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LIMIT BEHAVIOUR OF RANDOM WALKS ON $\mathbb{Z}^m$ WITH TWO-SIDED MEMBRANE

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Abstract. We study Markov chains on $\mathbb{Z}^m$, $m \geq 2$, that behave like a standard symmetric random walk outside of the hyperplane (membrane) $H = \{0\} \times \mathbb{Z}^{m-1}$. The exit probabilities from the membrane (penetration probabilities) H are periodic and also depend on the incoming direction to H, what makes the membrane H two-sided. Moreover, sliding along the membrane is allowed. We show that the natural scaling limit of such Markov chains is a m-dimensional diffusion whose first coordinate is a skew Brownian motion and the other $m-1$ coordinates is a Brownian motion with a singular drift controlled by the local time of the first coordinate at 0. In the proof we utilize a martingale characterization of the Walsh Brownian motion and determine the effective permeability and slide direction. Eventually, a similar convergence theorem is established for the one-sided membrane without slides and random iid penetration probabilities.

Résumé. ...

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1. Introduction

A multidimensional Brownian motion is a fundamental stochastic process that describes an idealized mathematical model of a free physical diffusion in a homogeneous medium. Having in mind the observations of pollen made by Brown or a later kinetic theory of gases, we can interpret a diffusion as a collective motion of independent random walkers whose distribution density in space obeys the isotropic Gaussian distribution.

However in real physical or biological systems, the space is often separated into compartments by *membranes* that impede or facilitate the passage of the walker and create an anisotropy in the walkers' collective motion. From the physical point of view, a membrane is a thin slice of a material whose physical properties (e.g. diffusivity, permeability) are different from the properties of the environment. Diffusions through membranes

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are observed in biological tissues where they control the transport of ions, water molecules and gases, or in porous and composite materials.

A rigorous mathematical justification of the interpretation of a diffusion as a limit of scaled random walks is given by the Functional Central Limit Theorem (FCLT, the Donsker–Prokhorov invariance principle). For $m \geq 1$, let $\{\mathrm{e}_1, \dots, \mathrm{e}_m\}$ be a standard basis in $\mathbb{R}^m$. Consider a symmetric random walk $Z = (Z(n))_{n\geq 0}$ on $\mathbb{Z}^m$ defined by identical one-step transition probabilities

$$\mathbf{P}\Big(Z(n+1) - Z(n) = \pm \mathrm{e}_k \Big| Z(0), \dots, Z(n)\Big) = \frac{1}{2m}, \quad k = 1, \dots m, \quad n \geq 0.$$

Then the FCLT yields the weak convergence in the uniform topology

$$\Big(\frac{Z([nt])}{\sqrt{n}}\Big)_{t\geq 0} \Rightarrow \Big(\frac{1}{\sqrt{m}} W(t)\Big)_{t\geq 0},$$

where W is a standard m-dimensional Brownian motion.

In this paper we consider a novel class of scaling limits of random walks in the presence of extended spatially non-homogeneous one- or two-sided barrier (membrane, interface). More precisely, in the m-dimensional space we consider an $(m-1)$-dimensional semi-permeable hyperplane that separates the space into two half-space compartments. A random walker performs a symmetric random walk outside of the membrane, however the probability of the passage through the membrane is determined by the hitting position of the membrane by the walker and by its incoming direction. Thus the probability to penetrate the membrane depends of the fact whether the walker has reached it from the "right" of from the "left". Furthermore we will consider two models in which the membrane has a spatially regular periodic or random structure.

In the first model, the penetration probabilities are periodic in space, so that the membrane reminds of a two-sided fabric. Hence, the scaled limit of such Markov chains outside of the membrane will be just a Brownian motion. The passage probability through the membrane will be obtained with the help of an appropriate averaging of the periodic individual penetration probabilities. Hence the limit process in the direction perpendicular to the membrane will be a skew Brownian motion. The other coordinates converge to a standard Brownian motion, maybe, with a singular drift controlled by the local time of the first coordinate at the origin.

Another case of a spatially regular structure considered in this paper is a membrane imitating a "random" perforated surface. In this case one can think of a random walk in a random environment: we assume that the membrane is one-sided and that the random penetration probabilities at each point are iid random variables with values in $[0, 1]$ with the mean value $\bar{p}$. As in the first model, the (quenched) scaled limit will be a skew Brownian motion in the direction perpendicular to the membrane and a standard Brownian motion in the rest $(m - 1)$ coordinates.

The proofs of these results are purely probabilistic and employ methods of homogenization and dynamics of singular stochastic differential equations. Our results give a path-wise picture of the diffusion through a two-side semi-permeable interface. In the physical language this corresponds to the Langevin–Smoluchowsky approach to diffusions. It should be emphasized that physical papers (see, e.g., Novikov et al. [24], Grebenkov et al. [7], Moutal and Grebenkov [21] and references therein) devoted to similar problems use analytical methods, mainly the analysis of the Fokker–Planck equation.

One-dimensional locally perturbed random walks were considered from different points of view in Harrison and Shepp [10], Minlos and Zhizhina [20], Pilipenko and Pryhod'ko [29], Pilipenko and Prikhod'ko [32], Pilipenko and Sakhanenko [30], Ngo and Paigné [23]. In this paper, we use the multidimensional martingale characterization approach previously considered in Iksanov and Pilipenko [11] in dimension one.

It should be noted that if transition probabilities of a multidimensional random walk are perturbed on a finite set or on a hyperplane of co-dimension 2, then under some natural assumptions its scaling limit is a Brownian motion, see Szász and Telcs [36], Yarotskii [38], and Paulin and Szász [25].

We also refer the reader's attention to the following related mathematical works. In the monograph by Portenko [34], the theory of diffusion processes with semipermeable membranes was developed. The research papers by Lejay [18] and Mandrekar and Pilipenko [19] considered thin layer perturbation of a Brownian motion that results in a Brownian motion with the so-called non-instantaneous "hard membrane". In papers Pilipenko and Khomenko [31], Pilipenko [28], the perturbation of transition probabilities of a random walk depended on the number of visits of the walk's current state. As a result, Donsker's scaling limit process was obtained as a solution to a stochastic equation with drift depending on a process's local time. Iksanov et al. [12] and Pilipenko and Prykhodko [33] considered one-dimensional random walks on a half line with reflection to the upper half line upon crossing zero. It was assumed that bounces off zero belong to a domain of attraction of a stable law. The limit process for scaled processes was a reflected Brownian motion with a non-local Feller–Wentzell boundary condition at zero.

The paper is organized as follows. Section 2 contains the setting and the main result for the model of two-sided periodic membrane. In Section 3 we introduce the Walsh Brownian motion that will be used in the proofs and give its convenient realization as a d-dimensional stochastic process that takes values on the positive coordinate half-axes. We will also formulate two martingale characterizations of the Walsh Brownian motion. Section 4 is devoted to the proof of Theorem 2.1. In the final Section 5 we show how our method can be applied to a model of a random one-sided membrane that possesses some ergodic properties.

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Notation. The weak convergence in the Skorokhod space $D([0,\infty),\mathbb{R}^m;J_1)$ is denoted by $\Rightarrow$. It should be noted, however, that all limit processes in this paper are continuous. The convergence in distribution of random variables is denoted by $\stackrel{\mathrm{d}}{\to}$.

2. Two-sided periodic membrane: the model and the main result

Let $m\geq 2$ and let $\{\mathrm{e}_1,\dots,\mathrm{e}_m\}$ be a standard basis in $\mathbb{R}^m$. Consider a Markov chain $Z=\{Z(n)\}_{n\geq 0}$ on $\mathbb{Z}^m$ that behaves as a simple random walk outside of the hyperplane $H:=\{0\}\times\mathbb{Z}^{m-1}$, i.e., for all $k=1,\dots,m$

$$\mathbf{P}\Big(Z(n+1)=z\pm \mathrm{e}_k\Big|Z(n)=z\Big)=\frac{1}{2m},\ z\notin H. \tag{2.1}$$

For each $n\geq 1$, we denote the first coordinate of the process Z by X, and the other $(m-1)$ coordinates by Y so that $Z=(X,Y)$.

We will interpret H as a semipermeable two-sided membrane that may let a particle pass from one half-space to another with certain probabilities that can depend on the crossing direction. Moreover, the particle can "slide" along the membrane.

The membrane has to be homogeneous, i.e., the transition probabilities from the membrane are periodic in space.

Notice that if the membrane is two-sided, then Z is not a Markov chain, generally. Indeed, its position upon leaving the membrane is determined by both the current location on the membrane and by the particle's incoming direction. Hence, in order to introduce a Markov structure we have to enlarge the state space by splitting the membrane H into two parts $\mathcal{H}^-$ and $\mathcal{H}^+$ corresponding to its "right" and "left" sides: we denote

$$\mathcal{H}^-:=\{-0\}\times\mathbb{Z}^{m-1},\quad \mathcal{H}^+:=\{+0\}\times\mathbb{Z}^{m-1},$$

and we set $\mathcal{H}:=\mathcal{H}^-\cup\mathcal{H}^+$.

Now we consider a Markov chain $\mathcal{Z}=(\mathcal{Z}(n))_{n\geq 0}=(\mathcal{X}(n),Y(n))_{n\geq 0}$ on the state space

$$\{\pm 0,\pm 1,\pm 2,\dots\}\times\mathbb{Z}^{m-1}.$$

Its second coordinate Y is the $(m-1)$-dimensional process from the original Markov chain Z whereas the process $\mathcal{X}$ is defined on the enlarged space $\{\pm 0, \pm 1, \pm 2, \dots\}$. The transition probabilities of the Markov chain $\mathcal{Z} = (\mathcal{X}, Y)$ are defined as follows.

Outside of the two-sided membrane $\mathcal{H}$ the satisfy (2.1), namely

$$\mathbf{P}\Big(\mathcal{Z}(n+1) = z \pm \mathrm{e}_k \,\Big|\, \mathcal{Z}(n) = z\Big) = \frac{1}{2m}, \ z \notin \mathcal{H},$$

where we agree that for each $y \in \mathbb{Z}^{m-1}$

$$\begin{aligned} (1,y) - \mathrm{e}_1 = (+0,y) \quad &\text{and} \quad (-1,y) + \mathrm{e}_1 = (-0,y), \\ (+0,y) + \mathrm{e}_1 = (1,y) \quad &\text{and} \quad (-0,y) + \mathrm{e}_1 = (1,y), \\ (+0,y) - \mathrm{e}_1 = (-1,y) \quad &\text{and} \quad (-0,y) - \mathrm{e}_1 = (-1,y). \end{aligned}$$

In other words, away from $\mathcal{H}$ the Markov chain $\mathcal{Z}$ is a symmetric random walk.

