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A PURSUIT PROBLEM FOR SQUARED BESSEL PROCESSES

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Abstract. In this note, we are interested in the probability that two independent squared Bessel processes do not cross for a long time. We show that this probability has a power decay which is given by the first zero of some hypergeometric function. We also compute along the way the distribution of the location where the crossing eventually occurs.

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1. Introduction

1.1. Statement of the main result

Let X and Y be two independent squared Bessel processes of respective dimensions $\alpha > 0$ and $\beta \geq 0$. We denote by $\mathbb{P}_{(x,y)}$ the law of the pair (X, Y) when starting from (x, y) and we assume that $0 \leq x < y$. In this note, we are interested in the probability that the process X remains below Y for a long time:

$$\mathbb{P}_{(x,y)} \left(\forall s \leq t,\, X_s < Y_s\right) \qquad \text{as } t \to +\infty.$$

This question is often labelled in the literature as a capture or a pursuit problem. In most papers, the set-up is a Brownian one and the leading process is called a lamb or a prisoner, while its pursuers are wolves or policemen, see for instance [1–3] and the references therein. One may also see this question as a persistence problem for the difference of two squared Bessel processes. Indeed, a natural way to study such a question is to introduce the stopping time

$$T = \inf\{t \geq 0,\, X_t = Y_t\}$$

and study its tail asymptotics since $\mathbb{P}_{(x,y)} \left(\forall s \leq t,\, X_s < Y_s\right) = \mathbb{P}_{(x,y)} \left(T > t\right)$. Unlike the sum of squared Bessel processes, the difference is no longer Markovian and studying such asymptotics is thus more delicate. We refer to the two surveys [4, 5] for an extensive background on persistence probabilities, as well as their links with many physic phenomena. When dealing with self-similar processes, one generally obtains a power-law decay. However, except in very specific situations, actually computing the exponent is usually complicated. The results in this paper are no exception: we will see that the persistence exponent of $X - Y$ is given by the first zero of a certain hypergeometric function.

Keywords and phrases: Bessel process, persistence probability, first passage time.

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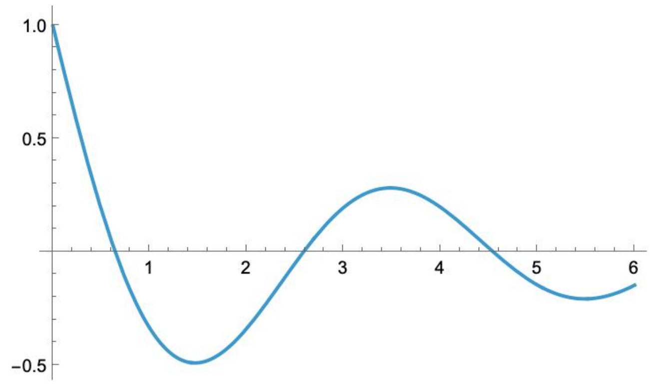

FIGURE 1. Graph of $F_{5,3}(s)$ for $s \in [0,6]$.

In order to state our results, let us define the hypergeometric function ${}_2F_1$ (see for instance [6], Chap. 9.1):

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\dots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ for $n \in \mathbb{N}$. We denote by $F_{\alpha,\beta}$ the hypergeometric function

$$ F_{\alpha,\beta}(s) = {}_2F_1\left[\begin{matrix} \frac{\alpha+\beta}{2} - 1 + s & -s \\ \frac{\alpha}{2} & \end{matrix}; \frac{1}{2}\right], \qquad s \geq 0. $$

We show Figure 1 a typical plot of the function $F_{\alpha,\beta}$.
Our first result gives the distribution of the position of X and Y when they cross each other.

Proposition 1.1. *Let us denote by θ the first positive zero of the hypergeometric function $F_{\alpha,\beta}$:*

$$ \theta = \inf \left\{ s \geq 0,\ F_{\alpha,\beta}(s) = 0 \right\}. $$

Under $\mathbb{P}_{(0,y)}$ the stopping time T is a.s. finite and the Mellin transform of X_T is given by:

$$ \mathbb{E}_{(0,y)}\left[X_T^s\right] = \frac{(2y)^s}{F_{\alpha,\beta}(s)}, \qquad s \in [0, \theta). $$

In particular, θ is equal to 1 when $\alpha = \beta$. Otherwise, $\theta < 1$ when $\alpha < \beta$, and $\theta > 1$ when $\alpha > \beta$.

It might be surprising to find that T is always finite whatever the value of $\alpha > 0$ and $\beta \geq 0$, in particular even if Y is transient (*i.e.* $\beta > 2$). A possible interpretation is the following law of the iterated logarithm, see [7], which states that a squared Bessel process Y of dimension $\beta > 2$ satisfies:

$$ \limsup_{t\to+\infty} \frac{Y_t}{2t\ln(\ln(t))} = 1 \quad \text{a.s.} \quad \text{and} \quad \limsup_{t\to+\infty} \frac{\inf_{s\geq t} Y_s}{2t\ln(\ln(t))} = 1 \quad \text{a.s.} $$

In particular, the limit sup of squared Bessel processes does not depend on the dimension. So in our set-up, even if β is large, the fluctuations of the future infimum of the "upper" process Y remain of the same order as those of the "lower" process X, which might explain why a crossing always occurs with probability one.

The hypergeometric function appearing in Proposition 1.1 may be computed explicitly in several situations.

1. For instance, when $\alpha = \beta$, we have from [8], p. 557, Formula (15.1.24):

$$F_{\alpha,\alpha}(s) = {}_2F_1\left[\begin{matrix}\alpha - 1 + s & -s \\ \frac{\alpha}{2} & \end{matrix}; \frac{1}{2}\right] = \sqrt{\pi}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s+\alpha}{2}\right)}, \qquad \text{and} \quad \theta = 1. \tag{1.1}$$

2. Another example is the case $\alpha + \beta = 4$ for which we obtain from Bailey's formula [8], p. 557, Formula (15.1.26):

$$F_{\alpha,4-\alpha}(s) = {}_2F_1\left[\begin{matrix}1 + s & -s \\ \frac{\alpha}{2} & \end{matrix}; \frac{1}{2}\right] = 2^{1-\frac{\alpha}{2}}\sqrt{\pi}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{2+2s+\alpha}{4}\right)\Gamma\left(\frac{\alpha-2s}{4}\right)}, \qquad \text{and} \quad \theta = \frac{\alpha}{2}.$$

3. As a last example, we assume that $\alpha = \beta + 2$. From [8], p. 557, Formula (15.1.25), we deduce that

$$F_{\beta+2,\beta}(s) = \frac{2\sqrt{\pi}}{\beta + 2s}\Gamma\left(\frac{\beta}{2} + 1\right)\left(\frac{1}{\Gamma\left(\frac{\beta+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\Gamma\left(\frac{\beta+s+1}{2}\right)\Gamma\left(-\frac{s}{2}\right)}\right).$$

   In this case, θ does not seem to admit an explicit expression.

We now show that θ actually controls the decay of T. The underlying idea, which was already successfully applied in [9, 10], is as follows. If T were independent of X, then, by scaling, the random variables X_T and $T \times X_1$ would have the same distribution. Since X_1 admits moments of all orders, this would imply that the finiteness of the fractional moments of X_T and T should be equivalent. This heuristic result turns out to be correct, as evidenced by the following theorem.

Theorem 1.2. *For $\lambda \geq 0$ and $0 \leq x < y$, we have the equivalence:*

$$\mathbb{E}_{(x,y)}\left[T^\lambda\right] < +\infty \qquad \Longleftrightarrow \qquad \lambda < \theta.$$

In particular, when $\alpha > \beta$, it holds

$$\mathbb{E}_{(x,y)}\left[T\right] = \frac{y - x}{\alpha - \beta}.$$

Remark 1.3. A consequence of this result is the monotonicity of θ in its arguments. Indeed, let us emphasize the dependence in α and β by writing $T = T_{\alpha,\beta}$. Using the comparison theorem for SDE [11], Chapter IX, Theorem 3.8 and a coupling argument, it is immediate to see that for $\lambda > 0$, the function $(\alpha, \beta) \mapsto \mathbb{E}[T^\lambda_{\alpha,\beta}]$ is decreasing in α and increasing in β. As a consequence, the same is true for θ, which is thus also non-decreasing in α and non-increasing in β.

We finally give an asymptotics for the tail decay of T. To this end, observe that from Lebedev [12], Section 9.4, the function $F_{\alpha,\beta}$ is an entire function. As a consequence, there exists $m \in \mathbb{N}$ such that the following factorization holds

$$F_{\alpha,\beta}(s) = (s - \theta)^m G_{\alpha,\beta}(s)$$

where $G_{\alpha,\beta}$ is such that $G_{\alpha,\beta}(\theta) \neq 0$.

