Exit-time of mean-field particles system

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Abstract

The current article is devoted to the study of a mean-field system of particles. The question that we are interested in is the behaviour of the exit-time of the first particle (and the one of any particle) from a domain $\mathcal{D}$ on $\mathbb{R}^d$ as the diffusion coefficient goes to 0. We establish a Kramers' type law. In other words, we show that the exit-time is exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H^N\right\}$, H^N being the exit-cost. We also show that this exit-cost converges to some quantity H.

Key words and phrases: Exit-problem ; Large deviations ; Interacting particle systems ; Mean-field systems

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1 Introduction

The paper is devoted to the resolution of the exit-time for some mean-field interacting particles system. Let us briefly present the model. For any $i \in \mathbb{N}^*$, $\{B_t^i \ : \ t \in \mathbb{R}_+\}$ is a Brownian motion on $\mathbb{R}^d$. The Brownian motions are assumed to be independent. Each particle evolves in a non-convex landscape V, that we call the confining potential. Moreover, each particle interacts with any other one. We assume that the interaction does only depend on the distance between the two particles. This interacting force is odd.

In fine, the system of equations that we are interested in is the following:

$$X_t^{i,N} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_s^{i,N} - X_s^{j,N}\right) ds\,, \quad \text{(I)}$$

N being arbitrarily large and σ being an arbitrarily small positive constant.

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We can see the N particles in $\mathbb{R}^d$ as one diffusion in $\mathbb{R}^{dN}$. Indeed, let us write $\mathcal{X}_t^N := \left(X_t^{1,N}, \cdots, X_t^{N,N}\right)$ and $\mathcal{B}_t^N := \left(B_t^1, \cdots, B_t^N\right)$. The process $\mathcal{B}^N$ is a dN-dimensional Wiener process. Equation (I) can be rewritten like so:

$$\mathcal{X}_t^N = \mathcal{X}_0^N + \sigma\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s^N\right)ds\,. \tag{II}$$

Here, the potential on $\mathbb{R}^{dN}$ is defined by $\Upsilon^N(X_1, \cdots, X_N) := \frac{1}{N}\sum_{i=1}^N V(X_i) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F(X_i - X_j)$ for any $(X_1, \cdots, X_N) \in \left(\mathbb{R}^d\right)^N$.

Consequently, the whole system of particles, $\{\mathcal{X}_t^N \ : \ t \in \mathbb{R}_+\}$, is just an homogeneous and reversible diffusion in $\mathbb{R}^{dN}$.

It is well-known (see [Mél96, Szn91, HT16]) that the understanding of the behaviour of the diffusion $X^{1,N}$ when N is large is linked to its hydrodynamical limit diffusion that is to say

$$\begin{cases} X_t^{1,\infty} = x_0 + \sigma B_t^1 - \int_0^t \nabla V\left(X_s^{1,\infty}\right)ds - \int_0^t \nabla F * \mu_s^\infty\left(X_s^{1,\infty}\right)ds \\ \mu_t^\infty = \mathcal{L}\left(X_t^{1,\infty}\right) \end{cases}. \tag{III}$$

We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ and the associated exit-times:

$$\tau_{\mathcal{D}}^i(\sigma, N) := \inf\left\{t \geq 0 \ : \ X_t^{i,N} \notin \mathcal{D}\right\}$$

which corresponds to the first exit-time of the particle number i and

$$\tau_{\mathcal{D}}(\sigma, N) := \inf\left\{\tau_{\mathcal{D}}^i(\sigma, N) \ : \ i \in [\![1; N]\!]\right\}\,.$$

Let us point out that we can not directly tract the Kramers'law for $\tau_{\mathcal{D}}(\sigma, N)$ from the Kramers'law satisfied by the $\tau_{\mathcal{D}}^i(\sigma, N)$. Indeed, there is no independence since there is interaction between the particles.

We study these exit-times in the small-noise limit with N large (but we do not take the limit as N goes to infinity).