Suppose also that the process $\mathcal{Z}$ does not stay on the membrane, i.e.,

$$\mathbf{P}\Big(\mathcal{Z}(n+1) \in \{-1,1\} \times \mathbb{Z}^{m-1} \,\Big|\, \mathcal{Z}(n) \in \mathcal{H}\Big) = 1.$$

However it is allowed that upon hitting the membrane (from the "left" or from the "right") the walker can "slide" along the membrane, so that its y-coordinate changes. We assume that transition probabilities from $\mathcal{H}$ have periodic structure.

Notation. Let $\mathbf{k} = (k_2, \dots, k_m)$ be a fixed $(m-1)$-tuple of natural numbers $k_2, \dots, k_m \geq 1$. Let U denote the box in $\mathbb{Z}^{m-1}$ defined by

$$U := [0, k_2 - 1] \times \cdots \times [0, k_m - 1].$$

For $y \in \mathbb{Z}^{m-1}$, $y = (y_2, \dots, y_m)$ we denote

$$y \,(\mathrm{mod}\, \mathbf{k}) := (y_2 \,(\mathrm{mod}\, k_2), \dots, y_m \,(\mathrm{mod}\, k_m)) \in U.$$

For each $j \in U$ we set

$$\mathcal{H}_j^{\pm} := \{(\pm 0, y) \in \mathcal{H}^{\pm} \colon y \,(\mathrm{mod}\, \mathbf{k}) = j\}.$$

Clearly,

$$\mathcal{H}^{\pm} = \bigcup_{j \in U} \mathcal{H}_j^{\pm}.$$

The following are our key assumptions concerning the transition probabilities of the random walk in the membrane.

$\mathbf{A}_{\text{periodic}}$. Periodicity of the transition probabilities. We assume that there exist $k_2, \dots, k_m \geq 1$ such that for all $l_2, \dots, l_m \in \mathbb{Z}$, for all $z_0 \in \mathcal{H}$, for all $z_1 \in \{-1, +1\} \times \mathbb{Z}^{m-1}$ we have

$$\begin{aligned} &\mathbf{P}\Big(\mathcal{Z}(n+1) = z_1 \,\Big|\, \mathcal{Z}(n) = z_0\Big) \\ &\qquad = \mathbf{P}\Big(\mathcal{Z}(n+1) = z_1 + k_2 l_2 \mathrm{e}_2 + \cdots + k_m l_m \mathrm{e}_m \,\Big|\, \mathcal{Z}(n) = z_0 + k_2 l_2 \mathrm{e}_2 + \cdots + k_m l_m \mathrm{e}_m\Big). \end{aligned} \tag{2.2}$$

$\mathbf{A}_\gamma$. To describe the transitions through the membrane, we denote by $0 \leq \tau_0 < \tau_1 < \cdots$ the successive arrivals of $\mathcal{Z}$ to $\mathcal{H}$ or, equivalently, of $\mathcal{X}$ to $\{-0, +0\}$.

On the finite state space $\{-0, +0\} \times U$ we consider an auxiliary embedded process $\hat{\mathcal{Z}} = (\hat{\mathcal{Z}}(n))_{n \geq 0}$ as follows defined by

$$\hat{\mathcal{Z}}(n) = \Big(\mathcal{X}(\tau_n), Y(\tau_n) \,(\mathrm{mod}\, \mathbf{k})\Big) = \begin{cases} (-0, j), & \text{if } Y(\tau_n) \,(\mathrm{mod}\, \mathbf{k}) = j \in U \text{ and } \mathcal{X}(\tau_n) = -0, \\ (+0, j), & \text{if } Y(\tau_n) \,(\mathrm{mod}\, \mathbf{k}) = j \in U \text{ and } \mathcal{X}(\tau_n) = +0, \quad n \geq 0. \end{cases} \tag{2.3}$$

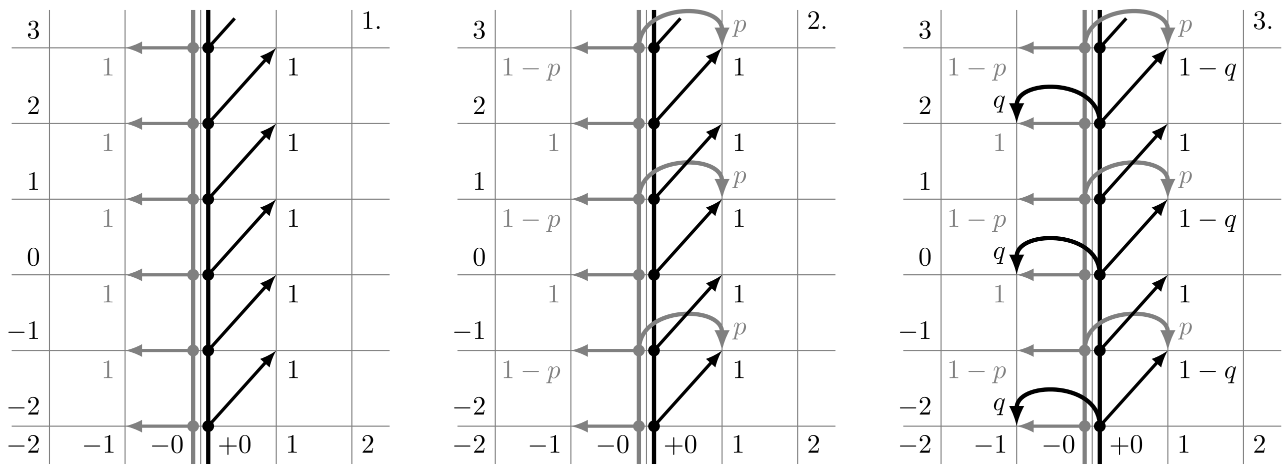

FIGURE 1. Examples of three possible membrane compositions. To guarantee permeability for the membranes 2. and 3. we assume that $p \in (0,1]$ and $p, q \in (0,1]$ respectively.

The process $\hat{\mathcal{Z}}$ is a finite Markov chain.

Since the original Markov chain Z is a symmetric random walk outside of the membrane, the states within the sets $\{-0\} \times U$ and $\{-0\} \times U$ are connected. The connectivity of these sets, however, is determined by the permeability property of the membrane. The examples on Fig. 1 illustrate the three possible situations.

1. The membrane $\mathcal{H}$ is two sided reflecting and no transition between the half-spaces is possible. Consequently the behaviour of $\mathcal{Z}$ on the left-hand space and on the left-hand space can be studied separately. In any case, $\hat{\mathcal{Z}}$ is an irreducible finite Markov chain either on $\{-0\} \times U$ or on $\{+0\} \times U$. For initial values $\mathcal{Z}(0) \in \mathcal{H}_\pm$ it will have invariant (limiting) distributions $\pi = \{\pi_{(\pm 0,j)}\}_{j\in U}$ supported on $\{\pm 0\} \times U$ respectively.

2. The membrane is semi-permeable in one direction. Assume for definiteness that transitions from $\mathcal{H}_-$ into $\{+0, 1, 2, \dots\} \times \mathbb{Z}^{m-1}$ are possible. In this case, all the states $\{-0\} \times U$ are inessential for the Markov chain $\hat{\mathcal{Z}}$ whereas the states $\{+0\} \times U$ form an irreducible class. Hence, $\hat{\mathcal{Z}}$ has a unique stationary distribution

$$\pi = \{\pi_{(+0,j)}\}_{j\in U}$$

supported on $\{+0\} \times U$.

3. Eventually, if the membrane is permeable in both directions there is a unique stationary distribution

$$\pi = \{\pi_{(-0,j)}, \pi_{(+0,j)}\}_{j\in U}$$

of $\hat{\mathcal{Z}}$ on $\{-0, +0\} \times U$.

For the limiting stationary distribution π (maybe depending on the initial value $\mathcal{Z}(0)$) and we introduce the *effective permeability*

$$\gamma := \sum_{j\in U} (\pi_{(+0,j)} - \pi_{(-0,j)}) \in [-1, 1]. \tag{2.4}$$

Note that $\gamma = \pm 1$ corresponding to "reflection" is possible only in cases 1 and 2.

$\mathbf{A}_c$. Finally, we describe the "slides" along the membrane. We denote

$$\alpha_{(\pm 0,j)} := \mathbf{E}\Big[Y(1) - Y(0) \,\Big|\, \mathcal{X}(0) = \pm 0, Y(0) = j\Big] \in \mathbb{R}^{m-1}, \quad j \in U,$$

the mean slide sizes along the "right" of the "left" side of the membrane, and assume that they are finite. Introduce the *effective slide* as

$$c := \mathbf{E}_\pi\Big[Y(1) - Y(0)\Big] = \sum_{j\in U}\Big(\pi_{(+0,j)}\cdot\alpha_{(+0,j)} + \pi_{(-0,j)}\cdot\alpha_{(-0,j)}\Big) \in \mathbb{R}^{m-1}. \tag{2.5}$$

By the strong law of large numbers for Markov chains (see, e.g. [3, Corollary 7.2.10]),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{I}\Big(\hat{\mathcal{Z}}(k) = (\pm 0, j)\Big) = \pi_{(\pm 0,j)} \quad \mathbf{P}_{\mathcal{Z}(0)}\text{-a.s.}$$

By the strong Markov property, it is easy to see that the process $(\mathcal{Z}(\tau_n), \mathcal{Z}(\tau_n+1))_{n\geq 0}$ is also a Markov chain and the strong law of large numbers yields again that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{X}(\tau_k+1) &= \mathbf{E}_\pi\mathcal{X}(1)\\ &= \sum_{j\in U}\Big[\pi_{(+0,j)}\Big(\mathbf{P}_{(+0,j)}(\mathcal{X}(1)=1) - \mathbf{P}_{(+0,j)}(\mathcal{X}(1)=-1)\Big)\\ &\qquad + \pi_{(-0,j)}\Big(\mathbf{P}_{(-0,j)}(\mathcal{X}(1)=1) - \mathbf{P}_{(-0,j)}(\mathcal{X}(1)=-1)\Big)\Big]\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{I}\Big(Y(\tau_k)\equiv j, \mathcal{X}(\tau_k+1)=\pm 1\Big) &= \mathbf{P}_\pi(Y(0)\equiv j, \mathcal{X}(1)=\pm 1)\\ &= \sum_{j\in U}\Big[\pi_{(+0,j)}\Big(\mathbf{P}_{(+0,j)}(\mathcal{X}(1)=1) - \mathbf{P}_{(+0,j)}(\mathcal{X}(1)=-1)\Big)\\ &\qquad + \pi_{(-0,j)}\Big(\mathbf{P}_{(-0,j)}(\mathcal{X}(1)=1) - \mathbf{P}_{(-0,j)}(\mathcal{X}(1)=-1)\Big)\Big].\end{aligned} \tag{2.7}$$

Notice that $\mathcal{X}(\tau_k+1) = 1$ if and only if $\mathcal{X}(\tau_{k+1}) = +0$ and $\mathcal{X}(\tau_k+1) = -1$ if and only if $\mathcal{X}(\tau_{k+1}) = -0$. So, (2.6) yields another representation for the effective permeability:

$$\gamma = \mathbf{E}_\pi\mathcal{X}(1).$$

Finally, an analogous argument yields the representation for the effective slide c:

$$c = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}(Y(\tau_k+1) - Y(\tau_k)) \quad \text{a.s.}$$

Now we are ready to formulate the main result of the paper. Let W_X and W_Y be independent standard Brownian motions in $\mathbb{R}$ and $\mathbb{R}^{m-1}$ respectively.

