

KARHUNEN–LOÈVE EXPANSION OF RANDOM MEASURES

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Abstract. We present an orthogonal expansion for real, function-regulated, second-order random measures over \mathbb{R}^d with measure covariance. Such an expansion, which can be seen as a Karhunen–Loève decomposition, consists in a series of deterministic real measures weighted by uncorrelated real random variables with the variances forming a convergent series. The convergence of the series is in a mean-square sense stochastically and against measurable and bounded test functions (with compact support if the random measure is not finite) in the measure sense, which implies set-wise convergence. This is proven taking advantage of the extra requirement of having a covariance measure over $\mathbb{R}^d \times \mathbb{R}^d$ describing the covariance structure of the random measure, for which we also provide a series expansion. These results cover for instance the cases of Gaussian White Noise, Poisson and Cox point processes, and can be used to obtain expansions for trawl processes.

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1. INTRODUCTION

Karhunen–Loève (KL) expansions are an important tool for the analysis of stochastic processes, both in theory and practice. In a general non-rigorous manner, a KL expansion consists in a series representation for a random object X taking values in a (real) vector space E , the representation being of the form

$$X = \sum_n X_n e_n, \quad (1.1)$$

where $(X_n)_n$ is a collection of uncorrelated real random variables with $\sum_n \sigma_{X_n}^2 < \infty$, $\sigma_{X_n}^2 = \text{Var}(X_n)$, and $(e_n)_n$ is a linearly independent collection of vectors in E . KL expansions are a theoretical basis for dimensionality reduction in functional data analysis [1] and for spectral approximations in stochastic PDE analysis [2]. When E is finite-dimensional, these expansions are commonly known as principal components analysis. When E is infinite-dimensional, the most studied case is when E is a separable Hilbert space, for which the vectors $(e_n)_n$ form an orthonormal basis [3]. The most basic scenario is when $X = (X(t))_{t \in [a,b]}$ is a mean-square continuous real stochastic process over a compact interval $[a, b] \subset \mathbb{R}$. In this case $E = L^2([a, b])$ is used as the underlying Hilbert space, which contains the continuous functions. The convergence of the series (1.1) must be specified, both in a stochastic sense as a series of random objects, and in the sense of the space E for which an appropriate topology must be specified. The stochastic convergence of KL expansions is taken to be in mean-square. The

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case with E Hilbert provides a direct topology on E (either the norm or weak topologies may be used). However, in some cases one can prove a stronger mode of convergence than the one associated to the Hilbert space. For example, for a mean-square continuous stochastic process over $[a, b]$, one can take $E = L^2([a, b])$, but Mercer's Theorem allows to conclude a stronger uniform-over- $[a, b]$ -mean-square convergence. We refer to [4–6] for the theory of KL expansions and their importance on functional probabilistic modelling.

This work focuses on KL expansions for random measures of a certain, very general kind. Precisely, Theorems 4.1 and 4.10 show that if M is a (zero mean) *measure-covariance second-order random measure* over \mathbb{R}^d that is function-regulated (i.e., M/f is finite for a given strictly positive function f , the case $f = 1$ covering the case of finite measures), then M admits the expansion

$$M = \sum_{n \in \mathbb{N}} X_n \mu_n, \quad (1.2)$$

where $(X_n)_{n \in \mathbb{N}}$ is a sequence of uncorrelated random variables with $\sum_n \sigma_{X_n}^2 < \infty$, and, for the indexes n such that $\sigma_{X_n}^2 > 0$, $(\mu_n)_n$ form a collection of linearly independent real measures.¹ Theorem 4.1 covers the case of M finite, which is where the real mathematical difficulty is present. Theorem 4.10 covers the function-regulated case, and it is essentially a corollary of Theorem 4.1. The convergence of (1.2) is in the sense

$$\mathbb{E}(|\langle M, \varphi \rangle - \sum_{j \leq n} X_j \langle \mu_j, \varphi \rangle|^2) \xrightarrow{n \rightarrow \infty} 0, \quad (1.3)$$

for every φ measurable and bounded when M is finite and every φ measurable, bounded and compactly supported when M is function-regulated. Here $\langle M, \varphi \rangle$ denotes the integral $\int_{\mathbb{R}^d} \varphi dM$.

The concept of a *measure-covariance second-order random measure* deserves an explanation. First, we focus on a second-order random measure, which is a stochastic process indexed by bounded Borel sets, $M = (M(A))_{A \in \mathcal{B}_B(\mathbb{R}^d)}$ such that $M(A) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ for every A , with $(\Omega, \mathcal{A}, \mathbb{P})$ some probability space, and the application $A \mapsto M(A)$ is σ -additive. Note that this does not imply that M is a *measure-valued random variable*, that is, for a given $\omega \in \Omega$ the function $A \mapsto M(A)(\omega)$ is not necessarily a real measure over \mathbb{R}^d , nor almost surely in ω . The measure-sample path definition is the one used by much of the current literature on random measures (see the introductory chapter in [7]), but it fails to cover very important cases such as Gaussian White Noise. Second-order random measures do contain Gaussian White Noise plus many other useful examples which we will mention further, but the literature on such random measures is more scarce; see [8–10] as examples of general works using this concept. Now, the extra important adjective *measure-covariance* comes from the very important assumption that there exists a measure C_M over $\mathbb{R}^d \times \mathbb{R}^d$, called here the *covariance measure*, which satisfies

$$\mathbb{C}ov(M(A), M(B)) = C_M(A \times B), \quad \forall A, B \in \mathcal{B}_B(\mathbb{R}^d). \quad (1.4)$$

In general, a second-order random measure M has its covariance structure determined by a *bi-measure*, that is, the function $(A, B) \mapsto \mathbb{C}ov(M(A), M(B))$ is a measure in one component when the other one is fixed. It is known [9], Section 2.2, Example 2 that a bi-measure is not generally identifiable with a measure over $\mathbb{R}^d \times \mathbb{R}^d$ as in (1.4). Therefore, the measure-covariance assumption is an extra regularity criterion. The existence of the total variation measure $|C_M|$ helps, as we shall see, to prove a semi-stochastic Fubini Theorem for random measures (Thm. 3.4, used mainly as an auxiliary result), and to prove the convergence mode (1.3) thanks to an adequate use of Lusin's Theorem. Assumption (1.4) is a mild condition and covers essentially every second-order random measure used in practice; we give some examples in Section 3.3. We refer to [11, 12] and [13], Chapter 5 as works where assumption (1.4) is used. Expansion (1.2) also implies an expansion for the covariance measure C_M , which is here specified in Proposition 4.8.

¹When $\sigma_{X_n}^2 = 0$, the object μ_n may not be a measure but it does not really intervene in expansion (1.2) since then $X_n = 0$.

Orthogonal expansions for second-order random measures have been explored in particular cases. For Gaussian White Noise and other orthogonal random measures, orthogonal expansions can be obtained quite immediately (see Sect. 3.3.1). Note also that a second-order random measure M can be interpreted as a generalized stochastic process (or random distribution, see [14, 15]) by focusing on the random variables $\langle M, \varphi \rangle$ for every φ smooth with compact support. For such stochastic objects, orthogonal expansions such as (1.2) are known. [16] covers the case of a generalized stochastic process over a bounded subset of \mathbb{R}^d , the objects $(\mu_n)_n$ being distributions in such a case. [17] explores the case of tempered random distributions over the whole space \mathbb{R}^d , in which second-order random measures regulated by polynomials are covered. The actual contribution of the present work is double: the demonstration that for a function-regulated measure-covariance second-order random measure M the objects $(\mu_n)_n$ in the KL expansion are *measures* and not general distributions, all of them being regulated by the same function that regulates M ; and the important convergence against measurable functions (1.3), which is stronger than convergence against smooth functions and implies set-wise convergence. In what concerns KL expansions of more abstract stochastic objects with values in more general topological spaces than the classical Hilbert space case, we refer to [18] for separable Banach spaces, [19] for separable Fréchet spaces, and [20] for compact topological groups.

This work is organized as follows. In Section 2 we give the basis of KL expansions with respect to a Hilbert space. In Section 3 we introduce random measures. Since the setting is not completely standard (use of δ -rings and random measures in a particular sense), Sections 3.1 and 3.2 introduce notations, basic notions and key properties of both deterministic and random measures over \mathbb{R}^d . Semi-stochastic Fubini Theorem 3.4 is also here presented. Section 3.3 gives important examples of widely studied random measures for which a KL expansion as here presented can be obtained² such as White Noise, orthogonal random measures, Poisson and Cox processes and the derivative of fractional Brownian motion with Hurst index $H \geq \frac{1}{2}$. In Section 4 we present the main Theorems 4.1 and 4.10, Proposition 4.8, and their proofs. We end in Section 5 with some concluding remarks and comments about these results and ideas for future research. Namely, we discuss: the Hilbert space E which contains the finite measures over \mathbb{R}^d implicitly used as basis for the KL expansions; the uniqueness of these expansions; details in the Gaussian case; applicability for non-function-regulated random measures; and ideas for obtaining KL expansions for some non-mean-square-continuous stochastic processes over \mathbb{R}^d , such as trawl processes.

Notations and conventions. $\mathbf{1}_A$ denotes the indicator function of the set A . $\|\cdot\|_\infty$ denotes the supremum norm. $\mathcal{D}(\mathbb{R}^d)$ denotes the space of (real) smooth compactly supported test-functions over \mathbb{R}^d used in Distribution Theory [21, 22]. Concerning measure theory over \mathbb{R}^d , we are inspired mainly by the *Bourbakian* approach to measure theory [23], so in particular a positive measure is taken to be a σ -additive positive function over the *bounded* Borel sets, not over the whole Borel σ -algebra, see more details in Section 3.1. The Lebesgue measure over \mathbb{R}^d is denoted $\ell^{\otimes d}$. All random variables are supposed to be defined over a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A stochastic process is understood as a family of random variables indexed by an arbitrary non-empty set. We do not make precise the laws of the random variables involved (the Gaussian case is a particular one which can be used as a reference example). Equality between random variables is always understood in an a.s. sense, and equality between stochastic processes is understood as one being a modification of the other.

2. KARHUNEN–LOÈVE EXPANSION

Let us give the details about KL expansions with respect to a Hilbert space. Let E be a real separable Hilbert space, with inner-product $(\cdot, \cdot)_E$. Let $X : E \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$ be a linear and continuous real mapping satisfying that there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}} \subset E$ such that

$$\sum_{n \in \mathbb{N}} \mathbb{E}(|X(e_n)|^2) < \infty. \quad (2.1)$$

²We do not explicitly obtain their expansions here, we only mention them as examples covered by the results.

If this holds, we say that X has a *traceable KL expansion with respect to E* . In such a case there exist an orthonormal basis of E , say $(f_n)_{n \in \mathbb{N}}$, and a sequence of uncorrelated random variables $(X_n)_{n \in \mathbb{N}}$, such that

$$X(e) = \sum_{n \in \mathbb{N}} X_n(f_n, e)_E, \quad \forall e \in E, \quad (2.2)$$

the convergence of the series being in a mean-square sense. Note that we have used the weak topology on E for the convergence criterion. The vectors $(f_n)_{n \in \mathbb{N}}$ are the eigenvectors of the *covariance operator* induced by the covariance of X : if $K_X : E \times E \rightarrow \mathbb{R}$ is the covariance kernel of X , that is

$$K_X(e, f) = \mathbb{E}(X(e)X(f)), \quad (2.3)$$

then K_X is bilinear, positive-semidefinite and continuous (since X is continuous). By Riesz Representation, for every $e \in E$ there exists an element $Q_X(e) \in E$ such that

$$K_X(e, f) = (Q_X(e), f)_E, \quad \forall f \in E. \quad (2.4)$$

The so-induced operator $Q_X : E \rightarrow E$ is called the covariance operator of X . This operator is linear, continuous, positive-semidefinite, and by (2.1) it is also trace-class [24], Theorem VI.18. Hence, it has a spectral decomposition in an orthonormal basis of eigenvectors $(f_n)_{n \in \mathbb{N}}$, with corresponding positive eigenvalues $(\sigma_{X_n}^2)_{n \in \mathbb{N}}$ which form a convergent series [24], Theorem VI.21:

$$\sigma_{X_n}^2 f_n = Q_X(f_n) \quad ; \quad \sum_{n \in \mathbb{N}} \sigma_{X_n}^2 < \infty. \quad (2.5)$$

The random variables $(X_n)_{n \in \mathbb{N}}$ are given by $X_n := X(f_n)$, for which we have $\mathbb{C}ov(X_n, X_m) = \sigma_{X_n}^2 \delta_{n,m}$.