Freidlin and Wentzell theory solves this question for time-homogeneous diffusion in finite dimension. See [DZ98, FW98] for a complete review. The main result is the following:

Theorem 1.1. *We consider a domain $\mathcal{G} \subset \mathbb{R}^k$, a potential U on $\mathbb{R}^k$ and a diffusion*

$$x_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla U\left(x_s^\sigma\right)ds\,.$$

We assume that $\mathcal{G}$ satisfies the following properties.

1. *The unique critical point of the potential U in the domain $\mathcal{G}$ is a_0. Furthermore, for any $y_0 \in \mathcal{G}$, for any $t \in \mathbb{R}_+$, we have $y_t \in \mathcal{G}$ and moreover $\lim_{t\to+\infty} y_t = a_0$ with*

$$y_t = y_0 - \int_0^t \nabla U(y_s)ds\,.$$

2. *For any $y_0 \in \partial\mathcal{G}$, y_t converges toward a_0.*

3. *The quantity $H := \inf_{z\in\partial\mathcal{G}} (U(z) - U(a_0))$ is finite.*

By $\tau_{\mathcal{G}}(\sigma)$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{G}$. Then, for any $\delta > 0$, we have

$$\lim_{\sigma\to 0} \mathbb{P}\left\{ e^{\frac{2}{\sigma^2}(H-\delta)} < \tau_{\mathcal{G}}(\sigma) < e^{\frac{2}{\sigma^2}(H+\delta)} \right\} = 1\,.$$

Furthermore, if $\mathcal{N} \subset \partial\mathcal{G}$ is such that $\inf_{z\in\mathcal{N}} U(z) > \inf_{z\in\partial\mathcal{G}} U(z)$, we know that the diffusion x^σ does not exit $\mathcal{G}$ by $\mathcal{N}$ with high probability:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{ x^\sigma_{\tau_{\mathcal{G}}(\sigma)} \in \mathcal{N} \right\} = 0\,.$$

We do not provide the proof which can be found in [DZ98].

In [Tug12], we have obtained a similar result (already obtained in [HIP08]) for the self-stabilizing diffusion (III). To do so, we establish a Kramers' type law for the first particle of the mean-field system of particles. In this previous work, both the confining potential and the interacting potential are assumed to be convex.

In the current paper, we remove the hypothesis of global convexity for the confining potential.

We proceed similarly than in [Tug12].

We now give a definition which is of crucial interest in large deviations for stochastic processes.

Definition 1.2. *Let $\mathcal{D}$ be an open domain of $\mathbb{R}^k$ and U be a potential of $\mathbb{R}^k$. In the following, we say that $\mathcal{D}$ is stable by the potential U if for any $\xi_0 \in \mathcal{D}$, for any $t \geq 0$, we have $\xi_t \in \mathcal{D}$ with*

$$\xi_t = \xi_0 - \int_0^t \nabla U(\xi_s)\, ds\,.$$

We now give the assumptions of the paper. First, we give the hypotheses on the confining potential V.

Assumption (A-1): *V is a $\mathcal{C}^2$-continuous function.*

Assumption (A-2): *For all $\lambda > 0$, there exists $R_\lambda > 0$ such that $\nabla^2 V(x) > \lambda$, for any $||x|| \geq R_\lambda$.*

Assumption (A-3) *The gradient ∇V is slowly increasing: there exist $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$, for all $x \in \mathbb{R}$.*

These three assumptions are used to prove the existence and the uniqueness of a solution.

Let us present now the assumptions on the interaction potential F:

Assumption (A-4): *There exists a function G from $\mathbb{R}_+$ to $\mathbb{R}$ such that $F(x) = G(||x||)$.*

This assumption also is used to obtain the existence and the uniqueness of a solution.

Assumption (A-5): *G is an even polynomial and convex function such that* $\deg(G) =: 2n \geq 2$ *and* $G(0) = 0$.