For $\gamma\in[-1,1]$ defined in (2.4), we consider a skew Brownian motion X^γ that is a unique strong solution of the SDE

$$X^\gamma(t) = \gamma L(t) + W_X(t), \quad t\geq 0. \tag{2.8}$$

where L is the symmetric two-sided local time of X^γ at zero, see Harrison and Shepp [10]. Other characterizations and properties of the skew Brownian motion can be found in the review by Lejay [17].

Furthermore, let

$$Y^c(t) := cL(t) + W_Y(t), \quad t \geq 0, \tag{2.9}$$

i.e., the process Y^c is a $(m-1)$-dimensional Brownian motion that slides in the direction c at the time instants when X^γ touches zero. Note that (2.9) is not a stochastic differential equation because the process L is already determined in (2.8).

For the Markov chain $\mathcal{Z} = (\mathcal{X}, Y)$ define the rescaled continuous time processes X_n and Y_n by

$$X_n(t) := \frac{\mathcal{X}([nt])\mathbb{I}(|\mathcal{X}([nt])| > 0)}{\sqrt{n}}, \quad Y_n(t) := \frac{Y([nt])}{\sqrt{n}}, \quad t \geq 0. \tag{2.10}$$

Note that the process X_n is a rescaled projection of $\mathcal{X}$ on $\mathbb{Z}^m$ with the states ± 0 identified as 0.

Theorem 2.1. *Let $\mathcal{Z}$ be a perturbed random walk on $\{\pm 0, \pm 1, \dots\} \times \mathbb{Z}^{m-1}$ satisfying the preceding assumptions. Then for any initial value $\mathcal{Z}(0)$, the weak convergence holds true:*

$$(X_n, Y_n) \Rightarrow \frac{1}{\sqrt{m}}(X^\gamma, Y^c), \quad n \to \infty,$$

where the processes X^γ and Y^c are defined in (2.8) *and* (2.9)*, and γ and c are defined in* (2.4) *and* (2.5) *respectively.*

The crux of Theorem 2.1 is transparent. Away from the membrane H, the limiting process (X^γ, Y^c) coincides with the m-dimensional Brownian motion $(W_X, W_Y)/\sqrt{m}$. The perturbation of the transition probabilities on the two-sided membrane results in the appearance of a singular drift in the x-direction perpendicular to the membrane. Hence the x-coordinate of the limiting process becomes a skew Brownian motion with the effective permeability parameter $\gamma \in [-1, 1]$. The y-coordinates are perturbed by a singular drift in the effective sliding direction $c \in \mathbb{R}^{m-1}$. This drift equals zero as long as the limiting process stays away from the membrane. However upon hitting the membrane, the limiting process performs a singular "sliding" in the direction c controlled by the local time at zero of the x-coordinate. Note that the two-sided membrane structure disappears in the limit as $n \to \infty$ and the limiting process (X^γ, Y^c) is a continuous Markovian process in $\mathbb{R}^m$.

We illustrate Theorem 2.1 by examples depicted on Fig. 1.

Example 2.2. 1. We have a one-periodic two-sided reflecting membrane. For initial values $\mathcal{Z}(0) \in \{\dots, -1, -0\} \times \mathbb{Z}^{m-1}$ we have convergence

$$(X_n, Y_n) \Rightarrow \frac{1}{\sqrt{2}}\Big(X^{-1}, Y^0\Big) \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{2}}\Big(-|W_X|, W_Y\Big)$$

with $\gamma = -1$ and $c = 0$, whereas for initial values $\mathcal{Z}(0) \in \{+0, +1, \dots\} \times \mathbb{Z}^{m-1}$ we have convergence

$$(X_n, Y_n) \Rightarrow \frac{1}{\sqrt{2}}\Big(X^1, Y^1\Big)$$

with $\gamma = 1$ and $c = 1$. Note that X^{-1} and X^1 are negative and positive reflected Brownian motions.

2. For initial values $\mathcal{Z}(0) \in \{\dots, -1, -0\} \times \mathbb{Z}^{m-1}$, the Markov chain $\mathcal{Z}$ leaves the half space in finite time with probability one, and never returns back. Hence the problem is essentially one-sided and for any initial value $\mathcal{Z}(0) \in \{\pm 0, \pm 1, \dots\} \times \mathbb{Z}^{m-1}$ we have convergence

$$(X_n, Y_n) \Rightarrow \frac{1}{\sqrt{2}}\Big(X^1, Y^1\Big)$$

with $\gamma = 1$ and $c = 1$.

3. In this case we have a two-periodic two-sided permeable membrane. Let

$$\tau = \inf\{k \geq 0\colon \mathcal{Z}(n) = (\mathcal{X}(n), Y(n)) \in \mathcal{H}\}$$

be the first hitting time of the Markov chain $\mathcal{Z}$ of the two-sided membrane $\mathcal{H}$. Since away of the membrane $\mathcal{H}$, the increments of $\mathcal{Z}$ coincide with the increments of a translation invariant two-dimensional symmetric random walk Z, we can easily calculate the probabilities of hitting the membrane in an even or an odd point:

$$\begin{aligned} \alpha &:= \mathbf{P}_{(1,0)}(Y(\tau)\equiv 0) = \mathbf{P}_{(1,1)}(Y(\tau)\equiv 1) = \mathbf{P}_{(-1,0)}(Y(\tau)\equiv 0) = \mathbf{P}_{(-1,1)}(Y(\tau)\equiv 1) = 2-\sqrt{2},\\ 1-\alpha &:= \mathbf{P}_{(1,0)}(Y(\tau)\equiv 1) = \mathbf{P}_{(1,1)}(Y(\tau)\equiv 0) = \mathbf{P}_{(-1,0)}(Y(\tau)\equiv 1) = \mathbf{P}_{(-1,1)}(Y(\tau)\equiv 0) = \sqrt{2}-1. \end{aligned} \tag{2.11}$$

With the help of (2.11) we calculate the transition probabilities of the embedded Markov chain $\hat{\mathcal{Z}}$ on $\{-0,+0\}\times\{0,1\}$:

$$\mathbb{P} = \begin{array}{c|cccc} & (-0,0) & (-0,1) & (+0,0) & (+0,1) \\ \hline (-0,0) & \alpha & 1-\alpha & 0 & 0 \\ (-0,1) & (1-p)(1-\alpha) & (1-p)\alpha & p(1-\alpha) & p\alpha \\ (+0,0) & q\alpha & q(1-\alpha) & (1-q)(1-\alpha) & (1-q)\alpha \\ (+0,1) & 0 & 0 & \alpha & 1-\alpha \end{array}$$

Solving the forward Kolmogorov equation $(\mathbb{P}^T - \mathrm{Id})\pi = 0$ we obtain the stationary law π of $\hat{\mathcal{Z}}$:

$$\begin{aligned} \pi_{(-0,0)} &= \frac{q(1-\alpha)+pq(2\alpha-1)}{2(p+q)(1-\alpha)+pq(2\alpha-1)}, \quad & \pi_{(-0,1)} &= \frac{q(1-\alpha)}{2(p+q)(1-\alpha)+pq(2\alpha-1)},\\ \pi_{(+0,0)} &= \frac{p(1-\alpha)}{2(p+q)(1-\alpha)+pq(2\alpha-1)}, \quad & \pi_{(+0,1)} &= \frac{p(1-\alpha)}{2(p+q)(1-\alpha)+pq(2\alpha-1)}, \end{aligned}$$

and calculate the effective permeability and the effective slide according to (2.4) and (2.5):

$$\begin{aligned} \gamma &= \frac{2(p-q)(1-\alpha)+pq(2\alpha-1)}{2(p+q)(1-\alpha)+pq(2\alpha-1)},\\ c &= \frac{2p(1-\alpha)}{2(p+q)(1-\alpha)+pq(2\alpha-1)}. \end{aligned} \tag{2.12}$$

Clearly, setting $q = 0$ we obtain $\gamma = 1$ and $c = 1$ as in the previous case 2. □

The idea of the proof of the Theorem 2.1 consists in a decomposition of the process $\mathcal{Z}$ into excursions starting and ending on the membrane $\mathcal{H}$. The excursions have a probability law of excursions of a symmetric random walk.

If the membrane is homogeneous ($k_2 = \cdots = k_m = 1$), then it is well know that the scaled limit of the x-coordinate is a skew Brownian motion, see Harrison and Shepp [10]. To control the slide in the y-component we only have to control the number of visits of the x-coordinate of the Markov chain to 0. However, if the membrane is periodic with a non-trivial period, then the sign of the x-coordinate of an excursion is selected in accordance with the transition probabilities (2.2), i.e., its sign depends on the position of the random walk at the last visit to the membrane. The slide along the y-subspace depends on that position too.

Due to the periodicity assumption (2.2), there is $d = 2|U| := 2k_2\cdots k_m$ different types of excursions between consecutive visits of the membrane. In order to treat this number of excursions at the same time we have to consider a natural generalization of the skew Brownian motion, namely, a Walsh Brownian motion. Hence we will show that the family of d one-sided random walks converge to a Walsh Brownian motion. This will allow us to derive the effective permeability and sliding parameters.

3. Walsh's Brownian motion and its martingale characterizations

Walsh's Brownian motion (WBM) was introduced in the Epilogue of Walsh [37] as a diffusion on d rays on a two-dimensional plane with the common origin. On each ray WBM, is a standard one-dimensional Brownian motion that however can change the ray upon hitting the origin, i.e., each ray is characterized by a weight $p_i > 0$, $i = 1, \ldots, d$, $p_1 + \cdots + p_d = 1$, that heuristically can be understood as a probability to go on the ray number i. Hence, the conventional WBM is a process X on the plane expressed in polar coordinates as $X = (R_t, \theta_t)$ where R is the reflecting Brownian motion and θ is a random process taking values on the set of d angles on $[0, 2\pi)$ and being constant during each excursion of R from 0. This representation has been used in various works on WBM including Barlow et al. [1], Freidlin and Sheu [5], Hajri [8], Hajri and Touhami [9], and Karatzas and Yan [15].