Let us study a particular example which we apply in this work. Let $(U(x))_{x \in \mathbb{R}^d}$ be a real mean-square continuous stochastic process over \mathbb{R}^d . Let $C_U(x, y) = \mathbb{E}(U(x)U(y))$ be its covariance function, which is continuous over $\mathbb{R}^d \times \mathbb{R}^d$. Let ν be a positive finite measure over \mathbb{R}^d such that

$$\int_{\mathbb{R}^d} C_U(x, x) d\nu(x) < \infty. \quad (2.6)$$

From Cauchy-Schwarz inequality and the positive-semidefiniteness of C_U , (2.6) implies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |C_U(x, y)| |\varphi(x)| |\phi(y)| d(\nu \otimes \nu)(x, y) < \infty, \quad \forall \varphi, \phi \in L^2(\mathbb{R}^d, \nu). \quad (2.7)$$

It is known that this condition allows to properly define stochastic integrals of the form³

$$\tilde{U}(\varphi) := \int_{\mathbb{R}^d} U(x) \varphi(x) d\nu(x), \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu). \quad (2.8)$$

³The case of integrals over a compact interval is covered for example by [25], Section 4.5, whose approach using Riemann sums can be extended to integrals over \mathbb{R}^d without major difficulties. For the sake of completeness, such a development is presented here in Appendix A.2 (use $\varphi d\nu$ as measure in Lemma A.3 for the case of integral (2.8)). A comment on the Bochner-integral construction, which requires a stronger condition than (2.7), is also exposed in such Appendix.

Hence, one can re-define U as a process indexed by elements of the separable Hilbert space $L^2(\mathbb{R}^d, \nu)$. The so-defined application $\tilde{U} : L^2(\mathbb{R}^d, \nu) \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$ is continuous. The covariance operator $Q_{\tilde{U}}$ is given by

$$Q_{\tilde{U}}(\varphi) = \int_{\mathbb{R}^d} C_U(\cdot, y)\varphi(y)d\nu(y), \quad (2.9)$$

which by (2.6) is trace-class [26]. \tilde{U} has then a traceable KL expansion with respect to $L^2(\mathbb{R}^d, \nu)$:

$$\int_{\mathbb{R}^d} U(x)\varphi(x)d\nu(x) = \sum_{n \in \mathbb{N}} X_n(f_n, \varphi)_{L^2(\mathbb{R}^d, \nu)}, \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu), \quad (2.10)$$

with $(f_n)_{n \in \mathbb{N}}$ the orthonormal basis of eigenfunctions of $Q_{\tilde{U}}$ and

$$X_n = \int_{\mathbb{R}^d} U(x)f_n(x)d\nu(x), \quad \forall n \in \mathbb{N}. \quad (2.11)$$

3. RANDOM MEASURES

The proofs of the claims exposed in this Section are present in Appendix A.

3.1. Reminders on measures over \mathbb{R}^d and their anti-derivatives

We denote $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -algebra of \mathbb{R}^d and $\mathcal{B}_B(\mathbb{R}^d)$ the δ -ring of bounded Borel subsets of \mathbb{R}^d . By a *measure over \mathbb{R}^d* , we mean a *real* application $\mu : \mathcal{B}_B(\mathbb{R}^d) \rightarrow \mathbb{R}$ which is σ -additive over $\mathcal{B}_B(\mathbb{R}^d)$. This implies that μ is locally-finite, but μ may not be defined over unbounded sets (some authors use the term *pre-measure* for this object [27]). The total-variation measure of μ , noted $|\mu|$, is the smallest positive measure such that $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathcal{B}_B(\mathbb{R}^d)$ [28], Chapter 6. If $|\mu|$ can be extended finitely and σ -additively to $\mathcal{B}(\mathbb{R}^d)$ (hence $|\mu|(\mathbb{R}^d) < \infty$), then μ is said to be finite, and it can be extended uniquely and σ -additively to $\mathcal{B}(\mathbb{R}^d)$. The space of measures (resp. finite measures) over \mathbb{R}^d is denoted $\mathcal{M}(\mathbb{R}^d)$ (resp. $\mathcal{M}_F(\mathbb{R}^d)$). The space of (real Borel) measurable functions over \mathbb{R}^d is denoted $\mathcal{M}(\mathbb{R}^d)$. $\mathcal{M}_B(\mathbb{R}^d)$ and $\mathcal{M}_{B,c}(\mathbb{R}^d)$ denote the subspaces of $\mathcal{M}(\mathbb{R}^d)$ consisting of bounded and bounded compactly supported functions respectively. A function $f \in \mathcal{M}(\mathbb{R}^d)$ is said to be integrable with respect to $\mu \in \mathcal{M}(\mathbb{R}^d)$ if $|f|$ is Lebesgue integrable with respect to $|\mu|$. In such a case, we note $\langle \mu, f \rangle := \int_{\mathbb{R}^d} f d\mu = \int_{\mathbb{R}^d} f(x)d\mu(x)$. We remark that the total variation measure $|\mu|$ can be expressed as

$$|\mu|(A) = \sup_{\varphi \in \mathcal{M}(\mathbb{R}^d), |\varphi| = \mathbf{1}_A} |\langle \mu, \varphi \rangle|, \quad \forall A \in \mathcal{B}_B(\mathbb{R}^d). \quad (3.1)$$

We recall the useful Lusin's Theorem, considered in a simplified version over \mathbb{R}^d [29], Theorem 7.10:

Theorem 3.1 (Lusin). *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $\psi \in \mathcal{M}(\mathbb{R}^d)$. Then, for every $\epsilon > 0$ there exists a closed set $E \subset \mathbb{R}^d$ such that ψ is continuous over E (with the subspace topology) and $|\mu|(E^c) < \epsilon$.*

One special property of measures over \mathbb{R}^d is that they are derivatives in distributional sense of regular functions. Moreover, if the measure is finite those primitives grow in a controlled manner. Consider thus the following (double) anti-derivative operator $\mathcal{O} : \mathcal{M}_F(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$:

$$\mathcal{O}(\mu)(\vec{x}) := \int_0^{\vec{x}} \mu((-\infty, \vec{u}])d\vec{u}, \quad \forall \vec{x} \in \mathbb{R}^d, \quad (3.2)$$

where we have used the abbreviated notations

$$\int_0^{\vec{x}} (\cdot) d\vec{u} := \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_d} (\cdot) du_d \dots du_2 du_1 \quad ; \quad (-\infty, \vec{x}] := (-\infty, x_1] \times (-\infty, x_2] \times \dots \times (-\infty, x_d], \quad (3.3)$$

for every $\vec{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$. Note that the function $\vec{u} \mapsto \mu((-\infty, \vec{u}])$ is bounded (since μ is finite) and càdlàg in each component when the others are fixed, therefore the iterated integrals in (3.2) are simple Riemann integrals and thus $\mathcal{O}(\mu)$ is a continuous function. The following bound holds for $\mathcal{O}(\mu)$:

$$|\mathcal{O}(\mu)(\vec{x})| \leq |x_1| \dots |x_d| |\mu|(\mathbb{R}^d), \quad \forall \vec{x} = (x_1, \dots, x_d) \in \mathbb{R}^d. \quad (3.4)$$

\mathcal{O} is an anti-derivative operator in the sense that $\frac{\partial^{2d} \mathcal{O}(\mu)}{\partial x_1^2 \dots \partial x_d^2} = \mu$ in distributional sense over \mathbb{R}^d , that is

$$\int_{\mathbb{R}^d} \mathcal{O}(\mu)(x) \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2}(x) dx = \int_{\mathbb{R}^d} \varphi(x) d\mu(x), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (3.5)$$

3.2. Measure-covariance random measures and properties

Definition 3.2. A centred measure-covariance second-order random measure (from now on *m-cov random measure*) over \mathbb{R}^d is a zero-mean stochastic process indexed by the bounded Borel sets $M = (M(A))_{A \in \mathcal{B}_B(\mathbb{R}^d)}$ such that there exists $C_M \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\mathbb{E}(M(A)M(B)) = C_M(A \times B), \quad \forall A, B \in \mathcal{B}_B(\mathbb{R}^d). \quad (3.6)$$

The first implication of Definition 3.2 is the following.

Proposition 3.3. *M is a σ -additive function from $\mathcal{B}_B(\mathbb{R}^d)$ to $L^2(\Omega, \mathcal{A}, \mathbb{P})$.*

In other words, M is an $L^2(\Omega, \mathcal{A}, \mathbb{P})$ -valued (locally finite) measure over \mathbb{R}^d . The extra adjective *m-cov* is added because of the identification of the covariance of M to the *covariance measure* C_M , which, as mentioned in the introduction, does not apply for a general second-order random measure [9], Chapter 2, Example 2. It is clear that covariance measures are symmetric in the sense $C_M(A \times B) = C_M(B \times A)$. It is possible to verify that $|C_M|$ is also a symmetric measure. Covariance measures are positive-semidefinite in the sense

$$\langle C_M, \varphi \otimes \varphi \rangle \geq 0, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d). \quad (3.7)$$

Conversely, every symmetric measure over $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (3.7) is the covariance measure of an m-cov random measure.⁴ If $\varphi \in \mathcal{M}(\mathbb{R}^d)$ is such that

$$\langle |C_M|, |\varphi| \otimes |\varphi| \rangle < \infty, \quad (3.8)$$

then the stochastic integral

$$\langle M, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x) dM(x) \quad (3.9)$$

can be uniquely defined as a random variable in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. This is just an example of the *Dunford-Schwartz* integral of φ with respect to the $L^2(\Omega, \mathcal{A}, \mathbb{P})$ -valued measure M ; see [9], Chapter 2 for an effective introduction, [30], Section IV.10 for the details, and [31], Proposition 3.3.1 for the sufficiency of condition (3.8). Also, the

⁴Construct a Gaussian m-cov random measure using Kolmogorov Extension Theorem.

French textbook [13], Chapter 5 covers in great detail, although with other terminology, the treatment of second-order random measures with a measure covariance and the integrals of deterministic function with respect to them. If φ and ϕ satisfy (3.8), then

$$\mathbb{E}(\langle M, \varphi \rangle \langle M, \phi \rangle) = \langle C_M, \varphi \otimes \phi \rangle. \quad (3.10)$$

The next theorem, which will play an auxiliary role, is called here *semi-stochastic Fubini Theorem*, since it provides sufficient conditions under which we can switch integral signs when one of the integrating measures is random and the other is not. Other stochastic Fubini theorems can be found in the literature but usually with diverse sample path, predictability or martingale-type conditions ([9], Theorem 7.4.10, [32], Section 4.5, [33]), which are not the focus here. The version here presented aims to provide conditions on C_M under which stochastic integrals can be defined with Riemann sums, without extra requirements on the sample paths of M .

Theorem 3.4 (Semi-stochastic Fubini). *Let M be an m -cov random measure over \mathbb{R}^d with covariance measure C_M and let $\mu \in \mathcal{M}(\mathbb{R}^m)$. Let $\psi \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^m)$ such that*

- (i) $\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m} |\psi(x, u)| |\psi(y, v)| d|C_M| \otimes |\mu| \otimes |\mu|(x, y, u, v) < \infty$.
- (ii) *The function $(u, v) \mapsto \langle |C_M|, |\psi|(\cdot, u) \otimes |\psi|(\cdot, v) \rangle$ is locally bounded and there exists $E \in \mathcal{B}(\mathbb{R}^m)$ with $|\mu|(E^c) = 0$ such that the function $(u, v) \mapsto \langle C_M, \psi(\cdot, u) \otimes \psi(\cdot, v) \rangle$ is continuous over $E \times E$.*

Then,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u). \quad (3.11)$$

Let us now focus on the case of finite random measures.

Definition 3.5. An m -cov random measure M over \mathbb{R}^d is said to be finite if its covariance measure C_M is finite.

When M is finite, its definition can be extended uniquely, finitely and σ -additively to the whole Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$, the random variable $M(\mathbb{R}^d)$ having finite variance. There is also an extra regularity property which holds for finite random measures.

Proposition 3.6. *Let M be an m -cov finite random measure over \mathbb{R}^d . Then, the function over $\mathbb{R}^d \times \mathbb{R}^d$*

$$(\vec{u}, \vec{v}) \mapsto C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) \quad (3.12)$$

is continuous over a set of the form $E \times E$, with $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\ell^{\otimes d}(E^c) = 0$.

