ for some suitable constant C' independent of $z \in \Lambda_n$. Therefore, we have

$$\int_{\mathbb{R}} \left(\sum_{y \in \Lambda_n, |y-x| \leq C2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p dx \leq \sum_{z \in \Lambda_n} 2^{-n} \left(\sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p.$$

Since the cardinality of $\{y \in \Lambda_n, |y-z| \leq C'2^{-n}\}$ is controlled by C' , it yields that

$$\left(\sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p \lesssim \sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n}} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p$$

and then a basic combinatorial argument gives that

$$\sum_{z \in \Lambda_n} 2^{-n} \sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n}} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p \lesssim \sum_{y \in \Lambda_n} 2^{-n} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right)^p = \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p}^p.$$

Hence, what we finally obtained is

$$\begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n \leq n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^{\frac{p}{2}}_n(d\lambda)} \\ & \lesssim \sum_{n \leq n_0} 2^{(n_0-n)\gamma} \int_{B(0, C'2^{-n})} 2^n \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p} dh. \end{aligned}$$

As a consequence, the $\ell^{\frac{p}{2}}(n_0 \in \mathbb{N}_0)$ -norm of (3.4) is bounded by

$$\begin{aligned}
& \left(\sum_{n_0 \geq 0} \left(\sum_{n \leq n_0} 2^{(n_0-n)\gamma} \int_{B(0, C'2^{-n})} 2^n \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p} dh \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \lesssim \left(\sum_{n_0 \geq 0} \sum_{n \leq n_0} 2^{(n_0-n)\gamma} \left(\int_{B(0, C'2^{-n})} 2^n \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p} dh \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \lesssim \left(\sum_{n \geq 0} \left(\int_{B(0, C'2^{-n})} 2^n \left(\left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)} \right)^{\frac{p}{2}} dh \right) \left(\left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p} \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\
& \lesssim \left(\left(\sum_{n \geq 0} \left(\int_{B(0, C'2^{-n})} 2^n \left(\left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)} \right)^{\frac{p}{2}} dh \right)^2 \right)^{\frac{1}{2}} \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{2}} \right)^{\frac{2}{p}} \\
& \lesssim \left(\sum_{n \geq 0} \int_{B(0, C'2^{-n})} 2^n \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)}^p dh \right)^{\frac{1}{p}} \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}} \\
& \lesssim \left(\int_{B(0, C')} \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)}^p \frac{dh}{|h|} \right)^{\frac{1}{p}} \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}} \\
& \lesssim \|f\|_{\gamma, p, p} \|\Pi\|_p,
\end{aligned}$$

where we used Jensen's inequality for the finite discrete measure $n \in \{0, \dots, n_0\} \mapsto 2^{(n_0-n)\gamma}$ (as $\gamma = 2\alpha - 1 < 0$) in the second line, Jensen's inequality for the finite measure $2^n dh$ on $B(0, C'2^{-n})$ in the third line, Hölder's inequality of the type $\sum |a_n b_n| \leq (\sum a_n^2)^{\frac{1}{2}} (\sum b_n^2)^{\frac{1}{2}}$ in the fourth line and again Jensen's inequality for $2^n dh$ on $B(0, C'2^{-n})$ in the sixth line. We also note that $\left(\int_{B(0, C')} \left\| \frac{|Y_{x+h} - Y_x|}{|h|^\alpha} \right\|_{L^p(dx)}^p \frac{dh}{|h|} \right)^{1/p}$ is the translation bound of the modelled distribution f (so that it can be controlled by $\|f\|_{\gamma, p, p}$) and by [HL17, Proposition 2.4] the term

$$\left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1) - \frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}}$$

is an equivalent Sobolev norm of $\dot{W} \in W_p^{\alpha-1}$ which is also the norm of Π in the sense of Definition 3.2.