In this paper prefer to embed the WBM into a d-dimensional Euclidean space as it was indicated in Walsh [37]. To this purpose, let E be the union of non-negative coordinate half-axes in $\mathbb{R}^d$, i.e.,

$$E = \{x \in \mathbb{R}^d \colon x_i \geq 0 \text{ and } x_i x_j = 0,\ i \neq j,\ i, j = 1, \ldots, d\}.$$

We fix probabilities $p_1, \ldots, p_d > 0$, $p_1 + \cdots + p_d = 1$, and also denote $q_i = 1 - p_i$, $i = 1, \ldots, d$. Let also $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Then adopting the Markovian characterization of the WBM from Barlow et al. [1], we say that WBM is a time-homogeneous continuous Feller Markov process $X = (X_1, \ldots, X_d)$ on E with the one-dimensional laws given by

$$\begin{aligned}
\mathbf{E}_0 \mathrm{e}^{\lambda_1 X_1(t) + \cdots + \lambda_d X_d(t)} &= \sum_{k=1}^d p_k \mathbf{E}_0 \mathrm{e}^{\lambda_k |W(t)|},\\
\mathbf{E}_x \mathrm{e}^{\lambda_1 X_1(t) + \cdots + \lambda_d X_d(t)} &= \mathbf{E}_{x_j}\Big[\mathbb{I}(t < \tau_0) \mathrm{e}^{\lambda_j W(t)}\Big] + \sum_{k=1}^d p_k \mathbf{E}_{x_j}\Big[\mathbb{I}(t \geq \tau_0) \mathrm{e}^{\lambda_k |W(t)|}\Big]\\
&= \mathbf{E}_{x_j}\Big[\sum_{k=1}^d p_k \mathrm{e}^{\lambda_k |W(t)|}\Big] + \mathbf{E}_{x_j}\Big[\mathbb{I}(t < \tau_0)\Big(\mathrm{e}^{\lambda_j W(t)} - \sum_{k=1}^d p_k \mathrm{e}^{\lambda_k W(t)}\Big)\Big],\\
x = (0, \ldots, x_j, \ldots, 0),&\ x_j > 0, \lambda \in \mathbb{C}^d,
\end{aligned}$$

W is a standard Brownian motion and $\tau_0 = \inf\{t \geq 0 \colon W_t = 0\}$. Notice that the last expectation can be considered as the expectation of a killed Brownian motion.

In Barlow et al. [1], the authors also gave the martingale characterization of the WBM realized as a process on the plane. In terms of the d-dimensional realization X of the WBM, their characterization takes the following form.

Theorem 3.1 (Propositon 3.1 and Theorem 3.2 in Barlow et al. [1])**.** *Let $X = (X_1(t), \ldots, X_d(t))_{t \geq 0}$ be an adapted continuous process. Then X is a WBM with parameters $p_1, \ldots, p_d > 0$ if and only if it satisfies the following conditions:*

(1) $X_i(t) \geq 0$ and $X_i(t) X_j(t) = 0$ for all $i \neq j$ and $t \geq 0$;

(2) for each $i = 1, \ldots, d$ the process

$$N_i(t) := q_i X_i(t) - p_i \sum_{j \neq i} X_j(t), \quad t \geq 0,$$

is a continuous martingale with respect to $\mathbb{F}$;

(3) for each $i = 1, \ldots, d$ the process

$$(N_i(t))^2 - \int_0^t \Big(q_i \mathbb{I}(X_i(s) > 0) - p_i \mathbb{I}(X_i(s) = 0)\Big)^2 \mathrm{d}s, \quad t \geq 0, \tag{3.1}$$

is a continuous martingale with respect to $\mathbb{F}$.

Note that since the product of the indicator functions in (3.1) is identically zero, we have

$$\langle N_i\rangle_t = \int_0^t \Big(q_i\,\mathbb{I}(X_i(s)>0) - p_i\,\mathbb{I}(X_i(s)=0)\Big)^2\,\mathrm{d}s$$
$$= \int_0^t \Big(q_i^2\,\mathbb{I}(X_i(s)>0) + p_i^2\,\mathbb{I}(X_i(s)=0)\Big)\,\mathrm{d}s.$$

Furthermore, it is clear, see Lemma 2.2 in Barlow et al. [1], that the radial process

$$R(t) := \sum_{i=1}^d X_i(t) = \max_{1\le i\le d} X_i(t)$$

is a reflecting Brownian motion, and hence it has a local time at 0 defined by:

$$L^X(t) := L^R(t) := \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\int_0^t \mathbb{I}\Big(\max_{1\le i\le d} X_i(s)\le\varepsilon\Big)\,\mathrm{d}s. \tag{3.2}$$

In the following theorem we give an equivalent martingale characterization of the WBM that better fits into the setting of this paper.

Theorem 3.2. *Let $X=(X_1(t),\dots,X_d(t))_{t\ge 0}$ and $\nu=(\nu(t))_{t\ge0}$ be adapted continuous processes. Then X is a WBM with parameters $p_1,\dots,p_d>0$, and ν is the local time of X at 0 if and only if they satisfy the following conditions:*

a) $X_i(t)\ge 0$ and $X_i(t)X_j(t)=0$ for all $i\ne j$, $t\ge 0$;
b) $\nu(0)=0$, ν is nondecreasing a.s., $\int_0^\infty \mathbb{I}(X(s)\ne 0)\,\mathrm{d}\nu(s)=0$ a.s.;
c) the processes $M_1,\dots,M_d$ defined by

$$M_i(t) := X_i(t) - p_i\nu(t),\quad t\ge 0, \tag{3.3}$$

are continuous square integrable martingales with respect to $\mathbb{F}$ with the predictable quadratic variations

$$\langle M_i\rangle_t = \int_0^t \mathbb{I}(X_i(s)>0)\,\mathrm{d}s. \tag{3.4}$$

d) $\int_0^\infty \mathbb{I}(X(s)=0)\,\mathrm{d}s = 0$ a.s.

Remark 3.3. There is a typo in §3 in Barlow et al. [1] (p. 281). One should remove the indicator $\mathbb{I}_{r>0}$ in the formula for $h_i(r,\theta)$, otherwise the zero process is also a solution of equations (3.2) or (3.3) from their paper. Note that condition (d) in our Theorem 3.2 prohibits 0 to be a sticky point of X.

Proof. We will show that conditions 1–3 of Theorem 3.1 and a)–d) of Theorem 3.2 are equivalent.
i) We show that a)–d) ⇒ 1–3.

First, it is obvious that a) implies 1.

By c), each process N_i is a martingale as a linear combination of martingales:

$$N_i(t) = q_iX_i(t) - p_i\sum_{j\ne i} X_j(t)$$
$$= q_i(M_i(t)+p_i\nu(t)) - p_i\sum_{j\ne i}(M_j(t)+p_j\nu(t))$$
$$= q_iM_i(t) - p_i\sum_{j\ne i} M_j(t).$$

Observe that quadratic covariations $\langle M_i, M_j\rangle$ vanish for $i \neq j$. Indeed, applying the Cauchy-type inequality for quadratic covariations (Proposition 15.10 in Kallenberg [14]) we get

$$\begin{aligned}\langle M_i, M_j\rangle_t &= \int_0^t \mathrm{d}\langle M_i, M_j\rangle_s = \int_0^t \Big(\mathbb{1}(X_i(s) > 0) + \mathbb{1}(X_i(s) = 0)\Big)\,\mathrm{d}\langle M_i, M_j\rangle_s \\ &\leq \Big(\int_0^t \mathbb{1}(X_i(s) > 0)\,\mathrm{d}\langle M_j\rangle_s \cdot \int_0^t \mathrm{d}\langle M_i\rangle_s\Big)^{1/2} + \Big(\int_0^t \mathbb{1}(X_i(s) = 0)\,\mathrm{d}\langle M_i\rangle_s \cdot \int_0^t \mathrm{d}\langle M_j\rangle_s\Big)^{1/2} \\ &\leq \Big(\int_0^t \mathbb{1}(X_i(s) > 0)\mathbb{1}(X_j(s) > 0)\,\mathrm{d}s \cdot t\Big)^{1/2} + \Big(\int_0^t \mathbb{1}(X_i(s) = 0)\mathbb{1}(X_i(s) > 0)\,\mathrm{d}s \cdot t\Big)^{1/2} \\ &= 0\end{aligned}$$

Consequently with the help of c) we obtain the martingale property 3. Indeed, by the Itô formula we get

$$\begin{aligned}(N_i(t))^2 &= (N_i(0))^2 + 2\int_0^t N_i(s)\,\mathrm{d}N_i(s) + \langle N_i\rangle_t \\ &= (N_i(0))^2 + 2q_i \int_0^t N_i(s)\,\mathrm{d}M_i(s) - 2p_i \sum_{j\neq i} \int_0^t N_i(s)\,\mathrm{d}M_j(s) \\ &+ q_i^2\langle M_i\rangle_t + p_i^2 \sum_{j\neq i}\langle M_j\rangle_t.\end{aligned}$$

We have

$$\begin{aligned}\int_0^t \Big(q_i \mathbb{1}(X_i(s) > 0) - p_i \mathbb{1}(X_i(s) = 0)\Big)^2 \mathrm{d}s &= \int_0^t \Big(q_i^2 \mathbb{1}(X_i(s) > 0) + p_i^2 \mathbb{1}(X_i(s) = 0)\Big)\,\mathrm{d}s \\ &= q_i^2\langle M_i\rangle_t + p_i^2 \sum_{j\neq i}\langle M_j\rangle_t,\end{aligned}$$

where we used (d) in the last equality. Hence we get the property 3.

Show that the process ν is the local time of X at zero. Let L^X be the local time of X at zero and let

$$R(t) = \max_{1\leq i\leq d} X_i(t) = \sum_{i=1}^d X_i(t)$$

be the radial part of the WBM X. By Lemma 2.2 in Barlow et al. [1], R is a reflecting Brownian motion with a local time L^R, and hence

$$R(t) - L^R(t)$$

is a Brownian motion. On the other hand

$$R(t) - \nu(t) = \sum_{i=1}^d M_i(t)$$

and $Z(t) := \sum_{i=1}^d M_i(t)$ is a continuous martingale with the bracket (we use 3.)

$$\langle Z\rangle_t = \sum_{i=1}^d \int_0^t \mathbb{1}(X_i(s) > 0)\,\mathrm{d}s = t,$$

thus, Z is a $\mathbb{F}$-Brownian motion. By the uniqueness of the semimartingale decomposition, $\nu = L^R$. Notice that $L^R = L^X$.

ii) We show that 1–3 $\Rightarrow$ a)–d).

Let X be a WBM. Denote $\nu := L^X$ its local time at 0. The properties a), b) and d) follow immediately.

