ESAIM: PS 30 (2026) 162–191
https://doi.org/10.1051/ps/2026001

ESAIM: Probability and Statistics
www.esaim-ps.org

ASYMPTOTICS FOR ABSOLUTE RUIN PROBABILITIES OF A DEPENDENT BIDIMENSIONAL RISK MODEL WITH SUBEXPONENTIAL CLAIMS

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Abstract. This paper considers a continuous-time bidimensional risk model with a constant interest force, where there exist dependence structures among the claim sizes of two business lines and the inter-arrival times of the claim sizes. When the claim sizes have subexponential distributions, some uniform asymptotics for the finite-time absolute ruin probabilities of the bidimensional risk model are established.

Mathematics Subject Classification. 62P05, 62E10, 91B30.

Received June 13, 2025. Accepted January 8, 2026.

1. Introduction

In this paper, we investigate a continuous-time bidimensional renewal risk model. At time $t \geq 0$, the bidimensional surplus process $\left(U_1\left(x_1, t\right), U_2\left(x_2, t\right)\right)^{\top}$ can be described as

$$\begin{pmatrix} U_1\left(x_1, t\right) \\ U_2\left(x_2, t\right) \end{pmatrix} = \mathrm{e}^{rt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \int_0^t c_1 \mathrm{e}^{r(t-s)} \mathrm{d}s \\ \int_0^t c_2 \mathrm{e}^{r(t-s)} \mathrm{d}s \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{N(t)} X_i^{(1)} \mathrm{e}^{r(t-\tau_i)} \\ \sum_{i=1}^{N(t)} X_i^{(2)} \mathrm{e}^{r(t-\tau_i)} \end{pmatrix}, \tag{1.1}$$

where $\left(x_1, x_2\right)^{\top}$ is a vector of initial surpluses, $\left(c_1, c_2\right)^{\top}$ is a non-negative constant premium rate vector and $r \geq 0$ is a constant interest rate. $\left\{\left(X_i^{(1)}, X_i^{(2)}\right)^{\top}, i \geq 1\right\}$ is a sequence of claim size vectors whose common arrival times $\{\tau_i, i \geq 1\}$ constitute a renewal claim-number process $\{N(t), t \geq 0\}$. That is $N(t) = \sup\{n \geq 1, \tau_n \leq t\}$, $t \geq 0$. The claim arrival times $\{\tau_i, i \geq 1\}$ are non-negative random variables with $\left\{\tau_i = \sum_{j=1}^{i} \theta_j, i \geq 1\right\}$, where the inter-arrival times $\{\theta_j, j \geq 1\}$ are independent and identically distributed (i.i.d.) random variables and are not degenerate at 0. Let $\lambda(t) = \mathbb{E}[N(t)] < \infty$, $t \geq 0$.

In recent decades, a considerable number of authors have turned their attention to bidimensional risk models with an independence structure. For example, [1–4] and so on. In these works, the claim sizes of each business line and the inter-arrival times form a sequence of i.i.d. random variables, respectively. Recently, dependence structures have been established between the claim sizes of two distinct businesses. [5] explored the bivariate

Keywords and phrases: Bidimensional risk model, absolute ruin probability, asymptotics, subexponential distribution.

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TABLE 1. Summary of copulas and dependence structures.

Article	**Name**	**Mathematical definition**	**Parameter**
[5]	Bivariate Farlie–Gumbel–Morgenstern (FGM) Copula	$C(u_1,u_2)=u_1u_2(1+\sigma(1-u_1)(1-u_2))$	$\sigma\in[-1,1]$.
[6]	Bivariate Sarmanov Distribution	$\mathbb{P}(\xi_1\in \mathrm{d}x_1,\xi_2\in \mathrm{d}x_2) = (1+\vartheta\phi_1(x_1)\phi_2(x_2)) V_1(\mathrm{d}x_1)V_2(\mathrm{d}x_2)$	$\vartheta\in(-\infty,\infty)$ satisfying $1+\vartheta\phi_1(x_1)\phi_2(x_2)\geq 0$, $x_i\in D_{\xi_i}$, $i=1,2$.
[7]	Strongly Tail Asymptotically Independent (SAI)	$\lim_{x_i\wedge x_j\to\infty}\frac{\mathbb{P}(\xi_i>x_i,\xi_j>x_j)}{\mathbb{P}(\xi_i>x_i)\mathbb{P}(\xi_j>x_j)}=\rho$, $1\leq i<j\leq n$	$\rho\in(0,\infty)$.
[17]	Widely Orthant Dependent (WOD)	WUOD: $\mathbb{P}(\bigcap_{i=1}^n\{\xi_i>x_i\}) \leq g_U(n)\prod_{i=1}^n\mathbb{P}(\xi_i>x_i)$, WLOD: $\mathbb{P}(\bigcap_{i=1}^n\{\xi_i\leq x_i\}) \leq g_L(n)\prod_{i=1}^n\mathbb{P}(\xi_i\leq x_i)$	$g_U(n)<\infty$, $g_L(n)<\infty$, $n\geq 1$.
[16]	Conditionally Independent	$\mathbb{P}(\xi_1\leq x_1,\xi_2\leq x_2\vert\mathcal{A}) = \mathbb{P}(\xi_1\leq x_1\vert\mathcal{A})\mathbb{P}(\xi_2\leq x_2\vert\mathcal{A})$	$\mathcal{A}$ is a σ-algebra.

Farlie–Gumbel–Morgenstern (FGM) distribution. [6] examined the bivariate Sarmanov distribution. [7] further investigated a general dependence structure. What's more, actuarial practice shows that there is a certain dependence between the claim sizes and their inter-arrival times. Related researches can be found in [8–11] and so on. [12] and [13] also studied the ruin probabilities of the continuous-time bidimensional risk models with a certain dependence structure between the claim sizes and their inter-arrival times, which inspire this paper.

In previous studies, the moment when the surplus process falls below zero is typically defined as the ruin time. However, as mentioned by [14], the boundary value of zero is not realistic in practice. A more reasonable boundary value should be defined as $-c/r$, where c is the constant premium rate and r is the constant interest rate. After the surplus drops below zero, the insurance company may not necessarily ruin and cease operations. Instead, the company may borrow funds at a certain interest rate to continue operating, while using its premium income to repay the debt. This type of ruin probability was defined as the absolute ruin probability by [15]. Recently, the absolute ruin probability has been studied by some scholars. For instance, [16] considered the absolute ruin probability of a one-dimensional renewal risk model with a constant premium rate and a constant interest rate, whose claim sizes are conditionally independent on some sigma algebra and the common distribution belongs to the intersection of the long-tailed distribution class and the dominatedly-varying-tailed distribution class. [17] considered the absolute ruin probability of a dependent compound renewal risk model with a constant premium rate and a constant interest rate, where the individual claim sizes are widely orthant dependent and the claim sizes have a common distribution belonging to three types of heavy-tailed distribution classes, respectively. [18] considered the problem of minimizing the absolute ruin probability of an insurance company. More studies on the absolute probability can be found in [15, 19, 20] and so on.

For the convenience of readers, Table 1 presents a brief summary of several commonly used copulas and dependence structures discussed above. The explanations of some notation used in Table 1 are given below. $\xi_i,\ i\geq 1$ are random variables with distributions $V_i,\ i\geq 1$, respectively. $C(u_1,u_2)$ is a bivariate copula with $u_i\in[0,1]$, $i=1,2$. $\phi_1(x_1)$ and $\phi_2(x_2)$ are two functions such that $\mathbb{E}[\phi_i(\xi_i)]=0$, $i=1,2$. $D_{\xi_i}=\{x_i\geq 0:\mathbb{P}(\xi_i\in(x_i-\delta,x_i+\delta))>0,\forall\delta>0\}$, $i=1,2$.

It follows from [14] and [15] that the finite-time absolute ruin probability at time $t \geq 0$ of a one-dimensional risk model can be defined as

$$\psi_k\left(x_k, t\right) = \mathbb{P}\left(\inf_{0 \leq s \leq t} U_k\left(x_k, s\right) < -\frac{c_k}{r} \middle| U_k\left(x_k, 0\right) = x_k\right), \quad k = 1, 2. \tag{1.2}$$

[15] pointed out that the finite-time absolute ruin probability at time $t \geq 0$ of one-dimensional risk model, *i.e.* (1.2), can be converted to

$$\psi_k\left(x_k, t\right) = \mathbb{P}\left(\sum_{p=1}^{N(t)} X_p^{(k)} \prod_{i=1}^{p} Y_i > x_k + \frac{c_k}{r}\right), \quad k = 1, 2, \tag{1.3}$$

where $\prod_{i=1}^{p} Y_i = \mathrm{e}^{-r\tau_p}$, $p \geq 1$ and $Y_i = \mathrm{e}^{-r\theta_i}$, $i \geq 1$.

Inspired by the above definition of the finite-time absolute ruin probability of a one-dimensional risk model and the four types of finite-time ruin probabilities of a bidimensional risk model, we define the corresponding four types of finite-time absolute ruin probabilities of a bidimensional risk model at time $t \geq 0$ as

$$\begin{aligned}&\Psi_{sim}\left(x_1, x_2; t\right)\\&= \mathbb{P}\left(U_1\left(x_1, s\right) < -\frac{c_1}{r} \text{ and } U_2\left(x_2, s\right) < -\frac{c_2}{r} \text{ for some } 0 \leq s \leq t \middle| U_k\left(x_k, 0\right) = x_k, k = 1, 2\right),\end{aligned} \tag{1.4}$$

$$\Psi_{and}\left(x_1, x_2; t\right) = \mathbb{P}\left(\bigcap_{k=1}^{2}\left\{\inf_{0 \leq s \leq t} U_k\left(x_k, s\right) < -\frac{c_k}{r}\right\} \middle| U_k\left(x_k, 0\right) = x_k, k = 1, 2\right), \tag{1.5}$$

$$\Psi_{or}\left(x_1, x_2; t\right) = \mathbb{P}\left(\bigcup_{k=1}^{2}\left\{\inf_{0 \leq s \leq t} U_k\left(x_k, s\right) < -\frac{c_k}{r}\right\} \middle| U_k\left(x_k, 0\right) = x_k, k = 1, 2\right), \tag{1.6}$$

and

$$\Psi_{sum}\left(x_1, x_2; t\right) = \mathbb{P}\left(\inf_{0 \leq s \leq t}\left\{\sum_{k=1}^{2} U_k\left(x_k, s\right)\right\} < -\sum_{k=1}^{2} \frac{c_k}{r} \middle| U_k\left(x_k, 0\right) = x_k, k = 1, 2\right). \tag{1.7}$$

The meaning of these four definitions is similar to the meaning of corresponding ruin probabilities given by [12]. That is, $\Psi_{sim}\left(x_1, x_2; t\right)$ represents the probability that absolute ruin happens in all business lines at the same time within the time period $[0, t]$. $\Psi_{and}\left(x_1, x_2; t\right)$ indicates the probability of absolute ruin in all business lines, though not necessarily simultaneously. $\Psi_{or}\left(x_1, x_2; t\right)$ reflects the probability that absolute ruin occurs in at least one business line, while $\Psi_{sum}\left(x_1, x_2; t\right)$ represents the probability of absolute ruin across the sum of all business lines. It is worth mentioning that it holds for all $x_1, x_2 \geq 0$ and $t \geq 0$ that

$$\Psi_{sim}\left(x_1, x_2; t\right) \leq \Psi_{and}\left(x_1, x_2; t\right) \leq \Psi_{or}\left(x_1, x_2; t\right),$$

and

$$\Psi_{sum}\left(x_1, x_2; t\right) \leq \Psi_{or}\left(x_1, x_2; t\right).$$

In summary, this paper introduces four definitions of finite-time absolute ruin probabilities for a bidimensional risk model. The focus is on the scenario where the claims from two lines of business exhibit a dependence structure, while also considering the dependence structure between the claims and the inter-arrival times. For the case of subexponential claims, the asymptotics of the four finite-time absolute ruin probabilities are derived. Compared with [1] and [17], [1] discussed a bidimensional risk model without an interest rate (*i.e.*, a bidimensional renewal risk model) and provided asymptotics for the finite-time ruin probabilities (not the finite-time absolute ruin probabilities), where the claims between the two business lines are independent, and the claims and the claim number process are independent. [17] investigated a one-dimensional compound renewal risk model with a constant interest rate and derived asymptotics for the finite-time absolute ruin probabilities, where the claims and the accident number process are independent.

The rest of this paper consists of five sections. Section 2 includes preliminaries on heavy-tailed distributions and some assumptions. The main results are given in Section 3. Section 4 provides some examples satisfying assumptions. Section 5 contains some numerical studies. The proofs of our results are presented in Section 6.

