VERY FAT GEOMETRIC GALTON-WATSON TREES

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Abstract. Let τ_n be a random tree distributed as a Galton-Watson tree with geometric offspring distribution conditioned on $\{Z_n = a_n\}$ where Z_n is the size of the n-th generation and $(a_n, n \in \mathbb{N}^*)$ is a deterministic positive sequence. We study the local limit of these trees τ_n as $n \to \infty$ and observe three distinct regimes: if $(a_n, n \in \mathbb{N}^*)$ grows slowly, the limit consists in an infinite spine decorated with finite trees (which corresponds to the size-biased tree for critical or subcritical offspring distributions), in an intermediate regime, the limiting tree is composed of an infinite skeleton (that does not satisfy the branching property) still decorated with finite trees and, if the sequence $(a_n, n \in \mathbb{N}^*)$ increases rapidly, a condensation phenomenon appears and the root of the limiting tree has an infinite number of offspring.

1. Introduction

A Galton-Watson (GW for short) process $(Z_n, n \geq 0)$ describes the size of an evolving population where, at each generation, every extant individual reproduces according to the same offspring distribution p independently of the rest of the population. The associated genealogical tree τ is called a GW tree. Let μ denote the mean number of offspring per individual, that is the mean of p. When p is non degenerate, a classical result states that if $\mu < 1$ (sub-critical case) or $\mu = 1$ (critical case), then the population becomes a.s. extinct (i.e. $Z_n = 0$ for some $n \geq 0$ a.s.) whereas if $\mu > 1$ (super-critical case), the population has a positive probability of non extinction.

Another classical result from Kesten's work [7] describes the local limit in distribution of a critical or subcritical GW tree conditioned on $\{Z_n > 0\}$ as $n \to \infty$, which can be seen as a critical or sub-critical GW tree conditioned on non-extinction. The limiting tree is the so-called sized-biased tree or Kesten tree, and it can also be viewed as a two-type GW tree.

There are other ways of conditioning the tree of being large: conditioning on having a large total population size, or a large number of leaves... In the critical case, all these conditionings lead to the same local limit, see [2] and the references therein. In the sub-critical case, a condensation phenomenon (i.e. a vertex with an infinite number of offspring at the limit) may happen. This phenomenon has been pointed out first in [6], see also [5], [1] or [8]. But even there, there can be only two different limiting trees, a size-biased GW tree or a condensation tree.

In order to have different limits, an idea is to condition the tree to be even bigger, i.e. to consider conditionings of the form $\{Z_n = a_n\}$ for some positive deterministic sequence $(a_n, n \in \mathbb{N}^*)$ possibly converging to infinity. Some results on branching processes conditioned on their limit behaviour already appeared in previous works, see for instance [10] where the distributions of the conditioned Yule process (which corresponds to a super-critical branching process) or a critical binary branching process are described via an infinitesimal generator and a martingale problem. The first study of local limits for GW trees with such a conditioning appears in [2] where it is proven that, if p is a critical offspring distribution with finite

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variance, then the tree conditioned on $\{Z_n = a_n\}$ converges in distribution to the associated sized-biased tree if and only if $\lim_{n\to\infty} a_n n^{-2} = 0$.

The goal of this paper is to study what happens beyond that condition and to consider the sub-critical and super-critical cases. We give a complete description of all the cases when the offspring distribution is a geometric distribution with a Dirac mass at 0 (in that case, the distribution of Z_n is explicit). We observe three regimes according to the speed of growth of $(a_n, n \in \mathbb{N}^*)$. We set:

$$c_n = \begin{cases} \mu^{-n} & \text{if } \mu < 1 \text{ (sub-critical case)}, \\ n^2 & \text{if } \mu = 1 \text{ (critical case)}, \\ \mu^n & \text{if } \mu > 1 \text{ (super-critical case)}, \end{cases}$$

and we shall consider that:

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \theta \in [0, +\infty].$$

Let $\tau^{0,0}$ denote the GW tree τ conditioned on the extinction event $\mathcal{E} = \bigcup_{n\in\mathbb{N}^*}\{Z_n = 0\}$. Notice that $\tau^{0,0}$ is distributed as τ in the sub-critical and critical cases.

- In the **Kesten regime** ($\theta = 0$), the limiting tree, τ^0, is the Kesten tree, which is a two-type GW tree, with an infinite spine corresponding to the individuals having an infinite progeny (called the survivor type), on which are grafted independent GW trees distributed as $\tau^{0,0}$ corresponding to individuals having a finite progeny (called extinction type).
- In the **Poisson regime** ($\theta \in (0, +\infty)$), the limiting tree, τ^θ, is no more a GW tree, but it still has two types, with a backbone without leaves corresponding to individuals having an infinite progeny (also called the survivor type), on which are grafted independent GW trees distributed as $\tau^{0,0}$. However, the backbone can not be seen as a GW tree, as it lacks the branching property. This is more like a random tree with a Poissonian immigration at each generation with rates depending on θ and with all the configurations having the same probability.
- In the **condensation regime** ($\theta = +\infty$), the limiting tree τ^∞ is again a two-type GW tree, with a backbone without leaves corresponding to individuals having an infinite progeny (also called the survivor type), on which are grafted independent GW trees distributed as $\tau^{0,0}$. The backbone can be seen as an inhomogeneous GW tree with the root having an infinite number of children (condensation regime), and super-critical offspring distribution at level $h > 0$ with finite mean μ_h which decreases to 1 as h goes to infinity.

We also prove that the family $(\tau^\theta, \theta \in [0, +\infty])$ is continuous in distribution (the most interesting cases are the continuity at 0 and $+\infty$), see Remark 5.2 and Proposition 6.3.

Remark 1.1. The main ingredient of the proofs is Equation (3) and hence is the limit of the ratio

$$\lim_{n\to+\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)}$$

which is closely related to the extremal space-time harmonic functions associated with the GW process, see [10]. This limit is computed in the Kesten regime at the end of the proof of Proposition 4.2, and at the end of the proof of Proposition 5.3 in the Poisson regime. In the condensation regime, this limit is 0. Notice that in this regime, the conditioned Galton-Watson process converges to a trivial process which is always equal to $+\infty$ (except at $n = 0$) but considering the genealogical tree gives a non-trivial limit.

Partial results in a more general setting for super-critical and some sub-critical cases are given in [3]: convergence of τ_n in the Kesten and the Poisson regimes for general offspring distributions, and in the condensation regime in the Harris case (offspring distribution with bounded support), the continuity in distribution of the family of limiting trees at $\theta = 0$ and some partial results at $\theta = +\infty$. Some similar results can also be derived for sub-critical offspring distributions under strong additional assumptions.

The rest of the paper is organized as follows: Section 2 introduces the framework of discrete trees with the notion of local convergence for sequences of trees, the GW trees and some properties of the geometric distribution. Section 3 describes the GW tree with geometric offspring distribution with some technical lemmas that are used in the proofs of the main theorems. Section 4 studies the Kesten regime, where the Kesten tree τ^0 is defined and the convergence in distribution of τ_n to τ^0 is stated (Proposition 4.2). In Section 5, the family of random trees $(\tau^\theta, \theta \in (0, +\infty))$ is introduced and a convergence result is obtained for the Poisson regime (Proposition 5.3) as well as the continuity in distribution of $(\tau^\theta, \theta \in (0, +\infty))$ at $\theta = 0$ (Remark 5.2). Finally, Section 6 introduces the condensation tree τ^∞, proves the convergence of τ_n to τ^∞ in the condensation regime (Proposition 6.4) and the continuity in distribution of $(\tau^\theta, \theta \in (0, +\infty))$ at $\theta = +\infty$ (Proposition 6.3).

2. Notations

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, by $\mathbb{N}^* = \{1, 2, \ldots\}$ the set of positive integers and $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. For any finite set E, we denote by $\sharp E$ its cardinal.

2.1. **The set of discrete trees.** We recall Neveu's formalism [9] for ordered rooted trees. Let $\mathcal{U} = \bigcup_{n\geq 0} (\mathbb{N}^*)^n$ be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. We also set $\mathcal{U}^* = \bigcup_{n\geq 1} (\mathbb{N}^*)^n = \mathcal{U} \backslash \{\emptyset\}$.

For $u \in \mathcal{U}$, let $|u|$ be the length or the generation of u defined as the integer n such that $u \in (\mathbb{N}^*)^n$. If u and v are two sequences of $\mathcal{U}$, we denote by uv the concatenation of two sequences, with the convention that $uv = vu = u$ if $v = \emptyset$.

The set of strict ancestors of $u \in \mathcal{U}^*$ is defined by:

$$\mathrm{Anc}(u) = \{v \in \mathcal{U},\ \exists w \in \mathcal{U}^*,\ u = vw\},$$

and for $\mathscr{S} \subset \mathcal{U}^*$, being non-empty, we set $\mathrm{Anc}(\mathscr{S}) = \bigcup_{u\in\mathscr{S}} \mathrm{Anc}(u)$.

A tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies :

- $\emptyset \in \mathbf{t}$.
- If $u \in \mathbf{t}$, then $\mathrm{Anc}(u) \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \bar{\mathbb{N}}$ such that, for every positive integer i, $ui \in \mathbf{t} \iff 1 \leq i \leq k_u(\mathbf{t})$.

We denote by $\mathbb{T}_\infty$ the set of trees. Let $\mathbf{t} \in \mathbb{T}_\infty$ be a tree. The vertex $\emptyset$ is called the root of the tree $\mathbf{t}$ and we denote by $\mathbf{t}^* = \mathbf{t}\backslash\{\emptyset\}$ the tree without its root. For a vertex $u \in \mathbf{t}$, the integer $k_u(\mathbf{t})$ represents the number of offspring (also called the out-degree) of the vertex $u \in \mathbf{t}$. By convention, we shall write $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. The height $H(\mathbf{t})$ of the tree $\mathbf{t}$ is defined by:

$$H(\mathbf{t}) = \sup\{|u|,\ u \in \mathbf{t}\} \in \bar{\mathbb{N}}.$$

For $n \in \mathbb{N}$, the size of the n-th generation of $\mathbf{t}$ is defined by:

$$z_n(\mathbf{t}) = \sharp\{u \in \mathbf{t}, |u| = n\}.$$

We denote by $\mathbb{T}_\mathrm{f}^*$ the subset of trees with finite out-degrees except the root's:

$$\mathbb{T}_\mathrm{f}^* = \{\mathbf{t} \in \mathbb{T}_\infty;\, \forall u \in \mathbf{t}^*,\ k_u(\mathbf{t}) < +\infty\}$$

and by $\mathbb{T}_\mathrm{f} = \{\mathbf{t} \in \mathbb{T}_\mathrm{f}^*;\, k_\emptyset(\mathbf{t}) < +\infty\}$ the subset of trees with finite out-degrees.