Let us now define the application of the anti-derivative operator \mathcal{O} to a finite m -cov random measure M . The application $\vec{u} \mapsto M((-\infty, \vec{u}])$ defines a stochastic process over \mathbb{R}^d whose covariance function is (3.12), being thus mean-square continuous outside a set of null Lebesgue measure and with bounded covariance. Thus, the stochastic integral (see Lem. A.2)

$$\mathcal{O}(M)(\vec{x}) := \int_0^{\vec{x}} M((-\infty, \vec{u}]) d\vec{u} \quad (3.13)$$

is well-defined through Riemann-like approximations. This process has covariance function

$$C_{\mathcal{O}(M)}(\vec{x}, \vec{y}) = \int_0^{\vec{x}} \int_0^{\vec{y}} C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) d\vec{v} d\vec{u}, \quad (3.14)$$

which is a continuous function over $\mathbb{R}^d \times \mathbb{R}^d$ (it is actually the function $\mathcal{O} \otimes \mathcal{O}(C_M)$), therefore $\mathcal{O}(M)$ is mean-square continuous. In addition one has the bound

$$|C_{\mathcal{O}(M)}(\vec{x}, \vec{y})| \leq |x_1| \dots |x_d| |y_1| \dots |y_d| |C_M|(\mathbb{R}^d \times \mathbb{R}^d), \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d. \quad (3.15)$$

Finally, an application of semi-stochastic Fubini Theorem 3.4 allows to conclude $\frac{\partial^{2d} \mathcal{O}(M)}{\partial x_1^2 \dots \partial x_d^2} = M$ in distributional sense over \mathbb{R}^d , that is, we have the equality between the stochastic integrals

$$\int_{\mathbb{R}^d} \mathcal{O}(M)(x) \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2}(x) dx = \int_{\mathbb{R}^d} \varphi(x) dM(x), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (3.16)$$

3.3. Examples of m-cov random measures

We provide some examples of commonly used random measures for which a KL expansion as it is presented here can be obtained.

3.3.1. White noise and other orthogonal random measures

A (non-necessarily Gaussian) White Noise over \mathbb{R}^d is a centred m-cov random measure $W = (W(A))_{A \in \mathcal{B}_B(\mathbb{R}^d)}$ with covariance given by

$$\text{Cov}(W(A), W(B)) = \ell^{\otimes d}(A \cap B). \quad (3.17)$$

The covariance measure of W satisfies $\langle C_W, \psi \rangle = \int_{\mathbb{R}^d} \psi(x, x) dx$ for every $\psi \in \mathcal{M}_{B,c}(\mathbb{R}^d \times \mathbb{R}^d)$. C_W is a measure concentrated on the hyperplane $\{x = y\} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid x = y\}$, sometimes denoted $\delta(x - y)$. White Noise is a particular case of an orthogonal random measure. An orthogonal random measure is a centred m-cov random measure $M = (M(A))_{A \in \mathcal{B}_B(\mathbb{R}^d)}$ such that there exists $\nu \in \mathcal{M}(\mathbb{R}^d)$ positive such that

$$\text{Cov}(M(A), M(B)) = \nu(A \cap B). \quad (3.18)$$

C_M is also concentrated on $\{x = y\}$ but with another weighting measure, having $\langle C_M, \psi \rangle = \int_{\mathbb{R}^d} \psi(x, x) d\nu(x)$. We denote in such a case $C_M = \nu \delta(x - y)$. Orthogonal random measures have the property of assigning null covariance when evaluated over disjoint sets, or when integrated against functions with disjoint support. These kinds of random measures appear in the spectral analysis of stationary random fields [34]. In the stronger case where M takes independent values at disjoint sets, M is sometimes called a *completely random measure* [35, 36], or an *independently scattered random measure* [37]. Lévy processes [38] can be seen as primitives of completely random measures (the *Lévy basis*), and therefore their derivatives in distributional sense are orthogonal random measures in the sense (3.18) if the increments of the Lévy process have finite variance. KL expansions for Lévy processes have been worked out for example in [39].

Orthogonal expansions for an orthogonal random measure M can be obtained with relative ease. Let $(f_n)_n$ to be an orthonormal basis of the space $L^2(\mathbb{R}^d, \nu)$. Then,

$$\langle M, \varphi \rangle = \sum_{n \in \mathbb{N}} \epsilon_n (f_n, \varphi)_{L^2(\mathbb{R}^d, \nu)}, \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu), \quad (3.19)$$

with $\epsilon_n = \langle M, f_n \rangle$. Note that in this case $\text{Var}(\epsilon_n) = 1$, therefore the expansion is not traceable contrary to the case of Section 2. This can be arranged, for example, by multiplying each ϵ_n by a coefficient $\sigma_n > 0$, with $\sum_n \sigma_n^2 < \infty$, and then take $\mu_n = f_n / \sigma_n$ as functions in the expansion. Note that in such a case, $(\mu_n)_n$ is not an orthonormal system of $L^2(\mathbb{R}^d, \nu)$ but of another more abstract Hilbert space, with respect to which M has a

traceable KL expansion (see further in Sect. 5.1). In order to identify expansion (3.19) as a KL expansion such as the here developed, f_n and μ_n must be interpreted as measures, not as functions.

Some orthogonal random measures provide the crucial example of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ -valued random measures that cannot be seen as random measures in the sense of random variables taking values in a space of measures or almost surely so. Over \mathbb{R}^d , independently scattered measure-valued random variables must necessarily be a point process [36]. In consequence, if M is a Gaussian orthogonal random measure such that the weighting measure ν is not purely a discrete measure, the sample paths of Gaussian orthogonal random measures have almost surely unbounded variation [40]. This includes the case of Gaussian White Noise, as it is widely known [41], Exercise 2.17.

3.3.2. Poisson and Cox point processes

A point process [42] is a stochastic process indexed by the bounded Borel sets $(M(A))_{A \in \mathcal{B}_B(\mathbb{R}^d)}$ which can be represented as

$$M(A) = \sum_{j \in \mathbb{N}} \delta_{X_j}(A), \quad (3.20)$$

where $(X_j)_{j \in \mathbb{N}}$ is a family of \mathbb{R}^d -valued random variables such that $M(A) < \infty$ almost surely. M is called an inhomogeneous Poisson process if for every disjoint collection of bounded Borel sets $(A_k)_k$, the random variables $(M(A_k))_k$ are independent Poisson random variables with $\mathbb{E}(M(A_k)) = \nu(A_k)$ for some positive measure $\nu \in \mathcal{M}(\mathbb{R}^d)$ (the *intensity* measure). From the independence at disjoint sets condition, the covariance structure of an inhomogeneous Poisson process is given by (3.18), and thus $M - \nu$ (that is, centering M) is an orthogonal random measure. In consequence, orthogonal expansions of the form (3.19) also hold for it, with $(f_n)_n$ interpreted as measures.

Now, let $\Lambda : \mathcal{B}_B(\mathbb{R}^d) \mapsto L^2(\Omega, \mathcal{A}, \mathbb{P})$ be a *positive* second-order random measure over \mathbb{R}^d , that is, $\Lambda(A) \geq 0$ for every $A \in \mathcal{B}_B(\mathbb{R}^d)$. It is known [43], Proposition 2.4 that in such a case, the covariance of Λ is always identified with a covariance measure $C_\Lambda \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$. In addition, the σ -additivity implies that the mean $\nu(A) := \mathbb{E}(\Lambda(A))$ defines a measure $\nu \in \mathcal{M}(\mathbb{R}^d)$. Now, define M such that, conditioned on Λ , M is a Poisson point process with intensity Λ . Then, M is another form of point process, commonly used in applications, called the Cox process [44]. In such a case $M - \nu$ is also an m-cov random measure, with covariance measure

$$C_M = \nu \delta(x - y) + C_\Lambda. \quad (3.21)$$

Thus, M has a richer covariance structure than a Poisson process, with an orthogonal random measure part $\nu \delta(x - y)$ plus an extra positive covariance C_Λ . The most popular Cox process among applications is the log-Gaussian Cox process [45], where the random intensity is given by

$$\Lambda(A) = \int_A e^{Z(x)} dx, \quad (3.22)$$

where Z is some mean-square continuous Gaussian process. Note that random measures constructed from the integrals of an enough regular stochastic process with respect to a deterministic measure such as in (3.22) also provide an example of m-cov random measures, see Appendix A.2.

3.3.3. Derivatives of fractional Brownian motion

Let $(B_H(t))_{t \geq 0}$ be a zero-mean \mathbb{R} -valued Gaussian process with covariance function

$$C_{B_H}(t, s) = \text{Cov}(B_H(t), B_H(s)) = \frac{t^{2H} + s^{2H} - |t - s|^{2H}}{2}, \quad (3.23)$$

where $H \in (0, 1)$. Then B_H is called a *fractional Brownian motion* and H is called the Hurst index. If $H = \frac{1}{2}$, B_H is a standard Brownian motion. Consider the case $H > \frac{1}{2}$. Let $\frac{d}{dt}B_H$ be the distributional derivative of B_H , whose covariance is given by [11]

$$\frac{\partial^2}{\partial t \partial s} C_{B_H} = H(2H - 1)|t - s|^{2H-2}, \quad (3.24)$$

which is not a continuous function but it is integrable over $[0, T] \times [0, T]$ for every $T > 0$. It follows that the covariance of $\frac{d}{dt}B_H$ can be identified with the measure

$$C_{\frac{d}{dt}B_H}(E) = H(2H - 1) \int_E \frac{d(x, y)}{|x - y|^{2-2H}}, \quad \forall E \in \mathcal{B}_B([0, \infty) \times [0, \infty)). \quad (3.25)$$

$\frac{d}{dt}B_H$ is thus another example of an m -cov random measure. Note that some authors call $\frac{d}{dt}B_H$ a long-range dependence process [46, 47]. It is known that the case $H > \frac{1}{2}$ is regular enough to develop an stochastic calculus around B_H without requiring specialized techniques, contrary to the Brownian motion case [48].

4. EXPANSION OF RANDOM MEASURES

Now that every required definition and basic result is established, we present the KL expansion for finite random measures, which is the main result of this work.

Theorem 4.1 (Karhunen–Loève expansion of finite random measures). *Let M be an m -cov finite random measure over \mathbb{R}^d . Then, there exists a sequence of pairwise uncorrelated random variables with summable variances $(X_n)_{n \in \mathbb{N}}$, and a linearly independent sequence of finite measures over \mathbb{R}^d , $(\mu_n)_{n \in \mathbb{N}}$ such that*

$$\langle M, \varphi \rangle = \sum_{n \in \mathbb{N}} X_n \langle \mu_n, \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d), \quad (4.1)$$

with the series convergence understood in the mean-square sense.

To prove Theorem 4.1, we associate the random measure M to a random object X which acts linearly on a Hilbert space such as in Section 2. Namely, we first apply the anti-derivative operator \mathcal{O} , getting $X := \mathcal{O}(M)$. X is a mean-square continuous stochastic process which can be identified with a mean-square continuous linear mapping over the separable Hilbert space $L^2(\mathbb{R}^d, \nu)$, being ν a conveniently defined measure. Thus, a KL expansion with respect to $L^2(\mathbb{R}^d, \nu)$ exists. Then, we derive X and its expansion to retrieve M and its expansion. This logic has been applied for the case of general tempered random distributions [17]. The particularity here is the measure structure of the objects $(\mu_n)_n$ and the convergence mode (4.1), which requires extra attention. The proof will be split into Lemmas. New notations will be introduced and kept along the Lemmas. The reader may recognize very similar arguments to the proof of the classical KL expansion for mean-square continuous stochastic processes over compact intervals [5], Section 37.5.

Lemma 4.2. *There exists $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ such that for every m -cov finite random measure M over \mathbb{R}^d the process $\mathcal{O}(M)$ has a KL expansion with respect to $L^2(\mathbb{R}^d, \nu)$.*

Proof of Lemma 4.2: Consider the polynomial function $p : \mathbb{R}^d \rightarrow \mathbb{R}^+$ given by $p(\vec{x}) = \prod_{j=1}^d (1 + |x_j|^2)^2$. Consider the finite measure over \mathbb{R}^d

$$d\nu(x) := \frac{dx}{p(x)}. \quad (4.2)$$

Let $C_{\mathcal{O}(M)}$ be the covariance function of $\mathcal{O}(M)$. From bound (3.15) we conclude

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} C_{\mathcal{O}(M)}(x, x) d\nu(x) \leq |C_M|(\mathbb{R}^d \times \mathbb{R}^d) \left(\int_{\mathbb{R}} \frac{t^2}{(1+t^2)^2} dt \right)^d < \infty. \quad (4.3)$$

$\mathcal{O}(M)$ thus has a traceable KL expansion with respect to $L^2(\mathbb{R}^d, \nu)$ (Sect. 2), having thus

$$\int_{\mathbb{R}^d} \mathcal{O}(M)(x) \varphi(x) d\nu(x) = \sum_{j \in \mathbb{N}} X_j \langle f_j, \varphi \rangle_{L^2(\mathbb{R}^d, \nu)} = \sum_{j \in \mathbb{N}} X_j \langle \nu, f_j \varphi \rangle, \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu), \quad (4.4)$$

being $(f_j)_{j \in \mathbb{N}}$ the orthonormal basis of $L^2(\mathbb{R}^d, \nu)$ given by the eigenfunctions of the covariance operator of $\mathcal{O}(M)$, and $(X_j)_{j \in \mathbb{N}}$ the associated uncorrelated random variables with variances $(\sigma_{X_j}^2)_{j \in \mathbb{N}}$, satisfying

$$\sigma_{X_j}^2 f_j = \int_{\mathbb{R}^d} C_{\mathcal{O}(M)}(\cdot, y) f_j(y) d\nu(y) \quad ; \quad X_j = \int_{\mathbb{R}^d} \mathcal{O}(M)(x) f_j(x) d\nu(x); \quad \sum_{j \in \mathbb{N}} \sigma_{X_j}^2 < \infty. \quad (4.5)$$

□

The measure ν in Lemma 4.2 is far from being unique: one can take any measure with fast-enough decreasing density so second-order primitives (in each component) of finite measures are integrable with respect to it (a Gaussian density works, for instance).

