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A DISCRETE-TIME MATSUMOTO–YOR THEOREM

Charlie Hérent[1,2,*]

Abstract. We study a random walk on the subgroup of lower triangular matrices of SL_2, with i.i.d. increments. We prove that the process of the lower corner of the random walk satisfies a Rogers–Pitman criterion to be a Markov chain if and only if the increments are distributed according to a Generalized Inverse Gaussian (GIG) law on their diagonals. For this, we prove a new characterization of these laws. We prove a discrete-time version of the Dufresne identity. We show how to recover the Matsumoto–Yor theorem by taking the continuous limit of the random walk.

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1. Introduction

1.1. Background and literature

Let $(B_t^{(\mu)} : t \geq 0)$ be a one-dimensional Brownian motion with drift $\mu \in \mathbb{R}$. Matsumoto–Yor's theorem (Thm. 1.6 from [1]) states that the continuous process $(\mathcal{Z}_t^{(\mu)} : t \geq 0)$ defined by

$$\mathcal{Z}_t^{(\mu)} := \mathrm{e}^{B_t^{(\mu)}} \int_0^t \mathrm{e}^{-2B_s^{(\mu)}} ds \tag{1.1}$$

is a diffusion process on $\mathbb{R}_+$ with an infinitesimal generator given by

$$\frac{1}{2} z^2 \frac{d^2}{\mathrm{d}z^2} + \left[\left(\frac{1}{2} + \mu \right) z + \left(\frac{K_{1-\mu}}{K_\mu} \right) \left(\frac{1}{z} \right) \right] \frac{d}{\mathrm{d}z} \tag{1.2}$$

where K_μ is a modified Bessel function of the second kind, also called Macdonald function. This result is a geometric version of Pitman's theorem [2] of 1975 which states that the stochastic process

$$(B_t^{(0)} - 2 \inf_{0 \leq s \leq t} B_s^{(0)} : t \geq 0)$$

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[1] Université Paris Cité, CNRS, MAP5, F-75006 Paris, France.
[2] Université Gustave Eiffel, CNRS, Institut Gaspard Monge, Champs-Sur-Marne, France.

[*] Corresponding author: `charlie.herent@ens-rennes.fr`

is distributed as the three-dimensional Bessel process. Indeed, from Matsumoto–Yor's theorem, with a Laplace approximation argument and using the scaling property of Brownian motion, we recover Pitman's theorem as it is done in [1].

Many generalizations of Pitman's theorem have appeared in last decades. In particular, Biane, Bougerol and O'Connell in [3] and [4] have extended Pitman's theorem in the context of semisimple Lie algebra and finite Coxeter groups. More recently, [5] and [6] have established Pitman type theorems in the context of affine Lie algebra. See also [7] for relations between Pitman's transformation and discrete integrable systems and [39] for generalization about Lévy processes.

Here, we focus on the interpretation of Matsumoto–Yor's theorem from the work of Chhaibi [8–11]. Chhaibi extends Matsumoto–Yor's theorem following the work of O'Connell [12] in the framework of random matrices. In the case of 2×2 matrices this extension reduces to the study of the following SDE in the Stratonovitch sense (Ex. 2.4 in [10]):

$$d\mathbf{B}_t = \mathbf{B}_t \circ \begin{pmatrix} \mathrm{d}B_t^{(\mu)} & 0 \\ \mathrm{d}t & -\mathrm{d}B_t^{(\mu)} \end{pmatrix} \text{ with } \mathbf{B}_0 = I_2 \tag{1.3}$$

whose solution is given by

$$\mathbf{B}_t = \begin{pmatrix} \mathrm{e}^{B_t^{(\mu)}} & 0 \\ \mathrm{e}^{B_t^{(\mu)}} \int_0^t \mathrm{e}^{-2B_s^{(\mu)}} ds & \mathrm{e}^{-B_t^{(\mu)}} \end{pmatrix} \tag{1.4}$$

Matsumoto–Yor's process (1.1) appearing in the lower corner of the matrix, is called the highest weight process for reasons explained in [8]. Chhaibi proved in [10] (Thm. 3.1) that the highest weight process is a diffusion with an explicit infinitesimal generator. The main object of this paper is the study of a discrete-time version of this diffusion.

The following result, often called *Dufresne identity*, states the equality in law for $\mu > 0$:

$$\int_0^{+\infty} \mathrm{e}^{-2B_s^{(\mu)}} ds \overset{\text{law}}{=} \frac{1}{2\xi} \tag{1.5}$$

where ξ is a random variable with $Gamma(\mu)$ distribution. This identity obtained in [13] plays also an important role in the work of Matsumoto–Yor in [1] and [14], and has many applications in finance and physics. See also in [15] and [16] some generalizations of this identity. We will prove a discrete-time version of this identity in our study. Note that the results obtained by [15] concern only positive definite matrices, whereas our work is a discrete-time version of Chhaibi's approach.

1.2. Contributions of this work

In this article, we consider a discrete-time version of the SDE (1.3). Let $(\gamma_n)_{n\in\mathbb{N}}$ be a family of identically distributed independent random variables with $\mathcal{C}^1$ density function and $(X_n)_{n\in\mathbb{N}^*}$, $(Z_n)_{n\in\mathbb{N}^*}$ be two discrete-time processes defined by the random walk $(b_n^{(\delta)})_{n\in\mathbb{N}}$, with $\delta \in \mathbb{R}^*$ a deterministic parameter:

$$b_{n+1}^{(\delta)} := b_n^{(\delta)} g_n^{(\delta)} \text{ with } b_0^{(\delta)} = I_2, \text{ hence } b_n^{(\delta)} = g_0^{(\delta)} \cdots g_{n-1}^{(\delta)} \tag{1.6}$$

where,

$$b_n^{(\delta)} := \begin{pmatrix} X_n & 0 \\ Z_n & X_n^{-1} \end{pmatrix} \text{ and } g_n^{(\delta)} := \begin{pmatrix} \gamma_n & 0 \\ \delta & \gamma_n^{-1} \end{pmatrix}.$$

The process $(Z_n)_{n\in\mathbb{N}^*}$ is the analog of highest weight process and is a function of the Markov process $(b_n^{(\delta)})_{n\in\mathbb{N}}$. Rogers–Pitman [17] gave a criterion for a function of a Markov process to be a Markov process. In the paper we give a necessary and sufficient condition on the increments to guarantee that this criterion is satisfied. More precisely we prove that $(Z_n)_{n\in\mathbb{N}^*}$ is a Markov chain and that there exists an intertwining relation between $(Z_n)_{n\in\mathbb{N}^*}$ and $(X_n)_{n\in\mathbb{N}^*}$ if and only if γ_i is distributed according to a Generalized Inverse Gaussian (GIG) law. We start by proving the sufficient condition, then we establish a new characterization of GIG distributions and deduce the necessary condition. GIG laws already appeared in the work of Matsumoto–Yor [40, 41] who showed that the process (1.1) is intertwined with $(B_t^{(\mu)} : t \geq 0)$ and the intertwining kernel may be expressed in terms of GIG laws as it is done in [14]. Chhaibi also obtained a generalization of this intertwining relation in [8].

Recall that every lower triangular matrix of SL_2 can be uniquely factorized in the form NA or AN with N a unipotent matrix and A a diagonal matrix,

$$\begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^{-1}z & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xz & 1 \end{pmatrix} \text{ with } x \in \mathbb{R}^*, z \in \mathbb{R}.$$

We will study these factorizations for the random walk (1.6). The N-part in the AN factorization of $(b_n^{(\delta)})_{n\in\mathbb{N}}$ is a Markov chain. We obtain the almost sure convergence of the N-part in NA factorization towards a random variable distributed according to the invariant probability measure of the N-part in AN factorization. We will compute this invariant probability measure and use it to obtain a discrete version of the Dufresne identity.

The Pitman transform appearing in Pitman's theorem, can be inverted using some additional information (Prop. 2.2 (iv) in [3]). From this one can infer reconstruction theorems, in the sense that, it is possible to recover an unconditioned Brownian motion from a conditioned one by applying a sequence of inverse Pitman's transform. We establish in our discrete-time context, a reconstruction theorem.

Using Lindeberg's theorem, we prove that our discrete-time process (1.6) converges in law towards (1.4) when δ goes to 0 choosing correctly the increments laws. Thus, we recover Matsumoto–Yor theorem and Chhaibi theorem (Thm. 3.1 in [10]) in the SL_2 case. For the proof of the convergence, we compute some asymptotic formulas for the moments of log(GIG) laws.

The generalisation of some results from the paper to the case of a higher dimension seems to be obtainable by following an analogous study and considering the correct laws for the increments. However, several results, such as the characterisation of the laws that play the role of GIG distributions, seem more difficult to obtain for higher dimensions, due to the technical nature of the proof used in the paper.

1.3. Organization of the paper

In the next section we give some formulas which will be used several times in the paper. In Section 3, we prove that the process $(Z_n)_{n\in\mathbb{N}^*}$ is a Markov chain when γ_i is distributed according to a GIG law. We give an explicit formula for its Markov kernel. We establish an intertwining relation between this kernel and the transition kernel of $(X_n)_{n\in\mathbb{N}^*}$. In Section 4, we prove a new characterization of GIG laws which gives a necessary condition to obtain an intertwining relation between $(Z_n)_{n\in\mathbb{N}^*}$ and $(X_n)_{n\in\mathbb{N}^*}$. In Section 5, we describe the invariant probability measure for the N-part in AN factorization of the random walk from which we deduce a discrete-time Dufresne identity. Section 6 is devoted to the reconstruction theorem. In Section 7, we establish that our random walk converges towards the continuous process described in the introduction. Finally, in Section 8, we use group theory to describe our random walk and explain a convergence theorem for the N-part in NA factorization of the random walk.