Step 2: Now we turn to the Term (3.5):

$$\left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n > n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L_{n_0}^{\frac{p}{2}}(d\lambda)}.$$

For $\lambda \in (2^{-n_0-1}, 2^{-n_0}]$ and $n > n_0$, we have

$$|\langle \psi_y^n, \eta_x^\lambda \rangle| \lesssim 2^{-\frac{n}{2} - rn} 2^{n_0(1+r)}$$

uniformly over all $y \in \Lambda_n$, $\eta \in \mathcal{B}^r$, $x \in \mathbb{R}$ and $n > n_0$. Moreover, this inner product can make contributions only when $|y - x| \leq C2^{-n_0}$ for some constant C . Hence, combining this with

Estimate (3.3) we get

$$\begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n>n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^{\frac{p}{2}}_0(d\lambda)} \\ & \lesssim \sum_{n>n_0} 2^{(n_0-n)(r+\alpha-1)} \left\| \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \int_{B(y, 2^{-n})} 2^{n_0} \frac{|Y_z - Y_x|}{2^{-n_0\alpha}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} dz \right\|_{L^{\frac{p}{2}}(dx)}. \end{aligned}$$

Since

$$\begin{aligned} & \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \int_{B(y, 2^{-n})} 2^{n_0} \frac{|Y_z - Y_x|}{2^{-n_0\alpha}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} dz \\ & \lesssim \int_{B(0, C'2^{-n_0})} 2^{n_0} \frac{|Y_{x+h} - Y_x|}{2^{-n_0\alpha}} dh \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right), \end{aligned}$$

holds for each $n > n_0$, we can deduce that

$$\begin{aligned} & \left\| \sum_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \int_{B(y, 2^{-n})} 2^{n_0} \frac{|Y_z - Y_x|}{2^{-n_0\alpha}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} dz \right\|_{L^{\frac{p}{2}}(dx)} \\ & \lesssim \int_{B(0, C'2^{-n_0})} 2^{n_0} \left(\int_{\mathbb{R}} \left(\frac{|Y_{x+h} - Y_x|}{2^{-n_0\alpha}} \right)^{\frac{p}{2}} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} dh, \end{aligned}$$

where we used the Minkowski's integral inequality as in Step 1. Then by Hölder's inequality, we find that

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left(\frac{|Y_{x+h} - Y_x|}{2^{-n_0\alpha}} \right)^{\frac{p}{2}} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \\ & \lesssim \left(\int_{\mathbb{R}} \left(\frac{|Y_{x+h} - Y_x|}{2^{-n_0\alpha}} \right)^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Next we consider the integral $\int_{\mathbb{R}} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p dx$. As before, we rewrite it as

$$\sum_{z \in \Lambda_n} \int_{x \in B(z, 2^{-n-1})} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p dx,$$

and observe the estimate

$$\begin{aligned} & \sum_{z \in \Lambda_n} \int_{x \in B(z, 2^{-n-1})} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p dx \\ & \leq \sum_{z \in \Lambda_n} 2^{-n} \sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n_0}} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p \end{aligned}$$

for some constant C' . Since the number of $y \in \Lambda_n$ such that $|y-z| \leq C'2^{-n_0}$ is of order 2^{n-n_0} for $n > n_0$ uniformly over all $z \in \Lambda_n$, we count every $y \in \Lambda_n$ for (a multiple of) 2^{n-n_0} times.

This implies that

$$\sum_{z \in \Lambda_n} 2^{-n} \sum_{y \in \Lambda_n, |y-z| \leq C'2^{-n_0}} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p \lesssim 2^{n-n_0} \sum_{y \in \Lambda_n} 2^{-n} \left(\frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p$$

and hence

$$\left(\int_{\mathbb{R}} \left(\max_{y \in \Lambda_n, |y-x| \leq C2^{-n_0}} \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right)^p dx \right)^{\frac{1}{p}} \lesssim 2^{(n-n_0)\frac{1}{p}} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p}.$$

So, finally we obtain that

$$\begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\sum_{n>n_0} \sum_{y \in \Lambda_n} \langle \mathcal{R}f - \Pi_x f(x), \psi_y^n \rangle \langle \psi_y^n, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^{\frac{p}{2}}(d\lambda)} \\ & \lesssim \sum_{n>n_0} 2^{(n_0-n)(r+\alpha-1-\frac{1}{p})} \left(\int_{h \in B(0, C'2^{-n_0})} 2^{n_0} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)} dh \right) \left(\left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p} \right). \end{aligned}$$