For each $i = 1, \dots, d$ consider the process

$$Y_i(t) = X_i(t) - \sum_{j \neq i} X_j(t), \quad t \geq 0. \tag{3.5}$$

and a rescaled martingale

$$\tilde{N}_i(t) := \frac{1}{p_i q_i} N_i(t) = \frac{1}{p_i} X_i(t) - \frac{1}{q_i} \sum_{j \neq i} X_j(t), \quad t \geq 0, \tag{3.6}$$

with the bracket

$$\langle \tilde{N}_i \rangle_t = \int_0^t \Big(\frac{1}{p_i^2} 1\!\!\mathrm{I}(X_i(s) > 0) + \frac{1}{q_i^2} 1\!\!\mathrm{I}(X_i(s) = 0) \Big) \,\mathrm{d}s.$$

Then obviously

$$\begin{aligned} 1\!\!\mathrm{I}(N_i(s) > 0) &= 1\!\!\mathrm{I}(\tilde{N}_i(s) > 0) = 1\!\!\mathrm{I}(X_i(s) > 0) = 1\!\!\mathrm{I}(Y_i(s) > 0), \\ 1\!\!\mathrm{I}(N_i(s) < 0) &= 1\!\!\mathrm{I}(\tilde{N}_i(s) < 0) = 1\!\!\mathrm{I}(Y_i(s) < 0). \end{aligned}$$

Consider the processes

$$W_i(t) := \int_0^t \Big(p_i 1\!\!\mathrm{I}(\tilde{N}_i(s) > 0) + q_i 1\!\!\mathrm{I}(\tilde{N}_i(s) < 0) \Big) \,\mathrm{d}\tilde{N}_i(s).$$

These process are continuous martingales with the bracket

$$\langle W_i \rangle_t = t,$$

hence they are Brownian motions. Denoting $a_i(x) := \frac{1}{p_i} 1\!\!\mathrm{I}(x > 0) + \frac{1}{q_i} 1\!\!\mathrm{I}(x < 0)$ we get that $\tilde{N}_i$ satisfies the SDE

$$\tilde{N}_i(t) = \tilde{N}_0 + \int_0^t a_i(\tilde{N}_i(s)) \,\mathrm{d}W_i(s),$$

and hence each N_i is the so-called oscillating Brownian motion, see Keilson and Wellner [16]. By Nakao's theorem, see Nakao [22], this SDE has a unique strong solution. Let $r_i(x) = p_i 1\!\!\mathrm{I}(x > 0) + q_i 1\!\!\mathrm{I}(x < 0)$. Taking into account (3.5), (3.6) and the property a) we get that

$$Y_i = r_i(\tilde{N}_i).$$

Repeating literally the calculations from Section 5.2 in Lejay [17] we get that

$$Y_i(t) = Y_i(0) + W_i(t) + (2p_i - 1) L^{Y_i}(t),$$

where $L^{Y_i}(t)$ is the symmetric local time of Y_i at 0.

Hence Y_i is a skew Brownian motion with parameter $(2p_i - 1)$. Moreover the radial process $R = |Y_i| = |X|$ is a reflected Brownian motion and

$$R(t) = |Y_i(t)| = R(0) + \int_0^t \operatorname{sgn}(Y_i) \,\mathrm{d}W_i(s) + L^{Y_i}(t).$$

Futhermore,

$$R(t) = R(0) + W(t) + L^R(t)$$

for some Brownian motion W. From the uniqueness of the decomposition of R as a semimartingale we get that $L^{Y_i} = L^R$ for all $i = 1, \dots, d$.

It follows from [35, Theorem 1.7, Chapter VI] that the right and left local times of Y_i at 0 are equal to

$$L^{Y_i}_{\text{left}}(t) = 2(1-p_i)L^{Y_i}(t), \quad L^{Y_i}_{\text{right}}(t) = 2p_i L^{Y_i}(t).$$

Finally Tanaka's formula yields

$$\begin{aligned} X_i(t) = \max\{Y_i(t), 0\} &= \\ &= \max\{Y_i(0), 0\} + \int_0^t \mathbb{I}(Y_i(s) > 0)\,\mathrm{d}W_i(s) + \frac{1}{2} L^{Y_i}_{\text{right}}(t) \\ &= \max\{Y_i(0), 0\} + \int_0^t \mathbb{I}(Y_i(s) > 0)\,\mathrm{d}W_i(s) + p_i L^R(t), \end{aligned}$$

and thus the process $M_i = X_i - p_i\nu$ is a continuous martingale with the bracket (3.4). □

Corollary 3.4. *Let $X = (X_1(t), \dots, X_d(t))_{t\geq 0}$ be a WBM with parameters $p_1, \dots, p_d > 0$ and let $I \subseteq \{1, \dots, d\}$. Let*

$$\gamma = \sum_{i\in I} p_i - \sum_{j\in I^c} p_j = 2\sum_{i\in I} p_i - 1 \in [-1, 1].$$

Then the process

$$X^\gamma(t) := \sum_{i\in I} X_i(t) - \sum_{j\in I^c} X_j(t), \quad t \geq 0,$$

is a skew Brownian motion with the parameter γ.

Proof. Without loss of generality assume that $I = \{1, \dots, k\}$ for some $0 \leq k \leq d$. By Theorem 3.2, the process

$$W(t) = \sum_{i=1}^k M_i(t) - \sum_{j=k+1}^d M_j(t)$$

is a continuous martingale. Taking into account (3.3) we get

$$W(t) = X^\gamma(t) - \gamma\nu(t).$$

Since the local times at 0 of X and X^γ coincide, we get

$$W(t) = X^\gamma(t) - \gamma L(t)$$

where L is the local time of X^γ at 0. The bracket of the martingale W equals

$$\langle W\rangle_t = \sum_{i=1}^d \int_0^t \mathbb{I}(X_i(s) > 0)\,\mathrm{d}s = t,$$

and thus W is a Brownian motion. In other words, X^γ satisfies the SDE $X^\gamma(t) = W(t) + \gamma L(t)$ and thus is a skew Brownian motion, see Harrison and Shepp [10]. □

4. Proof of Theorem 2.1

Consider the Markov chain $\mathcal{Z} = (\mathcal{X}, Y)$ on the enlarged state space $\{\pm 0, \pm 1, \dots\} \times \mathbb{Z}^{m-1}$. Let us decompose the process $\mathcal{X}$ into $2|U|$ non-negative excursions parameterized by the elements of the set $\{-0, +0\} \times U$. By σ_n, $n \geq 0$, denote the time instant of the last visit of the random walk $\mathcal{Z}$ to $\mathcal{H}$ before time n, i.e.,

$$\sigma_n := \max\Big\{k \leq n \colon \mathcal{X}(k) \in \{-0, +0\}\Big\}, \quad \sigma_0 = 0.$$

Consider $2|U|$ processes

$$\begin{aligned} X^{j,+}(n) &:= X(n)\mathbb{I}(Y(\sigma_n) \equiv j, X(n) > 0), \\ X^{j,-}(n) &:= |X(n)|\mathbb{I}(Y(\sigma_n) \equiv j, X(n) < 0), \quad j \in U, \end{aligned} \tag{4.1}$$

and introduce a $2|U|$-dimensional random sequence $\{(X^{j,+}(n), X^{j,-}(n))_{j\in U}\}_{n\geq 0}$ on $\mathbb{N}_0^{2|U|}$. Observe that all its coordinates are non-negative, only one coordinate of the vector $(X^{j,+}(n), X^{j,-}(n))_{j\in U}$ may be non-zero, and $X^{j,\pm}(\sigma_n) = 0$.

To study the limit behavior of the scaled process $X_n(\cdot)$ defined in (2.10) we will prove that the properly scaled $2|U|$-dimensional process $(X^{j,+}(n), X^{j,-}(n))_{j\in U}$ converges to a WBM with parameters $\{\pi_{(+0,j)}, \pi_{(-0,j)}\}_{j\in U}$. To this end, for each $j \in U$ we decompose the processes $X^{j,+}$ and $X^{j,-}$ into a sum of a martingale and a "local time in 0", namely we set

$$\begin{aligned} X^{j,\pm}(n) &= \sum_{k=1}^{n} (X^{j,\pm}(k) - X^{j,\pm}(k-1))\mathbb{I}(X^{j,\pm}(k-1) > 0) \\ &\quad + \sum_{k=1}^{n} \mathbb{I}(X^{j,\pm}(k-1) = 0, X^{j,\pm}(k) = 1) \\ &=: M^{j,\pm}(n) + L^{j,\pm}(n), \qquad M^{j,\pm}(0) = L^{j,\pm}(0) = 0. \end{aligned} \tag{4.2}$$

Since the $\mathcal{Z}$ is a symmetric random walk outside of $\mathcal{H}$, it follows from the construction that

$$\begin{aligned} &\mathbf{E}\Big[(X^{j,\pm}(k) - X^{j,\pm}(k-1))\mathbb{I}(X^{j,\pm}(k-1) > 0)\Big|\mathscr{F}_{k-1}\Big] \\ &\qquad = \mathbb{I}(X^{j,\pm}(k-1) > 0) \cdot \mathbf{E}\Big[X^{j,\pm}(k) - X^{j,\pm}(k-1)\Big|\mathscr{F}_{k-1}\Big] = 0 \end{aligned}$$

and the sequences $(M^{j,+}(n))_{n\geq 0}$ and $(M^{j,-}(n))_{n\geq 0}$ are martingales for any $j \in U$ with respect to filtration $(\mathscr{F}_n)_{n\geq 0}$ generated by $\mathcal{Z}$. Moreover, since

$$\begin{aligned} \langle M^{j,\pm}\rangle_n &= \sum_{k=1}^{n} \mathbf{E}\Big[(X^{j,\pm}(k) - X^{j,\pm}(k-1))^2 \cdot \mathbb{I}(X^{j,\pm}(k-1) > 0)\Big|\mathscr{F}_{k-1}\Big] \\ &= \frac{1}{m}\sum_{k=1}^{n} \mathbb{I}(X^{j,\pm}(k-1) > 0), \end{aligned}$$

the sequences

$$(M^{j,\pm}(n))^2 - \frac{1}{m}\sum_{k=1}^{n} \mathbb{I}(X^{j,\pm}(k-1) > 0), \quad n \geq 0,$$

are martingales too.