2. Preliminaries

2.1. Explicit expression of the random walk

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a family of identically distributed independent random variables with $\mathcal{C}^1$ density function and $(X_n)_{n\in\mathbb{N}^*}$, $(Z_n)_{n\in\mathbb{N}^*}$ be two discrete-time processes defined by the random walk $(b_n^{(\delta)})_{n\in\mathbb{N}}$, with $\delta \in \mathbb{R}^*$ a deterministic parameter:

$$b_{n+1}^{(\delta)} := b_n^{(\delta)} g_n^{(\delta)} \text{ with } b_0^{(\delta)} = I_2, \text{ hence } b_n^{(\delta)} = g_0^{(\delta)} \cdots g_{n-1}^{(\delta)} \tag{2.1}$$

where,

$$b_n^{(\delta)} := \begin{pmatrix} X_n & 0 \\ Z_n & X_n^{-1} \end{pmatrix} \text{ and } g_n^{(\delta)} := \begin{pmatrix} \gamma_n & 0 \\ \delta & \gamma_n^{-1} \end{pmatrix}.$$

By iteration of the relation (2.1), we have the following formula for $b_n^{(\delta)}$, with $n \in \mathbb{N}$,

$$b_n^{(\delta)} = \begin{pmatrix} \prod_{i=0}^{n-1} \gamma_i & 0 \\ \delta \sum_{k=0}^{n-1} \prod_{i=0}^{k-1} \gamma_i^{-1} \prod_{j=k+1}^{n-1} \gamma_j & \prod_{i=0}^{n-1} \gamma_i^{-1} \end{pmatrix}. \tag{2.2}$$

Let us remark that $b_n^{(\delta)}$ is conjugate to $b_n^{(1)}$, indeed, for $\delta > 0$:

$$P b_n^{(\delta)} P^{-1} = b_n^{(1)} \text{ with } P := \begin{pmatrix} \delta^{\frac{1}{2}} & 0 \\ 0 & \delta^{-\frac{1}{2}} \end{pmatrix}.$$

In the following, apart from Section 6, we set $\delta = 1$ to simplify some expressions and we will denote b_n instead of $b_n^{(1)}$ and g_n instead of $g_n^{(1)}$. This slightly changes the expressions of density functions that we will give but it does not change the Markov property and the intertwining relation that we prove in next sections.

From the expression (2.2), note that we have for $n \in \mathbb{N}^*$:

$$X_n = \prod_{i=0}^{n-1} \gamma_i \text{ and } Z_n = \sum_{k=0}^{n-1} \left(\prod_{i=0}^{k-1} \gamma_i^{-1} \right) \left(\prod_{j=k+1}^{n-1} \gamma_j \right), \text{ with } X_1 = \gamma_0 \text{ and } Z_1 = 1. \tag{2.3}$$

From (2.1), the processes $(Z_n)_{n\in\mathbb{N}^*}$ and $(X_n)_{n\in\mathbb{N}^*}$ satisfy for $k \geq 2$:

$$Z_k = \frac{X_k Z_{k-1} + 1}{X_{k-1}} \text{ and } X_k = \gamma_{k-1} X_{k-1}. \tag{2.4}$$

2.2. A change of variable

We consider the transformation, for $n \geq 2$:

$$\begin{array}{rcl} \Phi_n : \left(\mathbb{R}_+^*\right)^n & \to & \left(\mathbb{R}_+^*\right)^n \\ (y_0, \ldots, y_{n-1}) & \mapsto & (z_2, \ldots, z_n, x_n) \end{array} \tag{2.5}$$

where x_n and z_k are defined by:

$$x_n = \prod_{i=0}^{n-1} y_i \text{ and } z_k = \sum_{l=0}^{k-1} \left(\prod_{i=0}^{l-1} y_i^{-1} \right) \left(\prod_{j=l+1}^{k-1} y_j \right). \tag{2.6}$$

They satisfy the following recurrence formula, for $k \geq 2$:

$$z_k = \frac{x_k z_{k-1} + 1}{x_{k-1}} \text{ and } x_k = y_{k-1} x_{k-1}. \tag{2.7}$$

Lemma 2.1. *The transformation* $\Phi_n : \left(\mathbb{R}_+^*\right)^n \to \left(\mathbb{R}_+^*\right)^n$ *is a* C^1*-diffeomorphism and the Jacobian, in terms of* z *variables, associated to this transformation is given by*

$$\det(\mathcal{J}_n) = (-1)^{n-1} z_2 \cdots z_n. \tag{2.8}$$

Proof. To prove that Φ_n is indeed a C^1-diffeomorphism and compute the Jacobian, it is more convenient to consider intermediate changes of variables. First of all, we consider the change of variables with the expression (2.6):

$$(y_0, \ldots, y_{n-1}) \overset{\widetilde{\Phi}_0}{\mapsto} (x_1, \ldots, x_n)$$

which is clearly a C^1-diffeomorphism because x_i are products of y_i and the Jacobian of this change of variables is equal to $\prod_{i=1}^{n-1} x_i$. Then successively, according to the expressions (2.6), we proceed to the following changes of variables

$$(x_1, \ldots, x_n) \overset{\widetilde{\Phi}_1}{\mapsto} (z_2, x_2, \ldots, x_n) \overset{\widetilde{\Phi}_2}{\mapsto} (z_2, z_3, x_3, \ldots, x_n) \overset{\widetilde{\Phi}_3}{\mapsto} \cdots \overset{\widetilde{\Phi}_{n-1}}{\mapsto} (z_2, \ldots, z_n, x_n).$$

Thanks to formula (2.7), each change of variable $\widetilde{\Phi}_k$ is a C^1-diffeomorphism with Jacobian equal to, for all $k \in [\![1, n-1]\!]$:

$$\frac{\partial z_{k+1}}{\partial x_k} = \frac{-(x_{k+1} z_k + 1)}{x_k^2}.$$

The resulting change of variable is just $\Phi_n = \widetilde{\Phi}_{n-1} \circ \cdots \circ \widetilde{\Phi}_1 \circ \widetilde{\Phi}_0$, so the corresponding Jacobian is given by

$$\det(\mathcal{J}_n) = \prod_{i=1}^{n-1} x_k \prod_{k=1}^{n-1} \frac{-(x_{k+1} z_k + 1)}{x_k^2} = (-1)^{n-1} \prod_{k=1}^{n-1} \left(\frac{x_{k+1} z_k + 1}{x_k} \right) = (-1)^{n-1} z_2 \cdots z_n.$$

Where the last equality is obtained again from (2.7). □

Let us mention that the bijection Φ_n is related to a geometric version of the Robinson–Schensted–Knuth (RSK) correspondence. We refer to [18], [19] and [42] for more details about this correspondence.

3. Markov property of the discrete-time Matsumoto–Yor process $(Z_n)_{n \in \mathbb{N}^*}$ and intertwining relation

In this section, we consider Generalized Inverse Gaussian distributions $GIG(\lambda, a, b)$.

These distributions were introduced by Halphen in [20] who used these laws in hydrological problems. They have also been used by Good in his study of population frequencies [21].

Let us recall that the probability density function for a law $GIG(\lambda, a, b)$ is given by

$$x \in \mathbb{R}_+^* \mapsto \left(\frac{b}{a}\right)^{\lambda} \frac{1}{2K_\lambda(ab)} x^{\lambda-1} \mathrm{e}^{-\frac{1}{2}\left(\frac{a^2}{x} + b^2 x\right)}$$

where $a, b > 0$, $\lambda \in \mathbb{R}$ and K_λ is a Macdonald function. An integral representation of this function is given for $z > 0$ by

$$K_\lambda(z) = \frac{1}{2}\int_0^{+\infty} x^{\lambda-1} \mathrm{e}^{-\frac{z}{2}(x+\frac{1}{x})} \mathrm{d}x.$$

Let us recall the scaling property between GIG distributions:

$$\text{If } c > 0 \text{ and } X \sim GIG(\lambda, a, b), \text{ then } cX \sim GIG\left(\lambda, a\sqrt{c}, \frac{b}{\sqrt{c}}\right).$$

In this paper, we focus on the $GIG(\lambda, a, a)$ distribution with the probability density function

$$x \in \mathbb{R}_+^* \mapsto \frac{1}{2K_\lambda(a^2)} x^{\lambda-1} \mathrm{e}^{-\frac{a^2}{2}(x+\frac{1}{x})}.$$

In the following, we prove that the process $(Z_n)_{n\in\mathbb{N}^*}$ is a Markov chain when $(\gamma_n)_{n\in\mathbb{N}}$ is distributed according to the law $GIG(\lambda, a, a)$ with $a > 0$ and $\lambda \in \mathbb{R}$. We compute the transition kernel of $(Z_n)_{n\in\mathbb{N}^*}$ and as a by-product, we establish an intertwining relation between this kernel and the transition kernel of $(X_n)_{n\in\mathbb{N}^*}$.

The following expression is crucial for the proof of the Markov property, and it is obtained by induction on $n \geq 2$, using (2.4).

Lemma 3.1. *We have the following equality*

$$\forall n \geq 2, \quad \sum_{k=0}^{n-1}\left(\gamma_k + \frac{1}{\gamma_k}\right) = \frac{1}{Z_n}\left(X_n + \frac{1}{X_n}\right) + F_n(Z_2, \ldots, Z_n)$$

where $F_n(Z_2, \ldots, Z_n) = \sum_{k=1}^{n-1} \frac{Z_{k+1}^2 + Z_k^2 + 1}{Z_{k+1}Z_k}$.

From the expression above, we deduce the following joint density functions.

Proposition 3.2. *The joint density function of* $(Z_2, \ldots, Z_n, X_n)$ *is, for* $z_2, \ldots, z_n, x_n > 0$*:*

$$f_{(Z_2,\ldots,Z_n,X_n)}(z_2, \ldots, z_n, x_n) = \left(\frac{1}{2K_\lambda(a^2)}\right)^n \frac{x_n^{\lambda-1}}{z_2 \cdots z_n} \mathrm{e}^{-\frac{a^2}{2z_n}(x_n+\frac{1}{x_n}) - \frac{a^2}{2}F_n(z_2,\ldots,z_n)}. \tag{3.1}$$

The joint density function of $(Z_2, \ldots, Z_n)$ *is, for* $z_2, \ldots, z_n > 0$*:*

$$f_{(Z_2,\ldots,Z_n)}(z_2, \ldots, z_n) = \left(\frac{1}{2K_\lambda(a^2)}\right)^n \frac{2K_\lambda\left(\frac{a^2}{z_n}\right)}{z_2 \cdots z_n} \mathrm{e}^{-\frac{a^2}{2}F_n(z_2,\ldots,z_n)}. \tag{3.2}$$

The conditional density function of X_n *given* $\{Z_n = z_n, \ldots, Z_2 = z_2\}$ *is, for* $z_2, \ldots, z_n > 0$*:*

$$f_{X_n|Z_n=z_n,\ldots,Z_2=z_2}(x_n) = f_{X_n|Z_n=z_n}(x_n) = \frac{1}{2K_\lambda\left(\frac{a^2}{z_n}\right)} x_n^{\lambda-1} \mathrm{e}^{-\frac{a^2}{2z_n}(x_n+\frac{1}{x_n})}. \tag{3.3}$$

Proof. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function and f the common density function of γ_i's. Thanks to the independence of γ_i's and the change of variables Φ_n defined by (2.5) we get:

$$\begin{aligned}\mathbb{E}\left(h(Z_2,\dots,Z_n,X_n)\right) &= \int_{\left(\mathbb{R}_+^*\right)^n} h(\Phi_n(y_0,\dots,y_{n-1}))f(y_0)\cdots f(y_{n-1})\mathrm{d}y_0\cdots dy_{n-1}\\ &= \int_{\left(\mathbb{R}_+^*\right)^n} h(\Phi_n(y_0,\dots,y_{n-1}))\frac{\left(\prod_{i=0}^{n-1} y_i\right)^{\lambda-1}}{\left(2K_\lambda(a^2)\right)^n}\mathrm{e}^{-\frac{a^2}{2}\sum_{k=0}^{n-1}\left(y_k+\frac{1}{y_k}\right)}\mathrm{d}y_0\cdots dy_{n-1}.\end{aligned}$$

From (2.8) and Lemma 3.1 we deduce the joint density of $(Z_2,\dots,Z_n,X_n)$.