Corollary 1.4. *Let $\delta > 0$. There exist two positive constants κ_1 and κ_2 such that*

$$\frac{\kappa_1}{t^{\theta+\delta}} \leq \mathbb{P}_{(0,y)}\left(T \geq t\right) \leq \kappa_2 \frac{(\ln(t))^m}{t^\theta}, \qquad \text{as } t \to +\infty.$$

When $\alpha = \beta = n \in \mathbb{N}$, it is well-known that there exist two independent n-dimensional Brownian motions B and W such that $\|B\|_2^2 = X$ and $\|W\|_2^2 = Y$ where $\|\,\boldsymbol{.}\,\|_2$ denotes the standard Euclidean norm. Therefore, Theorem 1.2 and Corollary 1.4 state that the probability that B remains closer to the origin than W up to time t decays essentially as a power -1, independently of the dimension. In dimension 1, this is an immediate consequence of the fact that $S = (W+B)/\sqrt{2}$ and $D = (W-B)/\sqrt{2}$ are independent Brownian motions. Indeed, if we denote by $\tau_a^Z = \inf\{t \geq 0,\ Z_t = a\}$ the first hitting time of $a \in \mathbb{R}$ by a stochastic process Z, then

$$\begin{aligned} T &= \inf\{t \geq 0, B_t^2 = W_t^2\} \\ &= \inf\{t \geq 0, (W_t + B_t)(W_t - B_t) = 0\} = \inf\{t \geq 0, S_t\, D_t = 0\} = \tau_0^S \wedge \tau_0^D \end{aligned}$$

hence, since $S_0 = D_0 = \sqrt{y/2}$ under $\mathbb{P}_{(0,y)}$,

$$\mathbb{P}_{(0,y)}(T \geq t) = \mathbb{P}_{(0,y)}(\tau_0^S \geq t, \tau_0^D \geq t) = \left(\mathbb{P}_{(0,y)}(\tau_0^S \geq t)\right)^2 \underset{t\to+\infty}{\sim} \left(\sqrt{\frac{y}{2}}\sqrt{\frac{2}{\pi t}}\right)^2 = \frac{y}{\pi t}.$$

Remark 1.5. To complement Corollary 1.4, we note that the short-time asymptotics of T decays (at least) exponentially. Indeed, since the paths of X and Y are continuous, we have:

$$\mathbb{P}_{(0,y)}(T \leq t) = \mathbb{P}_{(0,y)}\left(T \leq t, \tau^X_{\inf_{[0,t]} Y_s} \leq t\right) \leq \mathbb{P}_{(0,y)}\left(\tau^X_{\inf_{[0,t]} Y_s} \leq t\right).$$

We then decompose this probability into two terms:

$$\begin{aligned} &\mathbb{P}_{(0,y)}\left(\tau^X_{\inf_{[0,t]} Y_s} \leq t,\ \inf_{[0,t]} Y_s > \frac{y}{2}\right) + \mathbb{P}_{(0,y)}\left(\tau^X_{\inf_{[0,t]} Y_s} \leq t,\ \inf_{[0,t]} Y_s \leq \frac{y}{2}\right) \\ &\qquad\qquad \leq \mathbb{P}_{(0,y)}\left(\tau^X_{y/2} \leq t\right) + \mathbb{P}_{(0,y)}\left(\tau^Y_{y/2} \leq t\right). \end{aligned}$$

The result now follows from the fact that the short-time asymptotics of the hitting times of squared Bessel processes decay exponentially. Specifically, if Z is a squared Bessel process with index ν started from $z \geq 0$, the Laplace transform of τ_a^Z is given by

$$\mathbb{E}_z\left[\mathrm{e}^{-\lambda \tau_a^Z}\right] = \begin{cases} \dfrac{z^{-\nu/2} I_\nu(\sqrt{2\lambda z})}{a^{-\nu/2} I_\nu(\sqrt{2\lambda a})} & \text{if } z \leq a, \\ \dfrac{z^{-\nu/2} K_\nu(\sqrt{2\lambda z})}{a^{-\nu/2} K_\nu(\sqrt{2\lambda a})} & \text{if } z \geq a \end{cases}$$

where I_ν and K_ν denote the usual modified Bessel functions. Using an integration by parts and letting $\lambda \to +\infty$, we deduce that

$$\int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathbb{P}_z\left(\tau_a^Z \leq t\right) dt \underset{\lambda\to+\infty}{\sim} \frac{z^{-\nu/2}}{a^{-\nu/2}} \frac{1}{\lambda} \mathrm{e}^{-\sqrt{2\lambda}|\sqrt{z} - \sqrt{a}|}.$$

Finally, applying the Tauberian theorem of exponential type (see [13], Thm. 4.2.9), we conclude that

$$\lim_{t\downarrow 0} t \ln\left(\mathbb{P}_z\left(\tau_a^Z \leq t\right)\right) = -\frac{(\sqrt{z}-\sqrt{a})^2}{2}.$$

1.2. The values of θ

When $\alpha = \beta$, we have already checked that $\theta = 1$. We now prove that θ is smaller or greater than 1, according as whether α is smaller or greater than β.

1. Assume that $\alpha < \beta$. Then, taking $s = 1$, the hypergeometric function simplifies to

$$F_{\alpha,\beta}(1) = {}_2F_1\left[\begin{matrix}\frac{\alpha+\beta}{2} - 1 \\ \frac{\alpha}{2}\end{matrix}; \frac{1}{2}\right] = 1 - \frac{1}{\alpha}\left(\frac{\alpha+\beta}{2}\right) < 0$$

hence, by continuity, since $F_{\alpha,\beta}(0) = 1$, its first zero θ lies in $(0,1)$.

2. Assume now that $\alpha > \beta$ and take $s \in (0,1)$. We have by definition, with $\gamma = \frac{\alpha+\beta}{2} - 1$:

$$F_{\alpha,\beta}(s) = 1 - \frac{s(\gamma+s)}{\alpha}\left(1 + \sum_{n=1}^{+\infty} \frac{(1-s)_n(\gamma+s+1)_n}{(\alpha/2+1)_n}\frac{1}{(n+1)!}\left(\frac{1}{2}\right)^n\right).$$

Now, on the one hand, if $\gamma + s \in (-1, 0)$, then all the terms are positive and $F_{\alpha,\beta}(s) > 0$. On the other hand, if $\gamma + s \geq 0$, by letting β increase up to α, we deduce that

$$F_{\alpha,\beta}(s) \geq F_{\alpha,\alpha}(s) > 0,$$

which is strictly positive from (1.1) since $s \in (0,1)$. As a consequence, θ is strictly greater than one since $F_{\alpha,\beta}(1) > 0$.

We provide Figure 2 a numerical simulation[1] of $(\alpha,\beta) \to \theta(\alpha,\beta)$ in which one can observe the monotonicity of θ in its arguments. Another interesting feature of this graph is that there appear to be different regimes as $\alpha \downarrow 0$, depending on whether $\beta > 2$ or $\beta < 2$. We will discuss these limits in Section 6.

The remainder of the paper is organized as follows. In Section 2, we compute the Mellin transform of X_T under $\mathbb{P}_{(0,y)}$. In Section 3, we show that the finiteness of the fractional moments of T does not depend on the starting points (x,y), allowing us to restrict our attention to the case $x = 0$. The proof of Theorem 1.2 is then given in Section 4.1, while that of Corollary 1.4 is provided in Section 5. Finally, Section 6 studies the limit of $\theta(\alpha,\beta)$ as $\alpha \downarrow 0$.

2. Computation of the Mellin transform of X_T

We start by proving Proposition 1.1. Recall that under $\mathbb{P}_{(x,y)}$ the pair (X,Y) is a solution of the SDE, see [11], Chapter XI:

$$X_t = x + 2\int_0^t \sqrt{X_s}dB_s + \alpha t \qquad \text{and} \qquad Y_t = y + 2\int_0^t \sqrt{Y_s}dW_s + \beta t, \qquad t \geq 0, \tag{2.1}$$

[1]The graph has been obtained using the *FindRoot* and *Hypergeometric2F1* functions in Mathematica.