Thanks to our choice of r (that is $r > |\alpha - 1 - \frac{1}{p}|$), for $\theta := r + \alpha - 1 - \frac{1}{p}$ the discrete measure

$$n \in \{n_0 + 1, \dots\} \mapsto 2^{(n_0-n)\theta}$$

has finite total mass independent of n_0 , hence by Jensen's inequality and we can get that

$$\begin{aligned} & \left(\sum_{n_0 \geq 0} \left(\sum_{n>n_0} 2^{(n_0-n)\theta} \left(\int_{h \in B(0, C'2^{-n_0})} 2^{n_0} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)} dh \right) \left(\left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p} \right) \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ & \lesssim \left(\sum_{n \geq 0} \left(\left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p} \right)^{\frac{p}{2}} \sum_{n_0=0}^n 2^{(n_0-n)\theta} \left(\int_{h \in B(0, C'2^{-n_0})} 2^{n_0} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)}^{\frac{p}{2}} dh \right) \right)^{\frac{2}{p}} \\ & \lesssim \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}} \left(\sum_{n \geq 0} \left(\sum_{n_0=0}^n 2^{(n_0-n)\theta} \int_{h \in B(0, C'2^{-n_0})} 2^{n_0} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)}^{\frac{p}{2}} dh \right)^2 \right)^{\frac{1}{p}} \\ & \lesssim \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}} \left(\sum_{n \geq 0} \sum_{n_0=0}^n 2^{(n_0-n)\theta} \int_{h \in B(0, C'2^{-n_0})} 2^{n_0} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)}^p dh \right)^{\frac{1}{p}} \\ & \lesssim \left(\sum_{n \geq 0} \left\| \frac{|\langle \dot{W}, \psi_y^n \rangle|}{2^{-n(\alpha-1)-\frac{n}{2}}} \right\|_{\ell_n^p}^p \right)^{\frac{1}{p}} \left(\int_{h \in B(0, C')} \left\| \frac{Y_{x+h} - Y_x}{|h|^\alpha} \right\|_{L^p(dx)}^p \frac{dh}{|h|} \right)^{\frac{1}{p}} \\ & \lesssim \|f\|_{\gamma, p, p} \|\Pi\|_p, \end{aligned}$$

where we used Jensen's inequality and Hölder's inequality in the same way as in Step 1. Hence, we showed that the $\ell_{n_0}^{\frac{p}{2}}$ -norm of (3.5) is also bounded by $\|f\|_{\gamma, p, p} \|\Pi\|_p$, as claimed. \square

With these two lemmas at hand, we are in a position to prove Theorem 3.8, which ensures the existence of a Sobolev rough path lift above a Sobolev path.

Proof of Theorem 3.8. Without loss of generality we set $T = 1$ keeping in mind that there is a smooth transformation between $[0, T]$ and $[0, 1]$. Moreover, we set $f_{s,t} := f_t - f_s$ for the increment of a function $f: [0, T] \rightarrow \mathbb{R}^d$, where $s, t \in [0, T]$.

In view of Lemma 3.9 and Lemma 3.10, there exists a distribution $\dot{Z} := \mathcal{R}\dot{Z} \in W_p^{\alpha-1}$ such that

$$\left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - \Pi_x \dot{Z}(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^\lambda} \lesssim 1.$$

Since $\Pi_x \dot{Z}(x) = Y_x \dot{W}$ for all $x \in \mathbb{R}$, it holds that

$$\left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - Y_x \dot{W}, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^\lambda} \lesssim 1.$$

Then by Fubini's theorem we obtain that

$$\begin{aligned} (3.6) \quad & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - Y_x \dot{W}, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^\lambda}^{\frac{p}{2}} = \int_0^1 \int_{\mathbb{R}} \left(\sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - Y_x \dot{W}, \eta_x^\lambda \rangle|}{\lambda^\gamma} \right)^{\frac{p}{2}} dx \frac{d\lambda}{\lambda} \\ & = \int_{\mathbb{R}} \int_0^1 \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - Y_x \dot{W}, \eta_x^\lambda \rangle|^{\frac{p}{2}}}{\lambda^{\frac{p}{2}+1}} d\lambda dx \\ & = \int_{\mathbb{R}} \int_x^{x+1} \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - Y_x \dot{W}, \eta_x^{(y-x)} \rangle|^{\frac{p}{2}}}{(y-x)^{\frac{p}{2}+1}} dy dx, \end{aligned}$$