2. Preliminaries and assumptions

Thereafter, all limit relations are for $x \to \infty$ or $(x_1, x_2) \to (\infty, \infty)$, unless otherwise specified. For two positive univariate or bivariate functions f and g, assume that $a = \liminf f/g \le \limsup f/g = b$. We write $f \gtrsim g$, if $a \ge 1$; write $f \lesssim g$, if $b \le 1$; write $f = o(g)$, if $b = 0$; write $f = O(g)$, if $b < \infty$; write $f \sim g$, if $a = b = 1$; and write $f \asymp g$, if $f = O(g)$ and $g = O(f)$, simultaneously. Furthermore, for two positive trivariate functions $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$, we say that the asymptotic relation $a(x_1, x_2, t) \sim b(x_1, x_2, t)$ holds uniformly for t in a nonempty set $\varDelta$ if

$$\lim_{(x_1,x_2)\to(\infty,\infty)} \sup_{t\in\varDelta} \left| \frac{a(x_1,x_2,t)}{b(x_1,x_2,t)} - 1 \right| = 0.$$

Typically, for a random variable ξ with a distribution V on $(-\infty, \infty)$, we write $\xi^+ = \xi \mathbf{1}_{\{\xi \ge 0\}}$, where $\mathbf{1}_{\{\xi\ge 0\}}$ denotes the indictor function of $\{\xi \ge 0\}$ and denote its tail by $\overline{V}(x) = 1 - V(x)$, $x \in (-\infty, \infty)$.

In this paper, assume that $\left\{ \left(X_i^{(1)}, X_i^{(2)}, Y_i \right); i \ge 1 \right\}$ is a sequence of i.i.d. random vectors with a generic random vector $\left(X^{(1)}, X^{(2)}, Y \right)$ whose marginal distributions are F_1, F_2 on $[0, \infty)$ and G on $(0, 1]$, respectively, where $Y = \mathrm{e}^{-r\theta}$ and θ is the generic random variable of $\{\theta_i; i \ge 1\}$. Define

$$\varLambda = \{t > 0, \lambda(t) > 0\} = \left\{ t > 0, \mathbb{P}\left(Y_1 \ge \mathrm{e}^{-rt}\right) > 0 \right\},$$

and

$$\varLambda_* = [0, T] \cap \varLambda$$

for all $T \in \varLambda$ for later use.

This paper will consider the claim sizes of two business lines and the inter-arrival times are dependent. We will need the following assumptions, which are introduced by [13].

Assumption 2.1. For $i = 1, 2$, there exists a function h_i such that

$$\mathbb{P}\left(X^{(i)} > x | Y = y \right) \sim \overline{F_i}(x) h_i(y) \tag{2.1}$$

holds uniformly for $y \in (0, 1]$, and

$$0 < A_1 := \min \left\{ \inf_{y\in(0,1]} h_i(y); i = 1, 2 \right\} \le \max \left\{ \sup_{y\in(0,1]} h_i(y); i = 1, 2 \right\} =: A_2 < \infty.$$

Clearly, (2.1) is equivalent that

$$\mathbb{P}\left(X^{(i)}>x|\theta=t\right)\sim\overline{F_i}\left(x\right)h_i\left(\mathrm{e}^{-rt}\right)=:\overline{F_i}\left(x\right)\mathcal{H}_i\left(t\right)$$

holds uniformly for $t\in[0,\infty)$.

Assumption 2.2. For $i,j=1,2,i\neq j$, there exists a bivariate function g_{ij} such that

$$\mathbb{P}\left(X^{(i)}>x|X^{(j)}=z,Y=y\right)\sim\overline{F_i}\left(x\right)g_{ij}\left(z,y\right) \tag{2.2}$$

holds uniformly for all $z\geq 0,y\in(0,1]$ and

$$\begin{aligned}0<B_1:=&\min\left\{\inf_{z\geq0,y\in(0,1]}g_{ij}\left(z,y\right);i,j=1,2,i\neq j\right\}\\ \leq&\max\left\{\sup_{z\geq0,y\in(0,1]}g_{ij}\left(z,y\right);i,j=1,2,i\neq j\right\}=:B_2<\infty.\end{aligned}$$

Assumption 2.3. There exists a function g such that

$$\mathbb{P}\left(X^{(1)}>x_1,X^{(2)}>x_2|Y=y\right)\sim\overline{F_1}\left(x_1\right)\overline{F_2}\left(x_2\right)g\left(y\right) \tag{2.3}$$

holds uniformly for $y\in(0,1]$.

It is obvious that (2.3) is equivalent that

$$\begin{aligned}\mathbb{P}\left(X^{(1)}>x_1,X^{(2)}>x_2|\theta=t\right)&\sim\overline{F_1}\left(x_1\right)\overline{F_2}\left(x_2\right)g\left(\mathrm{e}^{-rt}\right)\\&=:\overline{F_1}\left(x_1\right)\overline{F_2}\left(x_2\right)\mathcal{G}\left(t\right)\end{aligned}$$

holds uniformly for $t\in[0,\infty)$.

[11] and [13] point out that some concrete copulas satisfy the above assumptions. For example, the bivariate Farlie–Gumbel–Morgenstern(FGM) copula satisfies Assumption 2.1. The tri-dimensional Sarmanov copula and the tri-dimensional Frank copula satisfy Assumptions 2.2 and 2.3. For the definitions of these copulas, one can refer to [21].

In the rest of this section we will introduce some classes of heavy-tailed distributions. If a real-valued random variable ξ (or its distribution V) does not have a finite exponential moment, *i.e.*, for any $\lambda>0$, $\mathbb{E}\left[\mathrm{e}^{\lambda\xi}\right]=\infty$, then we call it heavy-tailed.

It is said that a distribution V on $(-\infty,\infty)$ belongs to the long-tailed distribution class, denoted by $V\in\mathscr{L}$, if its tail satisfies

$$\overline{V}\left(x+y\right)\sim\overline{V}\left(x\right)$$

for all $y\in(-\infty,\infty)$. The following property of the long-tailed distributions can be found in [11].

Proposition 2.4. *If $V\in\mathscr{L}$, then the function set*

$$\begin{aligned}\mathscr{H}\left(V\right)=\big\{&l\left(\cdot\right):(0,\infty)\to(0,\infty)\,|l\left(x\right)\to\infty,l\left(x\right)<x/2,l\left(x\right)x^{-1}\to0\text{ and}\\&\overline{V}\left(x+y\right)\sim\overline{V}\left(x\right)\text{ holds uniformly for all }|y|\leq l\left(x\right)\big\}\end{aligned}$$

is not empty. Besides, if $l\left(x\right)\in\mathscr{H}\left(V\right)$ then $c\cdot l\left(x\right)\in\mathscr{H}\left(V\right)$ for any $c>0$.

Another heavy-tailed distribution class is the subexponential distribution class $\mathscr{S}$. A distribution V on $[0,\infty)$ is said to be subexponential, denoted by $V \in \mathscr{S}$, if

$$\lim \frac{\overline{V^{*n}}(x)}{\overline{V}(x)} = n$$

for some (or, equivalently, for all) $n \geq 2$, where V^{*n} represents the n-fold convolution of V, $n \geq 2$. For a distribution V on $(-\infty,\infty)$, if $V(x)\mathbf{1}_{\{x\geq 0\}}$ is subexponential, then V is also said to be subexponential. It is well known that $\mathscr{S} \subset \mathscr{L}$ (see, *e.g.* [22, 23]). The following proposition shows a property of the subexponential distributions, which presents a weighted result for Theorem 3.6 of [23].

Proposition 2.5. *Let V be a distribution on $(-\infty,\infty)$. Suppose that $V \in \mathscr{S}$ and that a non-negative function l is such that $l(x) \to \infty$ and $l(x) < x/2, x \geq 0$. Let two distributions V_1 and V_2 be such that, for $i = 1, 2$, it holds that $V_i \in \mathscr{L}$ and $\overline{V_i}(x) \asymp \overline{V}(x)$. If η_1 and η_2 are two independent and non-negative random variables with distributions V_1 and V_2, respectively, then for any fixed $0 < a \leq b < \infty$, it holds uniformly for $(c_1, c_2) \in [a,b]^2$ that*

$$\mathbb{P}(c_1\eta_1 + c_2\eta_2 > x, c_1\eta_1 > l(x), c_2\eta_2 > l(x)) = o(\mathbb{P}(c_1\eta_1 > x) + \mathbb{P}(c_2\eta_2 > x)).$$

Proof. Since $l(x) < x/2, x \geq 0$, it holds for all $(c_1, c_2) \in [a,b]^2$ and $x > 0$ that

$$\begin{aligned}
&\mathbb{P}(c_1\eta_1 + c_2\eta_2 > x, c_1\eta_1 > l(x), c_2\eta_2 > l(x)) \\
=\ &\mathbb{P}(c_1\eta_1 + c_2\eta_2 > x) - \mathbb{P}(c_1\eta_1 + c_2\eta_2 > x, c_1\eta_1 \leq l(x)) \\
&-\mathbb{P}(c_1\eta_1 + c_2\eta_2 > x, c_2\eta_2 \leq l(x)) \\
=:\ &L_0 - L_1 - L_2.
\end{aligned} \tag{2.4}$$

By Lemma 1 of [24], it holds uniformly for $(c_1, c_2) \in [a,b]^2$ that

$$L_0 \sim \mathbb{P}(c_1\eta_1 > x) + \mathbb{P}(c_2\eta_2 > x). \tag{2.5}$$

For $(i, j) = (1, 2)$ or $(2, 1)$, it holds uniformly for $(c_1, c_2) \in [a,b]^2$ that

$$\begin{aligned}
L_j &\geq \mathbb{P}(c_i\eta_i > x, c_j\eta_j \leq l(x)) \\
&= \mathbb{P}(c_i\eta_i > x)\,\mathbb{P}(c_j\eta_j \leq l(x)) \\
&\geq \mathbb{P}(c_i\eta_i > x)\, V_j\left(\frac{l(x)}{b}\right) \\
&\sim \mathbb{P}(c_i\eta_i > x).
\end{aligned} \tag{2.6}$$

Then plugging (2.5) and (2.6) into (2.4) yields the desired result. This completes the proof of Proposition 2.5. □

3. Main results

To concisely present our main results, we adopt the following notation. Let

$$\begin{aligned}
\alpha(x_1, x_2; t) := \int_0^t \int_0^{t-u} \Bigg(&\overline{F_1}\left(\left(x_1 + \frac{c_1}{r}\right) \mathrm{e}^{r(u+v)}\right) \overline{F_2}\left(\left(x_2 + \frac{c_2}{r}\right) \mathrm{e}^{rv}\right) \\
&+\overline{F_1}\left(\left(x_1 + \frac{c_1}{r}\right) \mathrm{e}^{ru}\right) \overline{F_2}\left(\left(x_2 + \frac{c_2}{r}\right) \mathrm{e}^{r(u+v)}\right)\Bigg) \widetilde{\lambda_2}(dv)\, \widetilde{\lambda_1}(du),
\end{aligned}$$

$$\beta\left(x_1,x_2;t\right):=\int_0^t\overline{F_1}\left(\left(x_1+\frac{c_1}{r}\right)\mathrm{e}^{ru}\right)\overline{F_2}\left(\left(x_2+\frac{c_2}{r}\right)\mathrm{e}^{ru}\right)\widetilde{\widetilde{\lambda}}\left(du\right),$$

and

$$\gamma_k\left(x_k;t\right):=\int_0^t\overline{F_k}\left(\left(x_k+\frac{c_k}{r}\right)\mathrm{e}^{ru}\right)\widetilde{\lambda_k}\left(du\right),\quad k=1,2,$$

where

$$\widetilde{\lambda_k}(t)=\int_0^t\left(1+\lambda\left(t-u\right)\right)\mathcal{H}_k\left(u\right)\mathbb{P}\left(\theta\in du\right),\quad k=1,2, \tag{3.1}$$

and

$$\widetilde{\widetilde{\lambda}}(t)=\int_0^t\left(1+\lambda\left(t-u\right)\right)\mathcal{G}\left(u\right)\mathbb{P}\left(\theta\in du\right). \tag{3.2}$$

Theorems 3.1 and 3.2 provide asymptotics for four types of the finite-time absolute ruin probabilities of the bidimensional risk model (1.1) with dependence structures among the claim sizes of two business lines and the inter-arrival times of the claim sizes.

Theorem 3.1. *Consider the bidimensional risk model (1.1) satisfying Assumptions 2.1–2.3. If $F_k\in\mathscr{S},\ k=1,2$, then it holds uniformly for all $t\in\varLambda_*$ that*

$$\varPsi_{sim}\left(x_1,x_2;t\right)\sim\varPsi_{and}\left(x_1,x_2;t\right)\sim\alpha\left(x_1,x_2;t\right)+\beta\left(x_1,x_2;t\right), \tag{3.3}$$

and

$$\varPsi_{or}\left(x_1,x_2;t\right)\sim\gamma_1\left(x_1;t\right)+\gamma_2\left(x_2;t\right). \tag{3.4}$$

Theorem 3.2. *Consider the bidimensional risk model (1.1) satisfying Assumptions 2.1 and 2.2. If $F_k\in\mathscr{S},\ k=1,2$ and $\overline{F_1}\left(x\right)\asymp\overline{F_2}\left(x\right)$, then it holds uniformly for all $t\in\varLambda_*$ that*

$$\varPsi_{sum}\left(x_1,x_2;t\right)\sim\gamma_1\left(x_1+x_2+\frac{c_2}{r};t\right)+\gamma_2\left(x_1+x_2+\frac{c_1}{r};t\right). \tag{3.5}$$

The following remark is from Remark 2.1 of [12], which can explain the meanings of $\widetilde{\lambda_k}(t),\ k=1,2$ and $\widetilde{\widetilde{\lambda}}(t)$ mentioned in Theorems 3.1 and 3.2. Meanwhile, for the sake of clarity, we also present its proof.