Integrating with respect to the variable x_n in the formula (3.1) we obtain the density (3.2) using the integral representation of the Macdonald function.

The equality,

$$f_{X_n|Z_n=z_n,\dots,Z_2=z_2}(x_n) = \frac{1}{2K_\lambda\left(\frac{a^2}{z_n}\right)}x_n^{\lambda-1}\mathrm{e}^{-\frac{a^2}{2z_n}\left(x_n+\frac{1}{x_n}\right)}$$

is obtained from the quotient of (3.1) and (3.2). This last conditional density only depends on x_n and z_n, so we obtain

$$f_{X_n|Z_n=z_n,\dots,Z_2=z_2}(x_n) = f_{X_n|Z_n=z_n}(x_n).$$

□

Proposition 3.3. *The process $(Z_n)_{n\in\mathbb{N}^*}$ is a homogeneous Markov chain starting from $Z_1 = 1$ with transition kernel given, for $x > 0$, by*

$$Q(x,\mathrm{d}y) = \left(\frac{1}{2K_\lambda(a^2)}\right)\frac{K_\lambda\left(\frac{a^2}{y}\right)}{K_\lambda\left(\frac{a^2}{x}\right)}\frac{1}{y}\mathrm{e}^{-\frac{a^2(x^2+y^2+1)}{2xy}}\mathbb{1}_{\mathbb{R}_+^*}(y)\mathrm{d}y.$$

Proof. Using the density function (3.2) at rank n and $n-1$, we obtain the conditional density function for $z_2,\dots,z_n > 0$:

$$f_{Z_n|(Z_{n-1},\dots,Z_2)=(z_{n-1},\dots,z_2)}(z_n) = \left(\frac{1}{2K_\lambda(a^2)}\right)\frac{K_\lambda\left(\frac{a^2}{z_n}\right)}{K_\lambda\left(\frac{a^2}{z_{n-1}}\right)}\frac{1}{z_n}\mathrm{e}^{-\frac{a^2(z_n^2+z_{n-1}^2+1)}{2z_nz_{n-1}}}. \tag{3.4}$$

This last expression only depends on z_n and z_{n-1}, this implies the Markov property for the process $(Z_n)_{n\in\mathbb{N}^*}$ and the expression for the transition kernel. □

One can also give the transition kernel for the Markov chain $(X_n)_{n\in\mathbb{N}^*}$ starting from $X_1 = \gamma_0$, for $x > 0$:

$$P(x,\mathrm{d}y) = \left(\frac{1}{2K_\lambda(a^2)}\right)\frac{y^{\lambda-1}}{x^\lambda}\mathrm{e}^{-\frac{a^2}{2}\left(\frac{y}{x}+\frac{x}{y}\right)}\mathbb{1}_{\mathbb{R}_+^*}(y)\mathrm{d}y.$$

We will establish an intertwining relation between the transition kernels of $(X_n)_{n\in\mathbb{N}^*}$ and $(Z_n)_{n\in\mathbb{N}^*}$. Let us recall the definition:

Definition 3.4. Let P and Q be two Markov transition kernels on the measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ respectively. A Markov kernel Λ from F to E is a map

$$\Lambda : (u, A) \mapsto \Lambda(u, A) \text{ with } u \in F \text{ and } A \in \mathcal{E}$$

such that for each $u \in F$, $\Lambda(u, \cdot)$ is a probability on E, and for each $A \in \mathcal{E}$, $\Lambda(\cdot, A)$ belongs to the space of bounded measurable functions on F.

Then, the Markov kernel Λ intertwines P and Q if one has the relation:

$$\Lambda P = Q \Lambda$$

where the composition of kernels is defined by $\Lambda P(u, dv) := \int_E P(y, dv)\Lambda(u, \mathrm{d}y)$.

We refer to [22] and [23] for some examples of intertwining relations.

Let us recall that Rogers–Pitman criterion [17] gives conditions for a process that is a function of a Markov process to be Markov itself. One can use this criterion here to deduce the Markov property of the process $(Z_n)_{n \in \mathbb{N}^*}$ by giving the intertwining kernel Λ. This intertwining kernel is given, for all $n \in \mathbb{N}^*$, by:

$$\mathbb{P}\left(X_n \in \mathrm{d}x | Z_n, \cdots, Z_1\right) = \Lambda(Z_n, \mathrm{d}x) \text{ a.s.}$$

Here, from Proposition 3.2, the conditional law $\mathcal{L}(X_n | Z_n, \cdots, Z_1)$ is in fact $\mathcal{L}(X_n | Z_n)$. We recover the density function of a GIG law in (3.3) and we obtain:

Proposition 3.5. *The Markov transition kernels of the processes $(X_n)_{n \in \mathbb{N}^*}$ and $(Z_n)_{n \in \mathbb{N}^*}$ are intertwined by the Markov kernel Λ given by the law of X_n given Z_n which does not depend on n, more precisely, we have the following intertwining relation:*

$$\Lambda P = Q \Lambda$$

where, for $z > 0$:

$$\Lambda(z, \mathrm{d}x) = \frac{1}{2K_\lambda\left(\frac{a^2}{z}\right)} x^{\lambda - 1} \mathrm{e}^{-\frac{a^2}{2z}\left(x + \frac{1}{x}\right)} \mathbb{1}_{\mathbb{R}_+^*}(x) \mathrm{d}x.$$

Proof. We deduce the formula of Λ from (3.3). Then, one has for $z > 0$:

$$\Lambda P(z, \mathrm{d}x) = \int_{\mathbb{R}_+^*} \frac{1}{4K_\lambda\left(a^2\right) K_\lambda\left(\frac{a^2}{z}\right)} \frac{x^{\lambda-1}}{y} \mathrm{e}^{-\frac{a^2}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{1}{zy}\right)} \mathbb{1}_{\mathbb{R}_+^*}(x) \mathrm{d}y \mathrm{d}x.$$

Moreover, we have for $z > 0$:

$$\begin{aligned} Q\Lambda(z, \mathrm{d}x) &= \int_{\mathbb{R}_+^*} \Lambda(y, \mathrm{d}x) Q(z, \mathrm{d}y) \\ &= \int_{\mathbb{R}_+^*} \frac{1}{4K_\lambda\left(a^2\right) K_\lambda\left(\frac{a^2}{z}\right)} \frac{x^{\lambda-1}}{y} \mathrm{e}^{-\frac{a^2}{2}\left(\frac{x}{y} + \frac{1}{yx} + \frac{z}{y} + \frac{y}{z} + \frac{1}{zy}\right)} \mathbb{1}_{\mathbb{R}_+^*}(x) \mathrm{d}y \mathrm{d}x \\ &= \Lambda P(z, \mathrm{d}x) \end{aligned}$$

where the last equality is obtained by the change of variable $y := \frac{1}{u}(1 + zx)$. □

Let us mention that the intertwining relation can also be deduced from the work of Chhaibi [8] (Thm. 5.6.8).

4. A CHARACTERIZATION OF GENERALIZED INVERSE GAUSSIAN DISTRIBUTIONS (GIG)

There exist several characterizations of GIG distributions, see for example the work of Koudou, Ley, Letac, Seshadri, Matsumoto, Vallois and Yor in [14, 24–27]. A survey can be found in the paper [28]. We give in the following proposition a new characterization for these laws involving γ_0, γ_1 and γ_2. Let us recall that from (2.3) we get:

$$X_2 = \gamma_0\gamma_1,\ X_3 = \gamma_0\gamma_1\gamma_2,\ Z_2 = \gamma_0^{-1} + \gamma_1,\ Z_3 = \gamma_0^{-1}\gamma_1^{-1} + \gamma_0^{-1}\gamma_2 + \gamma_1\gamma_2.$$

Proposition 4.1. *Let $\gamma_0, \gamma_1, \gamma_2$ be three i.i.d. random variables with $\mathcal{C}^1$ density function supported on $\mathbb{R}_+$. If we have the equality of conditional laws, for $z, u > 0$:*

$$\mathcal{L}\left(X_3 | Z_3 = z, Z_2 = u\right) = \mathcal{L}\left(X_2 | Z_2 = z\right),$$

then, γ_0 is distributed according to the law $GIG(\lambda, a, a)$ with some $a > 0$ and $\lambda \in \mathbb{R}$.

Proof. Assume that $\gamma_0, \gamma_1, \gamma_2$ are i.i.d. random variables with a $\mathcal{C}^1$ density supported on $\mathbb{R}_+$ denoted by f. If the equality of conditional laws holds, then conditional density functions satisfy:

$$f_{X_2|Z_2=z} = f_{X_3|(Z_3,Z_2)=(z,u)}.$$

We have the following expressions by inverting Φ_2 from (2.5):

$$\gamma_0 = \frac{X_2+1}{Z_2} \text{ and } \gamma_1 = \frac{X_2 Z_2}{X_2+1}.$$

Moreover we know from (2.8) that $|\det(\mathcal{J}_2)| = \frac{1}{z}$ for the change of variable Φ_2^{-1}. We get the joint density,

$$f_{(Z_2,X_2)}(z,x) = \frac{1}{z} f\left(\frac{x+1}{z}\right) f\left(\frac{xz}{x+1}\right)$$

and hence we deduce the conditional density

$$f_{X_2|Z_2=z}(x) = \frac{f\left(\frac{x+1}{z}\right) f\left(\frac{xz}{x+1}\right)}{I(z)}$$

where

$$I(z) := \int_{\mathbb{R}} f\left(\frac{t+1}{z}\right) f\left(\frac{tz}{t+1}\right) \mathrm{d}t$$

depends only on the variable z.