We could remove the assumption of convexity of F but this would slightly modify the assumptions on the domain. Also, we need G to be a polynomial function so that we can use a good norm (linked to the degree of G in order to prove the stability of the domains).

We finish by giving the assumptions on the open domain $\mathcal{D}$:

Assumption (A-6): *The domain $\mathcal{D}$ contains only one critical point of V: a.*

Assumption (A-7): *The domain $\mathcal{D}$ is stable by the potential $V + F * \delta_a$.*

Indeed, heuristically, the potential $V + F * \left(\sum_{j=1}^{N} \delta_{X_t^{j,N}}\right)$ is close to the potential $V + F * \delta_a$ in the small-noise limit. So, we can link the study of the first particle (and of any particle) with the study of a classical diffusion with potential $V + F * \delta_a$. In order to apply Freidlin-Wentzell theory, we thus assume this hypothesis.

Assumption (A-8): *There exists $\rho > 0$ such that: $\langle \nabla V(x); x - a \rangle \geq \rho ||x - a||^2$ for any $x \in \overline{\mathcal{D}}$.*

This assumption allows us to prove that the domains that we will consider on $\mathbb{R}^{dN}$ are stable by the potential $N\Upsilon^N$.

Assumption (A-9): *There exists $\delta > 0$ such that for any $x \in \overline{\mathcal{D}}$: $V(x) - V(a) \geq \frac{\delta}{2}||x - a||^2$*

This simple hypothesis yields that the exit-cost of a ball of center $(a, \cdots, a)$ with any radius $\kappa > 0$ goes to infinity as N goes to infinity. Let us point out that this assumption is not of convexity type.

Assumption (A-10): *By putting $\varphi_t := x_0 - \int_0^t \nabla V(\varphi_s)\, ds$, then for any $t \geq 0$, $\varphi_t \in \mathcal{D}$.*

If Assumption (A-10) is not satisfied, we can easily prove that the exit-time is sub-exponential.

Assumption (A-11): *There exists an open domain $\mathcal{D}'$ which contains $\overline{\mathcal{D}}$ and which satisfies assumptions (A-6)–(A-10).*

This last assumption allows us to isolate the first particle.

Example 1.3. *We now give an example of potentials and domain satisfying Assumptions (A-1)–(A-11) in the one-dimensional case. We take $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$. Then, any domain of the form $]\xi; +\infty[$ with $\xi \in]0; 1[$ satisfies the assumptions with $\rho := \xi^2 + \xi > 0$ and $\delta := \left(\frac{1+\xi}{2}\right)^2$. Here, $a = 1$.*

The paper is organized as follows. We finish the introduction by introducing the norms that we use. In a second section, we give the material then we obtain the stability of the studied domains (on $\mathbb{R}^{dN}$) by $N\Upsilon^N$. In Section 3, we compute the exit-costs of the domains. Finally, main results are stated in Section 4.

On $\mathbb{R}^d$, we use the Euclidean norm.

Definition 1.4. *Let N be a positive integer. On $\mathbb{R}^{dN}$, we use the norm $||\,.\,||_N$ defined by*

$$||\mathcal{X}||_N^{2n} := \frac{1}{N}\sum_{i=1}^N ||X_i||^{2n} \,,$$

with $\mathcal{X} := (X_1, \cdots, X_N) \in \mathbb{R}^{dN}$. We remind the reader that $2n = \deg(G)$.

2 Preliminaries

We now add a last technical assumption on the domain and we will discuss how we can remove it.