Remark 3.3. For each $i=1,2$ and any $T\in\varLambda$, we introduce two random variables θ_i^* and θ_1^{**}, which are independent of $\{\theta_k,k\geq 2\}$ and have the proper distributions, respectively, given by

$$\mathbb{P}\left(\theta_i^*\in\mathrm{d}t\right):=\frac{\mathcal{H}_i\left(t\right)}{\mathbb{E}\left[\mathcal{H}_i\left(\theta\right)\mathbf{1}_{\{\theta\leq T\}}\right]}\mathbb{P}\left(\theta\in\mathrm{d}t\right),t\in\varLambda_*,$$

and

$$\mathbb{P}\left(\theta_1^{**}\in\mathrm{d}t\right):=\frac{\mathcal{G}\left(t\right)}{\mathbb{E}\left[\mathcal{G}\left(\theta\right)\mathbf{1}_{\{\theta\leq T\}}\right]}G\left(\mathrm{d}t\right),t\in\varLambda_*.$$

Construct three delayed renewal counting processes $\{N_i^*(t), t \geq 0\}$ with $\theta_i^*, \theta_k, k = 2, 3, ...$ and a mean function $\lambda_i^*(t), t \geq 0$ for each $i = 1, 2$ and $\{N^{**}(t), t \geq 0\}$ with $\theta_1^{**}, \theta_k, k = 2, 3, ...$ and a mean function $\lambda^{**}(t), t \geq 0$. It is easy to demonstrate that the following two relations hold

$$\widetilde{\lambda}_i(t) = \lambda_i^*(t)\,\mathbb{E}\left[\mathcal{H}_i(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right], \quad i = 1, 2, \quad t \in \Lambda_*,$$

and

$$\approx{\lambda}(t) = \lambda^{**}(t)\,\mathbb{E}\left[\mathcal{G}(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right], \quad t \in \Lambda_*.$$

Proof. Clearly, for $i = 1, 2$, it holds for all $t \in \Lambda_*$ that

$$\begin{aligned}
\lambda_i^*(t) &= \mathbb{E}\left[N^*(t)\right] \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(N^*(t) \geq n\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(\theta_i^* + \sum_{k=2}^{n} \theta_k \leq t\right) \\
&= \mathbb{P}\left(\theta_i^* \leq t\right) + \sum_{n=2}^{\infty} \mathbb{P}\left(\theta_i^* + \sum_{k=2}^{n} \theta_k \leq t\right) \\
&= \int_0^t \mathbb{P}\left(\theta_i^* \in du\right) + \int_0^t \sum_{n=2}^{\infty} \mathbb{P}\left(\sum_{k=2}^{n} \theta_k \leq t - u\right) \mathbb{P}\left(\theta_i^* \in du\right) \\
&= \int_0^t \left(1 + \lambda(t-u)\right) \mathbb{P}\left(\theta_i^* \in du\right).
\end{aligned}$$

Therefore, for $i = 1, 2$, it holds for all $t \in \Lambda_*$ that

$$\mathbb{P}\left(\theta_i^* \in du\right) \mathbb{E}\left[\mathcal{H}_i(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right] = \mathcal{H}_i(u)\,\mathbb{P}\left(\theta \in du\right),$$

and

$$\left(1 + \lambda(t-u)\right) \mathbb{P}\left(\theta_i^* \in du\right) \mathbb{E}\left[\mathcal{H}_i(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right] = \left(1 + \lambda(t-u)\right) \mathcal{H}_i(u)\,\mathbb{P}\left(\theta \in du\right).$$

Thus,

$$\begin{aligned}
\widetilde{\lambda}_i(t) &= \int_0^t \left(1 + \lambda(t-u)\right) \mathbb{P}\left(\theta_i^* \in du\right) \mathbb{E}\left[\mathcal{H}_i(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right] \\
&= \lambda_i^*(t)\,\mathbb{E}\left[\mathcal{H}_i(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right].
\end{aligned}$$

Similarly, we can obtain that for all $t \in \Lambda_*$,

$$\approx{\lambda}(t) = \lambda^{**}(t)\,\mathbb{E}\left[\mathcal{G}(\theta)\,\mathbf{1}_{\{\theta \leq T\}}\right].$$

This completes the proof of Remark 3.1. □

4. Some examples of Assumptions 2.1–2.3

This section presents several examples that satisfy Assumptions 2.1–2.3. We continue this discussion through copulas. For more details on copulas, one can refer to [21].

According to Sklar's theorem, if the joint distribution of $(X^{(1)}, X^{(2)}, Y)$, denoted by H, has continuous marginal distributions F_1, F_2 and G, then there exists some unique $C(u_1, u_2, u_3) : [0,1]^3 \mapsto [0,1]$ such that

$$H\left(s_1, s_2, s_3\right) = C\left(F_1\left(s_1\right), F_2\left(s_2\right), G\left(s_3\right)\right), \quad \left(s_1, s_2\right) \in [0,\infty)^2, s_3 \in (0,1].$$

The corresponding survival copula is defined as

$$\begin{aligned} \overline{H}\left(s_1, s_2, s_3\right) &:= \mathbb{P}\left(X^{(1)} > s_1, X^{(2)} > s_2, Y > s_3\right) \\ &= \hat{C}\left(\overline{F_1}\left(s_1\right), \overline{F_2}\left(s_2\right), \overline{G}\left(s_3\right)\right), \quad \left(s_1, s_2\right) \in [0,\infty)^2, s_3 \in (0,1]. \end{aligned}$$

Assume that both the copula $C(u_1, u_2, u_3)$ and the survival copula $\hat{C}(u_1, u_2, u_3)$ are absolutely continuous.

Firstly, it follows from Section 3 of [11] that if $h_1(y)$ and $h_2(y)$ defined in Assumption 2.2 exist, then they equal to

$$h_1\left(y\right) = \lim_{u_1 \to 0^+} \frac{\partial \hat{C}\left(u_1, 1, u_3\right)/\partial u_3}{u_1}\bigg|_{u_3 = \overline{G}(y)}, \quad y \in (0,1], \tag{4.1}$$

and

$$h_2\left(y\right) = \lim_{u_2 \to 0^+} \frac{\partial \hat{C}\left(1, u_2, u_3\right)/\partial u_3}{u_2}\bigg|_{u_3 = \overline{G}(y)}, \quad y \in (0,1]. \tag{4.2}$$

The uniformity required in Assumption 2.1 can be rewritten in terms of the survival copula $\hat{C}(u_1, u_2, u_3)$ as

$$\lim_{u_1 \to 0^+} \sup_{u_3 \in [0,1]} \left| \frac{\partial \hat{C}\left(u_1, 1, u_3\right)/\partial u_3}{u_1} \cdot \frac{1}{h_1\left(y\right)} - 1 \right| = 0,$$

and

$$\lim_{u_2 \to 0^+} \sup_{u_3 \in [0,1]} \left| \frac{\partial \hat{C}\left(1, u_2, u_3\right)/\partial u_3}{u_2} \cdot \frac{1}{h_2\left(y\right)} - 1 \right| = 0.$$

Secondly, if $g_{12}(z,y)$ and $g_{21}(z,y)$ defined in Assumption2.2 exist, then they equal to

$$g_{12}\left(z, y\right) = \lim_{u_1 \to 0^+} \frac{\partial^2 \hat{C}\left(u_1, u_2, u_3\right)/\partial u_2 \partial u_3}{u_1}\Big|_{u_2 = \overline{F_2}(z), u_3 = \overline{G}(y)}, \quad z \in [0,\infty), y \in (0,1], \tag{4.3}$$

and

$$g_{21}\left(z, y\right) = \lim_{u_2 \to 0^+} \frac{\partial^2 \hat{C}\left(u_1, u_2, u_3\right)/\partial u_1 \partial u_3}{u_2}\Big|_{u_1 = \overline{F_1}(z), u_3 = \overline{G}(y)}, \quad z \in [0,\infty), y \in (0,1]. \tag{4.4}$$

Similarly, the uniformity required in Assumption 2.2 can be rewritten as

$$\lim_{u_1\to 0^+}\sup_{u_2,u_3\in[0,1]}\left|\frac{\partial^2\hat{C}\left(u_1,u_2,u_3\right)/\partial u_2\partial u_3}{u_1}\cdot\frac{1}{g_{12}\left(z,y\right)}-1\right|=0,$$

and

$$\lim_{u_2\to 0^+}\sup_{u_1,u_3\in[0,1]}\left|\frac{\partial^2\hat{C}\left(u_1,u_2,u_3\right)/\partial u_1\partial u_3}{u_2}\cdot\frac{1}{g_{21}\left(z,y\right)}-1\right|=0.$$

Finally, if $g(y)$ defined in Assumption 2.3 exists, then it equals to

$$g\left(y\right)=\lim_{(u_1,u_2)\to(0^+,0^+)}\frac{\partial\hat{C}\left(u_1,u_2,u_3\right)/\partial u_3}{u_1u_2}|_{u_3=\overline{G}(y)},\quad y\in\left(0,1\right]. \tag{4.5}$$

The uniformity required in Assumption 2.3 can be rewritten as

$$\lim_{(u_1,u_2)\to(0^+,0^+)}\sup_{u_3\in[0,1]}\left|\frac{\partial\hat{C}\left(u_1,u_2,u_3\right)/\partial u_3}{u_1u_2}\cdot\frac{1}{g\left(y\right)}-1\right|=0.$$

In the following, we present two examples satisfying Assumptions 2.1–2.3, which is from Section 5 of [13].

Example 4.1. Assume that $(X^{(1)},X^{(2)},Y)$ follows a trivariate copula of the following form

$$C\left(u_1,u_2,u_3\right)=u_1u_2u_3\left(1+\sigma\left(1-u_1u_2\right)\left(1-u_3\right)\right),\quad u_1,u_2,u_3\in\left[0,1\right], \tag{4.6}$$

where $\sigma\in\left[-\frac{1}{3},\frac{1}{3}\right]$.

It follows from Section 5 of [13] that for $i=1,2$, $\left(X^{(i)},Y\right)$ has a common bivariate FGM copula

$$C\left(u_i,u_3\right)=u_iu_3\left(1+\sigma\left(1-u_i\right)\left(1-u_3\right)\right),\quad i=1,2.$$

Besides, by (4.1)–(4.5), we have for $z\in[0,\infty)\,,y\in(0,1]$ that

$$h_i\left(y\right)=h\left(y\right)=1+\sigma\left(2G\left(y\right)-1\right),i=1,2,$$

$$g_{ij}\left(z,y\right)=\frac{1+\sigma\left(1-4F_j\left(z\right)\right)\left(1-2G\left(y\right)\right)}{1+\sigma\left(1-2F_j\left(z\right)\right)}\left(1-2G\left(y\right)\right),\quad i,j=1,2,i\neq j,$$

and

$$g\left(y\right)=1+3\sigma\left(2G\left(y\right)-1\right).$$

Example 4.2. Assume that $(X^{(1)},X^{(2)},Y)$ follows a common trivariate Sarmanov copula

$$\begin{aligned}&\mathbb{P}\left(X^{(1)}\in dv,X^{(2)}\in \mathrm{d}z,Y\in \mathrm{d}y\right)\\&=\left(1+\sigma_{12}\phi_1\left(v\right)\phi_2\left(z\right)+\sigma_{13}\phi_1\left(v\right)\phi_3\left(y\right)+\sigma_{23}\phi_2\left(z\right)\phi_3\left(y\right)\right)F_1\left(dv\right)F_2\left(\mathrm{d}z\right)G\left(\mathrm{d}y\right),\end{aligned}$$

where σ_{ij}, $1 \le i < j \le 3$ are constants and $\phi_i(\cdot)$, $1 \le i \le 3$ are continuous functions satisfying

$$1+\sigma_{12}\phi_1(v)\,\phi_2(z)+\sigma_{13}\phi_1(v)\,\phi_3(y)+\sigma_{23}\phi_2(z)\,\phi_3(y)\ge 0,\quad v,z,y\in(-\infty,\infty),$$

and

$$\mathbb{E}\left[\phi_1\left(X^{(1)}\right)\right]=\mathbb{E}\left[\phi_2\left(X^{(2)}\right)\right]=\mathbb{E}\left[\phi_3(Y)\right]=0.$$

If $\lim \phi_i(x)=d_i$, $i=1,2$, then by (4.1)–(4.5), we have for $z\in[0,\infty),y\in(0,1]$ that

$$h_i(y)=1+\sigma_{i3}d_i\phi_3(y),\quad i=1,2,$$

$$g_{12}(z,y)=1+\frac{\sigma_{12}d_1\phi_2(z)+\sigma_{13}d_1\phi_3(y)}{1+\sigma_{23}\phi_2(z)\,\phi_3(y)},$$

$$g_{21}(z,y)=1+\frac{\sigma_{12}d_2\phi_1(z)+\sigma_{23}d_2\phi_3(y)}{1+\sigma_{13}\phi_1(z)\,\phi_3(y)},$$

and

$$g(y)=1+\sigma_{12}d_1d_2+\sigma_{13}d_1\phi_3(y)+\sigma_{23}d_2\phi_3(y).$$

5. Numerical studies

In this section, we present a specific example of a copula and distributions to illustrate $\alpha(x_1,x_2;t)$, $\beta(x_1,x_2;t)$ and $\gamma_i(x_i;t)$, $i=1,2$, which constitute the main components of the asymptotics for the finite-time absolute ruin probabilities in Theorems 3.1 and 3.2.