In the same way, we know by inverting Φ_3 from (2.5) that

$$\gamma_0 = \frac{Z_2X_3 + Z_3 + 1}{Z_2Z_3},\ \gamma_1 = \frac{Z_2^2X_3 + Z_2}{Z_2X_3 + Z_3 + 1},\ \gamma_2 = \frac{X_3Z_3}{Z_2X_3 + 1}$$

and from (2.8) we obtain $|\det(\mathcal{J}_3)| = \frac{1}{zu}$ for the change of variable Φ_3^{-1}, we deduce

$$f_{X_3|(Z_3,Z_2)=(z,u)}(x) = \frac{f\left(\frac{ux+z+1}{uz}\right) f\left(\frac{u^2x+u}{ux+z+1}\right) f\left(\frac{xz}{ux+1}\right)}{J(z,u)}$$

where

$$J(z,u) := \int_{\mathbb{R}} f\left(\frac{ut+z+1}{uz}\right) f\left(\frac{u^2t+u}{ut+z+1}\right) f\left(\frac{tz}{ut+1}\right) \mathrm{d}t$$

depends only on the variables z and u.

Thus, the equation $f_{X_2|Z_2=z}(x) = f_{X_3|(Z_3,Z_2)=(z,u)}(x)$ implies that

$$\frac{J(z,u)}{I(z)} = \frac{f\left(\frac{ux+z+1}{uz}\right) f\left(\frac{u^2x+u}{ux+z+1}\right) f\left(\frac{xz}{ux+1}\right)}{f\left(\frac{x+1}{z}\right) f\left(\frac{xz}{x+1}\right)}$$

does not depend on the variable x. The logarithmic derivative in x of this last quotient is therefore equal to 0. Let us denote by $g := \frac{f'}{f}$ the logarithmic derivative of f, we obtain the following functional equation:

$$\begin{aligned} \frac{1}{z} g\left(\frac{ux+z+1}{uz}\right) + \frac{u^2z}{(ux+z+1)^2} g\left(\frac{u^2x+u}{ux+z+1}\right) + \frac{z}{(ux+1)^2} g\left(\frac{xz}{ux+1}\right) \\ - \frac{1}{z} g\left(\frac{x+1}{z}\right) - \frac{z}{(x+1)^2} g\left(\frac{xz}{x+1}\right) = 0. \end{aligned} \tag{4.1}$$

Let $z := \frac{x+1}{x}$ and $u = \frac{1}{x^2}$, the above equation becomes:

$$x^2 g(2x^2) + \frac{1}{4x^2} g\left(\frac{1}{2x^2}\right) = g(1). \tag{4.2}$$

Setting $s := 2x^2$, we get:

$$sg(s) + \frac{1}{s} g\left(\frac{1}{s}\right) = 2g(1). \tag{4.3}$$

By the change of function $G(s) := sg(s)$, we obtain the following equation:

$$G(s) + G\left(\frac{1}{s}\right) = 2G(1). \tag{4.4}$$

The solutions of (4.4) are given by $G(x) = G(1) + \left(x - \frac{1}{x}\right)\varphi(x)$ where φ is some continuous function satisfying $\varphi(x) = \varphi\left(\frac{1}{x}\right)$. Thus, the solutions of (4.1) are of the form:

$$g(x) = \frac{g(1)}{x} + \left(1 - \frac{1}{x^2}\right)\varphi(x) \quad \text{with } \varphi(x) = \varphi\left(\frac{1}{x}\right). \tag{4.5}$$

We will prove that the only solutions of (4.1) correspond to the case φ constant. Indeed, if g is solution of (4.1), then $h(x) := g\left(\frac{1}{x}\right)$ is solution of:

$$\begin{aligned}\frac{1}{z}h\left(\frac{uz}{ux+z+1}\right)+\frac{u^2z}{(ux+z+1)^2}h\left(\frac{ux+z+1}{u^2x+u}\right)+\frac{z}{(ux+1)^2}h\left(\frac{ux+1}{xz}\right)\\ -\frac{1}{z}h\left(\frac{z}{x+1}\right)-\frac{z}{(x+1)^2}h\left(\frac{x+1}{xz}\right)=0.\end{aligned} \tag{4.6}$$

First, let us set $x=\frac{1}{2y}$, $u=2y^2-y$ and $z=y+\frac{1}{2}$. We get:

$$\frac{1}{y^2}h\left(\frac{2y^2-y}{2}\right)+\left(\frac{2y-1}{2}\right)^2h\left(\frac{2}{2y^2-y}\right)+\frac{1}{y^2}h(2y)-\frac{1}{y^2}h(y)=h(2). \tag{4.7}$$

Since it has been proved that the solutions of (4.1) are of the form (4.5), the function $h(x)=xh(1)+\left(1-x^2\right)\varphi(x)$ where $\varphi(x)=\varphi\left(\frac{1}{x}\right)$ must satisfy (4.7). Substituting this expression into the equation (4.7) where we replaced the letter y by the letter x, leads to the functional equation in φ:

$$(4x^2-1)\varphi(2x)+(1-x^2)\varphi(x)=3x^2\varphi(2) \quad \text{and} \quad \varphi(x)=\varphi\left(\frac{1}{x}\right). \tag{4.8}$$

We will prove in Lemma A.2 that the only continuous solutions of (4.8) are given by φ constant. To conclude, according to (4.5) this proves that the only continuous solutions of (4.1) are given by

$$g(x)=\frac{C_1}{x}+C_2\left(1-\frac{1}{x^2}\right)$$

where $C_1, C_2 \in \mathbb{R}$. Let us recall that $g := \frac{f'}{f}$. The solutions of the ordinary differential equation

$$f'(x)=\left(\frac{C_1}{x}+C_2\left(1-\frac{1}{x^2}\right)\right)f(x)$$

are given by $f(x)=Kx^{C_1}\mathrm{e}^{C_2\left(x+\frac{1}{x}\right)}$ where $K\in\mathbb{R}$, so we recover the density function of GIG law. ☐

Theorem 4.2. *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathcal{C}^1$ density function supported on $\mathbb{R}_+$. Then, there exists an intertwining kernel Λ such that, for all $n\in\mathbb{N}^*$:*

$$\mathbb{P}\left(X_n\in \mathrm{d}x|Z_n,\cdots,Z_1\right)=\Lambda(Z_n,\mathrm{d}x)\ \textit{a.s.}$$

if and only if the family $(\gamma_n)_{n\in\mathbb{N}}$ is distributed according to the law $GIG(\lambda,a,a)$ with some $a>0$ and $\lambda\in\mathbb{R}$.

Proof. Proposition 4.1 gives the necessary condition for the existence of an intertwining kernel. The converse was obtained in Proposition 3.5 where we considered the law $GIG(\lambda,a,a)$ for the increments. ☐

5. Invariant probability measure for the N-part and Dufresne identity

We consider the N-part of the random walk in both NA and AN factorizations. In 5.1 we will explain the link between the two. Then, in 5.2 we will obtain the invariant probability measure of the N-part in the AN factorization. Thanks to this, we will obtain a Dufresne identity involving the N-part of the NA factorization.

In this section, several results obtained remain true more generally for any random walk on group with NA factorization. For the convenience of the reader, we prove the results for $\delta = 1$.

5.1. Relations between both factorizations

Here, we only assume that $\gamma := (\gamma_i)_{i\geq 0}$ is a sequence of i.i.d. random variables with a probability density function and a finite first log-moment. Let $\mathcal{N}_n(\gamma)$ be the N-part of the random walk in the NA factorization and $\widetilde{\mathcal{N}}_n(\gamma)$ be the N-part of the random walk in the AN factorization. That is to say, with the notation $b_n(\gamma)$ instead of b_n defined in Section 2, we have

$$b_n(\gamma) = \mathcal{N}_n(\gamma)A_n(\gamma) = A_n(\gamma)\widetilde{\mathcal{N}}_n(\gamma) \text{ where } A_n(\gamma) := \begin{pmatrix} X_n & 0 \\ 0 & X_n^{-1} \end{pmatrix}.$$

Then, we denote by $N_n(\gamma)$ and $\widetilde{N}_n(\gamma)$ the matrix coefficients in the matrices:

$$\mathcal{N}_n(\gamma) = \begin{pmatrix} 1 & 0 \\ N_n(\gamma) & 1 \end{pmatrix} \text{ and } \widetilde{\mathcal{N}}_n(\gamma) = \begin{pmatrix} 1 & 0 \\ \widetilde{N}_n(\gamma) & 1 \end{pmatrix}.$$

We obtain the following expressions for $n \in \mathbb{N}^*$:

$$N_n(\gamma) = X_n^{-1} Z_n \text{ and } \widetilde{N}_n(\gamma) = X_n Z_n$$

We will see that the N-part in AN factorization $(\widetilde{\mathcal{N}}_n(\gamma))_{n\in\mathbb{N}^*}$ is a Markov chain. In general, $(N_n(\gamma))_{n\in\mathbb{N}^*}$ the N-part in NA factorization is not a Markov chain, but we have the following relation which is a classic tool to prove Dufresne-type identities, see [15] and [29] for instance. We prove this relation in our context.

Lemma 5.1. *We have the equality in law:*

$$\forall n \in \mathbb{N},\ N_n(\gamma) \overset{law}{=} \widetilde{N}_n\left(\gamma^{-1}\right).$$

Proof. The increments of $(b_n(\gamma))_{n\in\mathbb{N}}$ are denoted by $(g_n(\gamma))_{n\in\mathbb{N}}$ and we recall that for $k \in \mathbb{N}$:

$$g_k(\gamma) := \begin{pmatrix} \gamma_k & 0 \\ 1 & \gamma_k^{-1} \end{pmatrix}.$$

Let w_0 be the matrix

$$w_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $w_0 A_n(\gamma) w_0^{-1} = A_n(\gamma^{-1})$ and $w_0 \left[\mathcal{N}_n(\gamma)\right]^t w_0^{-1} = \mathcal{N}_n(\gamma)$, using NA factorization we get

$$w_0 \left[b_n(\gamma)\right]^t w_0^{-1} = A_n(\gamma^{-1})\mathcal{N}_n(\gamma). \tag{5.1}$$

From AN factorization, we have

$$b_n(\gamma^{-1}) = A_n(\gamma^{-1})\widetilde{\mathcal{N}}_n(\gamma^{-1}). \tag{5.2}$$

Moreover, for all $k \in \mathbb{N}$,

$$w_0[g_k(\gamma)]^t w_0^{-1} = \begin{pmatrix} \gamma_k^{-1} & 0 \\ 1 & \gamma_k \end{pmatrix} = g_k(\gamma^{-1}).$$

From

$$w_0 \left[b_n(\gamma)\right]^t w_0^{-1} = w_0 \left[g_{n-1}(\gamma)\right]^t w_0^{-1} \cdots w_0 \left[g_0(\gamma)\right]^t w_0^{-1} = g_{n-1}(\gamma^{-1}) \cdots g_0(\gamma^{-1}),$$

using that γ is a sequence of i.i.d. random variables,

$$g_{n-1}(\gamma^{-1}) \cdots g_0(\gamma^{-1}) \overset{\text{law}}{=} g_0(\gamma^{-1}) \cdots g_{n-1}(\gamma^{-1})$$

we deduce for all $n \in \mathbb{N}$,

$$w_0 \left[b_n(\gamma)\right]^t w_0^{-1} \overset{\text{law}}{=} b_n\left(\gamma^{-1}\right).$$

Hence, using the equality in law between (5.1) and (5.2), we obtain the equality $N_n(\gamma) \overset{\text{law}}{=} \widetilde{N}_n\left(\gamma^{-1}\right)$ from unicity of the AN factorization. ☐

In the following, we study the convergence of the process $\left(N_n(\gamma)\right)_{n\in\mathbb{N}^*}$.