where in the last equality we used the change-of-variable $\lambda = y - x$ for every $x \in \mathbb{R}$. Now we choose $\eta := \mathbf{1}_{[0,1]}$ such that $\eta_x^{(y-x)} = \frac{1}{y-x} \mathbf{1}_{[x,y]}$ for $y \in (x, x+1]$, and then follow the same arguments in the relevant proof of [Bra19, Theorem 4.6] (more precisely, a straight forward calculation reveals that [Bra19, Lemma 3.10] remains valid in the current Sobolev setup, from which one can easily establish the following bound for indicator function) to show that the bound (3.6) remains valid for this indicator function and consequently

$$\int_{\mathbb{R}} \int_x^{x+1} \frac{|\langle \dot{Z} - Y_x \dot{W}, \mathbf{1}_{[x,y]} \rangle|^{\frac{p}{2}}}{(y-x)^{\frac{p}{2}+1}} dy dx \lesssim \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \dot{Z} - \Pi_x \dot{Z}(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(dx)} \right\|_{L^\lambda}^{\frac{p}{2}} \lesssim 1.$$

Since $\dot{Z} \in W_p^{\alpha-1}$, the primitive Z of \dot{Z} , which is a distribution in W_p^α , is continuous due to the classical embedding theorem. Hence, we can immediately check that

$$\langle \dot{Z} - Y_x \dot{W}, \mathbf{1}_{[x,y]} \rangle = Z_{x,y} - Y_x W_{x,y}$$

(by approximation, of course) and conclude

$$(3.7) \quad \int_0^1 \int_x^1 \frac{|Z_{x,y} - Y_x W_{x,y}|^{\frac{p}{2}}}{(y-x)^{\alpha p + 1}} dy dx \lesssim 1.$$

Now we define $\mathbb{X}_{x,y}^{1,2} := Z_{x,y} - Y_x W_{x,y}$ on $\Delta := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$, Estimate (3.7) can be written as

$$\iint_{\Delta} \frac{|\mathbb{X}_{x,y}^{1,2}|^{\frac{p}{2}}}{(y-x)^{\alpha p + 1}} dy dx \lesssim 1.$$

Similarly, we can obtain the same bound for $\mathbb{X}_{x,y}^{2,1} := Z_{x,y} - W_x Y_{x,y}$, where now Z denotes the primitive of $\mathcal{R}\dot{Z}$ obtained from Lemma 3.9 for the same regularity structure as before with \dot{W} replaced by \dot{Y} such that $\Pi_x \dot{Y} = \dot{Y}$ and $\dot{Z}(x) := W_x \dot{Y}$. The notations $\mathbb{X}_{x,y}^{1,1} = Z_{x,y} - Y_x Y_{x,y}$

and $\mathbb{X}_{x,y}^{2,2} = Z_{x,y} - W_x W_{x,y}$ are then self-explanatory (although we use Z to denote different functions). Let $\mathbb{X}_{x,y} := \mathbb{X}_{x,y}^{i,j}$ for $x, y \in I$ and $i, j = 1, 2$, then Bound (3.7) guarantees that

$$(3.8) \quad \int_0^1 \int_0^1 \frac{|X_{x,y}|^p + |\mathbb{X}_{x,y}|^{\frac{p}{2}}}{|y-x|^{\alpha p+1}} dy dx \lesssim 1.$$

Moreover, we can immediately check that \mathbb{X} satisfies Chen's relation by construction. Now, we define $F = (F^{i,j})_{i,j=1,2}$, which is a continuous paths taking value in $\mathbb{R}^2 \otimes \mathbb{R}^2$ such that $F_x^{i,j} = F_x^{j,i} = \frac{1}{2}(X_{0,x}^i X_{0,x}^j - \mathbb{X}_{0,x}^{i,j} - \mathbb{X}_{0,x}^{j,i})$ for $x \in I$ and $i, j = 1, 2$ with $X^1 = Y$ and $X^2 = W$. Then it is easy to check that $F_{x,y}^{i,j} = \frac{1}{2}(X_{x,y}^i X_{x,y}^j - \mathbb{X}_{x,y}^{i,j} - \mathbb{X}_{x,y}^{j,i})$ for any $(x, y) \in \Delta$ and $\mathbf{X}_x := (X_{0,x}, \mathbb{X}_{0,x} + F_x)$ takes value in $G^2(\mathbb{R}^2)$ and

$$(3.9) \quad \int_0^1 \int_x^1 \frac{d_{cc}(\mathbf{X}_x, \mathbf{X}_y)^p}{(y-x)^{\alpha p+1}} dy dx \lesssim 1,$$

which indeed means that $\mathbf{X} \in W_p^\alpha([0, T]; G^2(\mathbb{R}^2))$. \square