We set

$$X_n^{j,\pm}(t) := \frac{X^{j,\pm}([nt])}{\sqrt{n}}, \quad M_n^{j,\pm}(t) := \frac{M^{j,\pm}([nt])}{\sqrt{n}}, \quad \nu_n^{j,\pm}(t) := \frac{L^{j,\pm}([nt])}{\sqrt{n}}. \tag{4.3}$$

Proposition 4.1. *There are continuous processes* $\{X_j^\pm\}_{j\in U}$*, continuous martingales* $\{M_j^\pm\}_{j\in U}$ *and a nondecreasing process* ν *such that*

$$(X_n^{j,\pm}, M_n^{j,\pm}, \nu_n^{j,\pm})_{j\in U} \Rightarrow (X_j^\pm, M_j^\pm, p_j^\pm \nu)_{j\in U}, \quad n\to\infty, \tag{4.4}$$

where

$$\begin{aligned} p_j^\pm &= \mathbf{E}_\pi \mathbb{I}(\mathcal{X}(1)=\pm 1, Y(0)=j) \\ &= \pi_{(+0,j)}\mathbf{P}_{(+0,j)}(\mathcal{X}(1)=\pm1) + \pi_{(-0,j)}\mathbf{P}_{(-0,j)}(\mathcal{X}(1)=\pm1), \quad j\in U. \end{aligned}$$

Moreover, the process $\sqrt{m}(X_j^\pm, M_j^\pm, p_j^\pm\nu)_{j\in U}$ *satisfies conditions of Theorem 3.2.*

To prepare the proof of Proposition 4.1 we notice that in problems of this type it is often helpful to start with the study of the radial process

$$S(n) := \sum_{j\in U}(X^{j,-}(n) + X^{j,+}(n)),\ n\geq 0.$$

The process $S=\{S(n)\}_{n\geq0}$ is a Markov chain on $\mathbb{N}\cup\{0\}$ with reflection at 0 whose steps in $\mathbb{N}$ have the distribution

$$\begin{aligned} &\mathbf{P}\Big(S(n+1)=j-1\Big|S(n)=j\Big) = \mathbf{P}\Big(S(n+1)=j+1\Big|S(n)=j\Big) = \frac{1}{2m}, \\ &\mathbf{P}\Big(S(n+1)=j\Big|S(n)=j\Big) = 1-\frac{1}{m}, \quad j\geq 1, \\ &\mathbf{P}\Big(S(n+1)=1\Big|S(n)=0\Big) = 1. \end{aligned}$$

Set

$$L(n) := \sum_{k=0}^{n-1}\mathbb{I}(X(k)=0) = \sum_{k=0}^{n-1}\mathbb{I}(S(k)=0) = \sum_{j\in U}(L_j^+(n)+L_j^-(n)), \quad n\geq 0, \tag{4.5}$$

$L(n)$ being the number of visits of S to the origin up to time n.

Lemma 4.2. *We have the weak convergence*

$$\Big(\frac{S([n\cdot])}{\sqrt{n}}, \frac{L([n\cdot])}{\sqrt{n}}\Big) \Rightarrow \frac{1}{\sqrt{m}}\Big(|B(\cdot)|, L^0(\cdot)\Big),\ n\to\infty,$$

where B *is a standard Brownian motion,* $L^0(\cdot)$ *is a local time of* B *at 0 defined by* (3.2)*.*

Proof. Whereas the convergence of the reflected random walks to a reflected Brownian motion is straightforward, certain work should be done to ensure the convergence of the local times.

Let $\{\xi_k\}_{k\geq1}$ be a sequence of iid random variables, $\mathbf{P}(\xi_k=\pm1)=\frac{1}{2m}, \mathbf{P}(\xi_k=0)=1-\frac{1}{m}$. Consider a random walk $Q(n)=\sum_{k=1}^n \xi_k$, $Q(0)=0$. We recursively construct a reflected sequence $\tilde{Q}$ setting

$$\begin{aligned} \tilde{Q}(0) &= 0, \quad N(0)=0, \\ N(n+1) &= \begin{cases} N(n)+1, & \text{if } \tilde{Q}(n)=0, \\ N(n), & \text{if } \tilde{Q}(n)>0, \end{cases} \\ \tilde{Q}(n) &= Q(n-N(n)) + N(n). \end{aligned}$$

Notice that $\{(\tilde{Q}(n), N(n))\}_{n\geq0} \stackrel{\mathrm{d}}{=} \{(S(n), L(n))\}_{n\geq0}$. It also clear that

$$N(n) \leq -\min_{0\leq k\leq n} Q(n) + 1,$$

so that $N(n)/n \to 0, n \to \infty$ a.s. Consequently,

$$\frac{[nt]-N([nt])}{n} \overset{\text{a.s.}}{\to} t, \quad n \to \infty,$$

uniformly on each interval $[0,T]$, $T>0$.

We have the following estimate of the modulus of continuity of $\tilde{Q}$ and N: for all $n,k \geq 1$

$$\begin{aligned}
\max_{\substack{|i-j|\leq k \\ 1\leq i\leq j\leq n}} |\tilde{Q}(k)-\tilde{Q}(l)| &\leq 2 \max_{\substack{|i-j|\leq k \\ 1\leq i\leq j\leq n}} |Q(k)-Q(l)|+2, \\
\max_{\substack{|i-j|\leq k \\ 1\leq i\leq j\leq n}} |N(k)-N(l)| &\leq \max_{\substack{|i-j|\leq k \\ 1\leq i\leq j\leq n}} \Big(|\tilde{Q}(k)-\tilde{Q}(l)|+|Q(k-N(k))-Q(l-N(l))|\Big) \\
&\leq 3 \max_{\substack{|i-j|\leq k \\ 1\leq i\leq j\leq n}} |Q(k)-Q(l)|+2.
\end{aligned}$$

These estimates together with the convergence $\sqrt{m}\frac{Q([n\cdot])}{\sqrt{n}} \Rightarrow B(\cdot)$ yield that the sequence

$$\Big\{\sqrt{m}\Big(\frac{\tilde{Q}([n\cdot])}{\sqrt{n}}, \frac{Q([n\cdot])}{\sqrt{n}}, \frac{N([n\cdot])}{\sqrt{n}}\Big)\Big\}_{n\geq 1}$$

is weakly relatively compact and any of its limit points (A,B,C) is a continuous process such that

a) $A(t)\geq 0$ and $A(t)=B(t)+C(t)$, $t\geq 0$,
b) B is a standard Brownian motion,
c) C is a nondecreasing process, $C(0)=0$, $\int_0^\infty \mathbb{1}(A(s)>0)\,\mathrm{d}C(s)=0$.

This means that the pair (A,C) is a solution to the Skorokhod problem for B, see for example Chapter 1 in Pilipenko [27], i.e.,

$$A(t)=B(t)-\min_{s\in[0,t]} B(s), \quad C(t)=-\min_{s\in[0,t]} B(s).$$

The joint convergence

$$\sqrt{m}\Big(\frac{\tilde{Q}([n\cdot])}{\sqrt{n}}, \frac{Q([n\cdot])}{\sqrt{n}}, \frac{N([n\cdot])}{\sqrt{n}}\Big) \Rightarrow (A,B,C), \quad n\to\infty,$$

follows from uniqueness of the solution to the Skorokhod problem.

Eventually, it is well known that $(A(t),C(t))_{t\geq 0} \overset{\mathrm{d}}{=} (|B(t)|, L^0(t))_{t\geq 0}$ by Lévy's theorem, see, e.g. [14, Corollary 19.3]. The Lemma is proved. □

Lemma 4.3. *The sequence $\{(X_n^{j,\pm}(\cdot), M_n^{j,\pm}(\cdot), \nu_n^{j,\pm}(\cdot))_{j\in U}\}_{n\geq 1}$ is weakly relatively compact in $D([0,T],\mathbb{R}^{6|U|})$ and any limit point is a continuous processes.*

Proof. Notice that the modulus of continuity (in the uniform topology) of any $X_n^{j,\pm}$ is dominated by doubled modulus of continuity of $\{S(n\cdot)/\sqrt{n}\}$, and the modulus of continuity of any $M_n^{j,\pm}$ is dominated by doubled modulus of continuity of $X_n^{j,\pm}$. The third coordinate is the difference of the first two. Hence the statement of this Lemma follows from Lemma 4.2. □

Proof of Proposition 4.1. To show convergence $\{(X_n^{j,\pm})_{j\in U}\}_{n\geq 0}$ to the WBM it suffices to verify that for any subsequence $\{(X_{n_k}^{j,\pm})_{j\in U}\}_{k\geq 0}$ there is a subsubsequence that converges to the WBM. Due to Lemma 4.3 without loss of generality we will assume that the sequence $\Big\{\Big(X_n^{j,\pm}(\cdot), M_n^{j,\pm}(\cdot), \nu_n^{j,\pm}(\cdot)\Big)_{j\in U}\Big\}_{n\geq 1}$ converges in distribution to a continuous process $\Big(X^{j,\pm}(\cdot), M^{j,\pm}(\cdot), \nu^{j,\pm}(\cdot)\Big)_{j\in U}$.

Let us check the conditions a)–d) of Theorem 3.2 for the process $\sqrt{m}\Big(X^{j,\pm}(\cdot), M^{j,\pm}(\cdot), \nu^{j,\pm}(\cdot)\Big)_{j\in U}$.

a) It follows from the construction that $X^{j,\pm}(t) \geq 0$, $t \in [0,T]$. Moreover, only one of these processes may be positive at any fixed time.

b) and d) The processes $\nu^{j,\pm}(\cdot)$ are non-decreasing a.s. and $\nu^{j,\pm}(0) = 0$. Lemma 4.2 yields that

$$\Big(|X(\cdot)|, \nu(\cdot)\Big) \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{m}}\Big(|B(\cdot)|, L^0(\cdot)\Big),$$

where

$$\nu(t) = \sum_{j\in U}(\nu^{j,-}(t) + \nu^{j,+}(t)).$$

Since $\int_0^T 1\!\!\mathrm{I}(B(t)=0)\,\mathrm{d}t = 0$ and $\int_0^T 1\!\!\mathrm{I}(|B(t)|>0)\,\mathrm{d}L^0(t) = 0$ a.s. for any $T > 0$, we have

$$\int_0^T 1\!\!\mathrm{I}(|X(t)|=0)\,\mathrm{d}t = 0,$$
$$\int_0^T 1\!\!\mathrm{I}(|X(t)|>0)\,\mathrm{d}\nu(t) = 0$$

almost surely.

c) It follows from the construction that

$$X^{j,\pm}(t) = M^{j,\pm}(t) + \nu^{j,\pm}(t) \text{ a.s.}$$

for all $j \in U, t \geq 0$. To show that

$$\nu^{j,\pm}(t) = p_j^{\pm}\nu(t) \text{ a.s.} \tag{4.6}$$

we recall the strong law of large numbers (2.7) for Markov chains. For any $t > 0$ the process $L([nt])$ defined in (4.5) increases to $+\infty$ a.s. as $n \to \infty$. Hence

$$\begin{aligned}\frac{\nu_n^{j,\pm}(t)}{\sum_{k\in U}\nu_n^{k,-}(t) + \sum_{k\in U}\nu_n^{k,+}(t)} &= \frac{1}{L([nt])}\sum_{k=0}^{L([nt])} 1\!\!\mathrm{I}(Y(\tau_k)\equiv j, \mathcal{X}(\tau_k+1)=\pm 1)\\ &\to \mathbf{E}_\pi 1\!\!\mathrm{I}(\mathcal{X}(1)=\pm 1, Y(0)=j)\\ &= \pi_{(+0,j)}\mathbf{P}_{(+0,j)}(\mathcal{X}(1)=\pm 1) + \pi_{(-0,j)}\mathbf{P}_{(-0,j)}(\mathcal{X}(1)=\pm 1) \text{ a.s.}\end{aligned}$$

The processes $M_n^{j,\pm}(\cdot)$, $j \in U$, are local martingales with respect to the filtration generated by the process $\{X_n^{j,\pm}(\cdot), M_n^{j,\pm}(\cdot)\}_{j\in U}$. Since the jumps of each $M_n^{j,\pm}(\cdot)$ are uniformly bounded, the limits $M^{j,\pm}$ are local martingales with respect to filtration generated by $\{X^{j,\pm}, M^{j,\pm}\}_{j\in U}$ due to Lemma 1.17 in Chapter IX of Jacod and Shiryaev [13]. Moreover, the limit processes are continuous due to Lemma 4.3.