From the expressions of X_n and Z_n in (2.3) we get for all $n \in \mathbb{N}^*$:

$$N_n(\gamma) = \sum_{k=0}^{n-1} \gamma_k^{-1} \left(\prod_{i=0}^{k-1} \gamma_i^{-1}\right)^2 \text{ and } \widetilde{N}_n(\gamma) = \sum_{k=0}^{n-1} \gamma_k \left(\prod_{j=k+1}^{n-1} \gamma_j\right)^2. \tag{5.3}$$

From the expression (5.3), we obtain

$$N_n(\gamma) = \sum_{k=0}^{n-1} \mathrm{e}^{-\log(\gamma_k) - 2\sum_{i=0}^{k-1} \log(\gamma_i)}. \tag{5.4}$$

Using the law of large number and an exponential decay argument we deduce the almost sure convergence when $\mathbb{E}\left(\log \gamma_0\right) > 0$.

The next proposition which is a classic result for $AX + B$ models (see [30], Cor. 2.1.2) gives the asymptotic distribution for the process $\left(N_n(\gamma)\right)_{n\in\mathbb{N}^*}$. We recall that $AX + B$ models refer to stochastic processes $(\mathcal{X}_n)_{n\in\mathbb{N}}$ satisfying a Kesten's stochastic recurrence equation of the form $\mathcal{X}_{n+1} = A_n \mathcal{X}_n + B_n$ with $n \in \mathbb{N}$.

Proposition 5.2. *Assume that* $\mathbb{E}\left(\log \gamma_0\right) > 0$. *The process* $\left(N_n(\gamma)\right)_{n\in\mathbb{N}^*}$ *converges almost surely, and the law of* $N_\infty(\gamma) := \lim_{n\to+\infty} N_n(\gamma)$ *is equal to the unique invariant probability measure of the process* $(\widetilde{N}_n(\gamma^{-1}))_{n\in\mathbb{N}^*}$.

We will explain in Section 8 that this result follows from a more general result valid for random walks on groups proved by Elie [31].

5.2. The case of GIG distributions

Now, we consider that

$$(\gamma_i)_{i\geq 0} \overset{\text{i.i.d.}}{\sim} GIG(\lambda, a, a).$$

Hence, we have $(\gamma_i^{-1})_{i\geq 0} \overset{\text{i.i.d.}}{\sim} GIG(-\lambda, a, a)$. In that case, we will denote $N_n^{(\lambda)}$ instead of $N_n(\gamma)$ and $\widetilde{N}_n^{(-\lambda)}$ instead of $\widetilde{N}_n\left(\gamma^{-1}\right)$ to simplify the notations. As we will see in next lemma, it is important to properly choose the sign of the parameter λ to ensure the convergence of the corresponding limiting processes according to Proposition 5.2.

Lemma 5.3. *The sign of $\mathbb{E}\left(\log\gamma_0\right)$ is the same as the sign of the parameter λ.*

Proof. By definition of $GIG(\lambda, a, a)$ density:

$$\mathbb{E}\left(\log\gamma_0\right) = \frac{1}{2K_\lambda(a^2)}\int_0^{+\infty} \log(x)x^{\lambda-1}\mathrm{e}^{-\frac{a^2}{2}\left(x+\frac{1}{x}\right)}\mathrm{d}x.$$

The sign of $\mathbb{E}\left(\log\gamma_0\right)$ is the same sign as the integral because the Macdonald function is positive. By cutting the integral into the two parts $]0,1]$ and $]1,+\infty[$ and doing the change of variable $x\mapsto\frac{1}{x}$ on the first one, we obtain

$$\mathbb{E}\left(\log\gamma_0\right) = \frac{1}{2K_\lambda(a^2)}\int_1^{+\infty} \frac{\log(x)}{x}\left(x^\lambda - x^{-\lambda}\right)\mathrm{e}^{-\frac{a^2}{2}\left(x+\frac{1}{x}\right)}\mathrm{d}x.$$

Hence, we deduce the statement of the lemma. □

Remark 5.4. When $\lambda = 0$, $(X_n)_{n\in\mathbb{N}^*}$ is recurrent. Using the law of large numbers, we deduce when $\lambda > 0$ (resp. $\lambda < 0$) that $X_n \overset{\text{a.s.}}{\to} +\infty$ (resp. 0) when $n\to+\infty$.

Now, we obtain the distribution of the asymptotic N-part of the random walk when $\lambda > 0$. As we will see later, $N_\infty^{(\lambda)} := \lim_{n\to+\infty} N_n^{(\lambda)}$ is an infinite sum of random variables whose law cannot be obtained directly. However, it is sufficient to obtain the invariant probability measure for $(\widetilde{N}_n^{(-\lambda)})_{n\in\mathbb{N}^*}$ as we will show.

Proposition 5.5. *The process $(\widetilde{N}_n^{(-\lambda)})_{n\in\mathbb{N}^*}$ is a homogeneous Markov chain starting from $\widetilde{N}_1^{(-\lambda)} = \gamma_0^{-1}$ with transition kernel given for $x > 0$ by*

$$\widetilde{K}(x, \mathrm{d}y) = \frac{1}{2K_\lambda(a^2)\sqrt{1+4xy}}\left(\frac{-1+\sqrt{1+4xy}}{2x}\right)^{-\lambda-1}\mathrm{e}^{-\frac{a^2}{2}\left(\frac{-1+\sqrt{1+4xy}}{2x}+\frac{2x}{-1+\sqrt{1+4xy}}\right)}\mathbb{1}_{\mathbb{R}_+^*}(y)\mathrm{d}y.$$

Moreover if $\lambda > 0$, the Markov chain $(\widetilde{N}_n^{(-\lambda)})_{n\in\mathbb{N}^*}$ is reversible with invariant probability measure

$$d\pi(x) = \frac{a^{2\lambda}}{2^\lambda\Gamma(\lambda)}x^{-\lambda-1}\mathrm{e}^{-\frac{a^2}{2x}}\mathbb{1}_{\mathbb{R}_+^*}(x)\mathrm{d}x \tag{5.5}$$

where Γ is Euler's gamma function. Thus, π is an inverse-gamma distribution.

Proof. From the recurrence formula (2.4) we obtain for all $n\geq 2$:

$$\widetilde{N}_n^{(-\lambda)} = \gamma_{n-1}^{-2}\widetilde{N}_{n-1}^{(-\lambda)} + \gamma_{n-1}^{-1}. \tag{5.6}$$

Then, it is clear that $(\widetilde{N}_n^{(-\lambda)})_{n\in\mathbb{N}^*}$ is a Markov chain since γ_{n-1}^{-1} is independent of $\widetilde{N}_{n-1}^{(-\lambda)}$.

The transition kernel $\widetilde{K}$ is obtained from (5.6) computing the law of $\gamma_0^2 x + \gamma_0$ when γ_0 follows $GIG(-\lambda, a, a)$ distribution and using the symmetric relation $K_\lambda = K_{-\lambda}$.

For the reversibility we consider the following calculation, for $x, y > 0$:

$$\frac{\widetilde{K}(x, \mathrm{d}y)}{\widetilde{K}(y, \mathrm{d}x)} = \left(\frac{y}{x}\right)^{-\lambda-1} \mathrm{e}^{-\frac{a^2}{2}\left(\frac{-1+\sqrt{1+4xy}}{2x} + \frac{2x}{-1+\sqrt{1+4xy}} - \frac{-1+\sqrt{1+4xy}}{2y} - \frac{2y}{-1+\sqrt{1+4xy}}\right)} \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Multiplying the numerator and the denominator of the second and fourth term in the argument of the exponential by $1 + \sqrt{1 + 4xy}$, it leads to simplification:

$$\frac{\widetilde{K}(x, \mathrm{d}y)}{\widetilde{K}(y, \mathrm{d}x)} = \left(\frac{y}{x}\right)^{-\lambda-1} \mathrm{e}^{-\frac{a^2}{2}\left(\frac{1}{y} - \frac{1}{x}\right)} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k(y)\mathrm{d}y}{k(x)\mathrm{d}x}$$

where k is defined by $k(x) := x^{-\lambda-1}\mathrm{e}^{-\frac{a^2}{2x}}\mathbb{1}_{\mathbb{R}_+^*}(x)$. If $\lambda > 0$, the integral of k on $\mathbb{R}_+$ converges, so we normalize k to obtain an invariant probability measure by:

$$d\pi(x) = \frac{k(x)}{\int_{\mathbb{R}_+} k(y)\mathrm{d}y}\mathrm{d}x = \frac{a^{2\lambda}}{2^\lambda \Gamma(\lambda)} x^{-\lambda-1}\mathrm{e}^{-\frac{a^2}{2x}}\mathbb{1}_{\mathbb{R}_+^*}(x)\mathrm{d}x.$$

□

From the expressions of $N_n^{(\lambda)}$ in (5.3) and using Proposition 5.2 and Proposition 5.5, it follows:

Theorem 5.6 (Dufresne identity)**.** *For $\lambda > 0$, the law of the random variable*

$$N_\infty^{(\lambda)} = \sum_{k=0}^{+\infty} \gamma_k^{-1} \left(\prod_{i=0}^{k-1} \gamma_i^{-1}\right)^2$$

is the inverse-gamma distribution π defined by (5.5)*.*

6. A DISCRETE-TIME RECONSTRUCTION THEOREM

In this section, we give a discrete-time reconstruction theorem in the sense that we have formulas to recover $(X_n)_{n\in\mathbb{N}}$ from $(Z_n)_{n\in\mathbb{N}}$.