Definition 2.1. 1. *$\mathbb{B}_\kappa^\infty(\overline{a})$ denotes the set of all the probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} ||x-a||^{2n}\mu(dx) \leq \kappa^{2n}$.*
2. *For all the measures μ, W_μ is equal to $V + F * \mu$.*
3. *For all $\nu \in \left(\mathbb{B}_\kappa^\infty(\overline{a})\right)^{\mathbb{R}_+}$ and for all $x \in \mathbb{R}^d$, we also introduce the dynamical system:*

$$\psi_t^\nu(x) = x - \int_0^t \nabla W_{\nu_s}\left(\psi_s^\nu(x)\right) ds \,.$$

Assumption 2.2. *If κ is sufficiently small, for any $\nu \in \left(\mathbb{B}_\kappa^\infty\right)^{\mathbb{R}_+}$, for any $x \in \mathcal{D}$, $\psi_t^\nu(x) \in \mathcal{D}$.*

Thanks to [Tug12], we know that if $\mathcal{D}$ satisfies Assumption (A-7) then, there exist two families of open domains $\left(\mathcal{D}_\xi^e\right)_{\xi>0}$ and $\left(\mathcal{D}_\xi^i\right)_{\xi>0}$ satisfying Assumption 2.2 and such that $\mathcal{D}_\xi^i \subset \mathcal{D} \subset \mathcal{D}_\xi^e$ and

$$\lim_{\xi\to 0}\sup_{z\in\partial\mathcal{D}_\xi^i} \mathrm{d}\left(z\,;\,\mathcal{D}^c\right) = \lim_{\xi\to 0}\sup_{z\in\partial\mathcal{D}_\xi^e} \mathrm{d}\left(z\,;\,\mathcal{D}\right) = 0\,.$$

Consequently, proving the Kramers' type law for a domain satisfying Assumptions (A-6)–(A-11) and Assumption 2.2 is sufficient to obtain it for a domain satisfying Assumptions (A-6)–(A-11).

Now, we give the two domains that we will study on $\mathbb{R}^{dN}$.

$$\mathcal{G}_N^1 := \left(\mathcal{D} \times \left(\mathcal{D}'\right)^{N-1}\right) \bigcap \mathbb{B}_\kappa^N(\overline{a}) \tag{1}$$

and

$$\mathcal{G}_N := \mathcal{D}^N \bigcap \mathbb{B}_\kappa^N(\overline{a})\,, \tag{2}$$

where

$$\mathbb{B}_\kappa^N(\overline{a}) := \left\{\mathcal{X} \in \left(\mathbb{R}^d\right)^N \;:\; \frac{1}{N}\sum_{k=1}^N ||X_k - a||^{2n} \leq \kappa^{2n}\right\}.$$

Here, $\overline{a} = (a, \cdots, a)$.

3 Stability of the domains by $N\Upsilon^N$

In the current work, we deal with the time-homogeneous diffusion $\mathcal{X}^N$,

$$\mathcal{X}_t^N = \overline{x_0} + \sigma\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s^N\right)ds\,.$$

with $\overline{x_0} := (x_0, \cdots, x_0)$ and Υ^N is a potential of $\mathbb{R}^{dN}$.

In [Tug12], we have used the fact that $\mathbb{B}_\kappa^N(\overline{a})$ is stable by $N\Upsilon^N$. Here, we are not able to prove this but we can circumvent the difficulty.

Proposition 3.1. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. Then, the domains $\mathcal{G}_N^1$ and $\mathcal{G}_N$ are stable by the potential $N\Upsilon^N$ providing that κ is sufficiently small.*

Proof. We will prove the proposition only for $\mathcal{G}_N := \mathcal{D}^N \bigcap \mathbb{B}_\kappa^N(\overline{a})$. Indeed, the technics are similar for $\mathcal{G}_N^1$. Set $\mathcal{X}_0 \in \mathcal{G}_N$. We consider the dynamical system

$$\mathcal{X}_t = \mathcal{X}_0 - N\int_0^t \nabla\Upsilon^N\left(\mathcal{X}_s\right)ds\,.$$

Let us assume that there exists $t > 0$ such that $\mathcal{X}_t \notin \mathcal{G}_N$. We consider t_0 the first time that $\mathcal{X}_{t_0} \notin \mathcal{G}_N$. Then, we have

$$\mathcal{X}_{t_0} = \mathcal{X}_0 - N\int_0^{t_0} \nabla\Upsilon^N\left(\mathcal{X}_s\right)ds\,.$$

For any $t \leq t_0$, $\mathcal{X}_t \in \mathbb{B}_\kappa^N(\overline{a})$: we deduce that the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is in $\mathbb{B}_\kappa^\infty(\overline{a})$. According to Assumption 2.2, we deduce that $\mathcal{X}_{t_0} \in \mathcal{D}^N$.