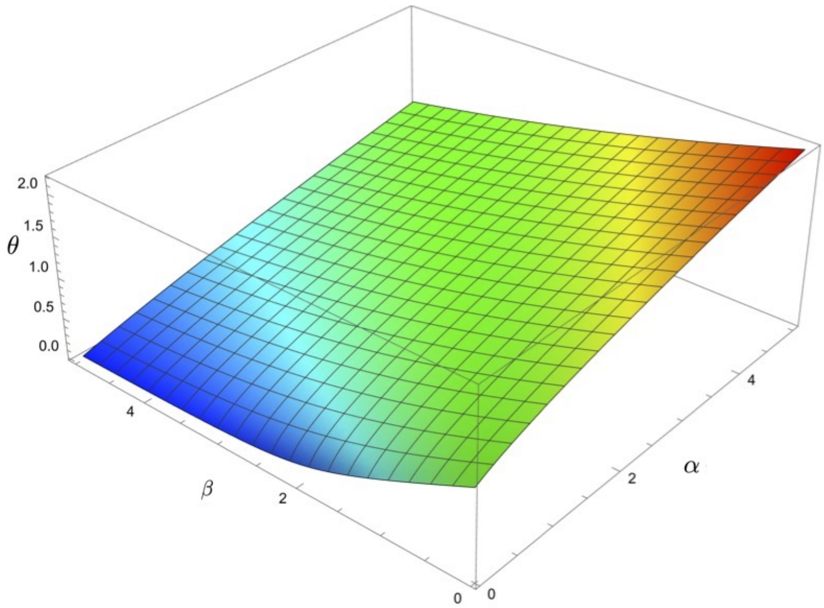

FIGURE 2. Simulation of $\theta(\alpha,\beta)$ for $(\alpha,\beta)\in(0,5)^2$.

where B and W are two independent Brownian motions. In the following, to simplify the notation, we shall write for a random variable Z and for $\gamma\in\mathbb{R}$:

$$\mathbb{E}[Z_+^\gamma]=\mathbb{E}[Z^\gamma 1_{\{Z>0\}}].$$

Take $r,\nu\in(0,1)$ such that $r+\nu>1$ and $\nu<\frac{\alpha+\beta}{2}$. Applying the Markov property and the scaling property of squared Bessel processes, we have

$$\begin{aligned}\int_0^{+\infty} t^{-r}\mathbb{E}_{(0,y)}\left[(X_t-Y_t)_+^{-\nu}\right]dt &= \int_0^{+\infty}\mathbb{E}_{(0,y)}\left[(u+T)^{-r}\mathbb{E}_{(X_T,X_T)}\left[(X_u-Y_u)_+^{-\nu}\right]1_{\{T<+\infty\}}\right]du\\ &= \int_0^{+\infty}\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}(sX_T+T)^{-r}1_{\{T<+\infty\}}\right]\mathbb{E}_{(1,1)}\left[(X_s-Y_s)_+^{-\nu}\right]\mathrm{d}s. \quad (2.2)\end{aligned}$$

Note that we cannot just take $r=0$, as both sides would be infinite. Indeed, in this case, for the left-hand side to be finite, the integral in t requires that $\nu>1$ while the expectation requires that $\nu<1$. We shall thus evaluate semi-explicitly both sides, show that the poles at $r+\nu=1$ cancel, and then use analytic continuation. To compute the positive parts in (2.2), recall the formula for any positive r.v. Z which admits some negative moments, see [10], Lemma 1:

$$\mathbb{E}\left[Z_+^{-\nu}\right]=\frac{\Gamma(1-\nu)}{\pi}\int_0^{+\infty}\lambda^{\nu-1}\mathbb{E}\left[\sin\left(\lambda Z+\frac{\pi}{2}\nu\right)\right]\mathrm{d}\lambda.$$

From [11], Chapter XI, p. 441 and analytic continuation, the Fourier transform of $X-Y$ is given, since X and Y are independent, by

$$\mathbb{E}_{(x,y)}\left[\mathrm{e}^{\imath\lambda(X_t-Y_t)}\right]=(1-2\imath\lambda t)^{-\alpha/2}(1+2\imath\lambda t)^{-\beta/2}\exp\left(\frac{\imath\lambda(x-y)-2\lambda^2t(x+y)}{1+4\lambda^2t^2}\right)$$

hence, after a change of variable

$$\frac{\pi}{\Gamma(1-\nu)}\mathbb{E}_{(x,y)}\left[(X_t-Y_t)_+^{-\nu}\right]$$
$$= t^{-\nu}\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}\nu}\int_0^{+\infty}\xi^{\nu-1}(1-2\imath\xi)^{-\alpha/2}(1+2\imath\xi)^{-\beta/2}\exp\left(\frac{\imath\xi(x-y)-2\xi^2(x+y)}{t(1+4\xi^2)}\right)\mathrm{d}\xi\right) \quad (2.3)$$

where $\Im$ denotes the imaginary part. On the one hand, taking $x=0$ in (2.3) and plugging this expression in (2.2), we deduce thanks to Fubini's theorem that the left-hand side of (2.2) equals

$$\frac{\Gamma(1-\nu)\Gamma(r+\nu-1)}{\pi y^{r+\nu-1}}\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}(1-r)}\int_0^{+\infty}\xi^{-r}(1-2\imath\xi)^{-\frac{\alpha}{2}}(1+2\imath\xi)^{r+\nu-1-\frac{\beta}{2}}\right)\mathrm{d}\xi.$$

Similarly, plugging (2.3) with $x=y=1$ in the right-hand side of (2.2) and applying Fubini's theorem, we obtain

$$\frac{\Gamma(1-\nu)\Gamma(r+\nu-1)}{\pi}\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}\nu}\int_0^{+\infty}\xi^{\nu-1}(1-2\imath\xi)^{-\alpha/2}(1+2\imath\xi)^{-\beta/2}G_{r,\nu}(\xi)\mathrm{d}\xi\right)$$

where $G_{r,\nu}$ is defined by

$$G_{r,\nu}(\xi):=\frac{1}{\Gamma(r+\nu-1)}\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}\int_0^{+\infty}(uX_T+T)^{-r}u^{-\nu}\mathrm{e}^{-\frac{4\xi^2}{u(1+4\xi^2)}}du\,1_{\{T<+\infty\}}\right].$$

Setting $\eta(\xi)=\dfrac{4\xi^2}{1+4\xi^2}$, using the change of variables $u=1/s$ and an integration by parts, the integral reads:

$$\int_0^{+\infty}(uX_T+T)^{-r}u^{-\nu}\mathrm{e}^{-\eta(\xi)/u}du=\int_0^{+\infty}(X_T+Ts)^{-r}s^{\nu+r-2}\mathrm{e}^{-s\eta(\xi)}\mathrm{d}s$$
$$=\frac{1}{r+\nu-1}\int_0^{+\infty}(X_T+Ts)^{-r}s^{\nu+r-1}\mathrm{e}^{-s\eta(\xi)}\left(r\frac{T}{X_T+Ts}+\eta(\xi)\right)\mathrm{d}s.$$

Note that in the integration by parts, the boundary parts cancel since $\nu+r>1$. As a consequence, using the functional equation of the Gamma function, $G_{r,\nu}$ equals

$$G_{r,\nu}(\xi)=\frac{1}{\Gamma(\nu+r)}\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}\int_0^{+\infty}s^{\nu+r-1}(X_T+Ts)^{-r}\mathrm{e}^{-\eta(\xi)s}\left(r\frac{T}{X_T+Ts}+\eta(\xi)\right)\mathrm{d}s\,1_{\{T<+\infty\}}\right].$$

Simplifying the Gamma terms, Equation (2.2) now reads

$$\frac{1}{y^{r+\nu-1}}\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}(1-r)}\int_0^{+\infty}\xi^{-r}(1-2\imath\xi)^{-\frac{\alpha}{2}}(1+2\imath\xi)^{r+\nu-1-\frac{\beta}{2}}\mathrm{d}\xi\right)$$
$$=\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}\nu}\int_0^{+\infty}\xi^{\nu-1}(1-2\imath\xi)^{-\frac{\alpha}{2}}(1+2\imath\xi)^{-\frac{\beta}{2}}G_{r,\nu}(\xi)\mathrm{d}\xi\right) \quad (2.4)$$

and this formula extends by analyticity to $r+\nu>0$. By monotone convergence, we then deduce that

$$G_{r,\nu}(\xi)\xrightarrow[r\downarrow 0]{}\frac{1}{\Gamma(\nu)}\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}\int_0^{+\infty}s^{\nu-1}\mathrm{e}^{-\eta(\xi)s}\eta(\xi)\mathrm{d}s\,1_{\{T<+\infty\}}\right]=\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}\,1_{\{T<+\infty\}}\right](\eta(\xi))^{1-\nu}.$$