It is left to show that

$$\langle M^{j,\pm}\rangle_t = \frac{1}{m}\int_0^t 1\!\!\mathrm{I}(X^{j,\pm}(s)>0)\,\mathrm{d}s \text{ a.s. for } j\in U,\ t\geq 0. \tag{4.7}$$

By Skorokhod's representation theorem there is a probability space and the copies

$$\Big\{\Big(\tilde{X}_n^{j,\pm}(\cdot), \tilde{M}_n^{j,\pm}(\cdot), \tilde{\nu}_n^{j,\pm}(\cdot)\Big)_{j\in U}\Big\}_{n\geq 1} \text{ and } \Big\{\Big(\tilde{X}^{j,\pm}(\cdot), \tilde{M}^{j,\pm}(\cdot), \tilde{\nu}^{j,\pm}(\cdot)\Big)_{j\in U}\Big\}$$

of

$$\left\{\left(X_n^{j,\pm}(\cdot), M_n^{j,\pm}(\cdot), \nu_n^{j,\pm}(\cdot)\right)_{j\in U}\right\}_{n\ge 1} \text{ and } \left\{\left(X^{j,\pm}(\cdot), M^{j,\pm}(\cdot), \nu^{j,\pm}(\cdot)\right)_{j\in U}\right\}$$

such that on any interval $[0,T]$ we have a.s. uniform convergence

$$\left(\tilde X_n^{j,\pm}(\cdot), \tilde M_n^{j,\pm}(\cdot), \tilde \nu_n^{j,\pm}(\cdot)\right)_{j\in U} \to \left(\tilde X^{j,\pm}(\cdot), \tilde M^{j,\pm}(\cdot), \tilde \nu^{j,\pm}(\cdot)\right)_{j\in U}, \quad n\to\infty. \tag{4.8}$$

To prove (4.7) it suffices to verify that with probability 1 the sequence

$$\left(\tilde M_n^{j,\pm}(t)\right)^2 - \frac{1}{m}\int_0^{[nt]/n} \mathbb{I}(\tilde X_n^{j,\pm}(s)>0)\,\mathrm{d}s$$

converges uniformly over $t\in[0,T]$ to

$$(\tilde M^{j,\pm}(t))^2 - \frac{1}{m}\int_0^{t} \mathbb{I}(\tilde X^{j,\pm}(s)>0)\,\mathrm{d}s.$$

Here we again use Lemma 1.17 in Chapter IX of Jacod and Shiryaev [13] and a localization procedure. It follows from (4.8) that we have to prove the convergence of the integrals only.

Let $\omega\in\Omega$ be such that (4.8) holds. If $s\in[0,T]$ is such that $\tilde X^{k,\mathfrak{s}}(s)>0$ for some $1\le k\le |U|$ and $\mathfrak{s}\in\{-,+\}$ then $\tilde X_n^{k,\mathfrak{s}}(s)>0$ for large n. Since only one of the processes $\{\tilde X^{j,\mathfrak{s}}(s)\}_{j\in U,\mathfrak{s}\in\{-,+\}}$ and only one of $\{\tilde X_n^{j,\mathfrak{s}}(s)\}_{j\in U,\mathfrak{s}\in\{-,+\}}$ may be non-zero we have convergence of the indicators for all $j\in U$:

$$\lim_{n\to\infty}\mathbb{I}(\tilde X_n^{j,\pm}(s)>0) = \mathbb{I}(\tilde X^{j,\pm}(s)>0). \tag{4.9}$$

The process $\tilde X = (\tilde X^{j,+}, \tilde X^{j,-})_{j\in U}$ spends zero time in 0 with probability 1 because $\sqrt{m}\sum_{j=1}^{|U|}(\tilde X^{j,+}+\tilde X^{j,-})$ is a reflected Brownian motion, see Lemma 4.2. Therefore for a.a. ω and a.a. $s\in[0,T]$ there is k and $\mathfrak{s}$ such that $\tilde X^{k,\mathfrak{s}}(s)>0$ and we have (4.9) for any index $(j,\pm)$, $j\in U$. So by the Fubini theorem and by the Lebesgue dominated convergence theorem for a.a. ω and all $j\in U$ we have convergence of the integrals

$$\lim_{n\to\infty}\int_0^t \mathbb{I}(\tilde X_n^{j,\pm}(s)>0)\,\mathrm{d}s = \int_0^t \mathbb{I}(\tilde X^{j,\pm}(s)>0)\,\mathrm{d}s.$$

This completes the proof of Proposition 4.1. □

To treat convergence of Y, similarly to the representation (4.2) for $\{X^{j,\pm}(n)\}$ we decompose the sequence $Y=\{Y(n)\}_{n\ge 0}$ into the sum

$$\begin{aligned} Y(n) &= \sum_{k=0}^{n-1}\Big(Y(k+1)-Y(k)\Big)\mathbb{I}(X(k)\ne 0) \\ &\quad + \sum_{j\in U}\sum_{k=0}^{n-1}\Big(Y(k+1)-Y(k)\Big)\mathbb{I}(\mathcal{X}(k)=\pm 0, Y(k)\equiv j) \\ &= M^Y(n) + D^Y(n), \quad n\ge 0, \end{aligned}$$

and we define

$$M_n^Y(t) := \frac{M^Y([nt])}{\sqrt n}, \quad D_n^Y(t) := \frac{D^Y([nt])}{\sqrt n}.$$

The following Proposition is proven analogously to the previous reasoning.

Proposition 4.4. *Let X and ν be as in Proposition 4.1. Then*

$$\Big(X_n(\cdot), \nu_n(\cdot), M_n^Y(\cdot), D_n^Y(\cdot)\Big) \Rightarrow \Big(X(\cdot), \nu(\cdot), \frac{1}{\sqrt{m}} W_Y(\cdot), c\nu(\cdot)\Big),$$

where W_Y is a $(m-1)$-dimensional Brownian motion independent of X, and c is defined in (2.5).

Proof of Theorem 2.1. We combine Propositions 4.1 and 4.4 together with Corollary 3.4. □

5. One-sided membrane with ergodic properties

The same method of decomposition of the perturbed Markov chain into a sum of excursions combined with the strong law of large numbers (2.6) can be applied for the analysis of a one-sided membrane that has ergodic properties.

As in Section 2, let $m \geq 2$ and let $\{\mathrm{e}_1, \dots, \mathrm{e}_m\}$ be a standard basis in $\mathbb{R}^m$. Consider a Markov chain $Z = (X, Y)$ on $\mathbb{Z}^m$ that behaves as a simple random walk outside of the hyperplane $H := \{0\} \times \mathbb{Z}^{m-1}$, i.e., (2.1) holds true.

Let $\{p_y\}_{y\in H} \subset [0,1]$. Now we interpret H as a semipermeable non-homogeneous membrane that may let a particle into one half-space with probabilities $\{p_y\}_{y\in H}$. More precisely, we assume that for each $z = (0, y) \in H$:

$$p_y = \mathbf{P}\Big(Z(n+1) = z + \mathrm{e}_1 \Big| Z(n) = z\Big) = 1 - \mathbf{P}\Big(Z(n+1) = z - \mathrm{e}_1 \,\Big|\, Z(n) = z\Big).$$

Note that the particle leaves the membrane in the direction orthogonal to H, i.e., $Y(n+1) = Y(n)$ for $Z(n) \in H$ with probability 1, and hence there is no slide along the membrane.

We assume that the membrane has the following ergodic property:
$\mathbf{A}_{\mathrm{SLLN}}(\beta)$: there is $\beta > 0$ such that

$$\lim_{A\to\infty} \frac{1}{|V(A,o)|} \sum_{y \in V(A,o)} p_y =: \bar{p} \in [0,1] \quad \text{a.s.},$$

where the limit is taken over all cubes $V(A, o) \subseteq H$ of volume $|V(A, o)|$ with side size larger than A and whose centre o is within distance A^β from the origin.

We give two clarifying examples for the assumption $\mathbf{A}_{\mathrm{SLLN}}(\beta)$.

Example 5.1. Assume that the family $\{p_y\}_{y\in H}$ has a periodic structure: there are $k_2, \dots, k_m \geq 1$ such that for all $l_2, \dots, l_m \in \mathbb{Z}$ and for all $y \in H$

$$p_y = p_{y + k_2 l_2 \mathrm{e}_2 + \dots + k_m l_m \mathrm{e}_m}.$$

Then $\{p_y\}$ clearly satisfy assumption $\mathbf{A}_{\mathrm{SLLN}}(\beta)$ for any $\beta > 0$ with

$$\bar{p} = \frac{1}{k_2 \cdots k_m} \sum_{i_2,\dots,i_m=1}^{k_2,\dots,k_m} p_{(i_2,\dots,i_m)}.$$

Example 5.2. Let $\{p_y\}_{y\in H}$ be i.i.d. random variables with values in $[0,1]$ defined on a probability space $(\Omega', \mathscr{F}', \mathbf{P}')$. Then for each fixed $\omega' \in \Omega'$ the family $\{p_y(\omega')\}$ defines a random "environment". Then the assumption $\mathbf{A}_{\mathrm{SLLN}}(\beta)$ with $\beta > 0$ is satified with

$$\bar{p} = \mathbf{E}' p_y.$$

To see this, let $\beta > 0$, and let $V(A,o)$ denote a cube with the size $A \in \mathbb{N}$ and the centre at $o \in \mathbb{Z}^{m-1}$. Then

$$\mathbf{P}\Big(\frac{1}{|V(A,o)|}\sum_{y\in V(A,o)} p_y \not\to \bar p\Big) = \mathbf{P}\Big(\bigcup_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{A=k}^{\infty}\bigcup_{|o|\le A^\beta}\Big\{\frac{1}{|V(A,o)|}\Big|\sum_{y\in V(A,o)}(p_y-\bar p)\Big| > \frac{1}{m}\Big\}\Big). \tag{5.1}$$

For each $m \ge 1$ and $A \ge 1$ we apply Hoeffding's inequality, see Chapter III, §5.8 in Petrov [26]:

$$\begin{aligned}\mathbf{P}\Big(\bigcup_{|o|\le A^\beta}\Big\{\frac{1}{|V(A,o)|}\Big|\sum_{y\in V(A,o)}(p_y-\bar p)\Big| > \frac{1}{m}\Big\}\Big) &\le (2A^\beta)^{m-1}\mathbf{P}\Big(\frac{1}{|V(A,o)|}\Big|\sum_{y\in V(A,o)}(p_y-\bar p)\Big| > \frac{1}{m}\Big)\\ &\le 2(2A^\beta)^{m-1}\mathrm{e}^{-2|V(A,o)|/m^2} = 2(2A^\beta)^{m-1}\mathrm{e}^{-2A^{m-1}/m^2}.\end{aligned}$$

Hence, the probability in (5.1) equals to 0.