For any $t \leq t_0$, $\mathcal{X}_t \in \mathcal{D}^N$. We deduce that

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\sum_{i=1}^N \left|\left|X_t^i - a\right|\right|^{2n} &= -\frac{2n}{N}\sum_{i=1}^N \left\langle X_t^i - a; \nabla V(X_t^i)\right\rangle \left|\left|X_t^i - a\right|\right|^{2n-2}\\ &\quad - \frac{2n}{N}\sum_{i=1}^N\sum_{j=1}^N \left\langle X_t^i - a; \nabla F(X_t^i - X_t^j)\right\rangle \left|\left|X_t^i - a\right|\right|^{2n-2}\,.\end{aligned}$$

However, due to the convexity of F, we can show that

$$\sum_{i=1}^N\sum_{j=1}^N \left\langle X_t^i - a; \nabla F(X_t^i - X_t^j)\right\rangle \left|\left|X_t^i - a\right|\right|^{2n-2}$$

is nonnegative. Indeed, we remark that

$$\begin{aligned}&\sum_{i=1}^N\sum_{j=1}^N \left\langle X_t^i - a; \nabla F(X_t^i - X_t^j)\right\rangle \left|\left|X_t^i - a\right|\right|^{2n-2}\\ &= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left\langle \left(X_t^i - a\right)\left|\left|X_t^i - a\right|\right|^{2n-2} - \left(X_t^j - a\right)\left|\left|X_t^j - a\right|\right|^{2n-2}; \nabla F(X_t^i - X_t^j)\right\rangle\,.\end{aligned}$$

It is then sufficient to show that $\left\langle X\left|\left|X\right|\right|^{2n-2}-Y\left|\left|Y\right|\right|^{2n-2};\nabla F(X-Y)\right\rangle\geq 0$ for any $X,Y\in\mathbb{R}^d$.

This last term is equal to

$$\frac{G'\left(||X-Y||\right)}{||X-Y||}\left\langle X\left|\left|X\right|\right|^{2n-2}-Y\left|\left|Y\right|\right|^{2n-2};\,X-Y\right\rangle\,.$$

Due to the convexity of G, we have $\frac{G'(||X-Y||)}{||X-Y||}\geq 0$ for any X,Y in $\mathbb{R}^d$. The Cauchy-Schwarz inequality implies

$$\left\langle X\left|\left|X\right|\right|^{2n-2}-Y\left|\left|Y\right|\right|^{2n-2};\,X-Y\right\rangle\geq\left(\left|\left|X\right|\right|^{2n-1}-\left|\left|Y\right|\right|^{2n-1}\right)\left(||X||-||Y||\right)\geq 0$$

for all $X,Y\in\mathbb{R}^d$.

We deduce:

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\left|\left|X_t^i-a\right|\right|^{2n}\leq-\frac{2n\rho}{N}\sum_{i=1}^{N}\left|\left|X_t^i-a\right|\right|^{2n}\,.$$

Consequently, $\mathcal{X}_{t_0}\in\mathbb{B}_\kappa^N(\overline{a})$. This is absurd. We deduce that $\mathcal{G}_N$ is stable by the potential $N\Upsilon^N$. $\square$

4 Exit-costs of the domains

We now compute the exit-costs of the two domains:

$$H_N^1(\kappa):=\inf_{Z\in\partial\mathcal{G}_n^1}\left(N\Upsilon^N(Z)-N\Upsilon^N(\overline{a})\right)$$

and

$$H_N(\kappa):=\inf_{Z\in\partial\mathcal{G}_n}\left(N\Upsilon^N(Z)-N\Upsilon^N(\overline{a})\right)\,.$$