As a consequence, letting $r \downarrow 0$ in (2.4), we obtain the formula

$$\frac{1}{y^{\nu-1}}\Im\left(\imath\int_0^{+\infty}(1-2\imath\xi)^{-\frac{\alpha}{2}}(1+2\imath\xi)^{\nu-1-\frac{\beta}{2}}\mathrm{d}\xi\right) = \mathbb{E}_{(0,y)}\left[X_T^{1-\nu}1_{\{T<+\infty\}}\right]\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}\nu}\int_0^{+\infty}\xi^{1-\nu}(1-2\imath\xi)^{\nu-1-\alpha/2}(1+2\imath\xi)^{\nu-1-\beta/2}\mathrm{d}\xi\right) \quad (2.5)$$

and it remains to compute both integrals. Let us start with the left-hand side. Taking the imaginary part in Lemma A.1 from the Appendix, we have

$$\Im\left(\imath\int_0^{+\infty}\xi^{\gamma-1}(1-2\imath\xi)^{-\lambda}(1+2\imath\xi)^{-\mu}\mathrm{d}\xi\right) = \frac{\Gamma(\gamma)\Gamma(\mu+\lambda-\gamma)}{\Gamma(\lambda)\Gamma(\mu)}2^{-\gamma}\left(\cos\left(\frac{\pi}{2}\gamma\right)B\left(\lambda,1-\gamma\right)2^{-\lambda}{}_2F_1\left[\begin{matrix}1-\mu & \lambda\\ \lambda-\gamma+1\end{matrix};\frac{1}{2}\right] + \cos\left(\frac{\pi}{2}\gamma\right)B\left(\mu,1-\gamma\right)2^{-\mu}{}_2F_1\left[\begin{matrix}1-\lambda & \mu\\ \mu-\gamma+1\end{matrix};\frac{1}{2}\right]\right)$$

where $B(x,y)$ denotes the usual Beta function. Letting the parameter $\gamma \uparrow 1$ and using the asymptotics $B(x,y) \underset{y\to 0}{\sim} \frac{1}{y}$ together with the identity ${}_2F_1\left[\begin{matrix}a & b\\ & b\end{matrix};\frac{1}{2}\right] = 2^{-a}$, we thus deduce that

$$\Im\left(\imath\int_0^{+\infty}(1-2\imath\xi)^{-\frac{\alpha}{2}}(1+2\imath\xi)^{\nu-1-\frac{\beta}{2}}\mathrm{d}\xi\right) = \frac{\pi\Gamma\left(\frac{\alpha+\beta}{2}-\nu\right)}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2}+1-\nu)}2^{\nu-\frac{\alpha+\beta}{2}-1}. \quad (2.6)$$

On the other side, multiplying Lemma A.1 in the Appendix by $\mathrm{e}^{\imath\frac{\pi}{2}\nu}$ and taking the imaginary part, we observe that only the first term will remain, *i.e.*

$$\Im\left(\mathrm{e}^{\imath\frac{\pi}{2}\nu}\int_0^{+\infty}\xi^{1-\nu}(1-2\imath\xi)^{\nu-1-\alpha/2}(1+2\imath\xi)^{\nu-1-\beta/2}\mathrm{d}\xi\right) = -\frac{\Gamma(2-\nu)\Gamma\left(\frac{\alpha+\beta}{2}-\nu\right)}{\Gamma\left(\frac{\beta}{2}+1-\nu\right)}2^{\nu-2}\sin(\pi\nu)\frac{\Gamma(\nu-1)}{\Gamma\left(\frac{\alpha}{2}\right)}2^{\nu-1-\frac{\alpha}{2}}{}_2F_1\left[\begin{matrix}\nu-\frac{\beta}{2} & \frac{\alpha}{2}+1-\nu\\ & \frac{\alpha}{2}\end{matrix};\frac{1}{2}\right]. \quad (2.7)$$

Plugging (2.6) and (2.7) in (2.5) and using the reflection formula for the Gamma function, *i.e.* $\Gamma(2-\nu)\sin(\pi\nu)\Gamma(\nu-1) = -\pi$ for $\nu\in(0,1)$, we finally conclude that

$$\mathbb{E}_{(0,y)}\left[X_T^{1-\nu}1_{\{T<+\infty\}}\right] = (2y)^{1-\nu}\frac{2^{1-\frac{\beta}{2}}}{{}_2F_1\left[\begin{matrix}\nu-\frac{\beta}{2} & \frac{\alpha}{2}+1-\nu\\ & \frac{\alpha}{2}\end{matrix};\frac{1}{2}\right]} \quad (2.8)$$

and the result follows from Euler's transformation

$${}_2F_1\left[\begin{matrix}\nu-\frac{\beta}{2} & \frac{\alpha}{2}+1-\nu\\ & \frac{\alpha}{2}\end{matrix};\frac{1}{2}\right] = 2^{1-\frac{\beta}{2}}{}_2F_1\left[\begin{matrix}\frac{\alpha+\beta}{2}-\nu & \nu-1\\ \frac{\alpha}{2} & \end{matrix};\frac{1}{2}\right] = 2^{1-\frac{\beta}{2}}F_{\alpha,\beta}(1-\nu).$$

In particular, letting $\nu\uparrow 1$ in (2.8), we deduce that $\mathbb{P}_{(0,y)}(T<+\infty)=1$, *i.e.*, that T is a.s. finite. $\square$

3. Some first estimates

3.1. From $\mathbb{P}_{(x,y)}$ to $\mathbb{P}_{(0,1)}$

Before tackling the proof of Theorem 1.2, we gather here some preliminary computations. We start by proving that the finiteness of $\mathbb{E}_{(x,y)}[T^\lambda]$ is equivalent to that of $\mathbb{E}_{(0,1)}[T^\lambda]$. Working with $x=0$ will be key, as we aim to compare the fractional moments of T with those of X_T given in Proposition 1.1.

Lemma 3.1. *Let $0\le x<y$ be fixed. There exist two positive constants κ_1,κ_2 such that for $0\le\lambda\le\theta$,*

$$\kappa_1\mathbb{E}_{(0,1)}\left[T^\lambda\right]-\kappa_2\le\mathbb{E}_{(x,y)}\left[T^\lambda\right]\le y^\lambda\mathbb{E}_{(0,1)}\left[T^\lambda\right].$$

Proof. Notice first that by a coupling argument $\mathbb{P}_{(x,y)}(T\ge t)\le\mathbb{P}_{(0,y)}(T\ge t)$ which implies by scaling that for $\lambda\ge 0$:

$$\mathbb{E}_{(x,y)}\left[T^\lambda\right]\le\mathbb{E}_{(0,y)}\left[T^\lambda\right]=y^\lambda\mathbb{E}_{(0,1)}\left[T^\lambda\right].$$

To prove a converse inequality, take $a\in(0,y)$ and recall that $\tau_y^Y=\inf\{t\ge0,\ Y_t=y\}$ denotes the first hitting time of y by the process Y. Then, applying the strong Markov property at the time τ_y^Y,

$$\begin{aligned}\mathbb{P}_{(0,a)}(T>t)&=\mathbb{P}_{(0,a)}(T>t,T\le\tau_y^Y)+\mathbb{P}_{(0,a)}(T>t,T>\tau_y^Y)\\&\le2\mathbb{P}_{(0,a)}(\tau_y^Y>t)+\mathbb{E}_{(0,a)}\left[\mathbb{P}_{(X_{\tau_y^Y},y)}(T>t-s)_{|s=\tau_y^Y}1_{\{T>\tau_y^Y\}}1_{\{\tau_y^Y\le t\}}\right]\end{aligned}$$

hence, for $\lambda\le\theta$, using the standard inequality $(a+b)^\lambda\le2^\lambda(a^\lambda+b^\lambda)$,

$$\begin{aligned}\mathbb{E}_{(0,a)}[T^\lambda]&\le2\mathbb{E}_{(0,a)}[(\tau_y^Y)^\lambda]+\int_0^y\int_0^{+\infty}\mathbb{E}_{(z,y)}[(T+s)^\lambda]\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\in\mathrm{d}z,\tau_y^Y\in\mathrm{d}s)\\&\le2\mathbb{E}_{(0,a)}[(\tau_y^Y)^\lambda]+2^\lambda\int_0^y\int_0^{+\infty}\left(\mathbb{E}_{(z,y)}[T^\lambda]+s^\lambda\right)\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\in\mathrm{d}z,\tau_y^Y\in\mathrm{d}s)\\&\le(2+2^\theta)\mathbb{E}_{(0,a)}[(\tau_y^Y)^\lambda]+2^\theta\int_0^y\mathbb{E}_{(z,y)}[T^\lambda]\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\in\mathrm{d}z).\end{aligned}\tag{3.1}$$

Recall then that, since $Y_0=a<y$, the r.v. τ_y^Y admits positive moments of all orders, see for instance [14], so the first term on the right-hand side is always finite. Now let $\delta\in(0,y)$. Since the function $z\to\mathbb{E}_{(z,y)}[T^\lambda]$ is decreasing on $[0,y]$, we have

$$\int_0^y\mathbb{E}_{(z,y)}[T^\lambda]\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\in\mathrm{d}z)\le\mathbb{E}_{(0,y)}[T^\lambda]\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\le\delta)+\mathbb{E}_{(\delta,y)}[T^\lambda].$$