Theorem 5.3. *Let assumption $\mathbf{A}_{\mathrm{SLLN}}(\beta)$ holds true for some $\beta > 1$. Then for any initial value $Z(0) \in \mathbb{Z}^m$ the weak convergence holds true:*

$$(X_n, Y_n) \Rightarrow \frac{1}{\sqrt{m}}(X^\gamma, Y^0), \quad n \to \infty,$$

where X^γ and Y^0 are defined in (2.8) *and* (2.9)*, with $\gamma = 2\bar p - 1$, and $c = 0$.*

Proof. 1. Without loss of generality assume that $Z(0) = 0$. As in Section 4 we decompose the Markov chain $Z = (X, Y)$ into the "left" and the "right" excursions. Since the membrane is one-sided, and there is no need of introducing the set U, the notation of Section 4 simplifies significantly. Similarly to (4.1) we define

$$\begin{aligned}X^+(n) &:= X(n)\cdot \mathbb{I}(X(n) > 0),\\ X^-(n) &:= |X(n)|\cdot \mathbb{I}(X(n) < 0),\end{aligned}$$

so that

$$X(n) = X^+(n) - X^-(n), \quad n \ge 0.$$

Then we decompose these processes similarly to (4.2) as

$$\begin{aligned}X^\pm(n) &= \sum_{k=1}^n \Big(X^\pm(k) - X^\pm(k-1)\Big)\cdot \mathbb{I}(X^\pm(k-1) > 0) + \sum_{k=1}^n \mathbb{I}(X^\pm(k-1) = 0, X^\pm(k) = 1)\\ &=: M^\pm(n) + L^\pm(n), \qquad M^\pm(0) = L^\pm(0) = 0.\end{aligned}$$

The processes $M^\pm$ are martingales, and $L^\pm$ are non-decreasing processes. Recall the processes $X_n(\cdot)$ and $Y_n(\cdot)$ defined in (2.10), and define additionally the scaled processes $X_n^\pm$, $M_n^\pm$, $\nu^\pm$ similarly to (4.3) (omitting the index j), so that

$$X_n(t) = X^+(t) - X^-(t), \quad M_n(t) = M^+(t) - M^-(t), \quad \nu_n(t) := \nu_n^+(t) + \nu_n^-(t).$$

Analogously to reasoning of the previous section (Propositions 4.1 and 4.4), we have that the sequence

$$\{(X_n^+, X_n^-, X_n, M_n^+, M_n^-, M_n, \nu_n^+, \nu_n^-, \nu_n, Y_n)\}_{n\ge 0}$$

is weakly relatively compact in $D(\mathbb{R}_+, \mathbb{R}^{10})$ each its limit point $(X^+, X^-, X, M^+, M^-, M, \nu^+, \nu^-, \nu, Y)$ is continuous, and

$$X(t) = X^+(t) - X^-(t), \quad M(t) = M^+(t) - M^-(t), \quad \nu(t) = \nu^+(t) + \nu^-(t),$$

where M^+ and M^- are local martingales with the brackets

$$\langle M^{\pm}\rangle_t = \frac{1}{m}\int_0^t 1\!\!1(X^{\pm}(s) > 0)\,\mathrm{d}s.$$

Moreover, the process $\sqrt{m}|X| = \sqrt{m}(X^+ + X^-)$ is a standard reflected Brownian motion, $\sqrt{m}\nu$ is its local time at 0, and the process $\sqrt{m}Y$ is a standard $(m-1)$-dimensional Brownian motion independent of X.

Hence to prove Theorem 5.3 it remains to show that

$$\frac{\nu^+(t)}{\nu(t)} = \bar{p} \quad \text{a.s. for all } t > 0. \tag{5.2}$$

To verify this, it suffices to prove that

$$\lim_{n\to\infty} \frac{\nu_n^+(t)}{\nu_n(t)} = \lim_{n\to\infty} \frac{L^+([nt])}{L([nt])} = \bar{p} \quad \text{a.s. for all } t > 0. \tag{5.3}$$

2. Let τ_k be the moment of kth visit of X to 0, $k \geq 0$, $\tau_0 = 0$, so that $Z(\tau_k) = (0, Y(\tau_k))$. By the strong Markov property we have

$$\mathrm{Law}(\tau_k - (\tau_{k-1}+1)) = \mathrm{Law}(\tau | X(0) = 1), \quad k \geq 1,$$

and

$$\mathrm{Law}(Y(\tau_k) - Y(\tau_{k-1})) = \mathrm{Law}(Y(\tau) | X(0) = 1, Y(0) = 0), \quad k \geq 1,$$

where τ is the first return time to zero of X,

$$\tau = \inf\{k \geq 1 \colon X(k) = 0\}.$$

It is well known that $\tau_k < \infty$ a.s. and $\tau_k \to +\infty$ a.s. as $k \to \infty$. Hence

$$Y(\tau_n) = \sum_{k=1}^{n} \Big(Y(\tau_k) - Y(\tau_{k-1})\Big), \quad n \geq 0,$$

is a random walk on $\mathbb{Z}^{m-1}$ whose jumps have the probability distribution $\mathrm{Law}(Y(\tau)|X(0) = 1, Y(0) = 0)$.

We claim that

$$\frac{Y(\tau_n)}{n} \xrightarrow{\mathrm{d}} S, \quad n \to \infty, \tag{5.4}$$

where S is a 1-stable random variable on $\mathbb{R}^{m-1}$ with the characteristic function $\mathbf{E}\mathrm{e}^{\mathrm{i}\langle u,S\rangle} = \mathrm{e}^{-|u|/\sqrt{m}}$, $u \in \mathbb{R}^{m-1}$. Indeed, for each $n \geq 1$ consider the symmetric random walk $\tilde{Z} = (\tilde{X}, \tilde{Y})$ on $\mathbb{Z}^m$ starting at $\tilde{Z}(0) = (n, 0, \ldots, 0)$. Denote

$$\begin{aligned}
\tilde{\tau}_0 &= 0,\\
\tilde{\tau}_1 &= \inf\{k \geq 1 \colon \tilde{X}(k) = n-1\},\\
&\ \cdots\\
\tilde{\tau}_n &= \inf\{k \geq 1 \colon \tilde{X}(n) = 0\}.
\end{aligned}$$

Then clearly

$$(\tau_1, \ldots, \tau_n, Y(\tau_1), \ldots, Y(\tau_n)) \stackrel{\mathrm{d}}{=} (\tilde{\tau}_1, \ldots, \tilde{\tau}_n, \tilde{Y}(\tilde{\tau}_1), \ldots, \tilde{Y}(\tilde{\tau}_n))$$

By the functional central limit theorem,

$$\mathrm{Law}\Big(\sqrt{m}\frac{\tilde{Z}([n^2\,\cdot])}{n}\Big|\tilde{Z}(0) = (n, 0, \ldots, 0)\Big) \Rightarrow \mathrm{Law}\Big(W(\cdot)\Big|W(0) = (1, 0\ldots, 0)\Big),$$

where W is a standard m-dimensional Brownian motion, and thus

$$\Big(n^2\tilde{\tau}_n, n^{-1}\sqrt{m}\tilde{Y}(\tilde{\tau}_n)\Big) \xrightarrow{\mathrm{d}} \Big(\tau^W, \big(W_2(\tau^W), \dots, W_m(\tau^W)\big)\Big),$$

where $\tau^W = \inf\{t \geq 0\colon W_1(t) = 0\}$. It is well known that $(W_2(\tau^W), \dots, W_m(\tau^W))$ is a 1-stable random vector, see, e.g., Theorem II.1.16 in Bass [2].

3. With (5.4) in hand, we apply Theorem 1.3 from Dolgopyat et al. [4], that states that under the assumption $\mathbf{A}_{\mathrm{SLLN}}(\beta)$ with $\beta > 1$

$$\frac{1}{n}\sum_{k=0}^{n-1} p_{Y(\tau_k)} \to \bar{p} \ \text{ a.s.}, \quad n \to \infty.$$

4. Now we are able to finish the proof of (5.2) Then

$$\begin{aligned} L^+(n) &= \sum_{k=0}^{n-1} \mathbb{I}(X(k) = 0, X(k+1) = 1) = \sum_{k\colon \tau_k \leq n-1} \mathbb{I}(X(\tau_k + 1) = 1), \\ L(n) &= \sum_{k=0}^{n-1} \mathbb{I}(X(k) = 0) = \max\{k \geq 1\colon \tau_k \leq n-1\}. \end{aligned}$$

Since $\tau_k \to +\infty$ a.s., $L(n) \to \infty$ a.s. as well as at least one of the processes $L^+(\cdot)$ and $L^-(\cdot)$.

Consider the process

$$Q^+(n) := \sum_{k=0}^{n-1} \mathbb{I}(X(\tau_k + 1) = 1),$$

and note that

$$\begin{aligned} \lim_{n\to\infty} \frac{L^+(n)}{L(n)} = \lim_{n\to\infty} \frac{Q^+(n)}{n} &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\Big(\mathbb{I}(X(\tau_k + 1) = 1) - p_{Y(\tau_k)}\Big) + \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} p_{Y(\tau_k)} \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\Big(\mathbb{I}(X(\tau_k + 1) = 1) - p_{Y(\tau_k)}\Big) + \bar{p}. \end{aligned}$$

The process

$$V(n) := \sum_{k=0}^{n-1}\Big(\mathbb{I}(X(\tau_k + 1) = 1) - p_{Y(\tau_k)}\Big)$$

is a martingale difference with

$$\mathbf{E}\Big[\mathbb{I}(X(\tau_k + 1) = 1) - p_{Y(\tau_k)}\Big] = 0 \quad \text{and} \quad \mathbf{E}\Big[\mathbb{I}(X(\tau_k + 1) = 1) - p_{Y(\tau_k)}\Big]^2 \leq 2.$$

Hence by the strong law of large numbers for martingales (Theorem 8b in Chapter II, §3 of Gikhman and Skorokhod [6]) we have convergence

$$\frac{V(n)}{n} \to 0 \ \text{ a.s.}, \quad n \to \infty,$$

what finishes the proof of Theorem 5.3. □

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