Assume that $N(t)$ is a homogeneous Poisson process with parameter $\hat{\lambda}$. Then $\lambda(t)=\hat{\lambda}t$, $t\in[0,\infty)$ and $\mathbb{P}(\theta\le t)=1-\mathrm{e}^{-\hat{\lambda}t}$, $t\in[0,\infty)$. Moreover, suppose that $F_i(x)=1-(1+x)^{-\kappa_i}$, $i=1,2$, $x\in[0,\infty)$ and $(X^{(1)},X^{(2)},\theta)$ follows a copula of the form given in (4.6). Hence, it holds for $t\in[0,\infty)$ that

$$\mathcal{H}_i(t)=\mathcal{H}(t)=1+\sigma-2\sigma\mathrm{e}^{-\hat{\lambda}t},\quad i=1,2,\tag{5.1}$$

and

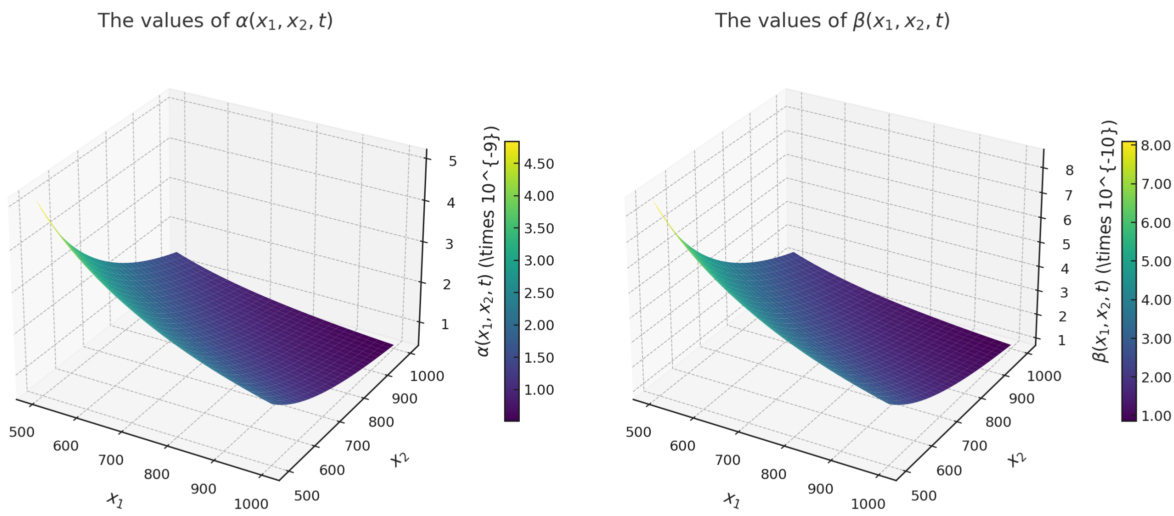
$$\mathcal{G}(t)=1+3\sigma-6\sigma\mathrm{e}^{-\hat{\lambda}t}.\tag{5.2}$$

Combining (5.1) and (5.2) with (3.1) and (3.2), it holds for $t\in[0,\infty)$ that

$$\widetilde{\lambda_k}(t)=\hat{\lambda}t-\frac{\sigma}{2}\left(1-\mathrm{e}^{-2\hat{\lambda}t}\right),\quad k=1,2,\tag{5.3}$$

and

$$\overset{\approx}{\lambda}(t)=\hat{\lambda}t-\frac{3\sigma}{2}\left(1-\mathrm{e}^{-2\hat{\lambda}t}\right).\tag{5.4}$$



(a) (b)

FIGURE 1. The values of $\alpha(x_1, x_2; t)$ and $\beta(x_1, x_2; t)$ with $\hat{\lambda} = 0.3$,$\kappa_1 = 1.5$, $\kappa_2 = 2$, $r = 0.02$, $c_1 = 0.5$, $c_2 = 0.7$ and $t = 20$.

Therefore, it holds for $t \in [0, \infty)$ that

$$\alpha(x_1, x_2; t) = \hat{\lambda}^2 \int_0^t \int_0^{t-u} \left(\left(1 + \zeta_1 \mathrm{e}^{r(u+v)}\right)^{-\kappa_1} \left(1 + \zeta_2 \mathrm{e}^{rv}\right)^{-\kappa_2} + \left(1 + \zeta_1 \mathrm{e}^{ru}\right)^{-\kappa_1} \left(1 + \zeta_2 \mathrm{e}^{r(u+v)}\right)^{-\kappa_2} \right) \cdot \left(1 - \sigma \mathrm{e}^{-2\hat{\lambda} v}\right) \left(1 - \sigma \mathrm{e}^{-2\hat{\lambda} u}\right) dv du,$$

$$\beta(x_1, x_2; t) = \hat{\lambda} \int_0^t \left(1 + \zeta_1 \mathrm{e}^{ru}\right)^{-\kappa_1} \left(1 + \zeta_2 \mathrm{e}^{ru}\right)^{-\kappa_2} \left(1 - 3\sigma \mathrm{e}^{-2\hat{\lambda} u}\right) du,$$

and

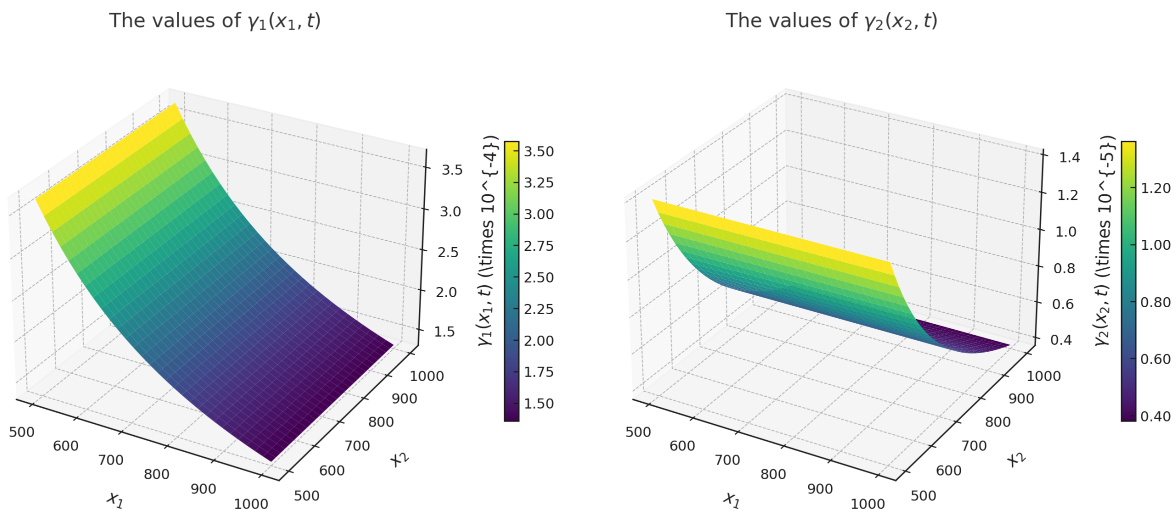
$$\gamma_i(x_i; t) = \hat{\lambda} \int_0^t \left(1 + \zeta_i \mathrm{e}^{ru}\right)^{-\kappa_i} \left(1 - \sigma \mathrm{e}^{-2\hat{\lambda} u}\right) du, \quad i = 1, 2,$$

where $\zeta_i = x_i + \frac{c_i}{r}, i = 1, 2$.

In what follows, we plot the values of $\alpha(x_1, x_2; t)$, $\beta(x_1, x_2; t)$ and $\gamma_i(x_i; t)$, $i = 1, 2$ with $\hat{\lambda} = 0.3$, $\kappa_1 = 1.5$, $\kappa_2 = 2$, $r = 0.02$, $c_1 = 0.5$, $c_2 = 0.7$, $t = 20$ when x_1 and x_2 are sufficiently large.

The values of $\alpha(x_1, x_2; t)$ and $\beta(x_1, x_2; t)$ are depicted in Figure 1. The surface of $\alpha(x_1, x_2; t)$ is smooth and monotonically decreasing as both x_1 and x_2 increase with values concentrated near the lower boundary. The surface of $\beta(x_1, x_2; t)$ also decreases smoothly and monotonically as both x_1 and x_2 increase with overall magnitude smaller than $\alpha(x_1, x_2; t)$.

The values of $\gamma_1(x_1; t)$ and $\gamma_2(x_2; t)$ are depicted in Figure 2. The surface of $\gamma_1(x_1; t)$ depends primarily on x_1, decreasing smoothly as x_1 increases. However, the surface of $\gamma_2(x_2; t)$ depending primarily on x_2, decreases smoothly as x_2 increases and has smaller magnitude than $\gamma_1(x_1; t)$.



(a) (b)

FIGURE 2. The values of $\gamma_1(x_1; t)$ and $\gamma_2(x_2; t)$ with $\hat{\lambda} = 0.3$,$\kappa_1 = 1.5$, $\kappa_2 = 2$, $r = 0.02$, $c_1 = 0.5$, $c_2 = 0.7$ and $t = 20$.

6. Proofs of main results

6.1. Lemmas

In this section, we write $\Omega_n(t) = \left\{(y_1, y_2, ..., y_n) \in [\mathrm{e}^{-rt}, 1]^n : \prod_{k=1}^n y_k \geq \mathrm{e}^{-rt}\right\}$ for every $t \in \Lambda_*$ and $n \geq 1$. Before giving the proofs of main results, we first present some lemmas. Lemma 6.1 will be used in the proof of Lemma 6.2.

Lemma 6.1. *Consider the bidimensional risk model (1.1) satisfying Assumptions 2.1 and 2.2. If $F_k \in \mathscr{S}, k = 1, 2$, then for any fixed $n \geq 1$, it holds uniformly for all $(y_1, y_2, \cdots, y_n) \in \Omega_n(t)$, $z_i \geq 0, 1 \leq i \leq n$ and $t \in \Lambda_*$ that*

$$
\begin{aligned}
P_1(n) &:= \mathbb{P}\left(\sum_{p=1}^n X_p^{(1)} \prod_{i=1}^p y_i > x_1 + \frac{c_1}{r} \middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \leq l \leq n\right) \\
&\sim \sum_{p=1}^n \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^p y_i > x_1 + \frac{c_1}{r} \middle| X_p^{(2)} = z_p, Y_p = y_p\right) =: P_2(n), \qquad (6.1)
\end{aligned}
$$

and for every $1 \leq p \leq n$, it holds uniformly for all $(y_1, y_2, ..., y_n) \in \Omega_n(t)$, $z_p \geq 0$ and $t \in \Lambda_*$ that

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{q=1}^n X_q^{(2)} \prod_{j=1}^q y_j > x_2 + \frac{c_2}{r} \middle| X_p^{(1)} = z_p, Y_l = y_l, 1 \leq l \leq n\right) \\
&\sim \sum_{q=1}^n \mathbb{P}\left(X_q^{(2)} \prod_{j=1}^q y_j > x_2 + \frac{c_2}{r} \middle| X_p^{(1)} = z_p, Y_q = y_q\right). \qquad (6.2)
\end{aligned}
$$

Proof. We will prove (6.1) by induction. Trivially, the assertion holds for $n = 1$. Now we assume that the assertion holds for some positive integer $n = m - 1$. We will show that (6.1) holds for $n = m$. Due to Proposition 2.4, there exists a function $a(\cdot) \in \mathscr{H}(F_1)$ such that $\overline{F_1}(x+y) \sim \overline{F_1}(x)$ holds uniformly for all $|y| \le a(x)$. By the way of the proof of Lemma 3.2 of [13], when $n = m$, we have for all $(y_1, y_2, \cdots, y_m) \in \Omega_m(t)$, $z_i \ge 0, 1 \le i \le m$, $x_1 \ge 0$ and $t \in \Lambda_*$ that

$$\begin{aligned}
&P_1(m)\\
=&\,\mathbb{P}\left(\sum_{p=1}^{m} X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r}, \sum_{p=1}^{m-1} X_p^{(1)} \prod_{i=1}^{p} y_i \le a\left(x_1 + \frac{c_1}{r}\right)\right.\\
&\left.\middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \le l \le m\right)\\
&+\mathbb{P}\left(\sum_{p=1}^{m} X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r}, \sum_{p=1}^{m-1} X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r} - a\left(x_1 + \frac{c_1}{r}\right)\right.\\
&\left.\middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \le l \le m\right)\\
&+\mathbb{P}\left(\sum_{p=1}^{m} X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r}, \sum_{p=1}^{m-1} X_p^{(1)} \prod_{i=1}^{p} y_i \in \left(a\left(x_1 + \frac{c_1}{r}\right), x_1 + \frac{c_1}{r} - a\left(x_1 + \frac{c_1}{r}\right)\right]\right.\\
&\left.\middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \le l \le m\right)\\
=:&\, I_1(x_1, m) + I_2(x_1, m) + I_3(x_1, m).
\end{aligned}$$