Let us now fix M as a given finite m-cov random measure.

Lemma 4.3. *For every j such that $\sigma_{X_j}^2 > 0$, the distribution $\frac{\partial^{2d} f_j}{\partial x_1^2 \dots \partial x_d^2}$ is in $\mathcal{M}_F(\mathbb{R}^d)$.*

Proof of Lemma 4.3: An arbitrary $f \in L^2(\mathbb{R}^d, \nu)$ determines a distribution over \mathbb{R}^d through the application $\varphi \mapsto \int_{\mathbb{R}^d} f(x) \varphi(x) dx = \langle fp\nu, \varphi \rangle$. Therefore, the derivatives $\frac{\partial^{2d} f_j}{\partial x_1^2 \dots \partial x_d^2}$ are well defined as distributions over \mathbb{R}^d . The eigenvalue-eigenfunction relation implies for $\sigma_{X_j}^2 > 0$

$$\begin{aligned} f_j(\vec{x}) &= \frac{1}{\sigma_{X_j}^2} \int_{\mathbb{R}^d} C_{\mathcal{O}(M)}(\vec{x}, \vec{y}) f_j(\vec{y}) d\nu(\vec{y}) \\ &= \frac{1}{\sigma_{X_j}^2} \int_{\mathbb{R}^d} \int_0^{\vec{x}} \int_0^{\vec{y}} C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) d\vec{v} d\vec{u} f_j(\vec{y}) d\nu(\vec{y}) \\ &= \frac{1}{\sigma_{X_j}^2} \int_0^{\vec{x}} \int_{\mathbb{R}^d} \int_0^{\vec{y}} C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) d\vec{v} f_j(\vec{y}) d\nu(\vec{y}) d\vec{u} \\ &= \frac{1}{\sigma_{X_j}^2} \int_0^{\vec{x}} \int_{\mathbb{R}^d} \mathcal{O}(C_M((-\infty, \vec{u}] \times \cdot))(\vec{y}) f_j(\vec{y}) d\nu(\vec{y}) d\vec{u}, \end{aligned} \quad (4.6)$$

where we used (deterministic) Fubini Theorem⁵ for changing the order of integration, and $C_M(A \times \cdot)$ stands for the measure $B \mapsto C_M(A \times B)$ for any $A \in \mathcal{B}(\mathbb{R}^d)$. Inspired by this, we define

$$\mu_j(A) := \frac{1}{\sigma_{X_j}^2} \int_{\mathbb{R}^d} \mathcal{O}(C_M(A \times \cdot))(\vec{y}) f_j(\vec{y}) d\nu(\vec{y}), \quad \forall A \in \mathcal{B}(\mathbb{R}^d). \quad (4.7)$$

⁵The classical Fubini Theorem for positive measures can be extended easily to the case of real measures over Euclidean spaces provided that the corresponding integrals using the total-variation of the measures involved are finite.

Given the property (3.4) of the operator \mathcal{O} , the function $\mathcal{O}(C_M(A \times \cdot))$ is in $L^2(\mathbb{R}^d, \nu)$ and the integral (4.7) is thus well-defined. Since \mathcal{O} is linear, the application $A \mapsto \mu_j(A)$ is additive. From bound (3.4) we have

$$|\mu_j(A)| \leq \frac{1}{\sigma_{X_j}^2} \underbrace{\int_{\mathbb{R}^d} |y_1| \dots |y_d| |f_j(\vec{y})| d\nu(\vec{y})}_{< \infty \text{ since } \vec{y} \mapsto |y_1| \dots |y_d| \in L^2(\mathbb{R}^d, \nu)} |C_M|(A \times \mathbb{R}^d). \quad (4.8)$$

Since $|C_M|$ is a finite measure, if we take any sequence of Borel sets $(A_n)_{n \in \mathbb{N}}$ such that $A_n \searrow \emptyset$ we have $|C_M|(A_n \times \mathbb{R}^d) \searrow 0$ and hence $|\mu_j(A_n)| \rightarrow 0$. This proves that μ_j is a measure over \mathbb{R}^d and it is also finite since C_M is finite. In addition, from (4.6) we have

$$f_j(\vec{x}) = \int_0^{\vec{x}} \mu_j((-\infty, \vec{u}]) d\vec{u} = \mathcal{O}(\mu_j)(\vec{x}). \quad (4.9)$$

Thus, f_j is nothing but $\mathcal{O}(\mu_j)$, therefore

$$\mu_j = \frac{\partial^{2d} f_j}{\partial x_1^2 \dots \partial x_d^2}. \quad (4.10)$$

□

When $\sigma_{X_j}^2 = 0$ the distribution $\frac{\partial^{2d} f_j}{\partial x_1^2 \dots \partial x_d^2}$ is not necessarily a measure, but such a case does not really intervene in the decomposition (4.4) ($X_j = 0$). For simplicity, we assume from now on that $\sigma_{X_j}^2 > 0$ for all $j \in \mathbb{N}$ (for the case where the sum (4.1) is finite we have nothing more to prove).

Lemma 4.4. *M has the following expansion*

$$\langle M, \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \mu_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (4.11)$$

Proof of Lemma 4.4: For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, one has $\varphi p \in L^2(\mathbb{R}^d, \nu)$. Thus, from expansion (4.4) we have

$$\langle \mathcal{O}(M), \varphi \rangle = \int_{\mathbb{R}^d} \mathcal{O}(M)(x) \varphi(x) p(x) d\nu(x) = \sum_{j \in \mathbb{N}} X_j \langle \nu, f_j \varphi p \rangle = \sum_{j \in \mathbb{N}} X_j \int_{\mathbb{R}^d} f_j(x) \varphi(x) dx. \quad (4.12)$$

Considering the derivative relations (3.16) and (4.10), we conclude

$$\begin{aligned} \langle M, \varphi \rangle &= \left\langle \frac{\partial^{2d} \mathcal{O}(M)}{\partial x_1^2 \dots \partial x_d^2}, \varphi \right\rangle \\ &= \left\langle \mathcal{O}(M), \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2} \right\rangle \\ &= \sum_{j \in \mathbb{N}} X_j \left\langle f_j, \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2} \right\rangle \\ &= \sum_{j \in \mathbb{N}} X_j \left\langle \frac{\partial^{2d} f_j}{\partial x_1^2 \dots \partial x_d^2}, \varphi \right\rangle = \sum_{j \in \mathbb{N}} X_j \langle \mu_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \end{aligned} \quad (4.13)$$

□

The goal of the following Lemmas is to extend the expansion (4.11) to $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$. We begin by computing an important covariance. The semi-stochastic Fubini Theorem 3.4 will be used here.

Lemma 4.5. *The following formula holds for every $A \in \mathcal{B}(\mathbb{R}^d)$:*

$$\mathbb{E}(M(A)X_j) = \sigma_{X_j}^2 \mu_j(A). \quad (4.14)$$

Proof of Lemma 4.5: By definition of $\mathcal{O}(M)$ and X_j (Eq. (4.5)) we have

$$\begin{aligned} X_j &= \int_{\mathbb{R}^d} \mathcal{O}(M)(\vec{y}) f_j(\vec{y}) d\nu(\vec{y}) \\ &= \int_{\mathbb{R}^d} \int_0^{\vec{y}} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) dM(\vec{s}) d\vec{u} f_j(\vec{y}) d\nu(\vec{y}) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) dM(\vec{s}) \theta_{\vec{y}}(\vec{u}) f_j(\vec{y}) d(\ell^{\otimes d} \otimes \nu)(\vec{u}, \vec{y}), \end{aligned} \quad (4.15)$$

where $\theta_{\vec{y}}$ is the function $\theta_{\vec{y}}: \mathbb{R}^d \rightarrow \{-1, 0, 1\}$ such that $\int_{\mathbb{R}^d} \theta_{\vec{y}}(\vec{u}) \varphi(\vec{u}) d\vec{u} = \int_0^{\vec{y}} \varphi(\vec{u}) d\vec{u}$ for every $\varphi \in C(\mathbb{R}^d)$ ⁶. We shall apply semi-stochastic Fubini Theorem 3.4 to switch integral signs in (4.15). Consider the measure λ over $\mathbb{R}^d \times \mathbb{R}^d$ given by $d\lambda(\vec{u}, \vec{y}) := \theta_{\vec{y}}(\vec{u}) f_j(\vec{y}) d(\ell^{\otimes d} \otimes \nu)(\vec{u}, \vec{y})$. λ is finite since by (deterministic) Fubini

$$\begin{aligned} |\lambda|(\mathbb{R}^d \times \mathbb{R}^d) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |\theta_{\vec{y}}(\vec{u}) f_j(\vec{y})| d(\ell^{\otimes d} \otimes \nu)(\vec{u}, \vec{y}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\theta_{\vec{y}}(\vec{u})| d\vec{u} |f_j(\vec{y})| d\nu(\vec{y}) \\ &\leq \int_{\mathbb{R}^d} |y_1| \dots |y_d| |f_j(\vec{y})| d\nu(\vec{y}) < \infty. \end{aligned} \quad (4.16)$$

For condition (i) we use that both C_M and λ are finite, so

$$\begin{aligned} \int_{(\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d)} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) \mathbf{1}_{(-\infty, \vec{v}]}(\vec{t}) d(|C_M| \otimes |\lambda| \otimes |\lambda|)((\vec{s}, \vec{t}), (\vec{u}, \vec{y}), (\vec{v}, \vec{z})) \\ \leq |C_M|(\mathbb{R}^d \times \mathbb{R}^d) [|\lambda|(\mathbb{R}^d \times \mathbb{R}^d)]^2 < \infty. \end{aligned} \quad (4.17)$$

For condition (ii), we have to study the function

$$((\vec{u}, \vec{y}), (\vec{v}, \vec{z})) \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) \mathbf{1}_{(-\infty, \vec{v}]}(\vec{t}) dC_M(\vec{s}, \vec{t}) = C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]). \quad (4.18)$$

This function is clearly bounded and it does not depend upon \vec{y}, \vec{z} , so it is continuous in such components. Moreover, from Proposition 3.6 it also follows that (4.18) is continuous over $(E \times \mathbb{R}^d) \times (E \times \mathbb{R}^d)$, being $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\ell^{\otimes d}(E^c) = 0$, and therefore such that $|\lambda|([E \times \mathbb{R}^d]^c) = 0$. Semi-stochastic Fubini Theorem can then be applied to switch the integral order in (4.15), obtaining

$$X_j = \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) dM(\vec{s}) d\lambda(\vec{u}, \vec{y}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s}) d\lambda(\vec{u}, \vec{y}) dM(\vec{s}). \quad (4.19)$$

⁶The function $\theta_{\vec{y}}$ is just the indicator function of $[0, \vec{y}]$ when the components of \vec{y} are all positive. When they are not, corresponding minus signs must be added in order to make the integrals coincide. In any case, $\theta_{\vec{y}}$ has compact support.

Using formula (3.10) and (deterministic) Fubini Theorem, we obtain

$$\begin{aligned}
\mathbb{E}(M(A)X_j) &= \mathbb{E}\left(\int_{\mathbb{R}^d} \mathbf{1}_A(\vec{t})dM(\vec{t}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s})d\lambda(\vec{u}, \vec{y})dM(\vec{s})\right) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_A(\vec{t}) \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(\vec{s})d\lambda(\vec{u}, \vec{y})dC_M(\vec{t}, \vec{s}) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_A(\vec{t})\mathbf{1}_{(-\infty, \vec{u}]}(\vec{s})dC_M(\vec{t}, \vec{s})d\lambda(\vec{u}, \vec{y}) \\
&= \int_{\mathbb{R}^d} \int_0^{\vec{y}} C_M(A \times (-\infty, \vec{u}])d\vec{u}f_j(\vec{y})d\nu(\vec{y}) \\
&= \int_{\mathbb{R}^d} \mathcal{O}(C_M(A \times \cdot))(\vec{y})f_j(\vec{y})d\nu(\vec{y}) \\
&= \sigma_{X_j}^2 \mu_j(A).
\end{aligned} \tag{4.20}$$

□

Lemma 4.6. *The bilinear form $\Lambda : \mathcal{M}_B(\mathbb{R}^d) \times \mathcal{M}_B(\mathbb{R}^d) \rightarrow \mathbb{R}$ given by*

$$\Lambda(\varphi, \phi) = \sum_{j=1}^{\infty} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle \tag{4.21}$$

is well-defined, the series being absolutely convergent.