Let $h, k \in \mathbb{N}^*$. We define $\mathbb{T}_\mathrm{f}^{(h)}$ the subset of finite trees with height h:

$$\mathbb{T}_\mathrm{f}^{(h)} = \{\mathbf{t} \in \mathbb{T}_\mathrm{f};\, H(\mathbf{t}) = h\}$$

and $\mathbb{T}_k^{(h)} = \{\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)};\, k_\emptyset(\mathbf{t}) = k\}$ the subset of finite trees with height equal to h and out-degree of the root equal to k. We also define the restriction operators r_h and $r_{h,k}$, for every $\mathbf{t} \in \mathbb{T}_\infty$, by:

$$r_h(\mathbf{t}) = \{u \in \mathbf{t};\ |u| \le h\} \quad \text{and} \quad r_{h,k}(\mathbf{t}) = \{\emptyset\} \cup \{u \in r_h(\mathbf{t})^*;\, u_1 \le k\},$$

where u_1 represents the first term of the sequence u if $u \neq \emptyset$. In other words, $r_h(\mathbf{t})$ represents the tree $\mathbf{t}$ truncated at height h and $r_{h,k}(\mathbf{t})$ represents the subtree of $r_h(\mathbf{t})$ where only the k-first offspring of the root are kept. Remark that, for $\mathbf{t} \in \mathbb{T}_\mathrm{f}$, if $H(\mathbf{t}) \ge h$ then $r_h(\mathbf{t}) \in \mathbb{T}_\mathrm{f}^{(h)}$ and if furthermore $k_\emptyset(\mathbf{t}) \ge k$ then $r_{h,k}(\mathbf{t}) \in \mathbb{T}_k^{(h)}$.

2.2. **Convergence of trees.** Set $\mathbb{N}_1 = \{-1\} \cup \bar{\mathbb{N}}$, endowed with the usual topology of the one-point compactification of the discrete space $\{-1\} \cup \mathbb{N}$. For a tree $\mathbf{t} \in \mathbb{T}_\infty$, recall that by convention the out-degree $k_u(\mathbf{t})$ of u is set to -1 if u does not belong to $\mathbf{t}$. Thus a tree $\mathbf{t} \in \mathbb{T}_\infty$ is uniquely determined by the sequence $(k_u(\mathbf{t}), u \in \mathcal{U})$ and then $\mathbb{T}_\infty$ is a subset of $\mathbb{N}_1^{\mathcal{U}}$. By Tychonoff theorem, the set $\mathbb{N}_1^{\mathcal{U}}$ endowed with the product topology is compact. Since $\mathbb{T}_\infty$ is closed it is thus compact. In fact, the set $\mathbb{T}_\infty$ is a Polish space (but we don't need any precise metric at this point). The convergence of sequences of trees is then characterized as follows. Let $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ be trees in $\mathbb{T}_\infty$. We say that $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} k_u(\mathbf{t}_n) = k_u(\mathbf{t})$ for all $u \in \mathcal{U}$. It is easy to see that:

- If $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ are trees in $\mathbb{T}_\mathrm{f}$, then we have $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} r_h(\mathbf{t}_n) = r_h(\mathbf{t})$ for all $h \in \mathbb{N}^*$.
- If $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ are trees in $\mathbb{T}_\mathrm{f}^*$, then we have $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} r_{h,k}(\mathbf{t}_n) = r_{h,k}(\mathbf{t})$ for all $h, k \in \mathbb{N}^*$.

Let T be a $\mathbb{T}_\mathrm{f}$-valued (resp. $\mathbb{T}_\mathrm{f}^*$-valued) random variable. It is easy to get that if a.s. $H(T) = +\infty$ (resp. a.s. $H(T) = +\infty$ and $k_\emptyset(T) = +\infty$), then the distribution of T is characterized by $\left(\mathbb{P}(r_h(T) = \mathbf{t});\, h \in \mathbb{N}^*,\, \mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}\right)$ (resp. $\left(\mathbb{P}(r_{h,k}(T) = \mathbf{t});\, h, k \in \mathbb{N}^*,\, \mathbf{t} \in \mathbb{T}_k^{(h)}\right)$). Using the Portmanteau theorem, we deduce the following results:

- Let $(T_n, n \in \mathbb{N})$ and T be $\mathbb{T}_\mathrm{f}$-valued random variables. Then we have the following characterization of the convergence in distribution if a.s. $H(T) = +\infty$:

$$T_n \xrightarrow[n\to\infty]{\textbf{(d)}} T \iff \lim_{n\to\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t}) \quad \text{for all } h \in \mathbb{N}^*,\, \mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}. \tag{1}$$

- Let $(T_n, n \in \mathbb{N})$ and T be $\mathbb{T}_\mathrm{f}^*$-valued random variables. Then we have the following characterization of the convergence in distribution if a.s. $H(T) = +\infty$, $k_\emptyset(T) = +\infty$:

$$T_n \xrightarrow[n\to\infty]{\textbf{(d)}} T \iff \lim_{n\to\infty} \mathbb{P}(r_{h,k}(T_n) = \mathbf{t}) = \mathbb{P}(r_{h,k}(T) = \mathbf{t}) \quad \text{for all } h, k \in \mathbb{N}^*,\, \mathbf{t} \in \mathbb{T}_k^{(h)}. \tag{2}$$

2.3. **GW trees.** Let $p = (p(n), n \in \mathbb{N})$ be a probability distribution on $\mathbb{N}$. A $\mathbb{T}_{\mathrm{f}}$-valued random variable τ is called a GW tree with offspring distribution p if for all $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}$ with $H(\mathbf{t}) \leq h$:

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t})).$$

The generation size process defined by $(Z_n = z_n(\tau), n \in \mathbb{N})$ is the so called GW process. We refer to [4] for a general study of GW processes. We set $\mathbb{P}_k$ the probability under which the GW process $(Z_n, n \in \mathbb{N})$ starts with $Z_0 = k$ individuals and write $\mathbb{P}$ for $\mathbb{P}_1$ so that:

$$\mathbb{P}_k(Z_n = a) = \mathbb{P}(Z_n^{(1)} + \cdots + Z_n^{(k)} = a),$$

where the $(Z^{(i)}, 1 \leq i \leq k)$ are independent copies of Z under $\mathbb{P}$.

We consider a sequence $(a_n, n \in \mathbb{N}^*)$ of elements in $\mathbb{N}^*$ and, when $\mathbb{P}(Z_n = a_n) > 0$, τ_n a random tree distributed as the GW tree τ conditionally on $\{Z_n = a_n\}$. Let $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$. We have by the branching property of GW-trees at height h, setting $k = z_h(\mathbf{t})$:

$$\mathbb{P}(r_h(\tau_n) = \mathbf{t}) = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \cdot \tag{3}$$

2.4. **Geometric distribution.** Let $\eta \in (0, 1]$ and $q \in (0, 1)$. We define the geometric $\mathcal{G}(\eta, q)$ distribution $p = (p(k), k \in \mathbb{N})$ by

$$\begin{cases} p(0) = 1 - \eta, \\ p(k) = \eta q (1-q)^{k-1} \quad \text{for } k \in \mathbb{N}^*. \end{cases} \tag{4}$$

We shall always consider that τ is a GW tree with geometric offspring distribution $\mathcal{G}(\eta, q)$.

The mean of $\mathcal{G}(\eta, q)$ is given by $\mu = \eta/q$ and its generating function f is given by:

$$f(s) = \frac{(1-\eta) - s(1-q-\eta)}{1 - s(1-q)}, \quad s \in [0, 1/(1-q)).$$

We set:

$$\gamma = \frac{1}{1-q} \quad \text{and} \quad \kappa = \frac{1-\eta}{1-q} \tag{5}$$

where γ is the radius of convergence of f and κ and 1 are the only fixed points of f on $[0, \gamma)$. If $\mu = 1$ then there is only one fixed point as $\kappa = 1$. We shall use frequently the following relations:

$$\gamma - \kappa = \mu(\gamma - 1) \quad \text{and, if } \mu \neq 1, \quad \gamma - 1 = \frac{\kappa - 1}{1 - \mu} \cdot \tag{6}$$

Notice that $\kappa \in [0, +\infty)$ and $\gamma \in (1, +\infty)$ allow to recover η and q as:

$$\eta = 1 - \frac{\kappa}{\gamma} \quad \text{and} \quad q = 1 - \frac{1}{\gamma} \cdot \tag{7}$$

For this reason, we shall also write $\mathcal{G}[\kappa, \gamma]$ for $\mathcal{G}(\eta, q)$. Notice that if $\mu < 1$, then $q > \eta$ and $\gamma > \kappa > 1$; and if $\mu > 1$, then $\eta > q$ and $\gamma > 1 > \kappa \geq 0$.

Since f is an homography, we get for $s \in [0, \gamma) \backslash \{1\}$:

$$\frac{f(s) - \kappa}{f(s) - 1} = \frac{1}{\mu} \frac{s - \kappa}{s - 1} \cdot \tag{8}$$

We set $f_1 = f$ and, for $n \in \mathbb{N}^*$, $f_{n+1} = f \circ f_n$. Notice that κ is a fixed point of f_n as it is a fixed point of f. We deduce from (8) and the second equality of (6) if $\mu \neq 1$ and by direct recurrence if $\mu = 1$, that f_n, for $n \in \mathbb{N}^*$, is the generating function of the geometric distribution $\mathcal{G}[\kappa, \gamma_n] = \mathcal{G}(\eta_n, q_n)$ with mean $\mu_n = \mu^n$ and, thanks to (7):

$$\eta_n = 1 - \frac{\kappa}{\gamma_n}, \quad q_n = 1 - \frac{1}{\gamma_n} \quad \text{with } \gamma_n = \begin{cases} \dfrac{\kappa - \mu^n}{1 - \mu^n} = 1 + (\gamma - 1)\dfrac{q^{n-1}(q-\eta)}{q^n - \eta^n} & \text{if } \mu \neq 1, \\ 1 + (\gamma - 1)\,\frac{1}{n} & \text{if } \mu = 1. \end{cases} \tag{9}$$

By convention, we set f_0 the identity function defined on $[0, +\infty)$ and $\gamma_0 = +\infty$ so that for all $n \in \mathbb{N}$, we have $\gamma_n = \lim_{r \to +\infty} f_n^{-1}(r)$ that is in short $\gamma_n = f_n^{-1}(\infty)$. We deduce that for all $n \geq \ell \geq 0$:

$$f_\ell(\gamma_n) = \gamma_{n-\ell}. \tag{10}$$

We derive some asymptotics for γ_n for large n. It is easy to deduce from (9) that:

$$\lim_{n\to\infty} \gamma_n = \max(1, \kappa) = \begin{cases} \kappa & \text{if } \mu \leq 1, \\ 1 & \text{if } \mu \geq 1. \end{cases} \tag{11}$$

Using (6), we get for large n:

$$(\gamma_n - \kappa)(\gamma_n - 1) = \begin{cases} \mu^n(\kappa-1)^2 + O(\mu^{2n}) & \text{if } \mu < 1, \\ (\gamma-1)^2 n^{-2} & \text{if } \mu = 1, \\ \mu^{-n}(\kappa-1)^2 + O(\mu^{-2n}) & \text{if } \mu > 1. \end{cases} \tag{12}$$

We derive from (9) the logarithm asymptotics of γ_n/γ_{n-h} for given $h \in \mathbb{N}^*$ and large n:

$$\log(\gamma_{n-h}/\gamma_n) = \log(\gamma_{n-h}) - \log(\gamma_n) = \begin{cases} \mu^{n-h}\left(1 - \mu^h\right)(\kappa-1)/\kappa + O(\mu^{2n}) & \text{if } \mu < 1, \\ (\gamma-1)hn^{-2} + O(n^{-3}) & \text{if } \mu = 1, \\ \mu^{-n}\left(\mu^h - 1\right)(1-\kappa) + O(\mu^{-2n}) & \text{if } \mu > 1. \end{cases} \tag{13}$$

We recall the following well-known equality which holds for all $k \in \mathbb{N}^*$ and $r \in (0,1)$:

$$\sum_{\ell \geq k} \binom{\ell-1}{k-1} r^\ell = \left(\frac{r}{1-r}\right)^k. \tag{14}$$

And we end this section with an elementary lemma.