As a consequence, plugging this last inequality in (3.1) and using the scaling property, we obtain the bound

$$\left(a^\lambda-2^\theta y^\lambda\mathbb{P}_{(0,a)}(X_{\tau_y^Y}\le\delta)\right)\mathbb{E}_{(0,1)}[T^\lambda]\le(2+2^\theta)\mathbb{E}_{(0,a)}[(\tau_y^Y)^\lambda]+2^\theta\mathbb{E}_{(\delta,y)}[T^\lambda]\tag{3.2}$$

in which we shall take δ small enough for the left-hand side to be strictly positive. Finally, it remains to write, applying the Markov property,

$$\mathbb{E}_{(x,y)}[(T-1)^\lambda1_{\{T>1\}}]\ge\mathbb{E}\left[1_{\{T>1,Y_1>y,X_1\le\delta\}}\mathbb{E}_{(X_1,Y_1)}[T^\lambda]\right]\ge\mathbb{E}\left[1_{\{T>1,Y_1>y,X_1\le\delta\}}\right]\mathbb{E}_{(\delta,y)}[T^\lambda]$$

hence, going back to (3.2), we have thus proven that for $0 \leq \lambda \leq \theta$,

$$\kappa_1 \mathbb{E}_{(0,1)}\left[T^\lambda\right] \leq \kappa_2 + \mathbb{E}_{(x,y)}[(T-1)^\lambda 1_{\{T>1\}}] \leq \kappa_2 + \mathbb{E}_{(x,y)}[T^\lambda]$$

where κ_1, κ_2 are two positive constants. This is the lower bound of Lemma 3.1. □

3.2. Maximal inequalities

To go from X_T to T, we shall rely on maximal inequalities for squared Bessel processes. Indeed, recall from DeBlassie [15] that there exist two constants $c_{\alpha,\lambda}$ and $C_{\alpha,\lambda}$ such that for any stopping time ζ (with respect to X):

$$c_{\alpha,\lambda}\mathbb{E}_{(x,y)}[(\zeta+x)^\lambda] \leq \mathbb{E}_{(x,y)}\left[\sup_{s\leq\zeta} X_s^\lambda\right] \leq C_{\alpha,\lambda}\mathbb{E}_{(x,y)}[(\zeta+x)^\lambda]. \tag{3.3}$$

As a consequence, taking $\zeta = T$, we first deduce that

$$\mathbb{E}_{(x,y)}\left[X_T^\lambda\right] \leq \mathbb{E}_{(x,y)}\left[\sup_{s\leq T} X_s^\lambda\right] \leq C_{\alpha,\lambda}\mathbb{E}_{(x,t)}[(T+x)^\lambda] \tag{3.4}$$

which implies that $\mathbb{E}_{(x,y)}\left[T^\lambda\right] = +\infty$ as soon as $\lambda \geq \theta$. Note that a converse inequality exists: from Pedersen [16], if $\lambda > 1-\alpha/2$ and ζ is a stopping time such that $\mathbb{E}_{(x,y)}[\zeta^\lambda] < +\infty$, then

$$\mathbb{E}_{(x,y)}\left[\sup_{s\leq\zeta} X_s^\lambda\right] \leq \left(\frac{2\lambda}{2\lambda-(2-\alpha)}\right)^{\frac{2\lambda}{2-\alpha}} \mathbb{E}_{(x,y)}\left[X_\zeta^\lambda\right]. \tag{3.5}$$

Unfortunately, to apply this inequality in our set-up, one needs the assumption that T admits moments of order up to θ, which is precisely what we wish to prove.

4. Proof of Theorem 1.2

We prove in this section that $\mathbb{E}_{(0,y)}[T^\lambda]$ is finite for $0 \leq \lambda < \theta$. As explained in the first section, the idea is to compare the moments of T with those of X_T. To do so, we construct martingales involving both T and X_T. Due to a technical difficulty, we will need to separate the cases $\alpha \leq \beta$ and $\alpha > \beta$, constructing two different martingales accordingly.

4.1. The case $\alpha \leq \beta$

Recall that in this case $\theta \in (0,1]$. We first show that $\theta > 1-\frac{\beta}{2}$. Indeed, since the paths of X and Y are continuous and non-negative, the path inequality $T \leq \tau_0^Y$ implies that $\mathbb{E}_{(0,1)}[T^\lambda] \leq \mathbb{E}_{(0,1)}[(\tau_0^Y)^\lambda]$. The right-hand side is finite if and only if $\beta < 2$ and $\lambda < 1-\frac{\beta}{2}$, see for instance [17]. As a consequence, we deduce from (3.4) the lower bound $\theta > 1-\frac{\beta}{2}$, with a strict inequality since $F_{\alpha,\beta}\left(1-\frac{\beta}{2}\right) = 2^{\frac{\beta}{2}-1} > 0$.

We now construct a martingale. Let us set for $z > 0$

$$\Psi_\lambda^{(\beta)}(z) := z^{\frac{\eta}{2}} K_\eta\left(\sqrt{2\lambda z}\right) = 2^{\frac{\eta}{2}-1}\lambda^{\frac{\eta}{2}} \int_0^{+\infty} \mathrm{e}^{-\lambda s}\mathrm{e}^{-\frac{z}{2s}} s^{\eta-1}\mathrm{d}s \tag{4.1}$$

where $\eta = 1 - \frac{\beta}{2} \in (-\infty, 1)$ and K_η denotes the usual McDonald's function. In particular, $\Psi_\lambda^{(\beta)}$ is a solution of the ordinary differential equation:

$$zf''(z) + \frac{\beta}{2} f'(z) - \frac{\lambda}{2} f(z) = 0 \tag{4.2}$$

such that $\lim\limits_{z\to+\infty} \Psi_\lambda^{(\beta)}(z) = 0$. Take $Y_0 = y > 0$ and $0 < \varepsilon < y$. Applying Itô's formula, we deduce from (4.2) that the process

$$M_t = \mathrm{e}^{-\lambda(t\wedge\tau_\varepsilon^Y)} \Psi_\lambda^{(\beta)} \left(Y_{t\wedge\tau_\varepsilon^Y}\right), \quad t \geq 0$$

is a bounded local martingale, hence a true martingale under $\mathbb{P}_{(0,y)}$. Then Doob's optional sampling theorem yields the identity

$$\mathbb{E}_{(0,y)} \left[\mathrm{e}^{-\lambda(T\wedge\tau_\varepsilon^Y)} \Psi_\lambda^{(\beta)} \left(Y_{T\wedge\tau_\varepsilon^Y}\right)\right] = \Psi_\lambda^{(\beta)}(y). \tag{4.3}$$

4.1.1. We first assume that $\eta < 0$

In this case, we may let $\lambda \downarrow 0$ in (4.3) and apply the monotone convergence theorem to obtain

$$\int_0^{+\infty} s^{\eta-1} \mathbb{E}_{(0,y)} \left[\mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}}\right] \mathrm{d}s = \int_0^{+\infty} s^{\eta-1} \mathrm{e}^{-\frac{y}{2s}} \mathrm{d}s \tag{4.4}$$

i.e. $\mathbb{E}_{(0,y)}\left[Y^\eta_{T\wedge\tau_\varepsilon^Y}\right] = y^\eta$. Then, subtracting $\mathbb{E}_{(0,y)}\left[\Psi_\lambda^{(\beta)}\left(Y_{T\wedge\tau_\varepsilon^Y}\right)\right]$ on both sides of (4.3), we obtain from (4.1) and (4.4), using an integration by parts:

$$\begin{aligned}
&\mathbb{E}_{(0,y)} \left[\int_0^{+\infty} \mathrm{e}^{-\lambda t} 1_{\{T\wedge\tau_\varepsilon^Y > t\}} dt \int_0^{+\infty} \mathrm{e}^{-\lambda s} \mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}} s^{\eta-1} \mathrm{d}s\right] \\
&\qquad = \frac{1}{\lambda} \int_0^{+\infty} \mathrm{e}^{-\lambda s}\ \mathbb{E}_{(0,y)} \left[\mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}} - \mathrm{e}^{-\frac{y}{2s}}\right] s^{\eta-1} \mathrm{d}s \\
&\qquad = \int_0^{+\infty} \mathrm{e}^{-\lambda s} \left(\int_s^{+\infty} \mathbb{E}_{(0,y)} \left[\mathrm{e}^{-\frac{y}{2u}} - \mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2u}}\right] u^{\eta-1} du\right) \mathrm{d}s.
\end{aligned}$$

Inverting these Laplace transforms yields the identity:

$$\int_0^t s^{\eta-1} \mathbb{E}_{(0,y)} \left[1_{\{T\wedge\tau_\varepsilon^Y > t-s\}} \mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}}\right] \mathrm{d}s = \int_t^{+\infty} s^{\eta-1} \mathbb{E}_{(0,y)} \left[\mathrm{e}^{-\frac{y}{2s}} - \mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}}\right] \mathrm{d}s =: H_\varepsilon(t). \tag{4.5}$$

We now compute the Mellin transform of H_ε. Recall the formula, for $a, b > 0$ and $\nu \in (0,1)$:

$$\int_0^{+\infty} t^{\nu-1} \left(\mathrm{e}^{-\frac{a}{t}} - \mathrm{e}^{-\frac{b}{t}}\right) dt = \Gamma(-\nu) \left(a^\nu - b^\nu\right). \tag{4.6}$$