Remark 3.11. *In the proof of Theorem 3.8 we have seen that the new Bound (3.1) was essential to obtain the Sobolev regularity of the rough path lift, see (3.9). This would not have been possible with the original bounds (cf. [HL17, Theorem 3.1] and [LPT21b, Theorem 2.11]) of the reconstruction operator relying on (standard) modelled distributions with Hölder bounds, which read in our case as*

$$\left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \mathcal{R}\dot{Z} - \Pi_x \dot{Z}(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^p(dx)} \right\|_{L_\lambda^p} \lesssim \|\Pi\|_p (1 + \|\Gamma\|_p) \left\| \dot{Z} \right\|_{\gamma, p, p}.$$

This bound leads only to the regularity estimate

$$(3.10) \quad \int_0^1 \int_0^1 \frac{|X_{x,y}|^p + |\mathbb{X}_{x,y}|^p}{|y-x|^{\alpha p+1}} dy dx \lesssim 1$$

and not to the required Estimate (3.8). Note, while the Estimate (3.10) gives the "right" regularity parameter of the second order term \mathbb{X} , the integrability parameter is not the required one (here: p instead of $p/2$).

Remark 3.12. *As we have seen, lifting a path to a rough path based on Hairer's theory of regularity theory requires essentially the reconstruction operator for modelled distributions with negative regularity, which leads to the expected non-uniqueness of the rough path lift, cf. [Hai14, CZ20, LPT21b].*

3.3. Continuity of the rough path lifting map. Let us conclude by showing that the method used to construct rough paths via Hairer's reconstruction theorem actually provides a *continuous* way to lift \mathbb{R}^d -valued Sobolev paths to Sobolev rough paths of the same regularity.

For this purpose the distance between two elements \mathbf{X}^1 and \mathbf{X}^2 in $W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ will be measured with respect to the inhomogeneous Sobolev metric. The *inhomogeneous Sobolev metric* $\rho_{W_p^\alpha}$ is defined by

$$\rho_{W_p^\alpha}(\mathbf{X}^1, \mathbf{X}^2) := \sum_{k=1,2} \rho_{W_p^\alpha}^{(k)}(\mathbf{X}^1, \mathbf{X}^2)$$

and for each k ,

$$\rho_{W_p^\alpha}^{(k)}(\mathbf{X}^1, \mathbf{X}^2) := \left(\int_0^T \int_0^T \frac{|\pi_k(\mathbf{X}_{s,t}^1 - \mathbf{X}_{s,t}^2)|^{p/k}}{|t-s|^{\alpha p+1}} ds dt \right)^{k/p}.$$

The inhomogeneous metrics play an important role in the theory of rough differential equations as, for instance, the Itô–Lyons map is continuous with respect to inhomogeneous metrics, cf. [LPT21a]. For a general discussion of inhomogeneous norms and distances in the rough path theory we refer to [FV10, Chapter 8].

The next theorem is a generalization of [FH14, Proposition 13.23] (see also [Bra19, Theorem 4.6]) from Hölder spaces to Sobolev spaces.

Theorem 3.13. *Let $\alpha \in (1/3, 1/2)$ and $p \in (1, +\infty]$ be such that $\alpha > 1/p$. Then, there exists a map*

$$L: W_p^\alpha([0, T]; \mathbb{R}^d) \rightarrow W_p^\alpha([0, T]; G^2(\mathbb{R}^d)), \quad \text{via } X \mapsto L(X) =: \mathbf{X},$$

such that \mathbf{X} is a Sobolev rough path lift of X and L is locally Lipschitz continuous with respect to the inhomogeneous Sobolev metric $\rho_{W_p^\alpha}$.

Proof. Without loss of generality we again assume that $d = 2$ and $T = 1$. Throughout the whole proof, we fix a wavelet analysis with father wavelet φ and mother wavelet ψ in \mathcal{C}_0^r with $r > |\alpha - 1 - \frac{1}{p}|$, which satisfy the desired properties (1)–(5) for wavelet analysis introduced in Subsection 3.2.