Lemma 2.1. *Let $(X_\ell, \ell \in \mathbb{N}^*)$ be independent random variables with distribution $\mathcal{G}(\eta, q) = \mathcal{G}[\kappa, \gamma]$. For $a \geq k \geq 1$:*

$$\mathbb{P}\left(\sum_{\ell=1}^k X_\ell = a\right) = \sum_{i=1}^k \binom{k}{i}\binom{a-1}{i-1}\kappa^{k-i}(\gamma-\kappa)^i(\gamma-1)^i\gamma^{-a-k}.$$

Proof. We have:

$$\begin{aligned} \mathbb{P}\left(\sum_{\ell=1}^k X_\ell = a\right) &= \sum_{i=1}^k \binom{k}{i}\mathbb{P}(X_1 = 0)^{k-i}\,\mathbb{P}\left(\sum_{\ell=1}^i X_\ell = a,\ X_\ell \geq 1 \text{ for } \ell \in \{1, \ldots, i\}\right) \\ &= \sum_{i=1}^k \binom{k}{i}(1-\eta)^{k-i}\binom{a-1}{i-1}(\eta q)^i(1-q)^{a-i}. \end{aligned} \tag{15}$$

Then use (7) to conclude. □

3. The geometric GW tree

Let τ be a GW tree with geometric $\mathcal{G}(\eta, q)$ offspring distribution p given by (4), with $\eta \in (0,1]$ and $q \in (0,1)$. Recall that $(Z_n, n \in \mathbb{N})$ is the associated GW process.

For $k \in \mathbb{N}^*$, we denote by $\mathbb{P}_k$ the distribution of the geometric GW forest composed of k independent GW trees with geometric offspring distribution $\mathcal{G}(\eta, q)$, and write $\mathbb{P}$ for $\mathbb{P}_1$. For convenience, we shall under $\mathbb{P}$ denote by $Z^{(k)} = (Z_n^{(k)}, n \in \mathbb{N})$ a GW process distributed as $Z = (Z_n, n \in \mathbb{N})$ under $\mathbb{P}_k$. For $n \in \mathbb{N}^*$, we set:

$$M_n = \gamma_1^{-Z_1} \, \gamma_n^{Z_n}. \tag{16}$$

Since Z_n has generating function f_n under $\mathbb{P}$, we deduce from (10) that $(M_n, n \in \mathbb{N}^*)$ is a martingale with $M_1 = 1$.

For $n > h \geq 1$, we set:

$$b_{n,h} = \left(\frac{\gamma_n}{\gamma_{n-h}} \right)^{a_n}. \tag{17}$$

We shall use the following formula when $\lim_{n\to\infty} b_{n,h}$ exists and belongs to $(0, \infty)$.

Lemma 3.1. *Let $n > h \geq 1$ and $k \in \mathbb{N}^*$. We have:*

$$\frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = b_{n,h} \sum_{i=1}^{k} \binom{k}{i} \kappa^{k-i} \, G_{n,h}(k,i), \tag{18}$$

with

$$G_{n,h}(k,i) = \binom{a_n - 1}{i-1} \frac{\gamma_n}{\gamma_{n-h}^k} \frac{(\gamma_{n-h} - \kappa)^i (\gamma_{n-h} - 1)^i}{(\gamma_n - \kappa)(\gamma_n - 1)}. \tag{19}$$

Proof. Let $n > h \geq 1$. Since Z_n has distribution $\mathcal{G}[\kappa, \gamma_n]$, we obtain thanks to (5):

$$\mathbb{P}(Z_n = a_n) = \eta_n q_n (1 - q_n)^{a_n - 1} = (\gamma_n - \kappa)(\gamma_n - 1)\gamma_n^{-a_n - 1}.$$

Using that Z_{n-h} is under $\mathbb{P}_k$ distributed as the sum of k independent random variables with distribution $\mathcal{G}[\kappa, \gamma_{n-h}]$, we deduce from Lemma 2.1 that:

$$\begin{aligned}
\frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} &= \sum_{i=1}^{k} \binom{k}{i} \binom{a_n - 1}{i-1} \kappa^{k-i} \frac{(\gamma_{n-h} - \kappa)^i (\gamma_{n-h} - 1)^i}{\gamma_{n-h}^{a_n + k}} \frac{\gamma_n^{a_n+1}}{(\gamma_n - \kappa)(\gamma_n - 1)} \\
&= b_{n,h} \sum_{i=1}^{k} \binom{k}{i} \kappa^{k-i} G_{n,h}(k,i).
\end{aligned}$$

This gives the result. □

We shall use the following formula when $\lim_{n\to\infty} b_{n,h} = 0$ and $\lim_{n\to\infty} a_n = +\infty$.

Lemma 3.2. *Let $n > h \geq 1$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$. We have, with $a_n \geq k = z_h(\mathbf{t})$:*

$$\mathbb{P}(r_{h,k_0}(\tau_n) = \mathbf{t}) = \frac{1-q}{\eta q} \mathbb{P}(r_h(\tau) = \mathbf{t}) \left(\gamma_h^k - R_{n,h}^1(k) - R_{n,h}^2(k) \right), \tag{20}$$

with $\alpha_n = (\gamma_{n-h} - \kappa)(\gamma_{n-h} - 1)$, $x_n = \gamma_n/\gamma_{n-h}$ *and:*

$$0 \leq R^1_{n,h}(k) \leq b_{n,h} \, \frac{\alpha_n}{1 - x_n} \, \max(1, \kappa)^{k-1} \, 2^{2k-1} \left(2 + \left(\frac{\alpha_n}{1 - x_n} \right)^{k-1} + (\alpha_n a_n)^{k-1} \right), \tag{21}$$

$$R^2_{n,h}(k) = (\kappa + 1 - \gamma) \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \cdot \tag{22}$$

Proof. Let $n > h \geq 1$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}^{(h)}_{k_0}$. We set $k = z_h(\mathbf{t})$. For every $1 \leq j \leq k_0$, we denote by $\mathbf{t}_j$ the subtree rooted at the j-th offspring of the root i.e.

$$u \in \mathbf{t}_j \iff ju \in \mathbf{t}.$$

In what follows, we denote by $\tilde{Z}^{(i)}$ a process distributed as $Z^{(i)}$ and independent of $Z^{(k)}$. We have:

$$\begin{aligned}
\mathbb{P}(r_{h,k_0}(\tau_n) = \mathbf{t}) &= \sum_{i=0}^{+\infty} p(i + k_0) \left[\prod_{j=1}^{k_0} \mathbb{P}(r_{h-1}(\tau) = \mathbf{t}_j) \right] \frac{\mathbb{P}(Z^{(k)}_{n-h} + \tilde{Z}^{(i)}_{n-1} = a_n)}{\mathbb{P}(Z_n = a_n)} \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t}) \sum_{i=0}^{+\infty} (1 - q)^i \frac{\mathbb{P}(Z^{(k)}_{n-h} + \tilde{Z}^{(i)}_{n-1} = a_n)}{\mathbb{P}(Z_n = a_n)}
\end{aligned}$$

by the branching property

$$\begin{aligned}
&= \mathbb{P}(r_h(\tau) = \mathbf{t}) \sum_{i=1}^{+\infty} \frac{1 - q}{\eta q} p(i) \sum_{\ell=0}^{a_n} \frac{\mathbb{P}(Z^{(k)}_{n-h} = \ell) \mathbb{P}(Z^{(i)}_{n-1} = a_n - \ell)}{\mathbb{P}(Z_n = a_n)} \\
&\qquad + \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\mathbb{P}(Z^{(k)}_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \\
&= \frac{1 - q}{\eta q} \mathbb{P}(r_h(\tau) = \mathbf{t})(A + B),
\end{aligned}$$

where

$$\begin{aligned}
A &= p(0) \frac{\mathbb{P}(Z^{(k)}_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} + \sum_{\ell=0}^{a_n} \mathbb{P}(Z^{(k)}_{n-h} = \ell) \sum_{i=1}^{+\infty} p(i) \frac{\mathbb{P}(Z^{(i)}_{n-1} = a_n - \ell)}{\mathbb{P}(Z_n = a_n)} \\
&= \sum_{\ell=0}^{a_n} \mathbb{P}(Z^{(k)}_{n-h} = \ell) \sum_{i=0}^{+\infty} p(i) \frac{\mathbb{P}(Z^{(i)}_{n-1} = a_n - \ell)}{\mathbb{P}(Z_n = a_n)}
\end{aligned}$$

and

$$B = \left(\frac{\eta q}{1 - q} - p(0) \right) \frac{\mathbb{P}(Z^{(k)}_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \cdot$$

We have:

$$A = \sum_{\ell=0}^{a_n} \mathbb{P}(Z^{(k)}_{n-h} = \ell) \frac{\mathbb{P}(Z_n = a_n - \ell)}{\mathbb{P}(Z_n = a_n)} = \sum_{\ell=0}^{a_n} \mathbb{P}(Z^{(k)}_{n-h} = \ell) \gamma_n^\ell = \left(f_{n-h} (\gamma_n)^k - R^1_{n,h}(k) \right),$$

where we used that $k_\emptyset(\tau)$ has distribution p for the first equality, that Z_n has distribution $\mathcal{G}[\kappa, \gamma_n]$ for the second one and thus $\mathbb{P}(Z_n = k) = \eta_n q_n \gamma_n^{-(k-1)}$, and for the last one that:

$$R^1_{n,h}(k) = \sum_{\ell > 0} \mathbb{P}(Z^{(k)}_{n-h} = \ell + a_n) \gamma_n^{\ell + a_n}.$$