Take $\lambda \in (0, \theta)$. Separating the cases $Y_{T\wedge\tau_\varepsilon^Y} \leq y$ and $Y_{T\wedge\tau_\varepsilon^Y} > y$ and using a change of variables, we may apply the Fubini-Tonelli theorem to obtain

$$\begin{aligned}\int_0^{+\infty} t^{\lambda-\eta-1} H_\varepsilon(t)dt &= \int_0^{+\infty} t^{\lambda-1}\int_1^{+\infty} u^{\eta-1}\mathbb{E}_{(0,y)}\left[\mathrm{e}^{-\frac{y}{2ut}} - \mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2ut}}\right] dudt \\ &= \Gamma(-\lambda)\int_1^{+\infty} u^{\eta-1}\mathbb{E}_{(0,y)}\left[\left(\frac{y}{2u}\right)^\lambda - \left(\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2u}\right)^\lambda\right] du \\ &= \Gamma(-\lambda)\frac{2^{-\lambda}}{\eta-\lambda}\left(\mathbb{E}_{(0,y)}\left[Y^\lambda_{T\wedge\tau_\varepsilon^Y}\right] - y^\lambda\right). \end{aligned} \tag{4.7}$$

We now come back to (4.5) and observe that:

$$\begin{aligned} H_\varepsilon(t) &\geq \mathbb{E}_{(0,y)}\left[1_{\{T\wedge\tau_\varepsilon^Y > t\}}\int_0^t s^{\eta-1}\mathrm{e}^{-\frac{Y_{T\wedge\tau_\varepsilon^Y}}{2s}}\mathrm{d}s\, 1_{\{Y_{T\wedge\tau_\varepsilon}\leq t\}}\right] \\ &\geq \mathbb{P}_{(0,y)}\left(T\wedge\tau_\varepsilon^Y > t, Y_{T\wedge\tau_\varepsilon^Y}\leq t\right)\int_0^t s^{\eta-1}\mathrm{e}^{-\frac{t}{2s}}\mathrm{d}s \\ &\geq t^\eta\left(\mathbb{P}_{(0,y)}\left(T\wedge\tau_\varepsilon^Y > t\right) - \mathbb{P}_{(0,y)}\left(Y_{T\wedge\tau_\varepsilon^Y} > t\right)\right)\int_0^1 u^{\eta-1}\mathrm{e}^{-\frac{1}{2u}}du. \end{aligned}$$

Integrating this relation against $(\lambda-\eta)t^{\lambda-\eta-1}$ on $(0,+\infty)$ for $\lambda\in(0,\theta)$, we deduce from (4.7) that there exists a constant K such that

$$\begin{aligned}\mathbb{E}_{(0,y)}\left[\left(T\wedge\tau_\varepsilon^Y\right)^\lambda\right] &\leq \mathbb{E}_{(0,y)}\left[Y^\lambda_{T\wedge\tau_\varepsilon^Y}\right]\left(1-\frac{2^{-\lambda}K}{\lambda-\eta}\Gamma(-\lambda)\right) + \frac{2^{-\lambda}K}{\lambda-\eta}\Gamma(-\lambda)y^\lambda \\ &\leq \left(\mathbb{E}_{(0,y)}\left[Y_T^\lambda\right]+\varepsilon^\lambda\right)\left(1-\frac{2^{-\lambda}K}{\lambda-\eta}\Gamma(-\lambda)\right) + \frac{2^{-\lambda}K}{\lambda-\eta}\Gamma(-\lambda)y^\lambda. \end{aligned}\tag{4.8}$$

The upper bound of Theorem 1.2 now follows by letting $\varepsilon\downarrow 0$ and applying the monotone convergence theorem. □

4.1.2. We now assume that $\eta\in(0,1)$

Then, the function $\Psi_\lambda^{(\beta)}$ is well-defined and finite at $z=0$. As a consequence, we may take directly $\varepsilon=0$ (*i.e.* remove the stopping time τ_ε^Y) when applying Doob's optional stopping theorem. However, we cannot let $\lambda\downarrow 0$ in (4.3) as both sides of (4.4) would be infinite. Nevertheless, from Proposition 1.1 and (4.6),

$$\int_0^{+\infty} s^{\eta-1}\left(\mathbb{E}_{(0,y)}\left[\mathrm{e}^{-\frac{Y_T}{2s}}\right]-\mathrm{e}^{-\frac{y}{2s}}\right)\mathrm{d}s = \Gamma(-\eta)\left(\mathbb{E}_{(0,y)}\left[Y_T^\eta\right]-y^\eta\right)=0$$

and the previous argument applies mutadis mutandis, taking $\lambda\in(\eta,\theta)$.

4.1.3. We now assume that $\eta=0$, (i.e. $\beta=2$)

To simplify, we will rely here on a coupling argument. Take $\varepsilon>0$ and recall from Remark 1.3 that for $\lambda>0$:

$$\mathbb{E}_{(0,y)}\left[T^\lambda_{\alpha,2}\right]\leq\mathbb{E}_{(0,y)}\left[T^\lambda_{\alpha,2+\varepsilon}\right].$$

Consequently, we deduce from the first part of the proof that $\mathbb{E}_{(0,y)}\left[T^\lambda_{\alpha,2}\right]<+\infty$ for any $\lambda<\theta(\alpha,2+\varepsilon)$. The result then follows by letting $\varepsilon\downarrow 0$, since θ is continuous in its parameters. □

Remark 4.1. Note that this argument no longer holds for $\alpha > \beta$. Indeed, in this latter case $\theta > 1$, but since the right-hand side of (4.8) has a pole at $\lambda = 1$ (coming from the Mellin transform of H_ε), we can only deduce that $\mathbb{E}_{(0,y)}[T^\lambda] < +\infty$ for $\lambda \in (0,1)$.

4.2. The case $\alpha > \beta$

Let us define

$$\gamma = \sup\{s \geq 0,\ \mathbb{E}_{(0,y)}[T^s] < +\infty\}.$$

We already know from (3.4) that $\gamma \leq \theta$ so let us assume that $\gamma < \theta$. We shall prove by contradiction that this is not possible.

4.2.1. First step

We first prove that if $\gamma < \theta$, then $\mathbb{E}_{(0,y)}[T^\gamma] < +\infty$, *i.e.* that the supremum in the definition of γ is attained. To do so, observe first that as explained in Remark 4.1, $\gamma \geq 1$. This may also be checked using the following argument: since $\alpha > \beta$, the SDE (2.1) defining X and Y implies that the process

$$M_{t\wedge T} = Y_{t\wedge T} - X_{t\wedge T} + (\alpha - \beta) t \wedge T = y + \int_0^{t\wedge T} \sqrt{Y_s} dW_s - \int_0^{t\wedge T} \sqrt{X_s} dB_s, \qquad t \geq 0$$

is a positive local martingale, hence a supermartingale. Applying Fatou's lemma, we thus have

$$(\alpha - \beta)\mathbb{E}_{(0,y)}[T] = \mathbb{E}_{(0,y)}\left[\liminf_{t\to+\infty} M_{t\wedge T}\right] \leq \liminf_{t\to+\infty} \mathbb{E}_{(0,y)}\left[M_{t\wedge T}\right] \leq \mathbb{E}_{(0,y)}\left[M_0\right] = y$$

which proves that T is integrable, *i.e.* $\gamma \geq 1$. In particular, $\gamma > 1 - \alpha/2$, which allows us to apply the maximal inequality of Pedersen [16] recalled in (3.5). Take $\varepsilon > 0$. By definition of γ, $\mathbb{E}_{(0,y)}[T^{\gamma-\varepsilon}] < +\infty$, hence we have:

$$\mathbb{E}_{(0,y)}\left[\sup_{s\leq T} X_s^{\gamma-\varepsilon}\right] \leq \left(\frac{2(\gamma-\varepsilon)}{2(\gamma-\varepsilon) - (2-\alpha)}\right)^{\frac{2(\gamma-\varepsilon)}{2-\alpha}} \mathbb{E}_{(0,y)}\left[X_T^{\gamma-\varepsilon}\right].$$

Using (3.3), we thus deduce from Fatou's lemma and the monotone convergence theorem that

$$c_{\alpha,\gamma}\mathbb{E}_{(0,y)}[T^\gamma] \leq c_{\alpha,\gamma} \liminf_{\varepsilon\to 0} \mathbb{E}_{(0,y)}[T^{\gamma-\varepsilon}] \leq \left(\frac{2\gamma}{2\gamma - (2-\alpha)}\right)^{\frac{2\gamma}{2-\alpha}} \left(1 + \mathbb{E}_{(0,y)}\left[X_T^\gamma 1_{\{X_T\geq 1\}}\right]\right) < +\infty$$

since $\gamma < \theta$. As a consequence, the supremum in the definition of γ is attained.