Proof of Lemma 4.6: Let us define the sequence of finite random measures

$$M_n = \sum_{j \leq n} X_j \mu_j, \quad n \in \mathbb{N}. \tag{4.22}$$

Their covariance measures C_{M_n} are given by

$$\begin{aligned}
C_{M_n}(A \times B) &= \mathbb{E}\left(\sum_{j \leq n} \sum_{k \leq n} X_j X_k \mu_j(A) \mu_k(B)\right) \\
&= \sum_{j \leq n} \sigma_{X_j}^2 \mu_j(A) \mu_j(B),
\end{aligned} \tag{4.23}$$

where we have used $\mathbb{E}(X_j X_k) = \sigma_{X_j}^2 \delta_{j,k}$. In addition, using Lemma 4.5 we conclude

$$\begin{aligned}
\mathbb{E}(M(A)M_n(B)) &= \mathbb{E}\left(\sum_{j \leq n} M(A)X_j \mu_j(B)\right) \\
&= \sum_{j \leq n} \mathbb{E}(M(A)X_j) \mu_j(B) \\
&= \sum_{j \leq n} \sigma_{X_j}^2 \mu_j(A) \mu_j(B) = C_{M_n}(A \times B).
\end{aligned} \tag{4.24}$$

Developing the expression $\mathbb{E}((M(A) - M_n(A))(M(B) - M_n(B)))$, one concludes from (4.23) and (4.24) that $M - M_n$ is an m-cov finite random measure with covariance

$$C_{M-M_n} = C_M - C_{M_n} = C_M - \sum_{j \leq n} \sigma_{X_j}^2 \mu_j \otimes \mu_j. \tag{4.25}$$

Since C_{M-M_n} is a finite covariance measure it must be positive definite, having

$$\langle C_M - C_{M_n}, \varphi \otimes \varphi \rangle = \langle C_{M-M_n}, \varphi \otimes \varphi \rangle \geq 0, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d), \quad (4.26)$$

which implies

$$\langle C_{M_n}, \varphi \otimes \varphi \rangle \leq \langle C_M, \varphi \otimes \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d). \quad (4.27)$$

Using Cauchy-Schwartz inequality, we conclude for every n

$$\begin{aligned} \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle| |\langle \mu_j, \phi \rangle| &\leq \sqrt{\sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2} \sqrt{\sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2} \\ &= \sqrt{\langle C_{M_n}, \varphi \otimes \varphi \rangle} \sqrt{\langle C_{M_n}, \phi \otimes \phi \rangle} \\ &\leq \sqrt{\langle C_M, \varphi \otimes \varphi \rangle} \sqrt{\langle C_M, \phi \otimes \phi \rangle} < \infty, \end{aligned} \quad (4.28)$$

which proves that the series (4.21) is absolutely convergent and thus the bilinear form Λ is well-defined. \square

The following Lemma is the crucial part where an argument essentially different from those found in the proof of the classical KL expansion is needed. Here, an adequate use of Lusin's Theorem will help us to conclude the convergence against measurable and bounded functions.

Lemma 4.7. *The following equality holds*

$$\Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d). \quad (4.29)$$

Proof of Lemma 4.7: Lemma 4.4 guarantees that (4.29) holds for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We will extend it to $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$. We begin by considering $\varphi \neq 0$ of the form $\varphi = \mathbf{1}_I$, where $I \subset \mathbb{R}^d$ is a rectangle $I = I_1 \times \dots \times I_d$, each I_j being an interval of \mathbb{R} . In such a case φ can be approximated point-wisely by a sequence of functions in $\mathcal{D}(\mathbb{R}^d)$, the sequence being dominated by $\|\varphi\|_\infty$. Let $\epsilon > 0$. Since C_M is a finite measure, from dominated convergence we can choose $\phi \in \mathcal{D}(\mathbb{R}^d)$ approaching φ so that $\|\phi\|_\infty \leq \|\varphi\|_\infty$ and so that

$$\langle |C_M|, |\varphi - \phi| \otimes |\varphi - \phi| \rangle < \frac{\epsilon^2}{64 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)}. \quad (4.30)$$

Now, from triangular inequality we have

$$\begin{aligned} |\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| &\leq \underbrace{|\Lambda(\varphi, \varphi) - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2|}_{(a)} + \underbrace{|\sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2 - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2|}_{(b)} \\ &+ \underbrace{|\sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2 - \langle C_M, \phi \otimes \phi \rangle|}_{(c)} + \underbrace{|\langle C_M, \phi \otimes \phi \rangle - \langle C_M, \varphi \otimes \varphi \rangle|}_{(d)}. \end{aligned} \quad (4.31)$$

By Lemmas 4.6 and 4.4, there exists n_0 such that both terms (a) and (c) are smaller than $\frac{\epsilon}{4}$ if $n \geq n_0$. For the term (d) we use the symmetry of C_M , the Cauchy-Schwarz inequality and inequality (4.30) to obtain

$$\begin{aligned}
(d) &= |\langle C_M, \varphi \otimes \varphi - \phi \otimes \phi \rangle| \\
&= |\langle C_M, (\varphi + \phi) \otimes (\varphi - \phi) \rangle| \\
&\leq \sqrt{\langle C_M, (\varphi + \phi) \otimes (\varphi + \phi) \rangle} \sqrt{\langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle} \\
&< \sqrt{|C_M|(\mathbb{R}^d \times \mathbb{R}^d) 4 \|\varphi\|_\infty^2} \sqrt{\frac{\epsilon^2}{64 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)}} = \frac{\epsilon}{4}.
\end{aligned} \tag{4.32}$$

On the other hand, for the term (b) we can do similarly, considering the covariance measure C_{M_n} (Eq. (4.23)) and inequality (4.27):

$$\begin{aligned}
(b) &= |\langle C_{M_n}, \varphi \otimes \varphi - \phi \otimes \phi \rangle| \\
&= |\langle C_{M_n}, (\varphi + \phi) \otimes (\varphi - \phi) \rangle| \\
&\leq \sqrt{\langle C_{M_n}, (\varphi + \phi) \otimes (\varphi + \phi) \rangle} \sqrt{\langle C_{M_n}, (\varphi - \phi) \otimes (\varphi - \phi) \rangle} \\
&\leq \sqrt{\langle C_M, (\varphi + \phi) \otimes (\varphi + \phi) \rangle} \sqrt{\langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle} < \frac{\epsilon}{4}.
\end{aligned} \tag{4.33}$$

We conclude that $|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| \leq \epsilon$ for every $\epsilon > 0$, and therefore $\Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle$ for every φ of the form $\varphi = \mathbf{1}_I$. By bilinearity of Λ , we can easily extend this result to every φ in the space

$$\mathcal{E} := \text{span}\{ \mathbf{1}_I \mid I \subset \mathbb{R}^d \text{ rectangle} \}. \tag{4.34}$$

Now, in order to extend this result to any $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$, we use Lusin's Theorem 3.1 applied to the space \mathbb{R}^d with the finite measure $|C_M|(\cdot \times \mathbb{R}^d)$. Given $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$, $\varphi \neq 0$, and given $\epsilon > 0$, there exists a closed set $E \subset \mathbb{R}^d$ such that φ is continuous over E (with the subspace topology) and such that

$$|C_M|(E^c \times \mathbb{R}^d) < \frac{\epsilon^2}{1536 \|\varphi\|_\infty^4 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)}. \tag{4.35}$$

Consider a typical Riemann-alike approximation of φ , done through a sequence of functions of the form

$$\phi_n := \sum_{j=1}^n \varphi(x_j^n) \mathbf{1}_{I_j^n}, \tag{4.36}$$

where for each n , $(I_j^n)_{j=1, \dots, n}$ is a collection of rectangles forming a partition of a subset K_n of \mathbb{R}^d , satisfying that $K_n \nearrow \mathbb{R}^d$ and $\max_{j=1, \dots, n} \text{diam}(I_j^n) \rightarrow 0$ as $n \rightarrow \infty$; and $x_j^n \in I_j^n$ is a tag-point chosen so $x_j^n \in E$ when $I_j^n \cap E \neq \emptyset$. Let $x \in E$. Denote j_n the index of the interval $I_{j_n}^n$ where x belongs to. By construction the sequence $(x_{j_n}^n)_n$ is in E , and $x_{j_n}^n \rightarrow x \in E$ as $n \rightarrow \infty$. Since φ is continuous over E with the subspace topology, we have $\phi_n(x) = \varphi(x_{j_n}^n) \rightarrow \varphi(x)$ as $n \rightarrow \infty$. In addition, one has $\|\phi_n\|_\infty \leq \|\varphi\|_\infty$. The sequence (4.36) converges thus point-wisely and dominated to φ over E . By dominated convergence (C_M is a measure), for every $\epsilon > 0$ we can select ϕ of the form (4.36) (so in \mathcal{E}) with $\|\phi\|_\infty \leq \|\varphi\|_\infty$ such that

$$\int_{E \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| < \frac{\epsilon^2}{128 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)}. \tag{4.37}$$

By splitting integrals, using elementary bounds and the symmetry of $|C_M|$, we have

$$\begin{aligned}
\langle |C_M|, |\varphi - \phi| \otimes |\varphi - \phi| \rangle &= \int_{E \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| + \int_{E \times E^c} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| \\
&\quad + \int_{E^c \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| + \int_{E^c \times E^c} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| \\
&\leq \frac{\epsilon^2}{128 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)} + 3 \cdot 4 \|\varphi\|_\infty^2 |C_M|(E^c \times \mathbb{R}^d) \\
&< \frac{\epsilon^2}{128 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)} + 12 \|\varphi\|_\infty^2 \frac{\epsilon^2}{1536 \|\varphi\|_\infty^4 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)} \\
&= \frac{\epsilon^2}{64 \|\varphi\|_\infty^2 |C_M|(\mathbb{R}^d \times \mathbb{R}^d)}.
\end{aligned} \tag{4.38}$$

With this set up, we can study the expression $|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle|$ for any $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$ by using *the same* splitting arguments exposed in (4.31), using $\phi \in \mathcal{E}$ constructed as above. Expressions (a) and (c) can be bounded by $\frac{\epsilon}{4}$ for $n \geq n_0$ for some n_0 . In expressions (b) and (d) we can also follow line by line the arguments (4.32) and (4.33) to bound both of them by $\frac{\epsilon}{4}$. We conclude once again that $|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| \leq \epsilon$ for every $\epsilon > 0$, and therefore, since φ is arbitrary,

$$\Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle = \sum_{j \in \mathbb{N}} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d). \tag{4.39}$$

□

Now we finish the proof of Theorem 4.1.

Proof of Theorem 4.1: With everything in place, we use Lemmas 4.6, 4.7 and expression (4.25) to obtain

$$\begin{aligned}
\mathbb{E}(|\langle M, \varphi \rangle - \sum_{j \leq n} X_j \langle \mu_j, \varphi \rangle|^2) &= \mathbb{E}(|\langle M - M_n, \varphi \rangle|^2) \\
&= \langle C_{M - M_n}, \varphi \otimes \varphi \rangle \\
&= \langle C_M - C_{M_n}, \varphi \otimes \varphi \rangle \\
&= \Lambda(\varphi, \varphi) - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2 \rightarrow 0, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d).
\end{aligned} \tag{4.40}$$

□

From Theorem 4.1 it follows that the covariance measure C_M has the expansion

$$\langle C_M, \varphi \otimes \phi \rangle = \sum_{j \in \mathbb{N}} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle, \quad \forall \varphi, \phi \in \mathcal{M}_B(\mathbb{R}^d). \tag{4.41}$$

There is a slightly stronger convergence mode for this expansion: if we fix φ (or ϕ) then the measure $\langle C_{M_n}, \varphi \otimes \cdot \rangle$ converges in absolute variation to $\langle C_M, \varphi \otimes \cdot \rangle$.