We have, with $\alpha_n = (\gamma_{n-h} - \kappa)(\gamma_{n-h} - 1)$ and $x_n = \gamma_n/\gamma_{n-h}$:

$$\begin{aligned}\mathbb{P}(Z_{n-h}^{(k)} = \ell + a_n)\gamma_n^{\ell+a_n} &= b_{n,h} \sum_{i=1}^{k} \binom{k}{i} \binom{\ell + a_n - 1}{i-1} \kappa^{k-i} (\gamma_{n-h} - \kappa)^i (\gamma_{n-h} - 1)^i \gamma_{n-h}^{-\ell-k} \gamma_n^{\ell} \\ &\leq b_{n,h}\, x_n^{\ell} \, \max(1,\kappa)^{k-1} \sum_{i=1}^{k} \binom{k}{i} \binom{\ell + a_n - 1}{i-1} \alpha_n^i,\end{aligned}$$

where we used Lemma 2.1 for the first equality and $\gamma_{n-h} \geq \max(1, \kappa)$ for the last. Using that $(x + y)^j \leq 2^{j-1}(x^j + y^j)$ for $j \in \mathbb{N}^*$ and $x, y \in (0, +\infty)$, we deduce that:

$$\binom{\ell + a_n - 1}{i-1} \leq \frac{2^{i-1}}{(i-1)!} \left(\ell^{i-1} + a_n^{i-1}\right).$$

We have the following rough bounds:

$$\begin{aligned}0 \leq R_{n,h}^1(k) &\leq b_{n,h} \, \max(1,\kappa)^{k-1} \, 2^{k-1} \sum_{i=1}^{k} \alpha_n^i \binom{k}{i} \sum_{\ell > 0} \left(\frac{\ell^{i-1}}{(i-1)!} x_n^{\ell} + a_n^{i-1} x_n^{\ell} \right) \\ &\leq b_{n,h} \, \frac{x_n \alpha_n}{1 - x_n} \, \max(1,\kappa)^{k-1} \, 2^{k-1} \sum_{i=1}^{k} \binom{k}{i} \left(\left(\frac{\alpha_n}{1 - x_n} \right)^{i-1} + (\alpha_n a_n)^{i-1} \right) \\ &\leq b_{n,h} \, \frac{\alpha_n}{1 - x_n} \, \max(1,\kappa)^{k-1} \, 2^{2k-1} \left(2 + \left(\frac{\alpha_n}{1 - x_n} \right)^{k-1} + (\alpha_n a_n)^{k-1} \right)\end{aligned}$$

where we used that $x_n \in (0, 1)$ as the sequence $(\gamma_m, m \in \mathbb{N}^*)$ is non-increasing and that $\sum_{\ell>0} \ell^{i-1} x^{\ell}/(i-1)! \leq x(1-x)^{i-1}$ for the last inequality but one. Then use (10), which gives $f_{n-h}(\gamma_n) = \gamma_h$, to get $A = \gamma_h^k - R_{n,h}^1(k)$ as well as (21).

We can rewrite the constant in B as

$$\frac{\eta q}{1-q} - p(0) = \frac{\eta q}{1-q} - (1 - \eta) = -(\kappa + 1 - \gamma),$$

so that $B = -R_{n,h}^2(k)$, see (22), and thus $A + B = \gamma_h^k - R_{n,h}^1(k) - R_{n,h}^2(k)$. This ends the proof. □

4. The Kesten regime or the not so fat case

4.1. **The Kesten tree.** In this section, we denote by τ a GW tree with geometric $p = \mathcal{G}(\eta, q)$ with $\eta, q \in (0, 1)$. It is well known that the extinction event $\mathcal{E} = \{H(\tau) < +\infty\}$ has probability $\mathfrak{c} = \min(1, \kappa)$. Moreover, as we assume $\eta < 1$, we have $\mathfrak{c} > 0$. We define the probability distribution $\mathfrak{p} = (\mathfrak{p}(n), n \in \mathbb{N})$ by:

$$\mathfrak{p}(n) = \mathfrak{c}^{n-1} p(n) \quad \text{for } n \in \mathbb{N}. \tag{23}$$

We denote by $\tau^{0,0}$ a random tree distributed as τ conditionally on the extinction event $\mathcal{E}$, that is a GW tree with offspring distribution $\mathfrak{p}$. We denote by $\mathfrak{m}$ the mean of $\mathfrak{p}$. If $\mu \leq 1$, then we have $\mathfrak{p} = p$, $\mathfrak{m} = \mu$, $\mathfrak{c} = 1$ and that $\tau^{0,0}$ is distributed as τ. If $\mu > 1$, then we have that $\mathfrak{p}$ is the geometric distribution $\mathcal{G}(q, \eta)$, $\mathfrak{m} = 1/\mu$ and $\mathfrak{c} = \kappa$.

Let $k \in \mathbb{N}^*$. We define the k-th order size-biased probability distribution of p as $p_{[k]} = (p_{[k]}(n), n \in \mathbb{N})$ defined by:

$$p_{[k]}(n) = \frac{n!}{(n-k)!f^{(k)}(1)}\, p(n) \quad \text{for } n \in \mathbb{N} \text{ and } n \geq k. \tag{24}$$

The generating function of $p_{[k]}$ is $f_{[k]}(s) = s^k f^{(k)}(s)/f^{(k)}(1)$. The probability distribution $p_{[1]}$ is the so-called size-biased probability distribution of p.

For the distribution $\mathcal{G}(\eta, q)$, we have $f^{(k)}(1) = k!\eta q^{-k}(1-q)^{k-1}$, so the k-th order size-biased probability distribution of p is given by:

$$p_{[k]}(n) = \binom{n}{k} q^{k+1}(1-q)^{n-k} \quad \text{for } n \in \mathbb{N} \text{ and } n \geq k. \tag{25}$$

We now define the so-called Kesten tree $\hat{\tau}^0$ associated with the offspring distribution p as a two-type GW tree where the vertices are either of type s (for survivor) or of type e (for extinction). It is then characterized as follows.

- The number of offspring of a vertex depends, conditionally on the vertices of lower or same height, only on its own type (branching property).
- The root is of type s.
- A vertex of type e produces only vertices of type e with offspring distribution $\mathfrak{p}$.
- The random number of children of a vertex of type s has the size-biased distribution of $\mathfrak{p}$ that is $\mathfrak{p}_{[1]}$ defined by (24) with $k = 1$. Furthermore, all of the children are of type e but one, uniformly chosen at random, which is of type s.

Informally the individuals of type s in $\hat{\tau}^0$ form an infinite spine on which are grafted independent GW trees distributed as $\tau^{0,0}$.

We define $\tau^0 = \mathrm{Ske}(\hat{\tau}^0)$ as the tree $\hat{\tau}^0$ when one forgets the types of the vertices. The distribution of τ^0 is given in the following classical result.

Lemma 4.1. *Let $p = \mathcal{G}(\eta, q)$ with $\eta, q \in (0,1)$. The distribution of τ^0 is characterized by: for all $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ with $k = z_h(\mathbf{t})$:*

$$\mathbb{P}(r_h(\tau^0)) = \mathbf{t}) = k\mathfrak{c}^{k-1}\mathfrak{m}^{-h}\, \mathbb{P}(r_h(\tau) = \mathbf{t}). \tag{26}$$

We give a short proof of this well-known result.

Proof. Since τ^0 belongs to $\mathbb{T}_{\mathrm{f}}$ and has infinite height, its distribution is indeed characterized by (26) for all $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ with $k = z_h(\mathbf{t})$.

Let $n \geq h \geq 1$, $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ and $v \in \mathbf{t}$ such that $|v| = h$. Let V be the vertex of type s at level h in $\hat{\tau}^0$. We have, with $k = z_h(\mathbf{t})$:

$$\begin{aligned}
\mathbb{P}(r_h(\tau^0) = \mathbf{t}, V = v) &= \prod_{u \in \mathbf{t} \setminus \mathrm{Anc}(\{v\});\, |u|<h} \mathfrak{p}(k_u(\mathbf{t})) \prod_{u \in \mathrm{Anc}(\{v\})} \frac{1}{k_u(\mathbf{t})}\, \mathfrak{p}_{[1]}(k_u(\mathbf{t})) \\
&= \mathfrak{m}^{-h}\mathfrak{c}^{\sum_{u\in r_{h-1}(\mathbf{t})}(k_u(\mathbf{t})-1)} \prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t})) \\
&= \mathfrak{m}^{-h}\mathfrak{c}^{k-1}\, \mathbb{P}(r_h(\tau) = \mathbf{t}),
\end{aligned}$$

where we used (24) (with $k = 1$, $n = k_u(\mathbf{t})$ and p replaced by $\mathfrak{p}$) and (23) (with $n = k_u(\mathbf{t})$) for the second equality and that $\sum_{u\in r_{h-1}(\mathbf{t})}(k_u(\mathbf{t}) - 1) = k - 1$ for the last one. Summing over all $v \in \mathbf{t}$ such that $|v| = h$ gives the result. □

4.2. **Convergence of the not so fat geometric GW tree.** We consider a sequence $(a_n, n \in \mathbb{N}^*)$ with $a_n \in \mathbb{N}^*$ and a random tree τ_n distributed as the GW tree τ with offspring distribution $p = \mathcal{G}(\eta, q)$ conditionally on $\{Z_n = a_n\}$. We have the following result.

Proposition 4.2. *Let $\eta \in (0,1)$ and $q \in (0,1)$. Assume that $\lim_{n\to\infty} a_n \mu^n = 0$ if $\mu < 1$, $\lim_{n\to\infty} a_n n^{-2} = 0$ if $\mu = 1$ or $\lim_{n\to\infty} a_n \mu^{-n} = 0$ if $\mu > 1$. Then we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{\text{(d)}} \tau^0.$$

The critical case, $\mu = 1$, appears in Corollary 6.2 of [2] for general offspring distribution with second moment.

Proof. Let $h \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$. Recall the definitions of $b_{n,h}$ in (17) and of $G_{n,h}$ in (19). According to Lemma 3.1, we have for $n > h \geq 1$ and $k \in \mathbb{N}^*$:

$$\frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = b_{n,h} \sum_{i=1}^{k} \binom{k}{i} \kappa^{k-i} G_{n,h}(k,i).$$

According to (17), we have $b_{n,h} = \exp\left(-a_n \log(\gamma_{n-h}/\gamma_n)\right)$. We deduce from (13) and the hypothesis on $(a_n, n \in \mathbb{N}^*)$ that $\lim_{n\to\infty} a_n \log(\gamma_{n-h}/\gamma_n) = 0$ and thus $\lim_{n\to\infty} b_{n,h} = 1$. We deduce from (19), (11) and (12) that, for $k \geq i > 1$, $\lim_{n\to\infty} G_{n,h}(k,i) = 0$ and for $k \geq 1$:

$$\lim_{n\to\infty} G_{n,h}(k,1) = \begin{cases} \kappa^{1-k}\mu^{-h} & \text{if } \mu < 1, \\ 1 & \text{if } \mu = 1, \\ \mu^h & \text{if } \mu > 1. \end{cases}$$

We deduce that:

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = \left\{ \begin{matrix} k\mu^{-h} & \text{if } \mu < 1 \\ k & \text{if } \mu = 1 \\ k\kappa^{k-1}\mu^h & \text{if } \mu > 1 \end{matrix} \right\} = k\mathfrak{c}^{k-1}\mathfrak{m}^{-h}.$$

Then, as a.s. $H(\tau^0) = +\infty$, we can use the characterization (1) of the convergence in $\mathbb{T}_\mathrm{f}$, as well as (3) and Lemma 4.1 to conclude.