Remark 6.1. Let $\lambda > 0$. From Remark 5.4, we deduce that $X_n \overset{\text{a.s.}}{\to} +\infty$ when $n \to +\infty$.
Since $Z_n = X_n N_n^{(\lambda)}$, it follows from Proposition 5.2 that $Z_n \overset{\text{a.s.}}{\to} +\infty$ when $n \to +\infty$.

Proposition 6.2. *Let $\lambda > 0$. The process $(Z_n)_{n\in\mathbb{N}^*}$ is independent of the random variable $N_\infty^{(\lambda)}$.*

Proof. Let $k \geq 2$ fixed and let $h : \mathbb{R}^{k-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two bounded continuous functions. By Proposition 5.2 and Lemma 5.3 one has $\lim_{n\to+\infty} N_n^{(\lambda)} = N_\infty^{(\lambda)}$ a.s. It follows that:

$$\begin{aligned}
\mathbb{E}\left(g(N_\infty^{(\lambda)})h(Z_2, \cdots, Z_k)\right) &= \lim_{n\to+\infty} \mathbb{E}\left(g(N_n^{(\lambda)})h(Z_2, \cdots, Z_k)\right) \\
&= \lim_{n\to+\infty} \mathbb{E}\left(\mathbb{E}\left(g(X_n^{-1}Z_n)h(Z_2, \cdots, Z_k)|Z_n, \cdots, Z_2\right)\right) \\
&= \lim_{n\to+\infty} \mathbb{E}\left(h(Z_2, \cdots, Z_k)\mathbb{E}\left(g(X_n^{-1}Z_n)|Z_n, \cdots, Z_2\right)\right).
\end{aligned}$$

From Proposition 3.2 or directly from Proposition 3.5, we deduce:

$$\mathbb{E}\left(g(X_n^{-1}Z_n)|Z_n,\cdots,Z_2\right)=\frac{1}{2K_\lambda(\frac{a^2}{Z_n})}\int_0^{+\infty}g(x^{-1}Z_n)x^{\lambda-1}\mathrm{e}^{-\frac{a^2}{2Z_n}\left(x+\frac{1}{x}\right)}\mathrm{d}x$$
$$=\frac{Z_n^\lambda}{2K_\lambda(\frac{a^2}{Z_n})}\int_0^{+\infty}g(u)u^{-\lambda-1}\mathrm{e}^{-\frac{a^2}{2Z_n}\left(\frac{Z_n}{u}+\frac{u}{Z_n}\right)}du.$$

where the second equality is obtained by a simple change of variable. The asymptotic when $z\to 0$ and $\lambda>0$ of Macdonald function is given by

$$K_\lambda(z)\sim\frac{1}{2}\Gamma(\lambda)\left(\frac{z}{2}\right)^{-\lambda},$$

see [32] formula 9.6.9. which can be obtained from the asymptotic expansion of modified Bessel function of first kind and its relation with second kind. Thus, thanks to Remark 6.1 we obtain:

$$\lim_{n\to+\infty}\frac{Z_n^\lambda}{2K_\lambda(\frac{a^2}{Z_n})}=\frac{a^{2\lambda}}{2^\lambda\Gamma(\lambda)}\ \text{a.s.}$$

Finally, from Dufresne's identity (Thm. 5.6) it follows:

$$\lim_{n\to+\infty}\mathbb{E}\left(g(X_n^{-1}Z_n)|Z_n,\cdots,Z_2\right)=\mathbb{E}\left(g(N_\infty^{(\lambda)})\right).$$

We get,

$$\mathbb{E}\left(g(N_\infty^{(\lambda)})h(Z_2,\cdots,Z_k)\right)=\mathbb{E}\left(g(N_\infty^{(\lambda)})\right)\mathbb{E}\left(h(Z_2,\cdots,Z_k)\right)$$

and the fact that the process $(Z_n)_{n\in\mathbb{N}^*}$ is independent of the random variable $N_\infty^{(\lambda)}$. ☐

The following identity is easily established by induction on p.

Proposition 6.3. *For all $n,p\geq 0$,*

$$X_n=\frac{Z_n}{N_{n+p}^{(\lambda)}}+Z_n\sum_{k=1}^p\frac{1}{Z_{n+k-1}Z_{n+k}}.\tag{6.1}$$

Factorizing by $\frac{Z_n}{N_{n+p}^{(\lambda)}}$ and then taking the logarithm, we deduce from the previous equality that for all $n,p\geq 0$:

$$\log(X_n)=\log\left(\frac{Z_n}{N_{n+p}^{(\lambda)}}\right)+\log\left(1+N_{n+p}^{(\lambda)}\sum_{k=1}^p\frac{1}{Z_{n+k-1}Z_{n+k}}\right).\tag{6.2}$$

This last equality can be interpreted as a reconstruction theorem. Indeed, let us recall the inversion of Pitman transform in the simplest case. We refer to [3] (Prop. 2.2) for a more general statement. Let us consider $\pi:[0,T]\to\mathbb{R}$ a continuous function starting from 0, *i.e.* $\pi(0)=0$. Let us define the Pitman transform $\mathcal{P}$ of π by the formula, for all $t\in[0,T]$:

$$\mathcal{P}\pi(t):=\pi(t)-2\inf_{0\leq s\leq t}\pi(s).$$

Then, the continuous function π can be recovered from the Pitman transform through the formula adding the information $\xi := -\inf_{0\le s\le T}\pi(s)$, for all $t\in[0,T]$:

$$\pi(t) := \mathcal{P}\pi(t) - 2\min\left(\xi,\ \inf_{t\le s\le T}\mathcal{P}\pi(s)\right). \tag{6.3}$$

The formula (6.2) is a geometric discrete-time analog of (6.3).

Theorem 6.4. *When $\lambda>0$ and for all $n\ge 0$,*

$$\log(X_n) = \log\left(\frac{Z_n}{N_\infty^{(\lambda)}}\right) + \log\left(1 + N_\infty^{(\lambda)}\sum_{k=1}^{+\infty}\frac{1}{Z_{n+k-1}Z_{n+k}}\right). \tag{6.4}$$

When $\lambda\le 0$ and for all $n\ge 0$,

$$\log(X_n) = \log(Z_n) + \log\left(\sum_{k=1}^{+\infty}\frac{1}{Z_{n+k-1}Z_{n+k}}\right). \tag{6.5}$$

Proof. When $\lambda>0$, letting p going to $+\infty$ thanks to Proposition 5.2 and Lemma 5.3, we obtain (6.4) from (6.2). In the case $\lambda\le 0$ we get that $\lim_{p\to+\infty}N_{n+p}^{(\lambda)} = +\infty$ a.s. using (5.4) and Remark 5.4, so in formula (6.1) taking the logarithm we obtain (6.5) □

The result (6.4) corresponds to the geometric analog, when π has a positive drift, of the following formula with $\xi' := -\inf_{0\le s}\pi(s)$, for all $t\ge 0$:

$$\pi(t) := \mathcal{P}\pi(t) - 2\min\left(\xi',\ \inf_{t\le s}\mathcal{P}\pi(s)\right). \tag{6.6}$$

The result (6.5) corresponds to the case $\inf_{0\le s}\pi(s) = -\infty$ when π has nonpositive drift of the formula (6.6) which becomes, for all $t\ge 0$:

$$\pi(t) := \mathcal{P}\pi(t) - 2\inf_{t\le s}\mathcal{P}\pi(s). \tag{6.7}$$

7. Convergence towards the continuous-time Matsumoto–Yor process

In this section we prove that for $T>0$, the sequence of processes $(b_{\lfloor nt\rfloor}^{(\delta_n)}, 0\le t\le T)_{n\in\mathbb{N}^*}$ converges weakly towards the continuous process (1.4). Here $\delta_n = \frac{1}{n}$ and the parameters of the GIG law for the increments will be dependent of n.

Let $(\gamma_j^{(a)})_{j\ge 0}$ be a sequence of independent and identically distributed random variables with the law $GIG(\lambda, a, a)$ and $S_n^{(a)}$ the random walk defined for all $n\ge 1$ by:

$$S_n^{(a)} := \sum_{j=0}^{n-1}\log\gamma_j^{(a)}.$$

Then, $t\in[0,T]\mapsto S_{\lfloor nt\rfloor}^{(a)}$ is a random variable with values in the Skorokhod space $\mathcal{D}([0,T],\mathbb{R})$.

To study the convergence of this random variable, we will use asymptotic formulas for the first moments of the law log(GIG).

Proposition 7.1. *Let $m \in \mathbb{N}$. When $a \to +\infty$,*

$$\mathbb{E}\left(\log^m \gamma_0^{(a)}\right) \sim \begin{cases} \frac{2^{\frac{m}{2}}}{a^m\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right) & \textit{if } m \textit{ is even} \\ \frac{\lambda 2^{\frac{m+1}{2}}}{a^{m+1}\sqrt{\pi}}\Gamma\left(\frac{m+2}{2}\right) & \textit{if } m \textit{ is odd.} \end{cases} \tag{7.1}$$

Proof. Using definition of the law $GIG(\lambda, a, a)$:

$$\mathbb{E}\left(\log^m \gamma_0^{(a)}\right) = \frac{1}{2K_\lambda(a^2)}\int_0^{+\infty} \log^m(x)x^{\lambda-1}\mathrm{e}^{-\frac{a^2}{2}(x+\frac{1}{x})}\mathrm{d}x.$$

The same calculation as in the proof of Lemma 5.3 gives:

$$2K_\lambda(a^2)\mathbb{E}\left(\log^m \gamma_0^{(a)}\right) = \int_1^{+\infty} \log^m(x)\left(x^\lambda + (-1)^m x^{-\lambda}\right)\mathrm{e}^{-\frac{a^2}{2}(x+\frac{1}{x})}x^{-1}\mathrm{d}x.$$

Putting $t = \frac{(x+x^{-1})}{2} - 1$ we obtain,

$$\mathrm{e}^{a^2}K_\lambda(a^2)\mathbb{E}\left(\log^m \gamma_0^{(a)}\right) = \int_0^{+\infty} f_m(t)\mathrm{e}^{-a^2t}\mathrm{d}t$$

where,

$$f_m(t) := \begin{cases} \frac{\mathrm{Argch}^m(t+1)\cosh(\lambda \mathrm{Argch}(t+1))}{\sinh(\mathrm{Argch}(t+1))} & \text{if } m \text{ is even} \\ \frac{\mathrm{Argch}^m(t+1)\sinh(\lambda \mathrm{Argch}(t+1))}{\sinh(\mathrm{Argch}(t+1))} & \text{if } m \text{ is odd.} \end{cases}$$

Now, it is easy to see that f_m is exponentially bounded, *i.e.* there exists $b_m \in \mathbb{R}$ for $m \in \mathbb{N}$ such that:

$$f_m(t) = O\left(\mathrm{e}^{b_m t}\right) \text{ when } t \to +\infty.$$

Therefore we have the following expansions when $t \to 0$,

$$f_m(t) \sim \begin{cases} (2t)^{\frac{m-1}{2}} & \text{if } m \text{ is even} \\ \lambda(2t)^{\frac{m}{2}} & \text{if } m \text{ is odd.} \end{cases}$$

Applying Lemma A.1, we obtain the asymptotic formulas for $K_\lambda(a^2)$ and for the moments when $a \to +\infty$. ☐

Corollary 7.2. *For all $T > 0$, the sequence of processes*

$$\left(S_{\lfloor nt \rfloor}^{(\sqrt{n})}, 0 \le t \le T\right)_{n\in\mathbb{N}^*}$$

converges weakly in $\mathcal{D}([0,T],\mathbb{R})$ equipped with Skorokhod distance, towards the drifted Brownian motion $(B_t^{(\lambda)} := B_t + \lambda t, t \ge 0)$ where $(B_t, t \ge 0)$ is a standard Brownian motion.