Proposition 4.8. *For every $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$, one has*

$$|\langle C_M, \varphi \otimes \cdot \rangle - \sum_{j \leq n} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \mu_j|(\mathbb{R}^d) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{4.42}$$

Proof of Proposition 4.8: Let $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$. We use expression (3.1) for the total-variation measure of $\langle C_{M-M_n}, \varphi \otimes \cdot \rangle$. Since (4.25) implies $\langle C_{M-M_n}, \phi \otimes \phi \rangle \leq \langle C_M, \phi \otimes \phi \rangle$ for every $\phi \in \mathcal{M}_B(\mathbb{R}^d)$, we have

$$\begin{aligned}
|\langle C_M - C_{M_n}, \varphi \otimes \cdot \rangle|(\mathbb{R}^d) &= |\langle C_{M-M_n}, \varphi \otimes \cdot \rangle|(\mathbb{R}^d) \\
&= \sup_{\phi \in \mathcal{M}(\mathbb{R}^d), |\phi|=1} |\langle C_{M-M_n}, \varphi \otimes \phi \rangle| \\
&\leq \sup_{\phi \in \mathcal{M}(\mathbb{R}^d), |\phi|=1} \sqrt{\langle C_{M-M_n}, \varphi \otimes \varphi \rangle} \sqrt{\langle C_{M-M_n}, \phi \otimes \phi \rangle} \\
&\leq \sqrt{\langle C_{M-M_n}, \varphi \otimes \varphi \rangle} \sup_{\phi \in \mathcal{M}(\mathbb{R}^d), |\phi|=1} \sqrt{\langle C_M, \phi \otimes \phi \rangle} \\
&\leq \sqrt{\langle C_{M-M_n}, \varphi \otimes \varphi \rangle} \sqrt{|C_M|(\mathbb{R}^d \times \mathbb{R}^d)} \xrightarrow{n \rightarrow \infty} 0.
\end{aligned} \tag{4.43}$$

□

To finish, we present the expansion of function-regulated random measures, which covers some non-finite random measures cases (such as White Noise or Poisson processes with non-finite intensity). Let us introduce the following definition.

Definition 4.9. A measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ (resp., an m -cov random measure M over \mathbb{R}^d) is said to be function-regulated if there exists a strictly positive and locally bounded measurable function f such that μ/f is finite (resp., such that M/f is finite).

It is not difficult to verify that M is regulated by f if and only if C_M is regulated by $f \otimes f$.

Theorem 4.10. Let M be an m -cov random measure over \mathbb{R}^d regulated by a function f . Then, there exist a sequence of pairwise uncorrelated random variables with summable variances $(X_n)_{n \in \mathbb{N}}$ and a linearly independent sequence of real measures $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$, all of them regulated by f , such that

$$\langle M, \varphi \rangle = \sum_{n \in \mathbb{N}} X_n \langle \mu_n, \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d), \tag{4.44}$$

with the series convergence understood in the mean-square sense. In addition, the covariance measure C_M satisfies

$$|\langle C_M, \varphi \otimes \cdot \rangle - \sum_{j \leq n} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \mu_j|(\mathbb{R}^d) \xrightarrow{n \rightarrow \infty} 0, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d), \forall K \subset \mathbb{R}^d \text{ compact}. \tag{4.45}$$

Proof of Theorem 4.10: We apply Theorem 4.1 to the finite random measure $\frac{1}{f}M$, obtaining thus

$$\langle \frac{1}{f}M, \phi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \nu_j, \phi \rangle, \quad \phi \in \mathcal{M}_B(\mathbb{R}^d), \tag{4.46}$$

with $\nu_j \in \mathcal{M}_F(\mathbb{R}^d)$ for every j . By posing $\mu_j = f\nu_j$ and using $\langle M, \varphi \rangle = \langle \frac{1}{f}M, f\varphi \rangle$ the result follows. The convergence for the covariance (4.45) is obtained following the same arguments as in Proposition 4.8. □

5. CONCLUDING REMARKS

The following remarks are meant to clarify some important points, expose some remarkable cases and provide ideas for extensions and applications of the results here obtained.

5.1. Hilbert space-based approach and uniqueness of the expansion

In general, KL expansions are based on a Hilbert space, either by taking the random object as a random variable with values in the Hilbert space, or by interpreting the object as a linear mapping over it, as in Section 2. Our case is no exception, although the space at stake is not yet explicitly shown. To make it precise, we consider the space of finite measures $\mathcal{M}_F(\mathbb{R}^d)$ endowed with the following positive-definite bilinear form

$$(\mu_1, \mu_2)_E \mapsto (\mathcal{O}(\mu_1), \mathcal{O}(\mu_2))_{L^2(\mathbb{R}^d, \nu)}, \quad (5.1)$$

where $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ is defined as in (4.2). By construction, $\mathcal{O} : \mathcal{M}_F(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d, \nu)$. From the continuity of $\mathcal{O}(\mu)$ together with the fact that ν has a strictly positive density, one has $\int_{\mathbb{R}^d} |\mathcal{O}(\mu)|^2 d\nu = 0 \iff \mathcal{O}(\mu) = 0$. Since $\mathcal{O}(\mu)$ is the primitive of the function $\vec{u} \mapsto \mu((-\infty, \vec{u}])$, $\mathcal{O}(\mu) = 0$ implies $\mu((-\infty, \vec{u}]) = 0$ \vec{u} -almost everywhere, and from right-continuity it must be null. We conclude $(\mu, \mu)_E = 0 \iff \mu = 0$, and thus (5.1) is a Hilbert product. The completion of $\mathcal{M}_F(\mathbb{R}^d)$ with this product is then an abstract separable Hilbert space E in which $\mathcal{M}_F(\mathbb{R}^d)$ is dense. This space consists of distributions which are derivatives of order $2d$ of elements in $L^2(\mathbb{R}^d, \nu)$. The KL expansion (4.1) can be obtained as the Hilbert-based expansion of M with respect to E . The measures $(\mu_n)_n$ are orthonormal in E , since $f_n = \mathcal{O}(\mu_n)$ (Eq. (4.9)). Following Section 2, one can identify M as a process linearly indexed on E by

$$M(\mu) = \int_{\mathbb{R}^d} \mathcal{O}(M)\mathcal{O}(\mu)d\nu, \quad \forall \mu \in E. \quad (5.2)$$

It is important to remark that the choice to link M with the Hilbert space $L^2(\mathbb{R}^d, \nu)$ is arbitrary and other Hilbert spaces can be used. In other words, one can find many different separable Hilbert spaces in which the space $\mathcal{M}_F(\mathbb{R}^d)$ can be embedded: $L^2(\mathbb{R}^d, \alpha)$ with a sufficiently fast decreasing density measure α other than ν ; a Sobolev space $H^\beta(\mathbb{R}^d)$ with $\beta > 0$ if one takes enough anti-derivatives of M to obtain a mean-square β -times differentiable stochastic process; a low fractional order Sobolev space $H^\beta(\mathbb{R}^d)$ with $\beta < 0$ by taking the Fourier transform of M and dividing by a corresponding polynomial; to mention only a few examples. In all these cases, an operator is applied to M in order to obtain a random linear mapping over the separable Hilbert space, its KL expansion is obtained, and then we retrieve M by applying some inverse operator. The resulting expansions are not the same, with different principal measures $(\mu_n)_n$, random coefficients $(X_n)_n$ and variances $(\sigma_{X_n}^2)_n$. The choice of $L^2(\mathbb{R}^d, \nu)$ is made here only for simplicity.

A consequence of this arbitrariness is that the KL expansion for M is not unique and it depends on the Hilbert space to which we can associate it. This also happens for regular stochastic processes. For example, take $X = (X(t))_{t \in [a, b]}$ to be a mean-square differentiable stochastic process. Then one can interpret X as a random object with values in $L^2([a, b])$, in $L^2([a, b], \alpha)$ with α some bounded density, or in the Sobolev space $H^1((a, b))$. In all these cases a KL expansion for X with respect to the chosen Hilbert space can be obtained and they consist of different eigenfunctions and random coefficients.

An interesting question, not addressed in this work, is whether the expansions for M have some common reference property which would allow us to speak about ‘the’ KL expansion of M . For instance, it is expected that if we change ν for another density measure α , the newly obtained principal measures $(\mu_n)_{n \in \mathbb{N}}$ will be absolutely continuous with respect to the ones obtained with ν . Regardless of this, we remark that the main contribution of this work is the proof of a stronger convergence mode than the one induced by the Hilbert space, here translated into a $\mathcal{M}_{B, c}(\mathbb{R}^d)^*$ -weak sense implying set-wise convergence, analogously to the classical uniform-mean-square convergence for the expansion of a mean-square continuous process over a compact interval. Another question to address is whether if such a convergence mode also remains present when changing the Hilbert space.

5.2. Gaussian case

If M is Gaussian, that is, if $(M(A_1), \dots, M(A_n))$ is a Gaussian vector for every $A_1, \dots, A_n \in \mathcal{B}_B(\mathbb{R}^d)$, then the variables X_j in the expansion of M are independent and Gaussian, since all the variables involved are

constructed linearly. In addition, the convergence of the series is also almost sure. This follows from a classical result on almost sure convergence of series of independent random variables with variances forming a convergent series [49], Section 12.2. In our case,

$$\sum_{n \in \mathbb{N}} \text{Var}(X_n \langle \mu_n, \varphi \rangle) = \sum_{n \in \mathbb{N}} \sigma_{X_n}^2 |\langle \mu_n, \varphi \rangle|^2 = \langle C_M, \varphi \otimes \varphi \rangle < \infty \implies \sum_{n \in \mathbb{N}} X_n \langle \mu_n, \varphi \rangle \text{ converges a.s.} \quad (5.3)$$

5.3. Expansions for non-regular processes and trawl processes

Theorems 4.1 and 4.10 can be used to obtain, as corollaries, diverse forms of KL expansions of non-regular stochastic processes over \mathbb{R}^d . Consider for instance a process of the form $Z(\vec{x}) = \langle M, \phi_{\vec{x}} \rangle$, with M being a function-regulated m-cov random measure and $\phi_{\vec{x}} \in \mathcal{M}(\mathbb{R}^d)$ for every \vec{x} , such that any of the Theorems 4.1 or 4.10 holds for any \vec{x} . Then, the following traceable KL expansion for Z holds:

$$Z(\vec{x}) = \sum_{n \in \mathbb{N}} X_n g_n(\vec{x}), \quad (5.4)$$

where $g_n(\vec{x}) = \langle \mu_n, \phi_{\vec{x}} \rangle$. Note that the convergence (5.4) is mean-square-point-wise, and this holds without requiring any particular regularity on Z (no mean-square continuity or measurability), since \vec{x} is acting just as an index parameter for $\phi_{\vec{x}}$ with respect to which no regularity is required.

One important example of so-defined stochastic processes are trawl processes [50, 51]. As mentioned in Section 3.3.1, a Lévy basis with finite variance is an example of m-cov random measure. If M is such a Lévy basis over $[0, \infty)$, then a process defined as

$$Z(t) = \langle M, \mathbf{1}_{A_t} \rangle, \quad t \geq 0, \quad (5.5)$$

is called a *trawl* process. For every t , $A_t \in \mathcal{B}([0, \infty))$ is called the trawl set. If M is finite, or if M is function-regulated and A_t is bounded for every t , then expansion (5.4) holds, providing thus a KL expansion for trawl processes.

5.4. General m-cov random measure case

It is not clear if a general m-cov random measure M can be regulated by a function f as it is required in Theorem 4.10. One thing that can always be done is to construct a KL decomposition *locally*. That is, for every $D \in \mathcal{B}_B(\mathbb{R}^d)$, M has a decomposition of the form (4.1) for every $\varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d)$ null outside D . This holds since one can focus on the compactly supported random measure $\mathbf{1}_D M$, which is finite and thus Theorem 4.1 applies. In this case, the measures μ_j and the random variables X_j depend upon the set D .

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No new data or code were created or analyzed in this study.