First, let us briefly summarize how to get a rough path lift by using Theorem 3.8: Let $X = (Y, W) \in W_p^\alpha([0, T]; \mathbb{R}^d)$ be given. As we have shown in the proof of Theorem 3.8, if we apply the Sobolev model introduced in Example 3.7 and define $f(t) := Y_t \dot{W}$, then it holds that $f \in \mathcal{D}_{p,p}^\gamma$ with $\gamma = 2\alpha - 1$, and the distribution $\mathcal{R}f \in W_p^{\alpha-1}$ defined as in (3.2) satisfies Bound (3.1). Furthermore, let $Z \in W_p^\alpha$ be the primitive of $\mathcal{R}f$, then $\mathbb{X}_{s,t}^{1,2} := Z_{s,t} - Y_s W_{s,t}$ for $s, t \in [0, 1]$ satisfies that

$$\int_0^1 \int_0^1 \frac{|\mathbb{X}_{s,t}^{1,2}|^{p/2}}{|t-s|^{\alpha p+1}} ds dt \lesssim 1.$$

Using the same way we can obtain other components $\mathbb{X}^{1,1}$, $\mathbb{X}^{2,1}$ and $\mathbb{X}^{2,2}$ such that $\mathbf{X}_t := (X_{0,t}, \mathbb{X}_{0,t} + F_t)$ is a rough path in $W_p^\alpha(G^2(\mathbb{R}^2))$ over X , where $F_t^{i,j} = F_t^{j,i} = \frac{1}{2}(X_{0,t}^i X_{0,t}^j - \mathbb{X}_{0,t}^{i,j} - \mathbb{X}_{0,t}^{j,i})$ for $t \in [0, 1]$ and $i, j = 1, 2$ with $X^1 = Y$ and $X^2 = W$.

Now we set $L(X) := \mathbf{X}$ and thus the map $L: W_p^\alpha([0, T]; \mathbb{R}^d) \rightarrow W_p^\alpha([0, T]; G^2(\mathbb{R}^d))$ is well-defined. It only remains to show that L is locally Lipschitz continuous with respect to the metric $\rho_{W_p^\alpha}$.

Step 1: Fix an $X = (Y, W)$ in $W_p^\alpha([0, T]; \mathbb{R}^2)$ and let $\tilde{X} = (\tilde{Y}, \tilde{W})$ be another element in $W_p^\alpha(\mathbb{R}^2)$. Let $\dot{\tilde{W}}$ be the derivative of \tilde{W} . We define a Sobolev model $(\tilde{\Pi}, \tilde{\Gamma})$ for the regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ given in Example 3.7 as following:

$$\tilde{\Pi}_t(\dot{\tilde{W}}) := \dot{\tilde{W}}, \quad \tilde{\Pi}_t(\mathbf{1}) := 1 \in \mathbb{R},$$

and $\tilde{\Gamma}_{s,t} = \text{Id}_{\mathcal{T}}$ for all $s, t \in \mathbb{R}$. Note that $(\tilde{\Pi}, \tilde{\Gamma})$ is the model used for constructing rough path lift over \tilde{X} . Hence, by defining $g(t) := \tilde{Y}_t \dot{\tilde{W}}$, we have $g \in \tilde{\mathcal{D}}_{p,p}^\gamma$, where $\tilde{\mathcal{D}}_{p,p}^\gamma$ is the space of modelled distributions associated to $(\tilde{\Pi}, \tilde{\Gamma})$. Then, as we stated above, if $\tilde{\mathcal{R}}g$ is defined as in (3.2) by changing f to g , Π to $\tilde{\Pi}$ and using the same wavelet basis, its primitive $\tilde{Z} \in W_p^\alpha$ satisfies that $\tilde{Z}_{s,t} - \tilde{Y}_s \tilde{W}_{s,t} = \tilde{\mathbb{X}}_{s,t}^{1,2}$, where $\tilde{\mathbb{X}} = (\tilde{\mathbb{X}}^{i,j})_{i,j=1,2}$ is the second level component of $L(\tilde{X}) = \tilde{\mathbf{X}}$ up to an addition of the function \tilde{F} which is the counterpart of the function F defined as above with \tilde{X} replacing X .