4.2.2. Second step

We shall now contradict the definition of γ by showing that T admits moments of order $\gamma + \varepsilon$ for some $\varepsilon > 0$ small enough. To do so, notice first, still from the SDE (2.1), that since $\alpha > \beta$, the process $(\alpha Y_{t\wedge T} - \beta X_{t\wedge T},\ t \geq 0)$ is a positive local martingale under $\mathbb{P}_{(0,y)}$. Let us set

$$A_t = \int_0^t (\beta^2 X_s + \alpha^2 Y_s) \mathrm{d}s, \qquad t \geq 0,$$

and observe that for $z > 0$, the bounded process

$$\exp\left(z(\beta X_{t\wedge T} - \alpha Y_{t\wedge T}) - \frac{z^2}{2} A_{t\wedge T}\right), \qquad t \geq 0, \tag{4.9}$$

is a positive martingale under $\mathbb{P}_{(0,y)}$. Applying Doob's optional stopping theorem and the dominated convergence theorem, we deduce that

$$\mathbb{E}_{(0,y)}\left[\mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T}\right] = \mathrm{e}^{-z\alpha y}. \tag{4.10}$$

The next lemma shows that the finiteness of the moments of T is equivalent to those of $\sqrt{A_T}$.

Lemma 4.2. *For $\lambda \in [0, \theta)$,*

$$\mathbb{E}_{(0,y)}[T^\lambda] < +\infty \qquad \Longleftrightarrow \qquad \mathbb{E}_{(0,y)}\left[A_T^{\frac{\lambda}{2}}\right] < +\infty.$$

Proof. On the one hand, applying Cauchy-Schwarz's inequality and Formula (3.3),

$$\begin{aligned}\mathbb{E}_{(0,y)}\left[A_T^{\frac{\lambda}{2}}\right] &\leq \mathbb{E}_{(0,y)}\left[T^{\frac{\lambda}{2}}\left(2\alpha^2 \sup_{s\leq T} Y_s\right)^{\frac{\lambda}{2}}\right] \\ &\leq \mathbb{E}_{(0,y)}\left[T^\lambda\right]^{\frac{1}{2}} (\sqrt{2}\alpha)^\lambda \mathbb{E}_{(0,y)}\left[\sup_{s\leq T} Y_s^\lambda\right]^{\frac{1}{2}} \leq (\sqrt{2}\alpha)^\lambda \sqrt{C_{\alpha,\lambda}} \mathbb{E}_{(0,y)}\left[T^\lambda\right]\end{aligned}$$

which proves that if the expectation of T^λ is finite, then so is the expectation of $A_T^{\frac{\lambda}{2}}$. On the other hand, going back to the SDE (2.1) defining X and applying the Burkholder-Davis-Gundy inequality, we deduce that there exists a constant K_λ such that

$$\begin{aligned}\alpha^\lambda \mathbb{E}_{(0,y)}[T^\lambda] &\leq \mathbb{E}_{(0,y)}\left[\left(X_T + \left|\int_0^T \sqrt{X_s} dB_s\right|\right)^\lambda\right] \\ &\leq K_\lambda \left(\mathbb{E}_{(0,y)}[X_T^\lambda] + \mathbb{E}_{(0,y)}\left[\left|\int_0^T X_s \mathrm{d}s\right|^{\frac{\lambda}{2}}\right]\right) \\ &\leq K_\lambda \left(\mathbb{E}_{(0,y)}[X_T^\lambda] + \frac{1}{\alpha^\lambda}\mathbb{E}_{(0,y)}\left[A_T^{\frac{\lambda}{2}}\right]\right)\end{aligned}$$

which concludes the proof of Lemma 4.2 since X_T admits moments of order $\lambda < \theta$. $\square$

Thanks to Lemma 4.2, it is thus enough to prove that $\mathbb{E}_{(0,y)}\left[A_T^{\frac{\gamma+\varepsilon}{2}}\right] < +\infty$ for some $\varepsilon > 0$ small enough such that $\gamma + \varepsilon < \theta$. We now set n equal to the integer part of γ: $n = \lfloor \gamma \rfloor$. Differentiating $n+1$ times Formula (4.10) with respect to z and applying Leibniz rule, we obtain

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (\alpha-\beta)^k \mathbb{E}_{(0,y)}\left[X_T^k A_T^{\frac{n+1-k}{2}} H_{n+1-k}(z\sqrt{A_T}) \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T}\right] = (\alpha y)^{n+1} \mathrm{e}^{-z\alpha y} \tag{4.11}$$

where the sequence $(H_k)_{k\geq 0}$ denotes the Hermite's polynomials, which are defined by:

$$H_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\frac{x^2}{2}}.$$

In particular, for any $k \in \mathbb{N}$, since H_k is a polynomial, there exists a finite constant c_k such that

$$\sup_{x\geq 0} |H_k(x)| \mathrm{e}^{-\frac{1}{4}x^2} < c_k.$$

Take ε small enough such that $n+1-\gamma-\varepsilon > 0$ and $\gamma+\varepsilon < \theta$. We now integrate (4.11) against $z^{n-\gamma-\varepsilon}$ on $(0,+\infty)$, applying repeatedly the Fubini-Tonelli theorem.

1. The right-hand side gives:

$$(\alpha y)^{n+1} \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathrm{e}^{-z\alpha y} \mathrm{d}z = \Gamma(n+1-\gamma-\varepsilon)(\alpha y)^{\gamma+\varepsilon}.$$

2. The term for $k = n+1$ gives:

$$(\alpha-\beta)^{n+1} \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)}\left[X_T^{n+1} \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T}\right] \mathrm{d}z$$
$$\leq \Gamma(n+1-\gamma-\varepsilon)(\alpha-\beta)^{\gamma+\varepsilon} \mathbb{E}_{(0,y)}\left[X_T^{\gamma+\varepsilon}\right] < +\infty.$$

3. The main sum, for $1 \leq k \leq n$, gives:

$$\begin{aligned}
&\left| \sum_{k=1}^n \binom{n+1}{k} (\alpha-\beta)^k \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)}\left[X_T^k A_T^{\frac{n+1-k}{2}} H_{n+1-k}(z\sqrt{A_T}) \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T}\right] \mathrm{d}z \right| \\
&\leq \sum_{k=1}^n \binom{n+1}{k} (\alpha-\beta)^k c_{n+1-k} \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)}\left[X_T^k A_T^{\frac{n+1-k}{2}} \mathrm{e}^{-\frac{z^2}{4}A_T}\right] \mathrm{d}z \\
&= \Gamma\left(\frac{n+1-\gamma-\varepsilon}{2}\right) 2^{n-\gamma-\varepsilon} \sum_{k=1}^n \binom{n+1}{k} (\alpha-\beta)^k c_{n+1-k} \mathbb{E}_{(0,y)}\left[X_T^k A_T^{\frac{\gamma+\varepsilon-k}{2}}\right].
\end{aligned}$$

   Furthermore, applying Hölder's inequality, the expectation in the sum on the right-hand side is smaller than

$$\mathbb{E}_{(0,y)}\left[X_T^k A_T^{\frac{\gamma+\varepsilon-k}{2}}\right] \leq \mathbb{E}_{(0,y)}\left[X_T^{\frac{\gamma k}{k-\varepsilon}}\right]^{\frac{k-\varepsilon}{\gamma}} \mathbb{E}_{(0,y)}\left[A_T^{\frac{\gamma}{2}}\right]^{\frac{\gamma+\varepsilon-k}{\gamma}} < +\infty \tag{4.12}$$

   which is finite provided that ε is small enough so that $\gamma < \theta(1-\varepsilon)$.

4. As for the last term $k=0$, let us take $\delta > 0$ and first write:

$$\begin{aligned}
&\left| \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)}\left[A_T^{\frac{n+1}{2}} H_{n+1}(z\sqrt{A_T}) \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T} 1_{\{\delta\sqrt{A_T}\leq X_T\}}\right] \mathrm{d}z \right| \\
&\leq c_{n+1} \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)}\left[A_T^{\frac{n+1}{2}} \mathrm{e}^{-z(\alpha-\beta)X_T} 1_{\{\delta\sqrt{A_T}\leq X_T\}}\right] \mathrm{d}z \\
&= c_{n+1} \frac{\Gamma(n+1-\gamma-\varepsilon)}{(\alpha-\beta)^{n+1-\gamma-\varepsilon}} \mathbb{E}_{(0,y)}\left[A_T^{\frac{n+1}{2}} X_T^{-n-1+\gamma+\varepsilon} 1_{\{\delta\sqrt{A_T}\leq X_T\}}\right] \\
&\leq c_{n+1} \frac{\Gamma(n+1-\gamma-\varepsilon)}{(\alpha-\beta)^{n+1-\gamma-\varepsilon}} \frac{1}{\delta^{n+1}} \mathbb{E}_{(0,y)}\left[X_T^{\gamma+\varepsilon}\right] < +\infty.
\end{aligned}$$