For $I_1(x_1, m)$, on the one hand, according to Proposition 2.4 and Assumption 2.2, it holds uniformly for all $(y_1, y_2, ..., y_m) \in \Omega_m(t)$, $z_i \ge 0, 1 \le i \le m$ and $t \in \Lambda_*$ that

$$\begin{aligned}
I_1(x_1, m) &\le \mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r} - a\left(x_1 + \frac{c_1}{r}\right) \middle| X_m^{(2)} = z_m, Y_m = y_m\right)\\
&\sim g_{12}(z_m, y_m)\,\mathbb{P}\left(X_m^{(1)} > \left(x_1 + \frac{c_1}{r}\right) \prod_{i=1}^{m} y_i^{-1} - a\left(x_1 + \frac{c_1}{r}\right) \prod_{i=1}^{m} y_i^{-1}\right)\\
&\sim g_{12}(z_m, y_m)\,\mathbb{P}\left(X_m^{(1)} > \left(x_1 + \frac{c_1}{r}\right) \prod_{i=1}^{m} y_i^{-1}\right)\\
&\sim \mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r} \middle| X_m^{(2)} = z_m, Y_m = y_m\right). \qquad (6.3)
\end{aligned}$$

On the other hand, it holds uniformly for all $(y_1, y_2, ..., y_m) \in \Omega_m(t)$, $z_i \ge 0, 1 \le i \le m$ and $t \in \Lambda_*$ that

$$\begin{aligned}
&I_1(x_1, m)\\
\ge&\,\mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r}, \sum_{p=1}^{m-1} X_p^{(1)} \prod_{i=1}^{p} y_i \le a\left(x_1 + \frac{c_1}{r}\right) \middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \le l \le m\right)\\
=&\,\mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r} \middle| X_m^{(2)} = z_m, Y_m = y_m\right)
\end{aligned}$$

$$\cdot\left(1-\mathbb{P}\left(\sum_{p=1}^{m-1}X_p^{(1)}\prod_{i=1}^{p}y_i>a\left(x_1+\frac{c_1}{r}\right)\middle|X_l^{(2)}=z_l,Y_l=y_l,1\le l\le m-1\right)\right)$$
$$\sim\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m}y_i>x_1+\frac{c_1}{r}\middle|X_m^{(2)}=z_m,Y_m=y_m\right),\tag{6.4}$$

where the last step holds since by the induction assumption and Assumption 2.2, it holds uniformly for all $(y_1,y_2,...,y_{m-1})\in\Omega_{m-1}(t)$, $z_i\ge 0, 1\le i\le m-1$ and $t\in\Lambda_*$ that

$$\begin{aligned}&\mathbb{P}\left(\sum_{p=1}^{m-1}X_p^{(1)}\prod_{i=1}^{p}y_i>a\left(x_1+\frac{c_1}{r}\right)\middle|X_l^{(2)}=z_l,Y_l=y_l,1\le l\le m-1\right)\\&\sim\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}y_i>a\left(x_1+\frac{c_1}{r}\right)\middle|X_p^{(2)}=z_p,Y_p=y_p\right)\\&=O(1)\overline{F_1}\left(a\left(x_1+\frac{c_1}{r}\right)\right)\to 0.\end{aligned}$$

For $I_2(x_1,m)$, on the one hand, similarly to the method in (4.2), according to the induction assumption and Assumption 2.2, it holds uniformly for all $(y_1,y_2,...,y_m)\in\Omega_m(t)$, $z_i\ge 0,1\le i\le m$ and $t\in\Lambda_*$ that

$$\begin{aligned}&I_2(x_1,m)\\&\le\mathbb{P}\left(\sum_{p=1}^{m-1}X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)\middle|X_l^{(2)}=z_l,Y_l=y_l,1\le l\le m-1\right)\\&\sim\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}\middle|X_p^{(2)}=z_p,Y_p=y_p\right)\\&=P_2(m-1).\end{aligned}\tag{6.5}$$

On the other hand, by the induction assumption, it holds uniformly for all $(y_1,y_2,...,y_m)\in\Omega_m(t)$, $z_i\ge 0,1\le i\le m$ and $t\in\Lambda_*$ that

$$\begin{aligned}I_2(x_1,m)&\ge\mathbb{P}\left(\sum_{p=1}^{m-1}X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}\middle|X_l^{(2)}=z_l,Y_l=y_l,1\le l\le m-1\right)\\&\sim\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}\middle|X_p^{(2)}=z_p,Y_p=y_p\right)\\&=P_2(m-1).\end{aligned}\tag{6.6}$$

Finally, we consider $I_3(x_1,m)$. Due to intergration by parts, it holds uniformly for all $(y_1,y_2,...,y_m)\in\Omega_m(t)$, $z_i\ge 0,1\le i\le m$ and $t\in\Lambda_*$ that

$$\begin{aligned}&I_3(x_1,m)\\&=\int_{a\left(x_1+\frac{c_1}{r}\right)}^{x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)}\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m}y_i>x_1+\frac{c_1}{r}-u\middle|X_m^{(2)}=z_m,Y_m=y_m\right)\end{aligned}$$

$$
\begin{aligned}
&\cdot\mathbb{P}\left(\sum_{p=1}^{m-1} X_p^{(1)}\prod_{i=1}^{p} y_i\in du\middle| X_l^{(2)}=z_l, Y_l=y_l, 1\le l\le m-1\right)\\
\le\ &\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i > x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)\middle| X_m^{(2)}=z_m, Y_m=y_m\right)\\
&\cdot\mathbb{P}\left(\sum_{p=1}^{m-1} X_p^{(1)}\prod_{i=1}^{p} y_i > a\left(x_1+\frac{c_1}{r}\right)\middle| X_l^{(2)}=z_l, Y_l=y_l, 1\le l\le m-1\right)\\
&+\int_{a\left(x_1+\frac{c_1}{r}\right)}^{x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)}\mathbb{P}\left(\sum_{p=1}^{m-1} X_p^{(1)}\prod_{i=1}^{p} y_i > x_1+\frac{c_1}{r}-u\middle| X_l^{(2)}=z_l, Y_l=y_l, 1\le l\le m-1\right)\\
&\cdot\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i\in du\middle| X_m^{(2)}=z_m, Y_m=y_m\right)\\
=:\ &I_{31}(x_1,m)+I_{32}(x_1,m). \qquad (6.7)
\end{aligned}
$$

Firstly, we consider $I_{31}(x_1,m)$. Due to Proposition 2.4, the induction assumption and Assumption 2.2, it holds uniformly for all $(y_1,y_2,...,y_m)\in\Omega_m(t)$, $z_i\ge 0, 1\le i\le m$ and $t\in\Lambda_*$ that

$$
\begin{aligned}
&I_{31}(x_1,m)\\
=\ &\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i > x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)\middle| X_m^{(2)}=z_m, Y_m=y_m\right)\\
&\cdot\mathbb{P}\left(\sum_{p=1}^{m-1} X_p^{(1)}\prod_{i=1}^{p} y_i > a\left(x_1+\frac{c_1}{r}\right)\middle| X_l^{(2)}=z_l, Y_l=y_l, 1\le l\le m-1\right)\\
\sim\ &g_{12}(z_m,y_m)\,\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i > x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)\right)\\
&\cdot\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p} y_i > a\left(x_1+\frac{c_1}{r}\right)\middle| X_p^{(2)}=z_p, Y_p=y_p\right)\\
\sim\ &g_{12}(z_m,y_m)\,\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i > x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)\right)\\
&\cdot\sum_{p=1}^{m-1} g_{12}(z_p,y_p)\,\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p} y_i > a\left(x_1+\frac{c_1}{r}\right)\right)\\
=\ &O(1)g_{12}(z_m,y_m)\,\mathbb{P}\left(X_m^{(1)}\prod_{i=1}^{m} y_i > x_1+\frac{c_1}{r}\right)\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)} > a\left(x_1+\frac{c_1}{r}\right)\right)\\
=\ &O(1)g_{12}(z_{m-1},y_{m-1})\,\mathbb{P}\left(X_{m-1}^{(1)}\prod_{i=1}^{m-1} y_i > x_1+\frac{c_1}{r}\right)\overline{F_1}\left(a\left(x_1+\frac{c_1}{r}\right)\right)\\
=\ &o\left(\sum_{p=1}^{m-1} g_{12}(z_p,y_p)\,\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p} y_i > x_1+\frac{c_1}{r}\right)\right)\\
=\ &o\left(\sum_{p=1}^{m-1}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p} y_i > x_1+\frac{c_1}{r}\middle| X_p^{(2)}=z_p, Y_p=y_p\right)\right)\\
=\ &o(P_2(m-1)). \qquad (6.8)
\end{aligned}
$$

Next ,we deal with $I_{32}(x_1, m)$. By the induction assumption, Assumption 2.2, Proposition 2.5 and $F_1 \in \mathscr{S}$, it holds uniformly for all $(y_1, y_2, ..., y_m) \in \Omega_m(t)$, $z_i \geq 0, 1 \leq i \leq m$ and $t \in \Lambda_*$ that

$$\begin{aligned}
& I_{32}(x_1, m) \\
\sim & \sum_{p=1}^{m-1} \int_{a\left(x_1+\frac{c_1}{r}\right)}^{x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r} - u \middle| X_p^{(2)} = z_p, Y_p = y_p\right) \\
& \cdot \mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i \in du \middle| X_m^{(2)} = z_m, Y_m = y_m\right) \\
\sim & \sum_{p=1}^{m-1} g_{12}(z_p, y_p) \int_{a\left(x_1+\frac{c_1}{r}\right)}^{x_1+\frac{c_1}{r}-a\left(x_1+\frac{c_1}{r}\right)} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r} - u\right) \\
& \cdot \mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i \in du \middle| X_m^{(2)} = z_m, Y_m = y_m\right) \\
= & \sum_{p=1}^{m-1} g_{12}(z_p, y_p) \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i + X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r},\right. \\
& \left. X_m^{(1)} \prod_{i=1}^{m} y_i \in \left(a\left(x_1 + \frac{c_1}{r}\right), x_1 + \frac{c_1}{r} - a\left(x_1 + \frac{c_1}{r}\right)\right] \middle| X_m^{(2)} = z_m, Y_m = y_m\right) \\
\leq & \sum_{p=1}^{m-1} g_{12}(z_p, y_p) \int_{a\left(x_1+\frac{c_1}{r}\right)}^{\infty} \mathbb{P}\left(X_m^{(1)} \prod_{i=1}^{m} y_i > \left(x_1 + \frac{c_1}{r} - u\right) \vee a\left(x_1 + \frac{c_1}{r}\right)\right. \\
& \left. \middle| X_m^{(2)} = z_m, Y_m = y_m\right) \cdot \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i \in du\right) \\
\sim & \sum_{p=1}^{m-1} g_{12}(z_p, y_p) g_{12}(z_m, y_m) \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i + X_m^{(1)} \prod_{i=1}^{m} y_i > x_1 + \frac{c_1}{r},\right. \\
& \left. X_p^{(1)} \prod_{i=1}^{p} y_i > a\left(x_1 + \frac{c_1}{r}\right), X_m^{(1)} \prod_{i=1}^{m} y_i > a\left(x_1 + \frac{c_1}{r}\right)\right) \\
= & o\left(\sum_{p=1}^{m-1} g_{12}(z_p, y_p) \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r}\right)\right) \\
= & o\left(\sum_{p=1}^{m-1} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i > x_1 + \frac{c_1}{r} \middle| X_p^{(2)} = z_p, Y_p = y_p\right)\right) \\
= & o(P_2(m-1)).
\end{aligned} \tag{6.9}$$

where Proposition 2.5 is used in the penultimate step. By (6.7)–(6.9), we can obtain that it holds uniformly for all $(y_1, y_2, ..., y_m) \in \Omega_m(t)$, $z_i \geq 0, 1 \leq i \leq m$ and $t \in \Lambda_*$ that

$$\begin{aligned}
I_3(x_1, m) &= o(P_2(m-1)) \\
&= o(P_2(m)).
\end{aligned} \tag{6.10}$$

Thus, we can obtain (6.1) holds for $n=m$. Besides, we have

$$\limsup_{t\in\Lambda_*}\sup_{(y_1,y_2,...,y_n)\in\Omega_n(t)}\sup_{(z_1,z_2,...,z_n)\in[0,\infty)^n}\left|\frac{P_1(m)}{P_2(m)}-1\right|=0.$$

Hence, we can obtain (6.1) holds uniformly for all $(y_1,y_2,...,y_n)\in\Omega_n(t)$, $z_i\geq 0, 1\leq i\leq n$ and $t\in\Lambda_*$ by (6.3)–(6.6) and (6.10).