Step 2: Next we will show that

$$(3.11) \quad \left(\int_0^1 \int_0^1 \frac{|\mathbb{X}_{s,t}^{1,2} - \tilde{\mathbb{X}}_{s,t}^{1,2}|^{p/2}}{|t-s|^{\alpha p+1}} ds dt \right)^{\frac{2}{p}} \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha}.$$

To this end, first of all we note that in view of the definitions of $\mathcal{R}f$ and $\tilde{\mathcal{R}}g$ (see (3.2)), for every $n \geq 0$, $x \in \mathbb{R}$ and $y \in \Lambda_n$, it holds that

$$\begin{aligned} & \langle \mathcal{R}f - \tilde{\mathcal{R}}g - \Pi_x f(x) + \tilde{\Pi}_x g(x), \psi_y^n \rangle \\ &= \int_{z \in B(y, 2^{-n})} 2^n \langle \Pi_y(f(z) - f(x) - g(z) + g(x)), \psi_y^n \rangle dz \\ & \quad + \int_{z \in B(y, 2^{-n})} 2^n \langle (\Pi_y - \tilde{\Pi}_y)(g(z) - g(x)), \psi_y^n \rangle dz. \end{aligned}$$

Then, by the definitions of the models (Π, Γ) and $(\tilde{\Pi}, \tilde{\Gamma})$ as well as the constructions of the modelled distributions f and g , we have

$$\Pi_y(f(z) - f(x) - g(z) + g(x)) = (Y_z - Y_x - \tilde{Y}_z + \tilde{Y}_x) \dot{W}$$

and

$$(\Pi_y - \tilde{\Pi}_y)(g(z) - g(x)) = (\tilde{Y}_z - \tilde{Y}_x)(\dot{W} - \dot{\tilde{W}}).$$

Hence, we obtain that

$$\begin{aligned} \langle \mathcal{R}f - \tilde{\mathcal{R}}g - \Pi_x f(x) + \tilde{\Pi}_x g(x), \psi_y^n \rangle &= \int_{z \in B(y, 2^{-n})} 2^n \langle (Y_z - Y_x - \tilde{Y}_z + \tilde{Y}_x) \dot{W}, \psi_y^n \rangle dz \\ & \quad + \int_{z \in B(y, 2^{-n})} 2^n \langle (\tilde{Y}_z - \tilde{Y}_x)(\dot{W} - \dot{\tilde{W}}), \psi_y^n \rangle dz. \end{aligned}$$

Then, following the arguments used in the proof of Lemma 3.10 we can derive that

$$\begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \mathcal{R}f - \tilde{\mathcal{R}}g - \Pi_x f(x) + \tilde{\Pi}_x g(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(\mathrm{d}x)} \right\|_{L^{\frac{p}{\lambda}}} \\ & \lesssim \|\Pi\|_p \|f - g\|_{\gamma, p, p} + \|\Pi - \tilde{\Pi}\|_p \|g\|_{\gamma, p, p}. \end{aligned}$$

Since $\|\Pi\|_p = \|\dot{W}\|_{W_p^{\alpha-1}}$, $\|\Pi - \tilde{\Pi}\|_p = \|\dot{W} - \dot{\tilde{W}}\|_{W_p^{\alpha-1}}$, $\|g\|_{\gamma, p, p} \lesssim \|\tilde{Y}\|_{W_p^\alpha}$ and $\|f - g\|_{\gamma, p, p} \lesssim \|Y - \tilde{Y}\|_{W_p^\alpha}$, the above inequality can be written as

$$(3.12) \quad \begin{aligned} & \left\| \left\| \sup_{\eta \in \mathcal{B}^r} \frac{|\langle \mathcal{R}f - \tilde{\mathcal{R}}g - \Pi_x f(x) + \tilde{\Pi}_x g(x), \eta_x^\lambda \rangle|}{\lambda^\gamma} \right\|_{L^{\frac{p}{2}}(\mathrm{d}x)} \right\|_{L^{\frac{p}{\lambda}}} \\ & \lesssim \|\dot{W}\|_{W_p^{\alpha-1}} \|Y - \tilde{Y}\|_{W_p^\alpha} + \|\dot{W} - \dot{\tilde{W}}\|_{W_p^{\alpha-1}} \|\tilde{Y}\|_{W_p^\alpha} \\ & \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha}, \end{aligned}$$

where in the third line we used [Tri10, Theorem 2.3.8].