Proposition 4.1. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. Then, the following limits hold:*

$$\lim_{N\to\infty}H_N^1(\kappa)=H^1(\kappa)\quad\text{and}\quad\lim_{\kappa\to 0}H^1(\kappa)=H\,,$$

with $H:=\inf_{z\in\mathcal{D}}V(z)+F(z-a)-V(a)$. *We also have:*

$$\lim_{N\to\infty}H_N(\kappa)=H(\kappa)\quad\text{and}\quad\lim_{\kappa\to 0}H(\kappa)=H\,.$$

Proof. We will prove the result for $H_N^1(\kappa)$. First, we remark that $\partial A\bigcap B=\left((\partial A)\bigcap\overline{B}\right)\bigcup\left((\partial B)\bigcap\overline{A}\right)$.

It has already been proved, see [Tug12], that

$$\lim_{N\to\infty}\inf_{Z\in\partial\left(\mathcal{D}\times(\mathcal{D}')^{N-1}\right)\bigcap\mathbb{B}_\kappa^N(\overline{a})}\left(N\Upsilon^N(Z)-N\Upsilon^N(\overline{a})\right)=H^1(\kappa)\,.$$

It is a straightforward task to show that the following holds:

$$\lim_{N\to\infty}\inf_{Z\in\partial\left(\mathcal{D}\times(\mathcal{D}')^{N-1}\right)\bigcap\overline{\mathbb{B}_\kappa^N(\overline{a})}}\left(N\Upsilon^N(Z)-N\Upsilon^N(\overline{a})\right)=H^1(\kappa)\,.$$

So, it is sufficient to prove

$$\lim_{N\to\infty}\inf_{Z\in\left(\overline{\mathcal{D}\times(\mathcal{D}')^{N-1}}\right)\bigcap\partial\mathbb{B}_\kappa^N(\overline{a})}\left(N\Upsilon^N(Z)-N\Upsilon^N(\overline{a})\right)=\infty\,.$$

It is immediate once we have remarked that for any

$$\mathcal{X}=(X_1,\cdots,X_N)\in\left(\overline{\mathcal{D}\times(\mathcal{D}')^{N-1}}\right)\bigcap\partial\mathbb{B}_\kappa^N(\overline{a})\,,$$

we have:

$$\begin{aligned}
N\Upsilon^N(X_1,\cdots,X_N)&=\sum_{i=1}^N V(X_i)+\frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N F(X_i-X_j)\\
&\geq\sum_{i=1}^N V(X_i)\\
&\geq\sum_{i=1}^N\left(V(a)+\frac{\delta}{2}\left|\left|X_i-a\right|\right|^2\right)\\
&\geq NV(a)+\frac{\delta}{2}\sum_{i=1}^N\left|\left|X_i-a\right|\right|^2\\
&\geq NV(a)+\frac{N^{\frac{1}{n}}\delta}{2}\left(\sum_{i=1}^N\left|\left|X_i-a\right|\right|^{2n}\right)^{\frac{1}{n}}\\
&\geq NV(a)+\frac{N^{\frac{1}{n}}\delta}{2}\kappa^2\,.
\end{aligned}$$

This achieves the proof since $\Upsilon^N(\overline{a})=V(a)$.

□

5 Main results

We now are able to obtain the Kramers' type law for the first particle or for any particle.

Theorem 5.1. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. By $\tau_{\mathcal{D}}^{1,N}(\sigma)$, we denote the first exit-time of the diffusion $X^{1,N}$ from the domain $\mathcal{D}$. If N is large enough, for any $\delta>0$, we have the following limit as σ goes to 0 :*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1-\delta\right)}\leq\tau_{\mathcal{D}}^{1,N}(\sigma)\leq e^{\frac{2}{\sigma^2}\left(H_N^1+\delta\right)}\right\}\longrightarrow1\,,$$

where

$$\lim_{N\to\infty} H_N^1 = H\,,$$

with $H := \inf_{z\in\mathcal{D}} V(z) + F(z-a) - V(a)$.