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APPENDIX A. PROOFS OF CLAIMS PRESENTED IN SECTION 3

A.1 Proof of Proposition 3.3

Let M be an m -cov random measure over \mathbb{R}^d with covariance measure C_M . Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint bounded Borel subsets of \mathbb{R}^d such that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{B}_B(\mathbb{R}^d)$. By symmetry of C_M , we have

$$\mathbb{E} \left(\left| M \left(\bigcup_{j \in \mathbb{N}} A_j \right) - \sum_{j \leq n} M(A_j) \right|^2 \right) = C_M \left(\bigcup_{j \in \mathbb{N}} A_j \times \bigcup_{j \in \mathbb{N}} A_j \right) - 2C_M \left(\bigcup_{j \leq n} A_j \times \bigcup_{j \in \mathbb{N}} A_j \right) + C_M \left(\bigcup_{j \leq n} A_j \times \bigcup_{j \leq n} A_j \right). \quad (\text{A.1})$$

Since C_M is a measure, by σ -additivity (A.1) must go to 0 as $n \rightarrow \infty$. □

A.2 Proof of semi-stochastic Fubini Theorem 3.4

The first issue with semi-stochastic Fubini Theorem 3.4 is the proper definition of the iterated integrals in (3.11). Namely, we require a canonical definition of the integral of a μ -almost everywhere mean-square continuous stochastic process with respect to a deterministic measure μ (right side of (3.11)). Here, we follow an approach using classical Riemann sums, as it is exposed in [25], Section 4.5 for $d = 1$ over a compact interval (there the measure is used through its bounded variation primitive). Here we need slightly more generality, so we develop explicitly such general definition, but the procedure is essentially the same as in [25]. We remark that a standard method for defining integrals of stochastic processes with respect to deterministic measures is using the Bochner integral [52]. However, Bochner integrability requires actually stronger conditions than the one required here, therefore we do not follow such approach.⁷

Lemma A.1 (Dominated convergence for double sequences). *Let (E, \mathcal{E}, μ) be a measure space ($\mu \geq 0$). Let $(f_{n,m})_{n,m \in \mathbb{N}}$ be a double-sequence of complex functions such that $\sup_{n,m \in \mathbb{N}} |f_{n,m}| \in \mathcal{L}^1(E, \mathcal{E}, \mu)$. Suppose that the double limit $\lim_{n,m \rightarrow \infty} f_{n,m}$ exists μ -almost everywhere. Then, the μ -almost everywhere defined function $f(x) = \lim_{n,m \rightarrow \infty} f_{n,m}(x)$ is in $\mathcal{L}^1(E, \mathcal{E}, \mu)$ and*

$$\lim_{n,m \rightarrow \infty} \int_E |f_{n,m} - f| d\mu = 0. \quad (\text{A.5})$$

Proof of Lemma A.1: ⁸ $f \in \mathcal{L}^1(E, \mathcal{E}, \mu)$ since $\sup_{n,m \in \mathbb{N}} |f_{n,m}| \in \mathcal{L}^1(E, \mathcal{E}, \mu)$. Suppose the double limit (A.5) is not 0. Then, there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$ there are $n_k, m_k \geq k$ such that

$$\int_E |f_{n_k, m_k} - f| d\mu \geq \epsilon. \quad (\text{A.6})$$

The sequence $f_k := f_{n_k, m_k}$ converges μ -almost everywhere to f , with $\sup_{k \in \mathbb{N}} |f_k|$ integrable. By traditional dominated convergence theorem [53], VI.9, $f_k \rightarrow f$ in $\mathcal{L}^1(E, \mathcal{E}, \mu)$. But this contradicts (A.6). \square

Lemma A.2. *Let $Z = (Z(x))_{x \in \mathbb{R}^d}$ be a centred second-order stochastic process with locally bounded covariance function. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Suppose Z is mean-square continuous outside a $|\mu|$ -null set. Let $A \in \mathcal{B}_B(\mathbb{R}^d)$. For every n , let $(I_j^n)_{j \in J_n} \subset \mathcal{B}(A)$ be a finite partition of A such that $\max_{j \in J_n} \text{diam}(I_j^n) \rightarrow 0$ as $n \rightarrow \infty$. Let $x_j^n \in I_j^n$ be an arbitrary tag-point for every (j, n) . Then the limit in mean-square*

$$\int_A Z(x) d\mu(x) := \lim_{n \rightarrow \infty} \sum_{j \in J_n} Z(x_j^n) \mu(I_j^n) \quad (\text{A.7})$$

exists and does not depend upon the choice of partitions $(I_j^n)_{j,n}$ or tag-points $(x_j^n)_{j,n}$. In addition, the application $A \mapsto \int_A Z(x) d\mu(x)$ is σ -additive on $\mathcal{B}_B(\mathbb{R}^d)$.

⁷For information, for defining a stochastic integral of the form $\int_{\mathbb{R}^d} Z(x) d\mu(x)$ with $\mu \in \mathcal{M}(\mathbb{R}^d)$ and Z a second-order process in the spirit of the Bochner integral, one requires, at least,

$$\int_{\mathbb{R}^d} \sqrt{C_Z(x, x)} d|\mu|(x) < \infty, \quad (\text{A.2})$$

where C_Z is the covariance function of Z . Such a condition is actually stronger than the one we require, namely (see Lem. A.3)

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |C_Z| d|\mu| \otimes |\mu| < \infty. \quad (\text{A.3})$$

The implication (A.2) \Rightarrow (A.3) follows from the Cauchy-Schwarz inequality. An example in which (A.3) holds but (A.2) does not is $d = 1$, μ the Lebesgue measure, and

$$C_Z(x, y) = \frac{e^{-|x-y|}}{xy} \mathbf{1}_{[1, \infty)}(x) \mathbf{1}_{[1, \infty)}(y). \quad (\text{A.4})$$

Also, in [13], Chapter 4 one can find a general Bochner-alike integral development for Hilbert-valued functions over general measurable spaces with respect to a finite measure, but there, condition (A.2) plays the essential role, rather than the weaker condition (A.3).

⁸The essentials of this proof has been borrowed from the StackExchange discussion <https://math.stackexchange.com/questions/448931/dominated-convergence-thm-dct-for-double-sequences>, consulted for the last time on March the 27th 2025.

Proof of Lemma A.2: For a given sequence of Riemann sums, we study the Cauchy gaps for $n, m \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E} \left(\left| \sum_{j \in J_n} Z(x_j^n) \mu(I_j^n) - \sum_{k \in J_m} Z(x_k^m) \mu(I_k^m) \right|^2 \right) &= \mathbb{E} \left(\left| \sum_{j \in J_n} \sum_{k \in J_m} (Z(x_j^n) - Z(x_k^m)) \mu(I_j^n \cap I_k^m) \right|^2 \right) \\
&= \sum_{j, j' \in J_n} \sum_{k, k' \in J_m} \mathbb{E} \left((Z(x_j^n) - Z(x_k^m))(Z(x_{j'}^n) - Z(x_{k'}^m)) \right) \mu(I_j^n \cap I_k^m) \mu(I_{j'}^n \cap I_{k'}^m) \\
&= \int_{A \times A} \underbrace{\sum_{j, j' \in J_n} \sum_{k, k' \in J_m} [C_Z(x_j^n, x_{j'}^n) - C_Z(x_j^n, x_{k'}^m) - C_Z(x_k^m, x_{j'}^n) + C_Z(x_k^m, x_{k'}^m)] \mathbf{1}_{I_j^n \cap I_k^m \times I_{j'}^n \cap I_{k'}^m}}_{f_{n,m} :=} d\mu \otimes \mu.
\end{aligned} \tag{A.8}$$

Let $E \in \mathcal{B}(\mathbb{R}^d)$ be a set such that $|\mu|(E^c) = 0$ and Z is mean-square continuous over E . C_Z is therefore continuous over $E \times E$. Therefore, by construction of the Riemann partitions we have $\lim_{n,m \rightarrow \infty} f_{n,m}(x, y) = 0$ for every $(x, y) \in (E \times E) \cap (A \times A)$. In addition, since C_Z is locally bounded and A is bounded, we have

$$\int_{A \times A} \sup_{n, m \in \mathbb{N}} |f_{n,m}| d(|\mu| \otimes |\mu|) \leq 4 \sup_{(x,y) \in A \times A} |C_Z(x, y)| |\mu|(A)^2 < \infty. \tag{A.9}$$

By Lemma A.1, (A.8) must go to 0 as $n, m \rightarrow \infty$. The sequence of Riemann sums is Cauchy and thus converges to a random variable in $L^2(\Omega, \mathcal{A}, \mathbb{P})$, noted as in (A.7). If we consider another sequence of Riemann sums with partitions $(\tilde{I}_j^n)_{j,n}$ and tag-points $(\tilde{x}_j^n)_{j,n}$, $j \in \tilde{J}_n$, then,

$$\sum_{j \in J_n} Z(x_j^n) \mu(I_j^n) - \sum_{\tilde{j} \in \tilde{J}_n} Z(\tilde{x}_{\tilde{j}}^n) \mu(\tilde{I}_{\tilde{j}}^n) = \sum_{j \in J_n} \sum_{\tilde{j} \in \tilde{J}_n} (Z(x_j^n) - Z(\tilde{x}_{\tilde{j}}^n)) \mu(I_j^n \cap \tilde{I}_{\tilde{j}}^n). \tag{A.10}$$

Applying $\mathbb{E}(| \cdot |^2)$ to (A.10), we can use the same splitting and dominated convergence arguments as in (A.8) and conclude that (A.10) converges to 0 in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Thus, the limit does not depend upon the partitions and tag-points.

Finally, using a particular Riemann partition of $A \cup B$ one concludes for A and B bounded

$$\begin{aligned}
\text{Cov} \left(\int_A Z(x) d\mu(x), \int_B Z(x) d\mu(x) \right) &= \lim_{n \rightarrow \infty} \text{Cov} \left(\sum_{j \in J_n} Z(x_j^n) \mu(I_j^n \cap A), \sum_{j' \in J_n} Z(x_{j'}^n) \mu(I_{j'}^n \cap B) \right) \\
&= \lim_{n \rightarrow \infty} \int_{A \times B} \sum_{j \in J_n} \sum_{j' \in J_n} C_Z(x_j^n, x_{j'}^n) \mathbf{1}_{I_j^n \times I_{j'}^n} d\mu \otimes \mu \\
&= \int_{A \times B} C_Z d\mu \otimes \mu,
\end{aligned} \tag{A.11}$$

which follows from dominated convergence. The application $A \mapsto \int_A Z(x) d\mu(x)$ thus has a covariance structure identified with the measure over $\mathbb{R}^d \times \mathbb{R}^d$ given by $D \mapsto \int_D C_Z d\mu \otimes \mu$, being thus σ -additive on $\mathcal{B}_B(\mathbb{R}^d)$ (Prop. 3.3). \square

For the case of an integral over an unbounded set, we define it through growing bounded sets.

Lemma A.3. *Let Z and μ be as in Lemma A.2. Suppose in addition*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |C_Z| d|\mu| \otimes |\mu| < \infty. \tag{A.12}$$

Let $A \in \mathcal{B}(\mathbb{R}^d)$. Then, for every sequence $(K_n)_{n \in \mathbb{N}} \subset \mathcal{B}_B(\mathbb{R}^d)$ with $K_n \nearrow \mathbb{R}^d$, the limit in mean-square

$$\int_A Z(x) d\mu(x) = \lim_{n \rightarrow \infty} \int_{K_n \cap A} Z(x) d\mu(x) \tag{A.13}$$

exists and is independent of the growing sequence $(K_n)_n$. In addition, the application $A \mapsto \int_A Z(x) d\mu(x)$ is σ -additive on $\mathcal{B}(\mathbb{R}^d)$.

Proof of Lemma A.3: For a growing sequence $(K_n)_n$, we use the additivity of the integral (Lem. A.2):

$$\left| \int_{K_n} Z(x) d\mu(x) - \int_{K_m} Z(x) d\mu(x) \right| = \left| \int_{K_n \vee m \setminus K_n \wedge m} Z(x) d\mu(x) \right|. \quad (\text{A.14})$$

Applying $\mathbb{E}(|\cdot|^2)$, using formula (A.11) we obtain

$$\int_{K_n \vee m \setminus K_n \wedge m \times K_n \vee m \setminus K_n \wedge m} C_Z d\mu \otimes \mu. \quad (\text{A.15})$$

Since $C_Z \in \mathcal{L}^1(\mathbb{R}^d \times \mathbb{R}^d, |\mu| \otimes |\mu|)$ (condition (A.12)), (A.15) goes to 0 as $n, m \rightarrow \infty$. Thus, the limit (A.13) exists as a limit of a Cauchy sequence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Now, if $(\tilde{K}_n)_n$ is another sequence of bounded Borel sets growing to \mathbb{R}^d , we have

$$\left| \int_{K_n} Z(x) d\mu(x) - \int_{\tilde{K}_n} Z(x) d\mu \right| = \left| \int_{K_n \Delta \tilde{K}_n} Z(x) d\mu(x) \right|, \quad (\text{A.16})$$

where $K_n \Delta \tilde{K}_n$ denotes the symmetric difference between K_n and \tilde{K}_n . The $\mathbb{E}(|\cdot|^2)$ of (A.16) is given by

$$\int_{K_n \Delta \tilde{K}_n \times K_n \Delta \tilde{K}_n} C_Z d\mu \otimes \mu, \quad (\text{A.17})$$

which goes to 0 as $n \rightarrow \infty$ since $C_Z \in \mathcal{L}^1(\mathbb{R}^d \times \mathbb{R}^d, |\mu| \otimes |\mu|)$. Hence, the limit does not depend upon the chosen sequence $(K_n)_n$. Finally, for any $A, B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$\begin{aligned} \text{Cov} \left(\int_A Z(x) d\mu(x), \int_B Z(x) d\mu(x) \right) &= \lim_{n \rightarrow \infty} \text{Cov} \left(\int_{A \cap K_n} Z(x) d\mu(x), \int_{B \cap K_n} Z(x) d\mu(x) \right) \\ &= \lim_{n \rightarrow \infty} \int_{A \cap K_n \times B \cap K_n} C_Z d\mu \otimes \mu \\ &= \int_{A \times B} C_Z d\mu \otimes \mu. \end{aligned} \quad (\text{A.18})$$

The application $D \mapsto \int_D C_Z d\mu$ defines a finite measure and therefore $A \mapsto \int_A Z(x) d\mu(x)$ defines a finite m-cov random measure over \mathbb{R}^d , being thus σ -additive over $\mathcal{B}(\mathbb{R}^d)$ (Prop. 3.3). \square

Proof of semi-stochastic Fubini Theorem 3.4: Condition (i) implies that the function $\int_{\mathbb{R}^m} \psi(\cdot, u) d\mu(u) \otimes \int_{\mathbb{R}^m} \psi(\cdot, v) d\mu(v)$ is in $L^1(\mathbb{R}^d \times \mathbb{R}^d, |C_M|)^9$. Hence, the integral of $\int_{\mathbb{R}^m} \psi(\cdot, u) d\mu(u)$ with respect to M over \mathbb{R}^d (left side of (3.11)) is a well-defined stochastic integral (condition (3.8)).