□

5. The Poisson regime or the fat case

5.1. **An infinite Poisson tree.** Let $\theta \in (0, +\infty)$. We consider a two-type random tree $\hat{\tau}^\theta$ where the vertices are either of type s (for survivor) or of type e (for extinction). We define $\tau^\theta = \mathrm{Ske}(\hat{\tau}^\theta)$ as the tree $\hat{\tau}^\theta$ when one forgets the types of the vertices of $\hat{\tau}^\theta$. We denote by $\mathscr{S}_h = \{u \in \tau^\theta;\, |u| = h \text{ and } u \text{ is of type s in } \hat{\tau}^\theta\}$ the set of vertices of $\hat{\tau}^\theta$ with type s at level $h \in \mathbb{N}$. Notice that $(\mathscr{S}_\ell, 0 \leq \ell < h) = \mathrm{Anc}(\mathscr{S}_h)$ and that $\hat{\tau}^\theta$ is completely characterized by τ^θ and $(\mathscr{S}_h, h \in \mathbb{N})$. Recall $\mathfrak{p}$ defined by (23) and the k-th order size-biased distribution, $p_{[k]}$, defined by (24). The random tree $\hat{\tau}^\theta$ is defined as follows.

- The root is of type s (i.e. $\mathscr{S}_0 = \{\emptyset\}$).
- The number of offspring of a vertex of type e does not depend on the vertices of lower or same height (branching property only for individuals of type e).
- A vertex of type e produces only vertices of type e with offspring distribution $\mathfrak{p}$ (as in the Kesten tree).

- For $h \in \mathbb{N}$, let $\Delta_h = \sharp\mathscr{S}_{h+1} - \sharp\mathscr{S}_h$ be the increase of number of vertices of type s between generations h and $h+1$. Conditionally on $r_h(\tau^\theta)$ and $(\mathscr{S}_\ell, 0 \leq \ell \leq h)$, Δ_h is distributed as a Poisson random variable with mean $\theta\zeta_h$, where:

$$\zeta_h = \begin{cases} \mu^{-h-1}(1-\mu)(\kappa-1)/\kappa & \text{if } \mu < 1, \\ (\gamma - 1) & \text{if } \mu = 1, \\ \mu^h(\mu-1)(1-\kappa) & \text{if } \mu > 1. \end{cases} \tag{27}$$

We denote by $\kappa^{\mathrm{s}}(u)$ the number of children of u of type s. Conditionally given $r_h(\tau^\theta)$, $(\mathscr{S}_\ell, 0 \leq \ell \leq h)$ and Δ_h, the vector $(\kappa^{\mathrm{s}}(u), u \in \mathscr{S}_h)$ is uniformly distributed on the set of vectors of positive integers $(n_i, 1 \leq i \leq \sharp\mathscr{S}_h)$ that sum to $\sharp\mathscr{S}_h + \Delta_h$, each configuration having hence probability $1/\binom{\sharp\mathscr{S}_{h+1}-1}{\sharp\mathscr{S}_h - 1} = 1/\binom{\sharp\mathscr{S}_{h+1}-1}{\Delta_h}$. (This breaks the branching property for the tree and the population process since the number of offspring of type s of a vertex depends on the size of the whole population of type s at the same level). Furthermore, conditionally on $r_h(\tau^\theta)$, $\mathscr{S}_h$ and $(\kappa^{\mathrm{s}}(v) = s_v \geq 1, v \in \mathscr{S}_h)$, the vertex $u \in \mathscr{S}_h$ has $\kappa^{\mathrm{e}}(u)$ vertices of type e such that $k_u(\tau^\theta) = \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u)$ has distribution $\mathfrak{p}_{[s_u]}$ and the s_u individuals of type s are chosen uniformly at random among the $k_u(\tau^\theta)$ children.

More precisely, for $h \in \mathbb{N}$, $n \in \mathbb{N}$, $u \in \mathscr{S}_h$, $k_u \geq s_u \geq 1$, $A_u \subset \{1, \ldots, k_u\}$ with $\sharp A_u = s_u$ and $\sum_{u \in \mathscr{S}_h} s_u = n + \sharp\mathscr{S}_h$, we have with $k = \sum_{u \in \mathscr{S}_h} k_u$:

$$\begin{aligned} &\mathbb{P}\left(\kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u) = k_u \text{ and } \mathscr{S}_{h+1} \cap \{u1, \ldots, uk_u\} = uA_u \ \ \forall u \in \mathscr{S}_h \,|\, r_h(\tau^\theta), \mathscr{S}_h\right) \\ &\quad = \frac{(\theta\zeta_h)^n}{n!}\, \mathrm{e}^{-\theta\zeta_h}\, \frac{1}{\binom{\sharp\mathscr{S}_h + n - 1}{n}} \prod_{u \in \mathscr{S}_h} \frac{1}{\binom{k_u}{s_u}}\, \mathfrak{p}_{[s_u]}(k_u) \\ &\quad = \frac{(\sharp\mathscr{S}_h - 1)!}{(\sharp\mathscr{S}_h + n - 1)!} (\theta(\gamma-1)\zeta_h)^n\, \mathrm{e}^{-\theta\zeta_h} \prod_{u \in \mathscr{S}_h} \mathfrak{p}(k_u) \begin{cases} \mu^{-\sharp\mathscr{S}_h} & \text{if } \mu \leq 1, \\ \mu^{\sharp\mathscr{S}_h} \left(\frac{\mu}{\kappa}\right)^n & \text{if } \mu > 1, \end{cases} \end{aligned} \tag{28}$$

where we used (25) and (23) as well as (7) for the last equality.

By construction, a.s. individuals of type s have a progeny which does not suffer extinction whereas individuals of type e have a progeny which suffers extinction. Since the individuals of type s do not satisfy the branching property, the random tree $\hat{\tau}^\theta$ is not a multi-type GW tree. We stress out that $\hat{\tau}^\theta$ truncated at level h can be recovered from $r_h(\tau^\theta)$ and $\mathscr{S}_h$ as all the ancestors of a vertex of type s are also of type s and a vertex of type s has at least one child of type s.

We have the following result.

Lemma 5.1. *Let $\eta \in (0, 1]$ and $q \in (0, 1)$. Let $\theta \in (0, +\infty)$. Let $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$. We have, with $k = z_h(\mathbf{t})$:*

$$\mathbb{P}(r_h(\tau^\theta) = \mathbf{t}) = \mathcal{H}(h, k, \theta)\, \mathbb{P}(r_h(\tau) = \mathbf{t}),$$

where $\mathcal{H}(h,k,\theta)$ *is equal to*

$$\mu^{-h}\,\mathrm{e}^{-\theta(\mu^{-h}-1)(\kappa-1)/\kappa}\sum_{i=1}^{k}\binom{k}{i}\frac{\left(\theta\mu^{-h}(\kappa-1)^2/\kappa\right)^{i-1}}{(i-1)!} \qquad \textit{if } \mu<1,$$

$$\mathrm{e}^{-\theta(\gamma-1)h}\sum_{i=1}^{k}\binom{k}{i}\frac{\left(\theta(\gamma-1)^2\right)^{i-1}}{(i-1)!} \qquad \textit{if } \mu=1,$$

$$\mu^{h}\,\mathrm{e}^{-\theta(\mu^{h}-1)(1-\kappa)}\sum_{i=1}^{k}\binom{k}{i}\kappa^{k-i}\frac{\left(\theta\mu^{h}(1-\kappa)^2\right)^{i-1}}{(i-1)!} \qquad \textit{if } \mu>1.$$

Remark 5.2. We deduce from Lemma 4.1 that $\tau^\theta \xrightarrow[\theta\to 0]{\textbf{(d)}} \tau^0$. Therefore the trees τ^θ appear as a generalization of the Kesten tree. We will also prove in Proposition 6.3 that a limit also exists when $\theta\to+\infty$.

Proof. We consider only the super-critical case. The sub-critical case and the critical case can be handled in a similar way.

Let $h\in\mathbb{N}^*$, $\mathbf{t}\in\mathbb{T}_{\mathrm{f}}^{(h)}$ and $S_h\subset\{u\in\mathbf{t};\,|u|=h\}$ be non empty. In order to shorten the notations, we set $\mathcal{A}=\mathrm{Anc}(S_h)$. Notice that $\mathcal{A}$ is tree-like. We set, for $\ell\in\{0,\dots,h-1\}$, $S_\ell=\{u\in\mathcal{A},\,|u|=\ell\}$ the vertices at level ℓ which have at least one descendant in S_h and $\Delta_\ell=\sharp S_{\ell+1}-\sharp S_\ell$. We recall that $\hat\tau^\theta$ truncated at level h can be recovered from $r_h(\tau^\theta)$ and $\mathscr{S}_h$. We compute $\mathcal{C}_{S_h}=\mathbb{P}(r_h(\tau^\theta)=\mathbf{t},\,\mathscr{S}_h=S_h)$. We have, using (28) and (27):

$$\begin{aligned}
\mathcal{C}_{S_h} &= \left[\prod_{u\in r_{h-1}(\mathbf{t}),u\notin\mathcal{A}}\mathfrak{p}(k_u(\mathbf{t}))\right] \\
&\qquad \prod_{\ell=0}^{h-1}\left[\frac{(\sharp S_\ell-1)!}{(\sharp S_{\ell+1}-1)!}(\theta(\gamma-1)\zeta_\ell)^{\Delta_\ell}\,\mathrm{e}^{-\theta\zeta_\ell}\left[\prod_{u\in S_\ell}\mathfrak{p}(k_u(\mathbf{t}))\right]\mu^{\sharp S_\ell}\left(\frac{\mu}{\kappa}\right)^{\Delta_\ell}\right] \\
&= \left[\prod_{u\in r_{h-1}(\mathbf{t})}\mathfrak{p}(k_u(\mathbf{t}))\right]\frac{\left(\frac{\theta(\gamma-1)(\mu-1)(1-\kappa)}{\kappa}\right)^{\sum_{\ell=0}^{h-1}\Delta_\ell}}{(\sharp S_h-1)!}\,\mathrm{e}^{-\theta\sum_{\ell=1}^{h-1}\zeta_\ell}\prod_{\ell=0}^{h-1}\mu^{(\ell+1)\Delta_\ell+\sharp S_\ell} \\
&= \left[\prod_{u\in r_{h-1}(\mathbf{t})}\kappa^{k_u(\mathbf{t})-1}\right]\left[\prod_{u\in r_{h-1}(\mathbf{t})}p(k_u(\mathbf{t}))\right]\frac{\left(\frac{\theta(1-\kappa)^2}{\kappa}\right)^{\sharp S_h-1}}{(\sharp S_h-1)!}\,\mathrm{e}^{-\theta(\mu^h-1)(1-\kappa)}\;\mu^{h\sharp S_h} \\
&= \kappa^{z_h(\mathbf{t})-\sharp S_h}\,\mathbb{P}(r_h(\tau)=\mathbf{t})\,\frac{\mu^h\left(\theta\mu^h(1-\kappa)^2\right)^{\sharp S_h-1}}{(\sharp S_h-1)!}\,\mathrm{e}^{-\theta(\mu^h-1)(1-\kappa)},
\end{aligned}$$