This method relies on the second part of the result of Freidlin-Wentzell theory (the one on the exit-location).

Proof. We do not give the detailed proof since it is similar to the ones in [Tug12]. Indeed, Proposition 3.1 and Proposition 4.1 imply that we have a Kramers' type law for the first exit-time from the domain $\mathcal{G}_n^1$ that is for any $\delta > 0$:

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)-\delta\right)} \leq \tau_{\mathcal{G}_n^1}(\sigma) \leq e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)+\delta\right)}\right\} \longrightarrow 1\,,$$

where $\tau_{\mathcal{G}_n^1}(\sigma)$ is the first exit-time from $\mathcal{G}_n^1$. But, we also have that for any $\mathcal{N} \subset \partial\mathcal{G}_n^1$ such that $\inf_{X\in\mathcal{N}} N\Upsilon^N(X) > H_N^1(\kappa)$, we obtain

$$\lim_{\sigma\to 0} \mathbb{P}\left(\mathcal{X}^N_{\tau_{\mathcal{G}_n^1}(\sigma)} \in \mathcal{N}\right) = 0\,.$$

Now, we have proved that the exit-cost from the ball of center $\overline{a}$ and radius κ was larger than the one from $\mathcal{G}_n^1$ by taking N sufficiently large. In the same way, we can prove that the exit cost of the particles 2 to N from the domain $\mathcal{D}'$ is larger than the one from $\mathcal{G}_n^1$. Consequently, with a probability close to 1 as σ goes to 0, we have that $X^{1,N}_{\tau_{\mathcal{G}_n^1}(\sigma)} \in \partial\mathcal{D}$.

Then, we have the following:

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)+\delta\right)} \leq \tau_{\mathcal{D}}^{1,N}(\sigma)\right\} \leq \mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)+\delta\right)} \leq \tau_{\mathcal{G}_n^1}(\sigma)\right\} + \mathbb{P}\left(X^{1,N}_{\tau_{\mathcal{G}_n^1}(\sigma)} \notin \partial\mathcal{D}\right)\,,$$

which goes to zero as σ goes to zero. In the same vein, as $\mathcal{G}_n^1 \subset \mathcal{D}\times\left(\mathbb{R}^d\right)^{N-1}$, we have immediately

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)-\delta\right)} \geq \tau_{\mathcal{D}}^{1,N}(\sigma)\right\} \leq \mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)-\delta\right)} \geq \tau_{\mathcal{G}_n^1}(\sigma)\right\} \longrightarrow 0\,.$$

This implies that for any $\kappa > 0$, if N is large enough:

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)-\delta\right)} \leq \tau_{\mathcal{D}}^{1,N}(\sigma) \leq e^{\frac{2}{\sigma^2}\left(H_N^1(\kappa)+\delta\right)}\right\} \longrightarrow 1\,,$$

Finally, the exit-cost $H_N^1(\kappa)$ does not depend on κ if N is large enough. □

We have a similar result for the exit of any particle:

Theorem 5.2. *We assume Hypotheses (A-1)–(A-11) and Assumption 2.2. By* $\tau_{\mathcal{D}}^N(\sigma)$*, we denote the first exit-time of the diffusion* $\mathcal{X}$ *from the domain* $\mathcal{D}^N$*. If* N *is large enough, for any* $\delta > 0$*, we have the following limit as* σ *goes to* 0 *:*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H_N-\delta\right)} \leq \tau_{\mathcal{D}}^N(\sigma) \leq e^{\frac{2}{\sigma^2}\left(H_N+\delta\right)}\right\} \longrightarrow 1\,,$$

where

$$\lim_{N\to\infty} H_N = H\,.$$

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