For the iterated integral at the right-hand side of (3.11), we set

$$Z(u) := \int_{\mathbb{R}^d} \psi(x, u) dM(x). \quad (\text{A.19})$$

The covariance function of Z is

$$C_Z(u, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \psi(y, v) dC_M(x, y). \quad (\text{A.20})$$

⁹ $\int_{\mathbb{R}^m} |\psi(\cdot, u)| d|\mu|(u) \otimes \int_{\mathbb{R}^m} |\psi(\cdot, v)| d|\mu|(v)$ is the Radon-Nikodym derivative of the measure over $\mathbb{R}^d \times \mathbb{R}^d$

$$D \mapsto \int_D \int_{\mathbb{R}^m \times \mathbb{R}^m} |\psi(x, u)| |\psi(y, v)| d(|\mu| \otimes |\mu|)(u, v) d|C_M|(x, y)$$

with respect to $|C_M|$.

By condition (ii), C_Z is well-defined and locally bounded, so Z is well-defined as a second-order process. From condition (i) we conclude $C_Z \in \mathcal{L}^1(\mathbb{R}^m \times \mathbb{R}^m, |\mu| \otimes |\mu|)$ and that C_Z is continuous over $E \times E$, being $E \in \mathcal{B}(\mathbb{R}^m)$ such that $|\mu|(E^c) = 0$. By Lemmas A.2 and A.3 the iterated stochastic integral

$$\int_{\mathbb{R}^m} Z(u) d\mu(u) = \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \psi(x, u) dM(x) \right] d\mu(u) \quad (\text{A.21})$$

is well defined *via* Riemann sums.

Let us now consider the variance of the difference between the iterated integrals

$$\begin{aligned} \text{Var} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right) &= \text{Var} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u) \right) \\ &+ \text{Var} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right) - 2\text{Cov} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u), \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right). \end{aligned} \quad (\text{A.22})$$

For the variances we have (Eqs. (A.20) and (3.10))

$$\text{Var} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u) \right) = \int_{\mathbb{R}^m \times \mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \psi(y, v) dC_M(x, y) d\mu \otimes \mu(u, v), \quad (\text{A.23})$$

$$\text{Var} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y). \quad (\text{A.24})$$

For the covariance in (A.22), we consider a sequence of bounded subsets $(K_n)_n$ growing to \mathbb{R}^m and, for each n , a sequence of Riemann partitions of K_n , say $(I_j^{\tilde{n}})_{j \in J_{\tilde{n}}}$, $\tilde{n} \in \mathbb{N}$ and $u_j^{\tilde{n}} \in I_j^{\tilde{n}}$ tag-points. Then we have

$$\begin{aligned} \text{Cov} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u), \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right) &= \lim_{n \rightarrow \infty} \lim_{\tilde{n} \rightarrow \infty} \sum_{j \in J_{\tilde{n}}} \text{Cov} \left(\int_{\mathbb{R}^d} \psi(x, u_j^{\tilde{n}}) dM(x), \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right) \mu(I_j^{\tilde{n}}) \\ &= \lim_{n \rightarrow \infty} \lim_{\tilde{n} \rightarrow \infty} \sum_{j \in J_{\tilde{n}}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u_j^{\tilde{n}}) \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dC_M(x, y) \mu(I_j^{\tilde{n}}) \\ &= \lim_{n \rightarrow \infty} \int_{K_n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dC_M(x, y) d\mu(v) \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dC_M(x, y) d\mu(v), \end{aligned} \quad (\text{A.25})$$

Now, since C_M is a measure, from condition (i) we can use the deterministic Fubini Theorem to argue that all the iterated integrals in (A.23), (A.24), (A.25) coincide. The variance (A.22) thus equals 0, showing that both iterated integrals coincide (the random variables involved are all zero-mean). \square

A.3 Proof of Proposition 3.6

We consider the following Lemma.

Lemma A.4. *Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. Then, the function $\vec{x} \mapsto \mu((-\infty, \vec{x}])$ is continuous outside a Lebesgue measure null set.*

Proof of Lemma A.4: We first remark that for $d = 1$ this holds immediately since the function $x \mapsto \mu((-\infty, x])$ is càdlàg and therefore it has an at most countable set of discontinuities (the atoms of μ). For $d > 1$ we proceed as follows. For each $j = 1, \dots, d$, consider the positive measure μ_j over \mathbb{R} defined by marginalizing the measure $|\mu|$ over all its components except the j -th one:

$$\mu_j(A) := |\mu|(\mathbb{R} \times \dots \times \underbrace{A}_{\text{position } j} \times \dots \times \mathbb{R}), \quad \forall A \in \mathcal{B}(\mathbb{R}). \quad (\text{A.26})$$

Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be the function $F(\vec{x}) = \mu((-\infty, \vec{x}])$. Then, by additivity

$$\begin{aligned}
|F(\vec{x} + \vec{h}) - F(\vec{x})| &\leq |F(x_1 + h_1, \dots, x_{d-1} + h_{d-1}, x_d + h_d) - F(x_1 + h_1, \dots, x_{d-1} + h_{d-1}, x_d)| \\
&\quad + |F(x_1 + h_1, \dots, x_{d-1} + h_{d-1}, x_d) - F(x_1 + h_1, \dots, x_{d-1}, x_d)| \\
&\quad + \dots \\
&\quad + |F(x_1 + h_1, x_2, \dots, x_d) - F(x_1, \dots, x_d)| \\
&= |\mu((-\infty, x_1 + h_1] \times \dots \times (-\infty, x_{d-1} + h_{d-1}] \times (x_d \wedge (x_d + h_d), x_d \vee (x_d + h_d)))| \\
&\quad + |\mu((-\infty, x_1 + h_1] \times \dots \times (x_{d-1} \wedge (x_{d-1} + h_{d-1}), x_{d-1} \vee (x_{d-1} + h_{d-1})) \times (-\infty, x_d])| \\
&\quad + \dots \\
&\quad + |\mu((x_1 \wedge (x_1 + h_1), x_1 \vee (x_1 + h_1)) \times (-\infty, x_2] \times \dots \times (-\infty, x_d])| \\
&\leq \mu_1((x_1 - |h_1|, x_1 + |h_1|)) + \dots + \mu_d((x_d - |h_d|, x_d + |h_d|)).
\end{aligned} \tag{A.27}$$

Since all the measures μ_j are over \mathbb{R} , they all have an at-most countable quantity of atoms. Let A_j be the set of atoms of μ_j . Note that if $x \notin A_j$, then $\mu_j((x - |h|, x + |h|)) \rightarrow 0$ as $h \rightarrow 0$. Set

$$E := \left[\bigcup_{j=1}^d \mathbb{R} \times \dots \times \underbrace{A_j}_{\text{position } j} \times \dots \times \mathbb{R} \right]^c. \tag{A.28}$$

Since E^c is a finite union of $\ell^{\otimes d}$ -null sets, we have $\ell^{\otimes d}(E^c) = 0$. From inequality (A.27) it follows that F is continuous over E . \square

Proof of Proposition 3.6: From Lemma A.4, there exists $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\ell^{\otimes d}(E^c) = 0$ and such that $\vec{u} \mapsto |C_M|(\mathbb{R}^d \times (-\infty, \vec{u}])$ is continuous over E . Using the symmetry of C_M and analogous inequalities as in (A.27), we obtain

$$\begin{aligned}
\left| C_M((-\infty, \vec{u} + \vec{h}_1] \times (-\infty, \vec{v} + \vec{h}_2]) - C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) \right| &\leq \left| C_M((-\infty, \vec{u}] \times (-\infty, \vec{v} + \vec{h}_2] \Delta (-\infty, \vec{v}]) \right| \\
&\quad + \left| C_M((-\infty, \vec{u} + \vec{h}_1] \Delta (-\infty, \vec{u}] \times (-\infty, \vec{v}]) \right| \\
&\leq |C_M|(\mathbb{R}^d \times (-\infty, \vec{u} + \vec{h}_1] \Delta (-\infty, \vec{u}]) \\
&\quad + |C_M|(\mathbb{R}^d \times (-\infty, \vec{v} + \vec{h}_2] \Delta (-\infty, \vec{v}]),
\end{aligned} \tag{A.29}$$

By continuity of $\vec{u} \mapsto |C_M|(\mathbb{R}^d \times (-\infty, \vec{u}])$, if $(\vec{u}, \vec{v}) \in E \times E$ then (A.29) goes to 0 as $(\vec{h}_1, \vec{h}_2) \rightarrow 0$. \square

A.4 Properties of the anti-derivative operator \mathcal{O}

The derivative property (3.5) is justified by a convolution argument which we make precise for $d = 1$, the case $d > 1$ is analogous but requires a more tedious notation. Let $\mu \in \mathcal{M}_F(\mathbb{R})$. We can rewrite $\mathcal{O}(\mu)$ as

$$\mathcal{O}(\mu) = \mathbf{1}_{[0, \infty)} * [(\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{[0, \infty)}] - \mathbf{1}_{(-\infty, 0)} * [(\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{(-\infty, 0)}], \tag{A.30}$$

where $*$ denotes the convolution operation. Using the properties of the convolution with respect to derivatives, that the Dirac measure δ is its identity element, and that $\frac{d}{dx} \mathbf{1}_{[0, \infty)} = \delta$, one has

$$\begin{aligned}
\frac{d}{dx} \left\{ \frac{d}{dx} \mathcal{O}(\mu) \right\} &= \frac{d}{dx} \left\{ \delta * [(\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{[0, \infty)}] - (-\delta) * [(\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{(-\infty, 0)}] \right\} \\
&= \frac{d}{dx} \left\{ (\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{[0, \infty)} + (\mathbf{1}_{[0, \infty)} * \mu) \mathbf{1}_{(-\infty, 0)} \right\} \\
&= \frac{d}{dx} \left\{ \mathbf{1}_{[0, \infty)} * \mu \right\} \\
&= \delta * \mu = \mu.
\end{aligned} \tag{A.31}$$

For the stochastic case, the derivative relation (3.16) comes from semi-stochastic Fubini Theorem 3.4: if $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we claim that we can exchange the order of integration and conclude

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{O}(M)(\vec{x}) \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2}(\vec{x}) d\vec{x} &= \int_{\mathbb{R}^d} \int_0^{\vec{x}} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty, \vec{u}]}(s) dM(s) d\vec{u} \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2}(\vec{x}) d\vec{x} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^{\vec{x}} \delta_s((-\infty, \vec{u}]) d\vec{u} \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2}(\vec{x}) d\vec{x} dM(s) \\
&= \int_{\mathbb{R}^d} \langle \mathcal{O}(\delta_s), \frac{\partial^{2d} \varphi}{\partial x_1^2 \dots \partial x_d^2} \rangle dM(s) \\
&= \int_{\mathbb{R}^d} \langle \delta_s, \varphi \rangle dM(s) = \int_{\mathbb{R}^d} \varphi(s) dM(s).
\end{aligned} \tag{A.32}$$

The arguments verifying that the hypotheses in semi-stochastic Fubini Theorem 3.4 hold are completely analogous to the case presented in the proof of Theorem 4.1 (see the proof of Lem. 4.5). We therefore omit them.