where we used for the third equality that $\sum_{\ell=0}^{h-1}\Delta_\ell=\sharp S_h-1$, $\sum_{\ell=1}^{h-1}\zeta_\ell=(\mu^h-1)(1-\kappa)$ and $\sum_{\ell=0}^{h-1}(\ell+1)\Delta_\ell+\sharp S_\ell=\sum_{\ell=0}^{h-1}(\ell+1)\sharp S_{\ell+1}-\ell\sharp S_\ell=h\sharp S_h$. Since $\mathcal{C}_{S_h}$ depends only of $\sharp S_h$, we shall write $\mathcal{C}_{\sharp S_h}$ for $\mathcal{C}_{S_h}$. Set $k=z_h(\mathbf{t})=\sharp\{u\in\mathbf{t};\,|u|=h\}$. Since $\sharp S_h\geq 1$ as the root if of type s, we obtain:

$$\mathbb{P}(r_h(\tilde\tau^\theta)=\mathbf{t})=\sum_{i=1}^{k}\sum_{S_h\subset\{u\in\mathbf{t};\,|u|=h\}}\mathbf{1}_{\{\sharp S_h=i\}}\,\mathcal{C}_{S_h}=\sum_{i=1}^{k}\binom{k}{i}\mathcal{C}_i=\mathbb{P}(r_h(\tau)=\mathbf{t})\,\mathcal{H}(h,k,\theta),$$

where we used the definition of $\mathcal{H}$ for the last equality. □

5.2. **Convergence of the fat geometric GW tree.** We consider a sequence $(a_n, n \in \mathbb{N}^*)$, with $a_n \in \mathbb{N}^*$ and τ_n a random tree distributed as the GW tree τ conditionally on $\{Z_n = a_n\}$. We have the following result.

Proposition 5.3. *Let $\eta \in (0,1]$, $q \in (0,1)$ and $\theta \in (0,+\infty)$. Assume that $\lim_{n\to\infty} a_n\mu^n = \theta$ if $\mu < 1$ or $\lim_{n\to\infty} a_n n^{-2} = \theta$ if $\mu = 1$ or $\lim_{n\to\infty} a_n\mu^{-n} = \theta$ if $\mu > 1$. Then we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{\text{(d)}} \tau^\theta.$$

Proof. Recall the definitions of $b_{n,h}$ in (17) and of $G_{n,h}$ in (19). According to Lemma 3.1, we have for $n > h \geq 1$ and $k \in \mathbb{N}^*$:

$$\frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = b_{n,h} \sum_{i=1}^{k} \binom{k}{i} \kappa^{k-i}\, G_{n,h}(k,i).$$

According to Definition (17), we have $b_{n,h} = \exp\left(-a_n \log(\gamma_{n-h}/\gamma_n)\right)$. We deduce from (13) and the hypothesis on $(a_n, n \in \mathbb{N}^*)$ that

$$\lim_{n\to\infty} -\log(b_{n,h}) = \begin{cases} \theta(\mu^{-h}-1)(\kappa-1)/\kappa & \text{if } \mu < 1,\\ \theta(\gamma-1)h & \text{if } \mu = 1,\\ \theta(\mu^h-1)(1-\kappa) & \text{if } \mu > 1. \end{cases}$$

We deduce from (19), (11) and (12), that for $h \in \mathbb{N}^*$, $k \geq i \geq 1$:

$$\lim_{n\to\infty} (i-1)!G_{n,h}(k,i) = \begin{cases} \left(\theta\mu^{-h}(\kappa-1)^2\right)^{i-1} \mu^{-h}\kappa^{1-k} & \text{if } \mu < 1,\\ \left(\theta(\gamma-1)^2\right)^{i-1} & \text{if } \mu = 1,\\ \left(\theta\mu^h(1-\kappa)^2\right)^{i-1} \mu^h & \text{if } \mu > 1. \end{cases}$$

Using definition of $\mathcal{H}$ in Lemma 5.1, we obtain that:

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = \mathcal{H}(h,k,\theta).$$

Then use the characterization of the convergence in $\mathbb{T}_{\mathrm{f}}$, (3) and Lemma 5.1 to conclude. □

6. The condensation regime or the very fat case

6.1. **An infinite geometric tree.** Recall γ_n defined in (9). For $n \in \mathbb{N}^*$, we define the probability $\tilde{p}_n = (\tilde{p}_n(k), k \in \mathbb{N})$ by:

$$\tilde{p}_n(k) = \frac{\gamma_{n+1}^k}{\gamma_n} p(k).$$

Thanks to (10), we get $\sum_{k\in\mathbb{N}} \tilde{p}_n(k) = f(\gamma_{n+1})\gamma_n^{-1} = 1$, so that $\tilde{p}$ is indeed a probability distribution on $\mathbb{N}$. For $n = 0$, we set $\tilde{p}_0$ the Dirac mass at $+\infty$, which is a degenerate probability measure on $\bar{\mathbb{N}}$.

We define τ^∞ as an inhomogeneous GW tree with reproduction distribution $\tilde{p}_h$ at generation $h \in \mathbb{N}$. In particular the root has an infinite number of children, whereas all the other individuals have a finite number of children. More precisely, for all $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$, we have:

$$\mathbb{P}(r_{h,k_0}(\tau^\infty) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})^*} \tilde{p}_{|u|}(k_u(\mathbf{t})), \tag{29}$$

where we recall that $\mathbf{t}^* = \mathbf{t} \setminus \{\emptyset\}$. Remark that a.s. $\tau^\infty \in \mathbb{T}_{\mathrm{f}}^*$.

We give a representation of the distribution of τ^∞ as the distribution of τ with a martingale weight.

Lemma 6.1. *Let $\eta \in (0,1]$ and $q \in (0,1)$. For all $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and F a non-negative function on $\mathbb{T}_\infty$, we have:*

$$\mathbb{E}\left[F(r_{h,k_0}(\tau^\infty))\right] = \frac{\mathbb{E}\left[F(r_h(\tau))\, M_h \mathbf{1}_{\{k_\emptyset(\tau)=k_0\}}\right]}{\mathbb{P}(k_\emptyset(\tau) = k_0)},$$

where $(M_h, h \in \mathbb{N}^*)$ is the martingale defined by (16). Equivalently, for all $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$, we have with $k = z_h(\mathbf{t})$:

$$\mathbb{P}\left(r_{h,k_0}(\tau^\infty) = \mathbf{t}\right) = \frac{1-q}{\eta q}\, \gamma_h^k\, \mathbb{P}\left(r_h(\tau) = \mathbf{t}\right). \tag{30}$$

Proof. Let $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$. Set $k = z_h(\mathbf{t})$. We have:

$$\begin{aligned}
\frac{1-q}{\eta q}\, \gamma_h^k\, \mathbb{P}\left(r_h(\tau) = \mathbf{t}\right) &= \frac{1-q}{\eta q} \left[\prod_{u \in \mathbf{t},\, |u|=h-1} \gamma_h^{k_u(\mathbf{t})}\right] \left[\prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t}))\right] \\
&= \frac{1-q}{\eta q}\, \gamma_1^{k_0} \left[\prod_{u \in r_{h-1}(\mathbf{t})^*} \gamma_{|u|}^{-1}\, \gamma_{|u|+1}^{k_u(\mathbf{t})}\right] \left[\prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t}))\right] \\
&= \frac{1-q}{\eta q}\, \gamma_1^{k_0} p(k_0) \left[\prod_{u \in r_{h-1}(\mathbf{t})^*} \tilde{p}_{|u|}(k_u(\mathbf{t}))\right] \\
&= \mathbb{P}(r_{h,k_0}(\tau^\infty) = \mathbf{t}),
\end{aligned}$$

where we used that $\sum_{u \in \mathbf{t},\, |u|=\ell} k_u(\mathbf{t}) = \sum_{u \in \mathbf{t},\, |u|=\ell+1} 1$ for the second equality and the definition of $p(k_0)$ and $\gamma_1 = \gamma$ as well as (29) for the last one. To conclude, notice also that thanks to the definition of $p(k_0)$ and $\gamma_1 = \gamma$ as well as (16), we have on $\{k_\emptyset(\tau) = k_0\}$:

$$\frac{1-q}{\eta q}\, \gamma_h^{z_h(\tau)} = \frac{M_h}{p(k_0)}.$$

□

We give an alternative description of τ^∞ as the skeleton of a two-type GW tree. We set for $n \in \mathbb{N}$:

$$\nu_n = 1 - \frac{\gamma_{n+1} - 1}{\gamma_1 - 1} = \begin{cases} \mu(1-\mu^n)\,(1-\mu^{n+1})^{-1} & \text{if } \mu \neq 1, \\ n(n+1)^{-1} & \text{if } \mu = 1. \end{cases}$$

We have $\nu_n \in [0,1)$. It is easy to check (using the first expression of ν_{n-1} for the first equality and the second expression for ν_{n-1} and ν_n for the second equality) that for all $n \in \mathbb{N}^*$:

$$\frac{1 - q\nu_{n-1}}{1-q} = \gamma_n \quad \text{and} \quad \frac{1}{\mu}(1-\nu_{n-1})\frac{\nu_n}{1-\nu_n} = 1. \tag{31}$$

We consider an inhomogeneous two-type GW tree $\hat{\tau}^\infty$ where the vertices are either of type s (for survivor) or of type e (for extinction). We define $\mathrm{Ske}(\hat{\tau}^\infty)$ as the tree $\hat{\tau}^\infty$ when one forgets the types of the vertices of $\hat{\tau}^\infty$. We denote by $\mathscr{S}_h = \{u \in \mathrm{Ske}(\hat{\tau}^\infty);\, |u| = h \text{ and } u \text{ is of type s in } \hat{\tau}^\infty\}$ the set of vertices of $\hat{\tau}$ with type s at level $h \in \mathbb{N}$. The random tree $\hat{\tau}^\infty$ is defined as follows:

- The number of offspring of a vertex depends, conditionally on the vertices of lower or same height, only on its own type (branching property).
- The root is of type s (i.e. $\mathscr{S}_0 = \{\emptyset\}$).
- A vertex of type e produces only vertices of type e with offspring distribution $\mathfrak{p}$ defined by (23).
- A vertex $u \in \hat{\tau}^\infty$ at level h of type s produces $\kappa^{\mathrm{s}}(u)$ vertices of type s with probability distribution $\mathcal{G}(1, \nu_h)$ (with the convention that if $h = 0$, then $\kappa^{\mathrm{s}}(\emptyset) = +\infty$) and $\kappa^{\mathrm{e}}(u)$ vertices of type e such that the type of the vertices $(ui, 1 \leq i \leq \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u))$ is a sequence of heads (type s) and tails (type e) where the probability to get an head is $q \vee \eta$ and a tail is $1 - q \vee \eta$, stopped just before the $(\kappa^{\mathrm{s}}(u)+1)$-th head. Equivalently, for $|u| \geq 1$, conditionally on $\kappa^{\mathrm{s}}(u) = s_u \geq 1$, the vertex u has $\kappa^{\mathrm{e}}(u)$ vertices of type e such that $k_u(\mathrm{Ske}(\hat{\tau}^\infty)) = \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u)$ has distribution $\mathfrak{p}_{[s_u]}$, defined in (25), and the s_u individuals of type s are chosen uniformly at random among the $k_u(\mathrm{Ske}(\hat{\tau}^\infty))$ children. More precisely, we have for $k_0 \in \mathbb{N}^*$ and $S_1 \subset \{1, \ldots, k_0\}$:

$$\mathbb{P}\left(\mathscr{S}_1 \cap \{1, \ldots, k_0\} = S_1\right) = (q \vee \eta)^{\sharp S_1} (1 - (q \vee \eta))^{k_0 - \sharp S_1},$$

  and for $h \geq 2$, $k \in \mathbb{N}^*$, $u \in \mathcal{U}$ with $|u| = h$, $s_u \in \{1, \ldots, k\}$, and $A \subset \{1, \ldots, k\}$ such that $\sharp A = s_u$:

$$\begin{aligned}\mathbb{P}\left(\kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u) = k,\, \mathscr{S}_{h+1} \cap \{u1, \ldots, uk\} = uA \,|\, r_h(\mathrm{Ske}(\hat{\tau}^\infty)),\, \mathscr{S}_h,\, u \in \mathscr{S}_h\right)& \\ = \nu_h (1 - \nu_h)^{s_u - 1} (q \vee \eta)^{s_u + 1} (1 - (q \vee \eta))^{k - s_u}&.\end{aligned}$$

By construction individuals of type s have a progeny which does not suffer extinction whereas individuals of type e have a.s. a finite progeny.

We stress out that $\hat{\tau}^\infty$, truncated at level h and when considering only the first k_0 children of the root, can be recovered from $r_{h,k_0}(\mathrm{Ske}(\hat{\tau}^\infty))$ and $\mathscr{S}_h$ as all the ancestors of a vertex of type s is also of a type s and a vertex of type s has at least one children of type s.

We have the following result.

Lemma 6.2. *Let $\eta \in (0, 1]$ and $q \in (0, 1)$. We have that τ^∞ is distributed as* $\mathrm{Ske}(\hat{\tau}^\infty)$.

Proof. We first suppose that $\eta \leq q$. In that case, $\mu \leq 1$ and we have $\mathfrak{p} = p$ and $q \vee \eta = q$.

Let $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$ and $S_h \subset \{u \in \mathbf{t};\, |u| = h\}$ which might be empty. In order to shorten the notations, we set $\mathcal{A} = \mathrm{Anc}(S_h)$ which is a tree if S_h is non-empty. For $u \in \mathcal{A}$, we set $s_u = \sharp\{i \in \mathbb{N};\, ui \in \mathcal{A} \cup S_h\}$ the number of children of u which have at least one descendant in S_h. We set, for $\ell \in \{0, \ldots, h-1\}$, $S_\ell = \{u \in \mathcal{A},\, |u| = \ell\}$ the vertices at level ℓ which have at least one descendant in S_h. Notice that $\sum_{u \in S_\ell} s_u = \sharp S_{\ell+1}$. Set $k = z_h(\mathbf{t})$. We compute $\mathcal{C}_{S_h} = \mathbb{P}(r_{h,k_0}(\mathrm{Ske}(\hat{\tau}^\infty)) = \mathbf{t},\, \mathscr{S}_h = S_h)$. If S_h is non-empty, we have:

$$\begin{aligned}
\mathcal{C}_{S_h} &= \left[\prod_{u\in r_{h-1}(\mathbf{t}),\, u\notin\mathcal{A}} p(k_u(\mathbf{t}))\right] q^{\sharp S_1}(1-q)^{k_0-\sharp S_1} \prod_{u\in\mathcal{A}^*} \nu_{|u|}(1-\nu_{|u|})^{s_u-1} q^{s_u+1}(1-q)^{k_u(\mathbf{t})-s_u} \\
&= \left[\prod_{u\in r_{h-1}(\mathbf{t})^*} p(k_u(\mathbf{t}))\right] q^{\sharp S_1}(1-q)^{k_0-\sharp S_1} \prod_{u\in\mathcal{A}^*} \frac{\nu_{|u|}}{1-\nu_{|u|}} \frac{1-q}{\eta} \left(\frac{q}{1-q}(1-\nu_{|u|})\right)^{s_u} \\
&= \frac{1-q}{\eta q}\mathbb{P}(r_h(\tau)=\mathbf{t}) \left(\frac{q}{1-q}\right)^{\sharp S_1} \prod_{\ell=1}^{h-1} \left(\frac{\nu_\ell}{1-\nu_\ell}\frac{1-q}{\eta}\right)^{\sharp S_\ell} \left(\frac{q}{1-q}(1-\nu_\ell)\right)^{\sharp S_{\ell+1}} \\
&= \frac{1-q}{\eta q}\mathbb{P}(r_h(\tau)=\mathbf{t}) \left(\frac{\nu_1}{1-\nu_1}\frac{q}{\eta}\right)^{\sharp S_1} \left(\frac{q}{1-q}(1-\nu_{h-1})\right)^{\sharp S_h} \prod_{\ell=2}^{h-1} \left(\frac{\nu_\ell}{1-\nu_\ell}\frac{q}{\eta}(1-\nu_{\ell-1})\right)^{\sharp S_\ell} \\
&= \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t}) \left(\frac{q}{1-q}(1-\nu_{h-1})\right)^{\sharp S_h},
\end{aligned}$$

where we used for the second equality that if $u \in \mathcal{A}$ and $\mathscr{S}_h = S_h$, then $k_{\mathrm{Ske}(\hat{\tau}^\infty)}(u) \geq 1$; and for the fifth the second equation from (31) as well as $\nu_1/(1-\nu_1) = \mu = \eta/q$ (which comes also from the second equation in (31) with $n = 0$). If S_h is empty, then we have:

$$\mathcal{C}_\emptyset = (1-q)^{k_0} \prod_{u\in r_{h-1}(\mathbf{t})^*} p(k_u(\mathbf{t})) = \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t}).$$

Notice that $\mathcal{C}_{S_h}$ depends on S_h only trough $\sharp S_h$. We deduce that:

$$\begin{aligned}
\mathbb{P}(r_{h,k_0}(\mathrm{Ske}(\hat{\tau}^\infty)) = \mathbf{t}) &= \sum_{i=0}^{k} \sum_{S_h\subset\{u\in\mathbf{t};\, |u|=h\}} \mathbf{1}_{\{\sharp S_h = i\}}\, \mathcal{C}_{S_h} \\
&= \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t}) \sum_{i=0}^{k} \binom{k}{i} \left(\frac{q}{1-q}(1-\nu_{h-1})\right)^{i} \\
&= \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t}) \left(1+\frac{q}{1-q}(1-\nu_{h-1})\right)^{k} \\
&= \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t}) \left(\frac{1-q\nu_{h-1}}{1-q}\right)^{k} \\
&= \frac{1-q}{\eta q}\,\mathbb{P}(r_h(\tau)=\mathbf{t})\gamma_h^k,
\end{aligned}$$

where we used the first equation from (31) for the last equality. Then we conclude using (30) from Lemma 6.1.

In the case $q < \eta$, we have that $\mathfrak{p}$ is the $\mathcal{G}(q, \eta)$ distribution. So the computations are the same, inverting the roles of q and η. □

As in Remark 5.2, we also have the convergence of the trees τ^θ introduced in Section 5.1 to the infinite geometric tree τ^∞ as $\theta \to +\infty$.

Proposition 6.3. *Let $\eta \in (0,1]$ and $q \in (0,1)$. Then we have the following convergence in distribution:*

$$\tau^\theta \xrightarrow[\theta\to\infty]{(d)} \tau^\infty.$$

Proof. We only deal with the supercritical case, the subcritical and critical cases can be handled in a similar way.

For $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_{\mathrm{f}}$ such that $k_\emptyset(\mathbf{t}) < \infty$, let us denote by $\mathbf{t} * \mathbf{t}'$ the tree obtained by grafting $\mathbf{t}$ and $\mathbf{t}'$ on the same root i.e.:

$$\mathbf{t} * \mathbf{t}' = \mathbf{t} \cup \{(u_1 + k_\emptyset(\mathbf{t}), u_2, \ldots, u_n),\ (u_1, \ldots, u_n) \in \mathbf{t}'^*\},$$

with the convention $\mathbf{t} * \mathbf{t}' = \mathbf{t}$ if $\mathbf{t}' = \{\emptyset\}$.