Proof. First of all, we prove the convergence in the sense of finite dimensional distributions.

First, we prove the convergence:

$$S_{\lfloor nt \rfloor}^{(\sqrt{n})} \overset{\text{law}}{\to} \mathbf{N}_t^{(\lambda)}$$

where for $t > 0$, $\mathbf{N}_t^{(\lambda)} \sim \mathcal{N}(t\lambda, t)$ is a normal distribution with mean $t\lambda$ and variance t.

We consider Lindeberg's theorem (Thm. 27.3 in [33]) on the triangular array with the random variables $V_{n,j}$ defined by:

$$V_{n,j} := \frac{\log \gamma_j^{(\sqrt{n})} - \mathbb{E}\left(\log \gamma_j^{(\sqrt{n})}\right)}{\sqrt{\sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbb{V}\left(\log \gamma_j^{(\sqrt{n})}\right)}} = \frac{\log \gamma_j^{(\sqrt{n})} - \mathbb{E}\left(\log \gamma_0^{(\sqrt{n})}\right)}{\sqrt{\lfloor nt \rfloor \mathbb{V}\left(\log \gamma_0^{(\sqrt{n})}\right)}}.$$

In order to apply the theorem, we establish the following Lyapounov's condition:

$$\lim_{n \to +\infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \mathbb{E} \left| V_{n,j} \right|^4 = 0.$$

Using the inequality $|a+b|^4 \leq 2^4(|a|^4 + |b|^4)$, we get:

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} \mathbb{E} \left| V_{n,j} \right|^4 = \frac{\mathbb{E}\left|\log \gamma_0^{(\sqrt{n})} - \mathbb{E}\left(\log \gamma_0^{(\sqrt{n})}\right)\right|^4}{\lfloor nt \rfloor \mathbb{V}^2\left(\log \gamma_0^{(\sqrt{n})}\right)} \leq \frac{2^5 \mathbb{E}\left(\log^4 \gamma_0^{(\sqrt{n})}\right)}{\lfloor nt \rfloor \mathbb{V}^2\left(\log \gamma_0^{(\sqrt{n})}\right)}.$$

Thanks to (7.1), the right-hand side of the inequality is asymptotically $O\left(\frac{1}{n}\right)$ when $n \to +\infty$, so the Lyapounov condition holds. By Lindeberg's theorem, we obtain when $n \to +\infty$ that:

$$\sum_{j=0}^{\lfloor nt \rfloor - 1} V_{n,j} \overset{\text{law}}{\to} \mathbf{N}_1^{(0)} \sim \mathcal{N}(0,1).$$

From (7.1) it follows when $n \to +\infty$:

$$S_{\lfloor nt \rfloor}^{(\sqrt{n})} = \sqrt{\lfloor nt \rfloor \mathbb{V}\left(\log \gamma_0^{(\sqrt{n})}\right)} \left(\sum_{j=0}^{\lfloor nt \rfloor - 1} V_{n,j}\right) + \lfloor nt \rfloor \mathbb{E}\left(\log \gamma_0^{(\sqrt{n})}\right) \overset{\text{law}}{\to} \mathbf{N}_t^{(\lambda)}.$$

Using the fact that $S_{\lfloor nt_{k-1} \rfloor}^{(\sqrt{n})}$ and $S_{\lfloor nt_k \rfloor}^{(\sqrt{n})} - S_{\lfloor nt_{k-1} \rfloor}^{(\sqrt{n})}$ are independent, we obtain for all $k \in \mathbb{N}^*$ and for all $0 < t_1 < t_2 < \cdots < t_k < T$ that:

$$\left(S_{\lfloor nt_1 \rfloor}^{(\sqrt{n})}, S_{\lfloor nt_2 \rfloor}^{(\sqrt{n})}, \ldots, S_{\lfloor nt_k \rfloor}^{(\sqrt{n})}\right) \overset{\text{law}}{\to} \left(\mathbf{N}_{t_1}^{(\lambda)}, \mathbf{N}_{t_2}^{(\lambda)}, \ldots, \mathbf{N}_{t_k}^{(\lambda)}\right)$$

which proves the convergence in finite dimensional distributions. The tightness is obtained in the same way as in Donsker's theorem. We refer to [34] (Thm. 14.1) for the proof for the Brownian motion without drift. □

Theorem 7.3. *Let $t > 0$ fixed, we have the following convergence for the random walk (2.2) with $\delta_n := \frac{1}{n}$ and $\gamma_i^{(\sqrt{n})} \sim GIG(\lambda, \sqrt{n}, \sqrt{n})$, when $n \to +\infty$:*

$$b_{\lfloor nt \rfloor}^{(\delta_n)} = \begin{pmatrix} X_{\lfloor nt \rfloor} & 0 \\ Z_{\lfloor nt \rfloor}^{(\delta_n)} & X_{\lfloor nt \rfloor}^{-1} \end{pmatrix} \overset{law}{\to} \begin{pmatrix} e^{B_t^{(\lambda)}} & 0 \\ e^{B_t^{(\lambda)}} \int_0^t e^{-2B_s^{(\lambda)}} ds & e^{-B_t^{(\lambda)}} \end{pmatrix}.$$

Proof. From Corollary 7.2 and by continuity of the exponential function, it is clear that $X_{\lfloor nt \rfloor} \overset{\text{law}}{\to} \mathrm{e}^{B_t^{(\lambda)}}$ and $X_{\lfloor nt \rfloor}^{-1} \overset{\text{law}}{\to} \mathrm{e}^{-B_t^{(\lambda)}}$ when $n \to +\infty$. Let $t > 0$ fixed, we have

$$\begin{aligned}
&\mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \int_0^t \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds \\
&\quad = \mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \left(\sum_{k=0}^{\lfloor nt \rfloor - 1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds + \int_{\frac{\lfloor nt \rfloor}{n}}^{t} \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds \right) \\
&\quad = \mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \left(\frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathrm{e}^{-2S_k^{(\sqrt{n})} - \log \gamma_k^{(\sqrt{n})}} + \left(t - \frac{\lfloor nt \rfloor}{n} \right) \mathrm{e}^{-2S_{\lfloor nt \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor nt \rfloor}^{(\sqrt{n})}} \right) \\
&\quad = Z_{\lfloor nt \rfloor}^{(\delta_n)} + \left(t - \frac{\lfloor nt \rfloor}{n} \right) \mathrm{e}^{-S_{\lfloor nt \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor nt \rfloor}^{(\sqrt{n})}}.
\end{aligned}$$

Then, it follows the convergence in probability when $n \to +\infty$:

$$\left| \mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \int_0^t \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds - Z_{\lfloor nt \rfloor}^{(\delta_n)} \right| \overset{\mathbb{P}}{\to} 0. \tag{7.2}$$

Therefore we have

$$\mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \int_0^t \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds = \int_0^t \mathrm{e}^{\sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor - 1} \log \gamma_k^{(\sqrt{n})} - S_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds.$$

Slightly adapting the proof of Corollary 7.2 we get that $\left(\sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor - 1} \log \gamma_k^{(\sqrt{n})}, 0 \le s \le t \right)_{n \in \mathbb{N}^*}$ converges weakly towards $\left(B_{t-s}^{(\lambda)}, 0 \le s \le t \right)$. Hence, using the independence between $S_{\lfloor ns \rfloor}^{(\sqrt{n})}$ and $\sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor - 1} \log \gamma_k^{(\sqrt{n})}$ and by continuity of the map $I : \mathcal{D}([0,T], \mathbb{R}) \to \mathbb{R}$ defined by the integral $I(f) := \int_0^t f(s) ds$ we obtain, when $n \to +\infty$:

$$\mathrm{e}^{S_{\lfloor nt \rfloor}^{(\sqrt{n})}} \int_0^t \mathrm{e}^{-2S_{\lfloor ns \rfloor}^{(\sqrt{n})} - \log \gamma_{\lfloor ns \rfloor}^{(\sqrt{n})}} ds \overset{\text{law}}{\to} \int_0^t \mathrm{e}^{B_{t-s}^{(\lambda)} - B_s^{(\lambda)}} ds.$$

Finally, using the stationary increments property of Brownian motion and (7.2), when $n \to +\infty$ we get that

$$Z_{\lfloor nt \rfloor}^{(\delta_n)} \overset{\text{law}}{\to} \mathrm{e}^{B_t^{(\lambda)}} \int_0^t \mathrm{e}^{-2B_s^{(\lambda)}} ds.$$

□

Remark 7.4. Let G be the subgroup of lower triangular matrices in SL_2. We obtain with standard arguments for tightness that the sequence of processes $(b_{\lfloor nt \rfloor}^{(\delta_n)}, 0 \le t \le T)_{n \in \mathbb{N}^*}$ converges weakly in $\mathcal{D}([0,T], G)$ towards the continuous process (1.4).

8. CONVERGENCE OF THE N-PART TOWARDS STATIONARY MEASURE

In this section, we give some elements of the theory of random walks on groups to state a result from [31] using the formalism of [35] to give another point of view on Proposition 5.2.