Now, invoking that $\mathcal{R}f - \tilde{\mathcal{R}}g - \Pi_x f(x) + \tilde{\Pi}_x g(x) = \dot{Z} - Y_x \dot{W} - (\dot{\tilde{Z}} - \tilde{Y}_x \dot{\tilde{W}})$, we can apply the same argument as for establishing (3.7) to the Estimate (3.12) to get that

$$\left(\int_0^1 \int_0^1 \frac{|Z_{s,t} - Y_s W_{s,t} - (\tilde{Z}_{s,t} - \tilde{Y}_s \tilde{W}_{s,t})|^{\frac{p}{2}}}{|t-s|^{\alpha p+1}} ds dt \right)^{\frac{2}{p}} \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha}.$$

Since $Z_{s,t} - Y_s W_{s,t} = \mathbb{X}_{s,t}^{1,2}$ and $\tilde{Z}_{s,t} - \tilde{Y}_s \tilde{W}_{s,t} = \tilde{\mathbb{X}}_{s,t}$, Estimate (3.11) has been established.

Step 3: The estimate from Step 2 gives that

$$\left(\int_0^1 \int_0^1 \frac{|\mathbb{X}_{s,t} - \tilde{\mathbb{X}}_{s,t}|^{p/2}}{|t-s|^{\alpha p+1}} ds dt \right)^{\frac{2}{p}} \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha},$$

which in turn implies that the same bound also holds true for $\left(\int_0^1 \int_0^1 \frac{|F_{s,t} - \tilde{F}_{s,t}|^{p/2}}{|t-s|^{\alpha p+1}} ds dt \right)^{\frac{2}{p}}$ by invoking the definitions of F and \tilde{F} . Hence, noting that

$$\pi_2(\mathbf{X}_{s,t} - \tilde{\mathbf{X}}_{s,t}) = \mathbb{X}_{s,t} - \tilde{\mathbb{X}}_{s,t} + F_{s,t} - \tilde{F}_{s,t}$$

we can deduce that

$$\rho_{W_p^\alpha}^{(2)}(L(X), L(\tilde{X})) = \left(\int_0^1 \int_0^1 \frac{|\mathbb{X}_{s,t} - \tilde{\mathbb{X}}_{s,t} - F_{s,t} + \tilde{F}_{s,t}|^{p/2}}{|t-s|^{\alpha p+1}} ds dt \right)^{\frac{2}{p}} \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha}.$$

Since $\rho_{W_p^\alpha}^{(1)}(L(X), L(\tilde{X})) = \|X - \tilde{X}\|_{W_p^\alpha}$, we finally obtain that

$$\rho_{W_p^\alpha}(L(X), L(\tilde{X})) \lesssim_{X, \tilde{X}} \|X - \tilde{X}\|_{W_p^\alpha}.$$

□

Remark 3.14. *While the proofs of Lemma 3.9 and 3.10 and Theorem 3.13 contain basically all the necessary ideas to prove the reconstruction theorem for modelled distributions of negative Sobolev regularity and Sobolev models (cf. [Hai14, Theorem 3.10] and [LPT21b, Theorem 2.11]), we decided not to set up the general theorem for two reasons: to prove Lyons–Victoir extension theorem is currently its main application and other applications might require a different definition of Sobolev type models, cf. Remark 3.4.*

On the other hand, one can of course prove the above reconstruction theorem without using the notations and notions from regularity structure. Here we still decide to illustrate the proof in terms of the language of regularity structure mainly because of a pedagogical purpose: as the construction of rough path lift over Hölder continuous paths via regularity structure is well-known, we believe it is easy for those readers who are familiar with regularity structure theory to understand our approach and realize the difference between the classical Hölder case and the current Sobolev setting.

Remark 3.15. *In general the rough path lift obtained in Theorem 3.13 does not coincide with the rough path lift defined by Riemann–Stieltjes integration even in the case of sufficiently regular \mathbb{R}^d -valued paths. This is due to the continuity assertion in the Theorem: if the lift coincided on piecewise affine curves (which do have Sobolev regularity) with the standard lift defined by Riemann–Stieltjes integration, then – by continuity – it would coincide on limits of such curves. This would yield in particular a rough path lift of Hölder curves of order $\alpha < 1/2$ continuous with respect to the Hölder norm and extending classical lifts, which is known to be impossible. However, there exists a class of rough differential equations where the solutions depend only on the driving \mathbb{R}^d -valued paths and not on their rough path lifts, see [LV07, Section 6]. For these rough differential equations the associated Itô–Lyons map depends, in a meaningful way, continuously only on the \mathbb{R}^d -valued driving paths due to continuous lifting map L provided in Theorem 3.13.*

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