By the similar proof of (6.1), we can obtain (6.2) holds by Assumptions 2.1 and 2.2. We omit the details of its proof. This completes the proof of Lemma 6.1. □

Lemma 6.2 will be used in the proof of Lemma 6.5.

Lemma 6.2. *Consider the bidimensional risk model (1.1) satisfying Assumptions 2.1 and 2.2. If $F_k\in\mathscr{S}, k=1,2$, then for any fixed $n\geq 1$, it holds uniformly for all $t\in\Lambda_*$ that*

$$\mathbb{P}\left(\sum_{p=1}^{n}X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1+\frac{c_1}{r},\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2+\frac{c_2}{r},N(t)=n\right)$$
$$\sim\sum_{p=1}^{n}\sum_{q=1}^{n}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1+\frac{c_1}{r},X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2+\frac{c_2}{r},N(t)=n\right).$$

Proof. For every $n\geq 1$, by Lemma 6.1, it holds uniformly for all $t\in\Lambda_*$ that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{p=1}^{n}X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1+\frac{c_1}{r},\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2+\frac{c_2}{r},N(t)=n\right)\\
&=\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\overline{\Delta_1}}\mathbb{P}\left(\sum_{p=1}^{n}X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}\middle|X_l^{(2)}=z_l,Y_l=y_l,1\leq l\leq n\right)\\
&\quad\cdot\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in\mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in\mathrm{d}y_l\right)\\
&\sim\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\overline{\Delta_1}}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r}\middle|X_p^{(2)}=z_p,Y_p=y_p\right)\\
&\quad\cdot\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in\mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in\mathrm{d}y_l\right)\\
&=\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}y_i>x_1+\frac{c_1}{r},\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}y_j>x_2+\frac{c_2}{r}\middle|Y_l=y_l,1\leq l\leq n\right)\\
&\quad\cdot\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in\mathrm{d}y_l\right)\\
&=\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\int_{\left(x_1+\frac{c_1}{r}\right)\prod_{i=1}^{p}y_i^{-1}}^{\infty}\mathbb{P}\left(\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}y_j>x_2+\frac{c_2}{r}\middle|X_p^{(1)}=z_p,Y_l=y_l,1\leq l\leq n\right)\\
&\quad\cdot\mathbb{P}\left(X_p^{(1)}\in\mathrm{d}z_p\middle|Y_p=y_p\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in\mathrm{d}y_l\right)
\end{aligned}$$

$$\sim \sum_{p=1}^{n}\sum_{q=1}^{n}\int\cdots\int_{\Omega_n(t)}\int_{\left(x_1+\frac{c_1}{r}\right)\prod_{i=1}^{p}y_i^{-1}}^{\infty}\mathbb{P}\left(X_q^{(2)}\prod_{j=1}^{q}y_j>x_2+\frac{c_2}{r}\middle|X_p^{(1)}=z_p,Y_p=y_p\right)$$
$$\cdot\mathbb{P}\left(X_p^{(1)}\in \mathrm{d}z_p\middle|Y_p=y_p\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)$$
$$=\sum_{p=1}^{n}\sum_{q=1}^{n}\mathbb{P}\left(X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1+\frac{c_1}{r},X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2+\frac{c_2}{r},N(t)=n\right),$$

where $\overline{\Delta_1}=\left\{(z_1,z_2,...,z_n)\in[0,\infty)^n:\sum_{q=1}^{n}z_q\prod_{j=1}^{q}y_j>x_2+\frac{c_2}{r},(y_1,y_2,...,y_n)\in\Omega_n(t)\right\}$. This completes the proof of Lemma 6.2. □

Lemma 6.3 is a restatement of Lemma 4.2 of [11], which is critical to prove Lemma 6.5.

Lemma 6.3. *Consider the bidimensional risk model (1.1) satisfying Assumption 2.1. If $F_k\in\mathscr{S}, k=1,2$, then for any fixed $n\geq 1$ and $k=1,2$, it holds uniformly for all $t\in\Lambda_*$ that*

$$\mathbb{P}\left(\sum_{p=1}^{n}X_p^{(k)}\prod_{i=1}^{p}Y_i>x_k+\frac{c_k}{r},N(t)=n\right)$$
$$\sim\sum_{p=1}^{n}\mathbb{P}\left(X_p^{(k)}\prod_{i=1}^{p}Y_i>x_k+\frac{c_k}{r},N(t)=n\right).$$

The following lemma can be obtained by using Lemma 3.3 of [12] with some obvious modifications.

Lemma 6.4. *Consider the bidimensional risk model (1.1) satisfying Assumptions 2.1–2.3. If $F_k\in\mathscr{S}, k=1,2$, then it holds uniformly for all $t\in\Lambda_*$ that*

$$\mathbb{P}\left(\sum_{p=1}^{N(t)}X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1+\frac{c_1}{r},\sum_{q=1}^{N(t)}X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2+\frac{c_2}{r}\right)\sim\alpha(x_1,x_2;t)+\beta(x_1,x_2;t).$$

Proof. It follows from Lemma 3.3 of [12] that

$$\mathbb{P}\left(\sum_{p=1}^{N(t)}X_p^{(1)}\prod_{i=1}^{p}Y_i>x_1,\sum_{q=1}^{N(t)}X_q^{(2)}\prod_{j=1}^{q}Y_j>x_2\right)$$
$$\sim\int_0^t\int_0^{t-u}\left(\overline{F_1}\left(x_1\mathrm{e}^{r(u+v)}\right)\overline{F_2}\left(x_2\mathrm{e}^{rv}\right)\right.$$
$$\left.+\overline{F_1}\left(x_1\mathrm{e}^{ru}\right)\overline{F_2}\left(x_2\mathrm{e}^{r(u+v)}\right)\right)\widetilde{\lambda_2}(dv)\widetilde{\lambda_1}(du)$$
$$+\int_0^t\overline{F_1}\left(x_1\mathrm{e}^{ru}\right)\overline{F_2}\left(x_2\mathrm{e}^{ru}\right)\approx\!\!\!\!\lambda(du)$$
$$=\alpha\left(x_1-\frac{c_1}{r},x_2-\frac{c_2}{r};t\right)+\beta\left(x_1-\frac{c_1}{r},x_2-\frac{c_2}{r};t\right). \tag{6.11}$$

It suffices to replace x_1 by $x_1+\frac{c_1}{r}$ and x_2 by $x_2+\frac{c_2}{r}$ in (6.11) to complete the proof of Lemma 6.4. □

Lemma 6.5 will be needed to prove Theorem 3.2.

Lemma 6.5. *Set $x = x_1 + x_2$ and $c = c_1 + c_2$. Under the conditions of Theorem 3.2, for any fixed $n \geq 1$, it holds uniformly for all $t \in \Lambda_*$ that*

$$\mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \sim \sum_{m=1}^{2}\sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right).$$

Proof. By Proposition 2.4, there exists a function $b_k(\cdot) \in \mathscr{H}(F_k), k = 1, 2$. Let $b(x) = \min\{b_1(x), b_2(x)\}, x > 0$. The probability can be split into three parts according to the value of $\sum_{q=1}^{n} X_q^{(2)} \prod_{j=1}^{q} Y_j$ belonging to $\left(0, b\left(x + \frac{c}{r}\right)\right]$, $\left(b\left(x + \frac{c}{r}\right), x + \frac{c}{r} - b\left(x + \frac{c}{r}\right)\right]$ and $\left(x + \frac{c}{r} - b\left(x + \frac{c}{r}\right), \infty\right)$ as

$$\mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) =: J_1(x, n, t) + J_2(x, n, t) + J_3(x, n, t). \tag{6.12}$$

We first consider the upper bound of (6.12). For $J_1(x, n, t)$, by Lemma 6.3, Proposition 2.4 and Assumption 2.1, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned} J_1(x, n, t) &\leq \mathbb{P}\left(\sum_{p=1}^{n} X_p^{(1)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r} - b\left(x + \frac{c}{r}\right), N(t) = n\right) \\ &\sim \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r} - b\left(x + \frac{c}{r}\right), N(t) = n\right) \\ &\sim \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right). \end{aligned} \tag{6.13}$$

We need the following notation to proceed with $J_2(x, n, t)$. Let X_1^* and X_2^* be two independent random variables with distributions F_1 and F_2, respectively, such that X_1^*, X_2^* and $\left(X^{(1)}, X^{(2)}, Y\right)$ are mutually independent. By $\overline{F_1}(x) \asymp \overline{F_2}(x)$, Lemma 6.2 and Proposition 2.5, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned} &J_2(x, n, t) \\ &= \int \cdots \int_{\Omega_n(t)} \int \cdots \int_{\Delta_2} \mathbb{P}\left(\sum_{p=1}^{n} X_p^{(1)} \prod_{i=1}^{p} y_i > x + \frac{c}{r} - \sum_{q=1}^{n} z_q \prod_{j=1}^{q} y_j \middle| X_l^{(2)} = z_l,\right. \\ &\quad \left. Y_l = y_l, 1 \leq l \leq n\right) \prod_{l=1}^{n} \mathbb{P}\left(X_l^{(2)} \in \mathrm{d}z_l \middle| Y_l = y_l\right) \prod_{l=1}^{n} \mathbb{P}(Y_l \in \mathrm{d}y_l) \\ &\sim \sum_{p=1}^{n} \int \cdots \int_{\Omega_n(t)} \int \cdots \int_{\Delta_2} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} y_i > x + \frac{c}{r} - \sum_{q=1}^{n} z_q \prod_{j=1}^{q} y_j \middle| X_p^{(2)} = z_p,\right. \\ &\quad \left. Y_p = y_p\right) \prod_{l=1}^{n} \mathbb{P}\left(X_l^{(2)} \in \mathrm{d}z_l \middle| Y_l = y_l\right) \prod_{l=1}^{n} \mathbb{P}(Y_l \in \mathrm{d}y_l) \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\overline{\Delta_2}}\mathbb{P}\left(X_1^*\prod_{i=1}^{p}y_i > x+\frac{c}{r}-\sum_{q=1}^{n}z_q\prod_{j=1}^{q}y_j\right)\\
&\quad\cdot\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in \mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)\\
&= O(1)\sum_{p=1}^{n}\mathbb{P}\left(X_1^*\prod_{i=1}^{p}Y_i+\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}Y_j > x+\frac{c}{r}, X_1^*\prod_{i=1}^{p}Y_i > b\left(x+\frac{c}{r}\right),\right.\\
&\quad\left.\sum_{q=1}^{n}X_q^{(2)}\prod_{j=1}^{q}Y_j > b\left(x+\frac{c}{r}\right), N\left(t\right)=n\right)\\
&= O(1)\sum_{p=1}^{n}\sum_{q=1}^{n}\mathbb{P}\left(X_1^*\prod_{i=1}^{p}Y_i+X_2^*\prod_{j=1}^{q}Y_j > x+\frac{c}{r}, X_1^*\prod_{i=1}^{p}Y_i > b\left(x+\frac{c}{r}\right),\right.\\
&\quad\left.X_2^*\prod_{j=1}^{q}Y_j > b\left(x+\frac{c}{r}\right), N\left(t\right)=n\right)\\
&= O(1)\sum_{p=1}^{n}\sum_{q=1}^{n}\int\cdots\int_{\Omega_n(t)}\mathbb{P}\left(X_1^*\prod_{i=1}^{p}y_i+X_2^*\prod_{j=1}^{q}y_j > x+\frac{c}{r}, X_1^*\prod_{i=1}^{p}y_i > b\left(x+\frac{c}{r}\right),\right.\\
&\quad\left.X_2^*\prod_{j=1}^{q}y_j > b\left(x+\frac{c}{r}\right)\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)\\
&= o\left(\sum_{m=1}^{2}\sum_{p=1}^{n}\mathbb{P}\left(X_m^*\prod_{i=1}^{p}Y_i > x+\frac{c}{r}, N\left(t\right)=n\right)\right)\\
&= o\left(\sum_{m=1}^{2}\sum_{p=1}^{n}\mathbb{P}\left(X_p^{(m)}\prod_{i=1}^{p}Y_i > x+\frac{c}{r}, N\left(t\right)=n\right)\right),
\end{aligned}
\tag{6.14}
$$

where $\overline{\Delta_2} = \Big\{(z_1, z_2, ..., z_n)\in[0,\infty)^n : \sum_{q=1}^{n} z_q\prod_{j=1}^{q} y_j \in \left(b\left(x_2+\frac{c_2}{r}\right), x_2+\frac{c_2}{r}-b\left(x_2+\frac{c_2}{r}\right)\right],\ (y_1, y_2, ..., y_n)\in \Omega_n\left(t\right)\Big\}$, Proposition 2.5 is used in the penultimate step and the last step holds since by Assumption 2.1, for every fixed $n\geq 1$, $1\leq p\leq n$ and $m=1,2$, it holds uniformly for all $(y_1, y_2, ..., y_n)\in\Omega_n\left(t\right)$ that

$$
\begin{aligned}
&\mathbb{P}\left(X_m^*\prod_{i=1}^{p}Y_i > x+\frac{c}{r}, N\left(t\right)=n\right)\\
&= \int\cdots\int_{\Omega_n(t)}\mathbb{P}\left(X_m^* > \left(x+\frac{c}{r}\right)\prod_{i=1}^{p}y_i^{-1}\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)\\
&= \int\cdots\int_{\Omega_n(t)}\mathbb{P}\left(X_p^{(m)} > \left(x+\frac{c}{r}\right)\prod_{i=1}^{p}y_i^{-1}\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)\\
&\leq A_1^{-1}\int\cdots\int_{\Omega_n(t)}\mathbb{P}\left(X_p^{(m)} > \left(x+\frac{c}{r}\right)\prod_{i=1}^{p}y_i^{-1}\right)h\left(y_p\right)\prod_{l=1}^{n}\mathbb{P}\left(Y_l\in \mathrm{d}y_l\right)\\
&\sim A_1^{-1}\mathbb{P}\left(X_p^{(m)}\prod_{i=1}^{p}Y_i > x+\frac{c}{r}, N\left(t\right)=n\right).
\end{aligned}
$$