Let G be the subgroup of SL_2 consisting of lower triangular matrices, defined by

$$G := \left\{ \begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^*, z \in \mathbb{R} \right\}.$$

Each element $h \in G$ can be decomposed as $h = n(h)a(h)$ or $h = a(h)\widetilde{n}(h)$ where for the matrix $h = \begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix}$, we defined:

$$n(h) := \begin{pmatrix} 1 & 0 \\ zx^{-1} & 1 \end{pmatrix}, \ \widetilde{n}(h) := \begin{pmatrix} 1 & 0 \\ zx & 1 \end{pmatrix} \text{ and } a(h) := \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}.$$

Here, G is a solvable Lie group such that $G = NA$ where A is an abelian subgroup and N is a nilpotent subgroup of G with

$$N := \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} \text{ and } A := \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{R}^* \right\}.$$

Moreover, G has a semidirect product structure defined on the set $N \times A$ with product law given by

$$h_1 h_2 = (n_1, a_2)(n_2, a_2) = (n_1(a_1 \odot n_2), a_1 a_2)$$

where $n_i := n(h_i)$, $a_i := a(h_i)$ and $\odot$ is the group action of A on N by inner automorphisms in G given, for $a \in A$ and $n \in N$, by

$$a \odot n = ana^{-1},$$

this means,

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ x^{-2}u & 1 \end{pmatrix}.$$

We obtain that G acts on N by

$$\begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} \diamond \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^{-2}u + zx^{-1} & 1 \end{pmatrix}. \tag{8.1}$$

Let $\delta \in \mathbb{R}^*$ be a deterministic parameter. Recall that

$$g_n := \begin{pmatrix} \gamma_n & 0 \\ \delta & \gamma_n^{-1} \end{pmatrix} \tag{8.2}$$

are some i.i.d. random elements and let us denote by μ their law in the group G. Here, we denote by g_n instead of $g_n^{(\delta)}$ the increments of the random walk $(b_n^{(\delta)})_{n \in \mathbb{N}}$ to simplify notation.

Then, we obtain by the recurrence relation (2.1), for all $n \in \mathbb{N}^*$:

$$b_n^{(\delta)} = g_0 g_1 \cdots g_{n-1}.$$

With the previous definitions, the abelian part of the random walk $(b_n^{(\delta)})_{n\in\mathbb{N}}$ is

$$a(b_n^{(\delta)}) = \exp\left(S_n\right) \text{ with } S_n := \sum_{i=0}^{n-1} \log(a(g_i)).$$

Let $\mathcal{N}_n := n(b_n^{(\delta)})$ be the N-part of the random walk in the NA factorization of G.

We assume that μ has a finite first log-moment and we denote by

$$\kappa := \mathbb{E}\left(\log a(g_0)\right) \tag{8.3}$$

the mean of the increments of S_n. Then, the mean of the random walk S_n is $n\kappa$. As we will see later, the convergence of the process $(\mathcal{N}_n)_{n\in\mathbb{N}}$ depends on the parameter κ.

Let α be the unique simple root for the group G. That is to say, if $d := \operatorname{diag}(a_1, a_2)$ we get

$$\alpha(d) = a_2 - a_1.$$

Hence, we obtain

$$\alpha(\kappa) = -2\mathbb{E}\left(\log \gamma_0\right). \tag{8.4}$$

Definition 8.1. A probability measure μ on a group G is said to be spread-out if there exists an integer p such that μ^{*p} is not singular with respect to a Haar measure on G.

Remark 8.2. As it is shown in [36], μ is spread-out on G if and only if there exists an integer q such that μ^{*q} dominates a multiple of a Haar measure on a non-empty open subset of the group G. Moreover, here a right Haar measure for the group G is given by

$$d_H \begin{pmatrix} x & 0 \\ z & x^{-1} \end{pmatrix} = \frac{\mathrm{d}x \cdot \mathrm{d}z}{x^2}. \tag{8.5}$$

Definition 8.3. Let H be a group acting on a locally compact space B by $H \times B \to B$, $(x, b) \mapsto x \cdot b$. The convolution of a probability measure μ on H with a probability measure ν on B is defined by

$$\mu * \nu(\phi) := \int_{H\times B} \phi(x \cdot b) d\mu(x) d\nu(b).$$

A measure ν on B will be called μ-stationary, if it satisfies

$$\mu * \nu = \nu.$$

Here, G acts on N by (8.1) and we consider the μ-stationary measure on N.

The following proposition is already included in the work of Elie [31] but we give the statement according to the work of Babillot [35] (Thm. 5.11) in the particular case of our random walk $(b_n^{(\delta)})_{n\in\mathbb{N}}$.

Proposition 8.4. *Assume that μ the probability measure defined in (8.2) is spread-out on $G = NA$ with finite first* log*-moment. Let κ defined by (8.3) such that $\alpha(\kappa) < 0$ (contractive mean). Then, the N-component $\mathcal{N}_n$ of $b_n^{(\delta)}$ converges almost surely, and the law ν of $\mathcal{N}_\infty = \lim_{n\to+\infty} \mathcal{N}_n$ is the unique μ-stationary measure on $N \simeq G/A$.*

Thanks to Proposition 8.4 we recover Proposition 5.2. Indeed, from (8.4), if $\mathbb{E}\left(\log \gamma_0\right) > 0$, then $\alpha(\kappa) < 0$. Moreover, N is identified with the homogeneous space G/A (which is locally compact) with origin $o = A$, so we consider Definition 8.3 with $H = G$ and $B = N$. From the recurrence formula (2.4) we obtain for all $n \geq 2$:

$$\widetilde{N}_n(\gamma^{-1}) = \gamma_{n-1}^{-2} \widetilde{N}_{n-1}(\gamma^{-1}) + \gamma_{n-1}^{-1}. \tag{8.6}$$

Taking $z = \delta = 1$ in the formula (8.1), we recover from (8.6) that the μ-stationary measure on N corresponds to the invariant probability measure of the process $(\widetilde{N}_n(\gamma^{-1}))_{n\in\mathbb{N}^*}$.

We can use Proposition 5.2 when we consider GIG laws. Indeed, the sign of $\alpha(\kappa) = -2\mathbb{E}\left(\log \gamma_0\right)$ is the same as $-\lambda$ due to Lemma 5.3. Moreover, the probability measure μ (8.2) for the increments g_i is spread-out when γ_i follows GIG law and δ follows Dirac distribution $\mathbb{1}_{\delta=1}$. For this, thanks to Remark 8.2 it is sufficient to find a power q such that μ^{*q} dominates a multiple of Haar measure (8.5). In fact, $q = 2$ works since μ^{*2} is given by the law of the random matrix:

$$\begin{pmatrix} \gamma_0\gamma_1 & 0 \\ \gamma_0^{-1} + \gamma_1 & \gamma_0^{-1}\gamma_1^{-1} \end{pmatrix}$$

and the pair $(\gamma_0\gamma_1, \gamma_0^{-1} + \gamma_1)$ has a density relative to Haar measure.

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Data availability statement

No new data/codes were created or analyzed in this study.

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Appendix A.

The following lemma is a slightly more general version of Watson's lemma originally proved in 1918 in [37]. We refer to [38] Section 4.1 for a modern proof of the statement below.

Lemma A.1 (Watson)**.** *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a locally integrable function satisfying two conditions:* (1) *There exists $b \in \mathbb{R}$ such that $f(t) = O(\mathrm{e}^{bt})$ when $t \to +\infty$.*
(2) *We have the following asymptotic expansion when $t \to 0^+$:*

$$f(t) \sim \sum_{n=0}^{+\infty} c_n t^{a_n}$$

where $(\Re(a_n))_{n\in\mathbb{N}}$ increases monotonically to $+\infty$ and such that $\Re(a_0) > -1$.
Then, we have the asymptotic expansion for the Laplace transform:

$$I(x) := \int_0^{+\infty} f(t)\mathrm{e}^{-xt}\mathrm{d}t \sim \sum_{n=0}^{+\infty} c_n \frac{\Gamma(a_n+1)}{x^{a_n+1}} \text{ when } x \to +\infty.$$

Lemma A.2. *The only continuous solutions on $\mathbb{R}_+$ of the functional equation:*

$$(4x^2-1)\varphi(2x) + (1-x^2)\varphi(x) = 3x^2\varphi(2)$$

which also satisfy the equation $\varphi(x) = \varphi\left(\frac{1}{x}\right)$ are constants.

Proof. Let us set $\beta(x) := \frac{x^2-4}{4(x^2-1)}$ and $\alpha(x) := \frac{3x^2}{4(x^2-1)}$. Replacing x by $\frac{x}{2}$ in the functional equation, we obtain for all $x \in \mathbb{R}_+ \setminus \{1\}$

$$\varphi(x) = \beta(x)\varphi\left(\frac{x}{2}\right) + \alpha(x)\varphi(2).$$

Iterating this equality, we get:

$$\varphi(x) = \left(\prod_{k=0}^{n} \beta\left(\frac{x}{2^k}\right)\right) \varphi\left(\frac{x}{2^{n+1}}\right) + \varphi(2) \sum_{k=0}^{n} \alpha\left(\frac{x}{2^k}\right) \prod_{j=0}^{k-1} \beta\left(\frac{x}{2^j}\right). \tag{A.1}$$

Furthermore, we get for all $k \geq 0$: $\beta\left(\frac{x}{2^k}\right) = \frac{(x-2^{k+1})(x+2^{k+1})}{4(x-2^k)(x+2^k)}$. Hence, we obtain

$$\forall n \geq 0: \ \prod_{k=0}^{n} \beta\left(\frac{x}{2^k}\right) = \frac{x^2 - 4^{n+1}}{4^{n+1}(x^2-1)}.$$

Thus, on the one hand,

$$\lim_{n \to +\infty} \prod_{k=0}^{n} \beta\left(\frac{x}{2^k}\right) = -\frac{1}{x^2-1}$$

and on the other hand,

$$\lim_{n \to +\infty} \sum_{k=0}^{n} \alpha\left(\frac{x}{2^k}\right) \prod_{j=0}^{k-1} \beta\left(\frac{x}{2^j}\right) = \lim_{n \to +\infty} \frac{3x^2}{4(x^2-1)} \sum_{k=0}^{n} \left(\frac{1}{4}\right)^k = \frac{x^2}{x^2-1}.$$

Thus, letting n going to $+\infty$ in (A.1), we obtain by continuity

$$\forall x \in \mathbb{R}_+ \setminus \{1\}, \ \varphi(x) = -\frac{1}{x^2-1}\varphi(0) + \frac{x^2}{x^2-1}\varphi(2). \tag{A.2}$$

This last equality combined to the equality $\varphi(x) = \varphi\left(\frac{1}{x}\right)$ gives $\varphi(0) = \varphi(2)$. Substituting this into (A.2) we get: $\forall x \in \mathbb{R}_+ \setminus \{1\}, \ \varphi(x) = \varphi(2)$. By continuity this last equality holds for all $x \in \mathbb{R}_+$. $\square$