We denote by $\mathbb{T}_{\mathrm{f}}^{(\leq h)}$ the subset of $\mathbb{T}_{\mathrm{f}}$ of trees with height less than or equal to h. Let $h, k_0 > 0$ and let $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$. Then using Lemma 5.1 with $k = z_h(\mathbf{t})$ and $k' = z_h(\mathbf{t}')$, we have:

$$\begin{aligned}
&\mathbb{P}(r_{h,k_0}(\tau^\theta) = \mathbf{t})\\
&= \sum_{\mathbf{t}' \in \mathbb{T}_{\mathrm{f}}^{(\leq h)}} \mathbb{P}(r_h(\tau^\theta) = \mathbf{t} * \mathbf{t}')\\
&= \sum_{\mathbf{t}' \in \mathbb{T}_{\mathrm{f}}^{(\leq h)}} \mu^h \,\mathrm{e}^{-\theta(\mu^h - 1)(1-\kappa)} \sum_{i=1}^{k+k'} \binom{k+k'}{i} \kappa^{k+k'-i} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \mathbb{P}(r_h(\tau) = \mathbf{t} * \mathbf{t}').
\end{aligned}$$

Let us remark that, if $\mathbf{t}' \neq \{\emptyset\}$, then

$$\begin{aligned}
\mathbb{P}(r_h(\tau) = \mathbf{t} * \mathbf{t}') &= \frac{\mathbb{P}(r_h(\tau) = \mathbf{t})}{p(k_\emptyset(\mathbf{t}))} \frac{\mathbb{P}(r_h(\tau) = \mathbf{t}')}{p(k_\emptyset(\mathbf{t}'))} p(k_\emptyset(\mathbf{t}) + k_\emptyset(\mathbf{t}'))\\
&= \frac{1-q}{\eta q} \mathbb{P}(r_h(\tau) = \mathbf{t}) \mathbb{P}(r_h(\tau) = \mathbf{t}').
\end{aligned}$$

Since $\mathbb{P}(r_h(\tau^\theta) = \mathbf{t})$ converges to 0 as θ increases to infinity, we deduce that for $\theta \to +\infty$:

$$\mathbb{P}(r_{h,k_0}(\tau^\theta) = \mathbf{t}) = \frac{1-q}{\eta q} \mu^h \mathbb{P}(r_h(\tau) = \mathbf{t}) \,\mathrm{e}^{-\theta(\mu^h - 1)(1-\kappa)}\ A_1 + o(1),$$

with

$$A_1 = \sum_{\mathbf{t}' \in \mathbb{T}_{\mathrm{f}}^{(\leq h)} \setminus \{\emptyset\}} \sum_{i=1}^{k+k'} \binom{k+k'}{i} \kappa^{k+k'-i} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \mathbb{P}(r_h(\tau) = \mathbf{t}').$$

We have, using for the third equality that Z_h has distribution $\mathcal{G}[\kappa, \gamma_h]$, that:

$$\begin{aligned}
A_1 &= \sum_{k'=0}^{+\infty} \sum_{i=1}^{k+k'} \binom{k+k'}{i} \kappa^{k+k'-i} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \sum_{\{\mathbf{t}' \in \mathbb{T}_{\mathrm{f}}^{(\leq h)},\, z_h(\mathbf{t}') = k'\}} \mathbb{P}(r_h(\tau) = \mathbf{t}')\\
&= \sum_{k'=0}^{+\infty} \sum_{i=1}^{k+k'} \binom{k+k'}{i} \kappa^{k+k'-i} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \mathbb{P}(Z_h = k')\\
&= \sum_{k'=0}^{+\infty} \sum_{i=1}^{k+k'} \binom{k+k'}{i} \kappa^{k+k'-i} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \left(1 - \frac{1}{\gamma_h}\right) \left(1 - \frac{\kappa}{\gamma_h}\right) \frac{1}{\gamma_h^{k'-1}}\\
&= \left(1 - \frac{1}{\gamma_h}\right) \left(1 - \frac{\kappa}{\gamma_h}\right) (A_2 + A_3),
\end{aligned}$$

where

$$A_2 = \sum_{i=k+1}^{+\infty} \left(\sum_{k'=i-k}^{+\infty} \binom{k+k'}{i} \left(\frac{\kappa}{\gamma_h}\right)^{k'-1} \right) \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \kappa^{k-i+1}$$

and

$$A_3 = \sum_{i=1}^{k} \left(\sum_{k'=0}^{+\infty} \binom{k+k'}{i} \left(\frac{\kappa}{\gamma_h}\right)^{k'-1} \right) \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \kappa^{k-i+1}.$$

Using (14) and $\kappa/\gamma_h < 1$, we get $\lim_{\theta\to+\infty} \mathrm{e}^{-\theta(\mu^h-1)(1-\kappa)}\ A_3 = 0$. Using (14), we also have:

$$\begin{aligned} A_2 &= \sum_{i=k+1}^{+\infty} \frac{1}{\left(1-\frac{\kappa}{\gamma_h}\right)^{i+1}} \left(\frac{\kappa}{\gamma_h}\right)^{i-k-1} \frac{(\theta\mu^h(1-\kappa)^2)^{i-1}}{(i-1)!} \kappa^{k-i+1} \\ &= \frac{\gamma_h^{k+2}}{(\gamma_h-\kappa)^2}\, \mathrm{e}^{\frac{(\theta\mu^h(1-\kappa)^2)}{\gamma_h-\kappa}} + O(\theta^k). \end{aligned}$$

Then, as $(\gamma_h - 1)/(\gamma_h - \kappa) = \mu^{-h}$ and $(1-\kappa)/(\gamma_h - \kappa) = 1 - \mu^{-h}$, we get that:

$$\lim_{\theta\to+\infty} \mathrm{e}^{-\theta(\mu^h-1)(1-\kappa)}\ A_1 = \lim_{\theta\to+\infty} \mathrm{e}^{-\theta(\mu^h-1)(1-\kappa)} \left(1-\frac{1}{\gamma_h}\right)\left(1-\frac{\kappa}{\gamma_h}\right) A_2 = \mu^{-h}\gamma_h^k.$$

We deduce that:

$$\lim_{\theta\to+\infty} \mathbb{P}(r_{h,k_0}(\tau^\theta) = \mathbf{t}) = \frac{1-q}{\eta q}\, \gamma_h^k\, \mathbb{P}(r_h(\tau) = \mathbf{t}).$$

Using (30), this gives the result. □

6.2. **Convergence of the very fat geometric GW tree.** We consider a sequence $(a_n, n \in \mathbb{N}^*)$, with $a_n \in \mathbb{N}^*$ and τ_n a random tree distributed as the GW tree τ conditionally on $\{Z_n = a_n\}$. We have the following result.

Proposition 6.4. *Let $\eta \in (0, 1]$ and $q \in (0, 1)$. Assume that $\lim_{n\to\infty} a_n\mu^n = +\infty$ if $\mu < 1$ or $\lim_{n\to\infty} a_n n^{-2} = +\infty$ if $\mu = 1$ or $\lim_{n\to\infty} a_n\mu^{-n} = +\infty$ if $\mu > 1$. Then we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^\infty.$$

Proof. First notice that a.s. $H(\tau^\infty) = +\infty$. Then, using the characterization (2) for the convergence in distribution in $\mathbb{T}_\mathrm{f}^*$, the result is a direct consequence of (20) in Lemma 3.2 and of (30) in Lemma 6.1, provided that $\lim_{n\to\infty} R^i_{n,h}(k) = 0$ for $i \in \{1, 2\}$, $h \geq 2$ and $k \in \mathbb{N}^*$, where $R^i_{n,h}$ are defined in (21) and (22).

According to (17) and the definitions in Lemma 3.2, we have $b_{n,h} = \exp\left(-a_n \log(\gamma_{n-h}/\gamma_n)\right)$, $\alpha_n = (\gamma_{n-h} - \kappa)(\gamma_{n-h} - 1)$ and $x_n = \gamma_n/\gamma_{n-h}$. Since $\kappa > 1$ (resp. $\gamma > 1$, resp. $\kappa < 1$) if $\mu < 1$ (resp. $\mu = 1$, resp. $\mu > 1$), and since $h \geq 1$, we deduce from (9), (12) and (13) that $\log(\gamma_{n-h}/\gamma_n)$, α_n and $1 - x_n$ are of the same order μ^{-n} (resp. n^{-2}, resp. μ^n). In particular $\lim_{n\to\infty} \alpha_n/(1 - x_n)$ exists and is finite. Because of the hypothesis on $(a_n, n \in \mathbb{N}^*)$, we deduce that $\lim_{n\to\infty} a_n \log(\gamma_{n-h}/\gamma_n) = +\infty$ and thus $\lim_{n\to\infty} b_{n,h} = 0$ as well as $\lim_{n\to\infty} b_{n,h}\,(\alpha_n a_n)^{k-1} = 0$ as $a_n \log(\gamma_{n-h}/\gamma_n)$ and $\alpha_n a_n$ are of the same order. This gives $\lim_{n\to\infty} R^1_{n,h}(k) = 0$

Since $p(k)\mathbb{P}_k(Z_{n-h}=a_n) \leq \sum_{i\in\mathbb{N}} p(i)\mathbb{P}_i(Z_{n-h}=a_n) = \mathbb{P}(Z_{n-h+1}=a_n)$, we deduce that:

$$\begin{aligned}\frac{\mathbb{P}_k(Z_{n-h}=a_n)}{\mathbb{P}(Z_n=a_n)} &\leq \frac{1}{p(k)}\,\frac{\mathbb{P}(Z_{n-h+1}=a_n)}{\mathbb{P}(Z_n=a_n)}\\ &= \frac{1}{p(k)}\; b_{n,h-1}\frac{(\gamma_{n-h+1}-\kappa)(\gamma_{n-h+1}-1)}{(\gamma_n-\kappa)(\gamma_n-1)}\frac{\gamma_n}{\gamma_{n-h+1}},\end{aligned}$$

where we used that Z_ℓ has distribution $\mathcal{G}[\kappa,\gamma_\ell]$ and (9) for the last equality. According to the previous paragraph, we have $\lim_{n\to\infty} b_{n,h-1}=0$ as $h\geq 2$. Furthermore, using (13), we get that:

$$\lim_{n\to\infty}\frac{(\gamma_{n-h+1}-\kappa)(\gamma_{n-h+1}-1)}{(\gamma_n-\kappa)(\gamma_n-1)}\frac{\gamma_n}{\gamma_{n-h+1}} = \mu^{-h+1}.$$

This implies that $\lim_{n\to\infty}\mathbb{P}_k(Z_{n-h}=a_n)/\mathbb{P}(Z_n=a_n)=0$ and thus $\lim_{n\to\infty} R^2_{n,h}(k)=0$. This finishes the proof. □

References

[1] R. ABRAHAM and J. DELMAS. Local limits of conditioned Galton-Watson trees: the condensation case. *Elec. J. of Probab.*, 19:Article 56, 1–29, 2014.

[2] R. ABRAHAM and J. DELMAS. Local limits of conditioned Galton-Watson trees: the infinite spine case. *Elec. J. of Probab.*, 19:Article 2, 1–19, 2014.

[3] R. ABRAHAM and J.-F. DELMAS. Asymptotic properties of expansive Galton-Watson trees. *Electron. J. Probab.*, 24:Article 15, 1–51, 2019.

[4] K. ATHREYA and P. NEY. *Branching processes*. Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.

[5] S. JANSON. Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation. *Probab. Surv.*, 9:103–252, 2012.

[6] T. JONNSSON and S. STEFANSSON. Condensation in nongeneric trees. *J. Stat. Phys.*, 142:277–313, 2011.

[7] H. KESTEN. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(4):425–487, 1986.

[8] I. KORTCHEMSKI. Limit theorems for conditioned non-generic Galton-Watson trees. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 51(2):489–511, 2015.

[9] J. NEVEU. Arbres et processus de Galton-Watson. *Ann. de l'Inst. Henri Poincaré*, 22:199–207, 1986.

[10] L. OVERBECK. Martin boundaries of some branching processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 30(2):181–195, 1994.

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