For $J_3(x,n,t)$, similarly to the method of proof of $J_1(x,n,t)$, it holds uniformly for all $t \in \Lambda_*$ that

$$J_3(x,n,t) \lesssim \sum_{q=1}^{n} \mathbb{P}\left(X_q^{(2)} \prod_{j=1}^{q} Y_j > x + \frac{c}{r}, N(t) = n\right). \tag{6.15}$$

By (6.13)–(6.15), it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&J_1(x,n,t) + J_3(x,n,t) \\
&\lesssim \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(1)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) + \sum_{q=1}^{n} \mathbb{P}\left(X_q^{(2)} \prod_{j=1}^{q} Y_j > x + \frac{c}{r}, N(t) = n\right) \\
&= \sum_{m=1}^{2} \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right),
\end{aligned} \tag{6.16}$$

and

$$J_2(x,n,t) = o\left(\sum_{m=1}^{2} \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right)\right). \tag{6.17}$$

Hence, by (6.12), (6.16) and (6.17), it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{m=1}^{2} \sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\lesssim \sum_{m=1}^{2} \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right).
\end{aligned} \tag{6.18}$$

Next, we consider the lower bound of (6.12). Due to Bonferroni Inequality, Lemma 6.3 and (6.14), it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{m=1}^{2} \sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\geq \sum_{m=1}^{2} \mathbb{P}\left(\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\quad - \mathbb{P}\left(\sum_{p=1}^{n} X_p^{(1)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, \sum_{q=1}^{n} X_q^{(2)} \prod_{j=1}^{q} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\gtrsim \sum_{m=1}^{2} \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) - J_2(x,n,t) \\
&\sim \sum_{m=1}^{2} \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right),
\end{aligned}$$

which, combining with (6.18), yields the desired result. This completes the proof of Lemma 6.5. □

The following lemma can be obtained by using Lemma 3.1 of [12] with some obvious modifications, which is used in the proof of Theorem 3.2.

Lemma 6.6. *Under the conditions of Theorem 3.2, if* $F_k \in \mathscr{S}$, $k=1,2$, *then for arbitrarily fixed* $\varepsilon>0$, *there exist two positive constants* K_1 *and* K_2 *such that*

$$\mathbb{P}\left(\sum_{p=1}^{n} X_p^{(1)} y_1 > x \middle| X_l^{(2)} = z_l, Y_l = y_l, 1 \le l \le n\right) \le K_1 (1+\varepsilon)^n \sum_{p=1}^{n} \mathbb{P}\left(X_p^{(1)} y_1 > x \middle| X_p^{(2)} = z_p, Y_p = y_p\right),$$

and

$$\mathbb{P}\left(\sum_{q=1}^{n} X_q^{(2)} y_1 > x \middle| X_l^{(1)} = z_l, Y_l = y_l, 1 \le l \le n\right) \le K_2 (1+\varepsilon)^n \sum_{q=1}^{n} \mathbb{P}\left(X_q^{(2)} y_1 > x \middle| X_q^{(2)} = z_q, Y_q = y_q\right)$$

hold uniformly for all $(y_1, y_2, ..., y_n) \in \Omega_n(t)$, $z_p \ge 0, 1 \le p \le n$, $t \in \Lambda_*$, $x \ge 0$ *and* $n \ge 1$.

6.2. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1: Firstly, we consider the asymptotics for $\Psi_{and}(x_1, x_2; t)$. According to (1.3), (1.5) and Lemma 6.4, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\Psi_{and}(x_1, x_2; t) \\
&= \mathbb{P}\left(\sum_{p=1}^{N(t)} X_p^{(1)} \prod_{i=1}^{p} Y_i > x_1 + \frac{c_1}{r}, \sum_{q=1}^{N(t)} X_q^{(2)} \prod_{j=1}^{q} Y_j > x_2 + \frac{c_2}{r}\right) \\
&\sim \alpha(x_1, x_2; t) + \beta(x_1, x_2; t).
\end{aligned} \tag{6.19}$$

Then, we consider the asymptotics for $\Psi_{sim}(x_1, x_2; t)$. By (1.3), (1.4) and Lemma 6.4, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\Psi_{sim}(x_1, x_2; t) \\
&\ge \mathbb{P}\left(\sum_{p=1}^{N(t)} X_p^{(1)} \prod_{i=1}^{p} Y_i - c_1 \int_0^t \mathrm{e}^{-rs} \mathrm{d}s > x_1 + \frac{c_1}{r} \mathrm{e}^{-rt},\right. \\
&\qquad \left.\sum_{q=1}^{N(t)} X_q^{(2)} \prod_{j=1}^{q} Y_j - c_2 \int_0^t \mathrm{e}^{-rs} \mathrm{d}s > x_2 + \frac{c_2}{r} \mathrm{e}^{-rt}\right) \\
&= \mathbb{P}\left(\sum_{p=1}^{N(t)} X_p^{(1)} \prod_{i=1}^{p} Y_i > x_1 + \frac{c_1}{r}, \sum_{q=1}^{N(t)} X_q^{(2)} \prod_{j=1}^{q} Y_j > x_2 + \frac{c_2}{r}\right) \\
&\sim \alpha(x_1, x_2; t) + \beta(x_1, x_2; t).
\end{aligned} \tag{6.20}$$

Therefore, by $\Psi_{sim}(x_1, x_2; t) \le \Psi_{and}(x_1, x_2; t)$, (6.19) and (6.20), (3.3) holds uniformly for all $t \in \Lambda_*$.

In the following, we consider the asymptotics for $\Psi_{or}(x_1, x_2; t)$. By (1.3) and Theorem 2.1 of [11], for each $k = 1, 2$, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0\le s\le t} U_k(x_k, s) < -\frac{c_k}{r}\right) \\
&= \mathbb{P}\left(\sum_{p=1}^{N(t)} X_p^{(k)} \prod_{i=1}^{p} Y_i > x_i + \frac{c_i}{r}\right) \\
&\sim \gamma_k(x_k; t), \quad k = 1, 2.
\end{aligned} \tag{6.21}$$

Moreover, by Assumption 2.2, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\alpha(x_1, x_2; t) \\
&\le \int_0^t \int_0^{t-u} \left(\overline{F_1}\left(\left(x_1 + \frac{c_1}{r}\right)\mathrm{e}^{ru}\right)\overline{F_2}\left(\left(x_2 + \frac{c_2}{r}\right)\mathrm{e}^{rv}\right)\right. \\
&\quad \left. + \overline{F_1}\left(\left(x_1 + \frac{c_1}{r}\right)\mathrm{e}^{ru}\right)\overline{F_2}\left(\left(x_2 + \frac{c_2}{r}\right)\mathrm{e}^{rv}\right)\right)\widetilde{\lambda_2}(dv)\,\widetilde{\lambda_1}(du) \\
&\le 2\gamma_1(x_1; t)\,\gamma_2(x_2; t)
\end{aligned} \tag{6.22}$$

and

$$\beta(x_1, x_2; t) = O\left(\overline{F_2}\left(x_2 + \frac{c_2}{r}\right)\gamma_1(x_1; t)\right). \tag{6.23}$$

It is easy to obtain that it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\Psi_{or}(x_1, x_2; t) \\
&= \sum_{k=1}^{2} \mathbb{P}\left(\inf_{0\le s\le t} U_k(x_k, s) < -\frac{c_k}{r}\right) - \Psi_{and}(x_1, x_2; t).
\end{aligned} \tag{6.24}$$

By (6.22), it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&\frac{\alpha(x_1, x_2; t)}{\sum_{k=1}^{2} \gamma_k(x_k; t^*)} \\
&\le 2\gamma_2(x_1; t) \\
&\le 2\int_0^T \overline{F_2}\left(x_2 + \frac{c_2}{r}\right)\widetilde{\lambda_2}(dv) \to 0
\end{aligned} \tag{6.25}$$

and by (6.23), it holds uniformly for all $t \in \Lambda_*$ that

$$\frac{\beta(x_1, x_2; t)}{\sum_{k=1}^{2} \gamma_k(x_k; t)} = O\left(\overline{F_2}\left(x_2 + \frac{c_2}{r}\right)\right) \to 0. \tag{6.26}$$

Hence, by (6.19), (6.21) and (6.24)–(6.26), we can obtain that it holds uniformly for all $t \in \Lambda_*$ that

$$\Psi_{or}(x_1, x_2; t) \sim \sum_{k=1}^{2} \gamma_k(x_k; t).$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.2: Set $x = x_1 + x_2$ and $c = c_1 + c_2$. Choose some large positive integer N and write

$$\begin{aligned}
&\Psi_{sum}(x_1, x_2; t) \\
&= \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{N(t)} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}\right) \\
&= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&=: E_1(x,t) + E_2(x,t). \qquad (6.27)
\end{aligned}$$

First, we consider $E_1(x,t)$. By Lemmas 6.5 and 6.3, it holds uniformly for all $t \in \Lambda_*$ that

$$\begin{aligned}
&E_1(x,t) \\
&= \sum_{n=1}^{N} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\sim \sum_{n=1}^{N}\sum_{m=1}^{2}\sum_{p=1}^{n} \mathbb{P}\left(X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\sim \sum_{m=1}^{2}\sum_{n=1}^{N} \mathbb{P}\left(\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\sim \gamma_1\left(x + \frac{c_2}{r}; t\right) + \gamma_2\left(x + \frac{c_1}{r}; t\right), \qquad (6.28)
\end{aligned}$$

where the last step can be obtained by the proof of Theorem 2.1 of [11].

Next, we consider $E_2(x,t)$. By Proposition 2.4, there exist functions $d_k(\cdot) \in \mathscr{H}(F_k), k = 1, 2$. Let $\mathrm{d}(x) = \min\{d_1(x), d_2(x)\}, x > 0$. Inspired by the proof of Lemma 3.4 of [4], we can do the following decomposition.

$$\begin{aligned}
&E_2(x,t) \\
&= \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} \prod_{i=1}^{p} Y_i > x + \frac{c}{r}, N(t) = n\right) \\
&\le \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} Y_1 > x + \frac{c}{r}, N(t) = n\right) \\
&= \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} Y_1 > x + \frac{c}{r}, \sum_{q=1}^{n} X_q^{(2)} Y_1 \le \mathrm{d}\left(x + \frac{c}{r}\right), N(t) = n\right) \\
&\quad + \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} Y_1 > x + \frac{c}{r}, \sum_{q=1}^{n} X_q^{(2)} Y_1 > \mathrm{d}\left(x + \frac{c}{r}\right),\right. \\
&\quad \left.\sum_{p=1}^{n} X_p^{(1)} Y_1 \le \mathrm{d}\left(x + \frac{c}{r}\right), N(t) = n\right)
\end{aligned}$$

$$+\sum_{n=N+1}^{\infty}\mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n}X_p^{(m)}Y_1>x+\frac{c}{r},\sum_{q=1}^{n}X_q^{(2)}Y_1>\mathrm{d}\left(x+\frac{c}{r}\right),\right.$$
$$\left.\sum_{p=1}^{n}X_p^{(1)}Y_1>\mathrm{d}\left(x+\frac{c}{r}\right),N\left(t\right)\right)=n\Bigg)$$
$$=:E_{21}\left(x,t\right)+E_{22}\left(x,t\right)+E_{23}\left(x,t\right). \tag{6.29}$$

Then, we deal with $E_{21}\left(x,t\right)$. Applying Lemma 6.6, Assumptions 2.1 and 2.2, for every $\varepsilon>0$, there exists some positive constant K_1 such that for all $t\in\Lambda_*$,

$$\begin{aligned}
&E_{21}\left(x,t\right)\\
&=\sum_{n=N+1}^{\infty}\mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n}X_p^{(m)}Y_1>x+\frac{c}{r},\sum_{q=1}^{n}X_q^{(2)}Y_1\le\mathrm{d}\left(x+\frac{c}{r}\right),N\left(t\right)=n\right)\\
&\le\sum_{n=N+1}^{\infty}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\sum_{q=1}^{n}z_qy_1\le\mathrm{d}\left(x+\frac{c}{r}\right)}\mathbb{P}\left(\sum_{p=1}^{n}X_p^{(1)}y_1>x+\frac{c}{r}-\mathrm{d}\left(x+\frac{c}{r}\right)\right.\\
&\quad\left|X_l^{(2)}=z_l,Y_l=y_l,1\le l\le n\right)\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in\mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}G\left(\mathrm{d}y_l\right)\\
&\le K_1\sum_{n=N+1}^{\infty}\left(1+\varepsilon\right)^n\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\sum_{q=1}^{n}z_qy_1\le\mathrm{d}\left(x+\frac{c}{r}\right)}\mathbb{P}\Big(X_p^{(1)}y_1>x+\frac{c}{r}-\mathrm{d}\left(x+\frac{c}{r}\right)\\
&\quad\left|X_p^{(2)}=z_p,Y_p=y_p\right)\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in\mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}G\left(\mathrm{d}y_l\right)\\
&\lesssim B_2K_1\sum_{n=N+1}^{\infty}\left(1+\varepsilon\right)^n\sum_{p=1}^{n}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\sum_{q=1}^{n}z_qy_1\le\mathrm{d}\left(x+\frac{c}{r}\right)}\overline{F_1}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)\\
&\quad\cdot\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(2)}\in\mathrm{d}z_l\middle|Y_l=y_l\right)\prod_{l=1}^{n}G\left(\mathrm{d}y_l\right)\\
&\le B_2K_1A_1^{-1}\sum_{n=N+1}^{\infty}\left(1+\varepsilon\right)^n n\int\cdots\int_{\Omega_n(t)}\overline{F_1}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)\prod_{l=1}^{n}G\left(\mathrm{d}y_l\right)\\
&\le B_2K_1A_1^{-1}\sum_{n=N+1}^{\infty}\left(1+\varepsilon\right)^n n\int_{\mathrm{e}^{-rt}}^{1}\overline{F_1}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_1\left(y_1\right)\mathbb{P}\left(N\left(t+\frac{\ln y_1}{r}\right)=n-1\right)\\
&\quad\cdot G\left(\mathrm{d}y_1\right)\\
&\le B_2K_1A_1^{-1}\sum_{n=N+1}^{\infty}\left(1+\varepsilon\right)^n n\mathbb{P}\left(=n-1\right)\int_{\mathrm{e}^{-rt}}^{1}\overline{F_1}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_1\left(y_1\right)G\left(\mathrm{d}y_1\right)\\
&=B_2K_1A_1^{-1}\mathbb{E}\left[\left(1+\varepsilon\right)^{N(t)+1}\left(N\left(t\right)+1\right)\mathbf{1}_{\{N(t)\ge N\}}\right]\\
&\quad\cdot\int_{\mathrm{e}^{-rt}}^{1}\overline{F_1}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_1\left(y_1\right)G\left(\mathrm{d}y_1\right).
\end{aligned}\tag{6.30}$$

By Theorem 1 of [25], there always exists some $b_1 > 1$ such that $\mathbb{E}\left[b_1^{N(t)}\right] < \infty$. Thus, for small $\varepsilon, \delta > 0$, we can find some N large enough such that

$$E_{21}(x,t) \lesssim \frac{\delta}{2}\gamma_1\left(x+\frac{c_2}{r};t\right). \tag{6.31}$$

For $E_{22}(x,t)$, applying a similar method of deriving (6.30), by Lemma 6.6, Assumptions 2.1 and 2.2, for ε mentioned above, there exists some positive constant K_2 such that, for all $t \in \Lambda_*$,

$$\begin{aligned}
&E_{22}(x,t)\\
&= \sum_{n=N+1}^{\infty}\mathbb{P}\Bigg(\sum_{m=1}^{2}\sum_{p=1}^{n}X_p^{(m)}Y_1 > x+\frac{c}{r}, \sum_{q=1}^{n}X_q^{(2)}Y_1 > \mathrm{d}\left(x+\frac{c}{r}\right),\\
&\quad \sum_{p=1}^{n}X_p^{(1)}Y_1 \le \mathrm{d}\left(x+\frac{c}{r}\right), N(t)=n\Bigg)\\
&\le \sum_{n=N+1}^{\infty}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\sum_{p=1}^{n}z_py_1\le \mathrm{d}\left(x+\frac{c}{r}\right)}\mathbb{P}\left(\sum_{q=1}^{n}X_q^{(2)}y_1 > x+\frac{c}{r}-\mathrm{d}\left(x+\frac{c}{r}\right)\right.\\
&\quad \left.\middle| X_l^{(1)}=z_l, Y_l=y_l, 1\le l\le n\right)\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(1)}\in \mathrm{d}z_l \middle| Y_l=y_l\right)\prod_{l=1}^{n}G(\mathrm{d}y_l)\\
&\lesssim B_2K_2\sum_{n=N+1}^{\infty}(1+\varepsilon)^n\sum_{q=1}^{n}\int\cdots\int_{\Omega_n(t)}\int\cdots\int_{\sum_{p=1}^{n}z_py_1\le \mathrm{d}\left(x+\frac{c}{r}\right)}\overline{F_2}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)\\
&\quad \cdot\prod_{l=1}^{n}\mathbb{P}\left(X_l^{(1)}\in \mathrm{d}z_l\middle| Y_l=y_l\right)\prod_{l=1}^{n}G(\mathrm{d}y_l)\\
&\le B_2K_2A_1^{-1}\sum_{n=N+1}^{\infty}(1+\varepsilon)^n n\int\cdots\int_{\Omega_n(t)}\overline{F_2}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)\prod_{l=1}^{n}G(\mathrm{d}y_l)\\
&\le B_2K_2A_1^{-1}\sum_{n=N+1}^{\infty}(1+\varepsilon)^n n\int_{\mathrm{e}^{-rt}}^{1}\overline{F_2}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_2(y_1)\mathbb{P}\left(N\left(t+\frac{\ln y_1}{r}\right)=n-1\right)\\
&\quad \cdot G(\mathrm{d}y_1)\\
&\le B_2K_2A_1^{-1}\sum_{n=N+1}^{\infty}(1+\varepsilon)^n n\mathbb{P}(N(t)=n-1)\int_{\mathrm{e}^{-rt}}^{1}\overline{F_2}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_2(y_1)G(\mathrm{d}y_1)\\
&= B_2K_2A_1^{-1}\mathbb{E}\left[(1+\varepsilon)^{N(t)+1}(N(t)+1)\mathbf{1}_{\{N(t)\ge N\}}\right]\\
&\quad \cdot\int_{\mathrm{e}^{-rt}}^{1}\overline{F_2}\left(\left(x+\frac{c}{r}\right)y_1^{-1}\right)h_2(y_1)G(\mathrm{d}y_1).
\end{aligned} \tag{6.32}$$

Again applying Theorem 1 of [25], there always exists some $b_2 > 1$ such that $\mathbb{E}\left[b_2^{\widehat{N}(t)}\right] < \infty$. Thus, for small $\varepsilon, \delta > 0$, we can find some N large enough such that

$$E_{22}(x,t) \lesssim \frac{\delta}{2}\gamma_2\left(x+\frac{c_1}{r};t\right). \tag{6.33}$$

Finally, for $E_{23}(x,t)$, employing a similar method to the one used in the proof of (6.14), by Lemma 6.6, Proposition 2.5, Assumptions 2.1 and 2.2, for ε mentioned above, there exist two positive constants K_1 and K_2 such that, for all $t \in \Lambda_*$,

$$
\begin{aligned}
& E_{23}(x,t) \\
&= \sum_{n=N+1}^{\infty} \mathbb{P}\left(\sum_{m=1}^{2}\sum_{p=1}^{n} X_p^{(m)} Y_1 > x+\frac{c}{r}, \sum_{q=1}^{n} X_q^{(2)} Y_1 > \mathrm{d}\left(x+\frac{c}{r}\right),\right. \\
&\quad \left.\sum_{p=1}^{n} X_p^{(1)} Y_1 > \mathrm{d}\left(x+\frac{c}{r}\right), N(t)=n\right) \\
&= \sum_{n=N+1}^{\infty} \int\cdots\int_{\Omega_n(t)} \int\cdots\int_{\sum_{p=1}^{n} z_p y_1 > \mathrm{d}\left(x+\frac{c}{r}\right)} \\
&\quad \mathbb{P}\left(\sum_{q=1}^{n} X_q^{(2)} y_1 > \left(x+\frac{c}{r}-\sum_{p=1}^{n} z_p y_1\right) \vee \mathrm{d}\left(x+\frac{c}{r}\right) \middle| X_l^{(1)} = z_l, Y_l = y_l, 1 \le l \le n\right) \\
&\quad \cdot \prod_{l=1}^{n} \mathbb{P}\left(X_l^{(1)} \in \mathrm{d}z_l \middle| Y_l = y_l\right) \prod_{l=1}^{n} G(\mathrm{d}y_l) \\
&\le K_1 \sum_{n=N+1}^{\infty} (1+\varepsilon)^n \sum_{q=1}^{n} \int\cdots\int_{\Omega_n(t)} \int\cdots\int_{\sum_{p=1}^{n} z_p y_1 > \mathrm{d}\left(x+\frac{c}{r}\right)} \\
&\quad \mathbb{P}\left(X_q^{(2)} y_1 > \left(x+\frac{c}{r}-\sum_{p=1}^{n} z_p y_1\right) \vee \mathrm{d}\left(x+\frac{c}{r}\right) \middle| X_q^{(1)} = z_q, Y_q = y_q\right) \\
&\quad \cdot \prod_{l=1}^{n} \mathbb{P}\left(X_l^{(1)} \in \mathrm{d}z_l \middle| Y_l = y_l\right) \prod_{l=1}^{n} G(\mathrm{d}y_l) \\
&\lesssim B_2 K_1 \sum_{n=N+1}^{\infty} (1+\varepsilon)^n n \int\cdots\int_{\Omega_n(t)} \int\cdots\int_{\sum_{q=1}^{n} z_q y_1 > \mathrm{d}\left(x+\frac{c}{r}\right)} \\
&\quad \overline{F_2}\left(\left(x+\frac{c}{r}-\sum_{p=1}^{n} z_p y_1\right) y_1^{-1} \vee \mathrm{d}\left(x+\frac{c}{r}\right) y_1^{-1}\right) \\
&\quad \cdot \prod_{l=1}^{n} \mathbb{P}\left(X_l^{(1)} \in \mathrm{d}z_l \middle| Y_l = y_l\right) \prod_{l=1}^{n} G(\mathrm{d}y_l) \\
&\lesssim B_2^2 K_1 K_2 \sum_{n=N+1}^{\infty} (1+\varepsilon)^{2n} n^2 \mathbb{P}(N(t)=n-1) \\
&\quad \cdot \int_{\mathrm{e}^{-rt}}^{1} \mathbb{P}\left(X_1^* y_1 + X_2^* y_1 > x+\frac{c}{r}, X_1^* y_1 > \mathrm{d}\left(x+\frac{c}{r}\right), X_2^* y_1 > \mathrm{d}\left(x+\frac{c}{r}\right)\right) G(\mathrm{d}y_1) \\
&\le B_2^2 K_1 K_2 A_1^{-2} \sum_{n=N+1}^{\infty} \varepsilon (1+\varepsilon)^{2n} n^2 \mathbb{P}(N(t)=n-1) \\
&\quad \cdot \sum_{m=1}^{2} \int_{\mathrm{e}^{-rt}}^{1} \overline{F_m}\left(\left(x+\frac{c}{r}\right) y_1^{-1}\right) h_m(y_1) G(\mathrm{d}y_1) \\
&= B_2^2 K_1 K_2 A_1^{-2} \mathbb{E}\left[\varepsilon (1+\varepsilon)^{2(N(t)+1)} (N(t)+1)^2 \mathbf{1}_{\{N(t) \ge N\}}\right] \\
&\quad \cdot \sum_{m=1}^{2} \int_{\mathrm{e}^{-rt}}^{1} \overline{F_m}\left(\left(x+\frac{c}{r}\right) y_1^{-1}\right) h_m(y_1) G(\mathrm{d}y_1),
\end{aligned}
\tag{6.34}
$$

where X_1^* and X_2^* are indicated in the proof of (6.14). Similarly, by Theorem 1 of [25], there always exists some $b_3 > 1$ such that $\mathbb{E}\left[b_3^{N(t)}\right] < \infty$. Thus, for small $\varepsilon, \delta > 0$, we can find some N large enough such that

$$E_{23}(x,t) \lesssim \frac{\delta}{2}\left(\gamma_1\left(x+\frac{c_2}{r};t\right)+\gamma_2\left(x+\frac{c_1}{r};t\right)\right). \tag{6.35}$$

By (6.29), (6.31), (6.33) and (6.35), we can obtain

$$E_2(x,t) \lesssim \delta\left(\gamma_1\left(x+\frac{c_2}{r};t\right)+\gamma_2\left(x+\frac{c_1}{r};t\right)\right), \tag{6.36}$$

which, combining with (6.28) and by the arbitrariness of δ, yields the desired result. This completes the proof of Theorem 3.2. □

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 12471448). The authors sincerely thank the anonymous referees and editor for the insightful and constructive comments that have substantially improved the quality of this paper.

Data availability statement

No new data/codes were created or analyzed in this study.

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