Plugging all the terms together in Equation (4.11), we have thus proven that for $\varepsilon > 0$ small enough

$$\left| \int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)} \left[A_T^{\frac{n+1}{2}} H_{n+1}(z\sqrt{A_T}) \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \mathrm{d}z \right| < +\infty. \tag{4.13}$$

We now study further this last expression, and assume, without loss of generality, that

$$\int_0^{+\infty} z^{n-\gamma-\varepsilon} H_{n+1}(z) \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z > 0. \tag{4.14}$$

Applying the Fubini-Tonelli theorem and a change of variables, we have:

$$\begin{aligned} &\int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)} \left[A_T^{\frac{n+1}{2}} H_{n+1}(z\sqrt{A_T}) 1_{\{H_{n+1}(z\sqrt{A_T})>0\}} \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \mathrm{d}z \\ &\qquad = \mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \int_0^{+\infty} z^{n-\gamma-\varepsilon} H_{n+1}(z) 1_{\{H_{n+1}(z)>0\}} \mathrm{e}^{-z(\alpha-\beta)\frac{X_T}{\sqrt{A_T}} - \frac{z^2}{2}} \mathrm{d}z \right] \\ &\qquad \geq \mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \int_0^{+\infty} z^{n-\gamma-\varepsilon} H_{n+1}(z) 1_{\{H_{n+1}(z)>0\}} \mathrm{e}^{-z(\alpha-\beta)\delta - \frac{z^2}{2}} \mathrm{d}z. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_0^{+\infty} z^{n-\gamma-\varepsilon} \mathbb{E}_{(0,y)} \left[A_T^{\frac{n+1}{2}} |H_{n+1}(z\sqrt{A_T})| 1_{\{H_{n+1}(z\sqrt{A_T})<0\}} \mathrm{e}^{-z(\alpha-\beta)X_T - \frac{z^2}{2}A_T} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \mathrm{d}z \\ &\qquad \leq \mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \int_0^{+\infty} z^{n-\gamma-\varepsilon} |H_{n+1}(z)| 1_{\{H_{n+1}(z)<0\}} \mathrm{e}^{-\frac{z^2}{2}} \mathrm{d}z. \end{aligned}$$

As a consequence, we deduce from (4.13) that

$$\mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] \int_0^{+\infty} z^{n-\gamma-\varepsilon} |H_{n+1}(z)| \mathrm{e}^{-\frac{z^2}{2}} \left(1_{\{H_{n+1}(z)>0\}} \mathrm{e}^{-z(\alpha-\beta)\delta} - 1_{\{H_{n+1}(z)<0\}} \right) \mathrm{d}z < +\infty.$$

But, from (4.14) the integral in z is strictly positive for δ small enough. As a consequence, we have obtained that

$$\mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] < +\infty$$

and the result follows from the observation that

$$\mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} \right] \leq \mathbb{E}_{(0,y)} \left[A_T^{\frac{\gamma+\varepsilon}{2}} 1_{\{\delta\sqrt{A_T} > X_T\}} \right] + \frac{1}{\delta^{\gamma+\varepsilon}} \mathbb{E}_{(0,y)} \left[X_T^{\gamma+\varepsilon} 1_{\{\delta\sqrt{A_T} \leq X_T\}} \right] < +\infty.$$

From Lemma 4.2, this contradicts the definition of γ, hence we conclude that $\gamma = \theta$. □

4.2.3. The expectation of T

We show in this section how to compute the expectation of T (which is finite since $\alpha > \beta$, *i.e.* $\theta > 1$) by using the martingale $(Y_t - X_t - (\alpha-\beta)t,\, t \geq 0)$. Let $n > 0$ and recall the definition of the stopping time $\tau_n^Y = \inf\{t \geq 0,\ Y_t = n\}$. From Doob's optional theorem:

$$\mathbb{E}_{(x,y)} \left[Y_{t\wedge T\wedge\tau_n^Y} - X_{t\wedge T\wedge\tau_n^Y} \right] - (\alpha-\beta) \mathbb{E}_{(x,y)} \left[t \wedge T \wedge \tau_n^Y \right] = y - x.$$

Using the monotone and dominated convergence theorems, we deduce:

$$\begin{aligned}\mathbb{E}_{(x,y)}\left[T\right] &= \frac{y-x}{\alpha-\beta} - \frac{1}{\alpha-\beta}\lim_{n\to+\infty}\mathbb{E}_{(x,y)}\left[Y_{T\wedge\tau_n^Y} - X_{T\wedge\tau_n^Y}\right]\\ &= \frac{y-x}{\alpha-\beta} - \frac{1}{\alpha-\beta}\lim_{n\to+\infty}\mathbb{E}_{(x,y)}\left[(n-X_{\tau_n^Y})1_{\{\tau_n^Y\le T\}}\right]\end{aligned}$$

and it remains to show that the limit equals 0. Let us take $0<\varepsilon<\theta-1$. Applying the Markov inequality together with the maximal inequality (3.3), we obtain:

$$\begin{aligned}0\le \mathbb{E}_{(x,y)}\left[(n-X_{\tau_n^Y})1_{\{\tau_n^Y\le T\}}\right] &\le n\mathbb{P}_{(x,y)}\left(\tau_n^Y\le T\right)\\ &= n\mathbb{P}_{(x,y)}\left(\sup_{s\le T} Y_s\ge n\right)\\ &\le n^{-\varepsilon}\mathbb{E}_{(x,y)}\left[\sup_{s\le T} Y_s^{1+\varepsilon}\right]\le C_{\alpha,1+\varepsilon}n^{-\varepsilon}\mathbb{E}_{(x,y)}\left[(T+x)^{1+\varepsilon}\right]\xrightarrow[n\to+\infty]{} 0\end{aligned}$$

since the last expectation is finite by the first part of the proof. □

5. Proof of Corollary 1.4

Before proving Corollary 1.4, we first study the asymptotics of X_T. Recall to this end that from Proposition 1.1 and from the analyticity of $F_{\alpha,\beta}$, there exists $m\in\mathbb{N}$ such that

$$\mathbb{E}_{(0,y)}\left[X_T^{\theta-s}\right] = \frac{(2y)^{\theta-s}}{(-s)^m G_{\alpha,\beta}(\theta-s)} \tag{5.1}$$

where $G_{\alpha,\beta}$ is such that $G_{\alpha,\beta}(\theta)\neq 0$.

5.1. Asymptotics of the tail distribution of X_T

Lemma 5.1. *Let $\delta>0$. There exist two positive constants κ_1 and κ_2 such that*

$$\frac{\kappa_1}{t^{\theta+\delta}}\le \mathbb{P}_{(0,y)}\left(X_T>t\right)\le \kappa_2\frac{(\ln(t))^m}{t^\theta},\qquad \text{as } t\to+\infty.$$

Proof. The upper bound is a direct consequence of the Markov inequality, for $t\ge 2$,

$$\mathbb{P}_{(0,y)}\left(X_T>t\right)\le \frac{\mathbb{E}_{(0,y)}\left[X_T^{\theta-\frac{1}{\ln(t)}}\right]}{t^{\theta-\frac{1}{\ln(t)}}} = \frac{e}{t^\theta}\mathbb{E}_{(0,y)}\left[X_T^{\theta-\frac{1}{\ln(t)}}\right]$$

together with Formula (5.1)

$$\mathbb{E}_{(0,y)}\left[X_T^{\theta-\frac{1}{\ln(t)}}\right] = \frac{(2y)^{\theta-\frac{1}{\ln(t)}}}{(-1)^m G_{\alpha,\beta}(\theta-\frac{1}{\ln(t)})}(\ln(t))^m. \tag{5.2}$$

To get the lower bound, we shall write the Mellin transform of X_T as a Laplace transform:

$$\frac{1}{\theta-s}\mathbb{E}_{(0,y)}[X_T^{\theta-s}] = \int_0^{+\infty} z^{\theta-s+1}\mathbb{P}_{(0,y)}\left(X_T>z\right)\mathrm{d}z$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \mathrm{e}^{-sz}\mathrm{e}^{\theta z}\mathbb{P}_{(0,y)}\left(X_T > \mathrm{e}^z\right)\mathrm{d}z \\
&= \int_{-\infty}^{0} \mathrm{e}^{-sz}\mathrm{e}^{\theta z}\mathbb{P}_{(0,y)}\left(X_T > \mathrm{e}^z\right)\mathrm{d}z + \int_{0}^{+\infty} \mathrm{e}^{-sz}\mathrm{e}^{\theta z}\mathbb{P}_{(0,y)}\left(X_T > \mathrm{e}^z\right)\mathrm{d}z.
\end{aligned}
$$

We now let $s \downarrow 0$. Applying the monotone convergence theorem, the first integral on the right-hand side converges towards $\int_{-\infty}^0 \mathrm{e}^{\theta z}\mathbb{P}_{(0,y)}\left(X_T > \mathrm{e}^z\right)\mathrm{d}z \leq \dfrac{1}{\theta}$. As a consequence, applying Karamata's Tauberian theorem [13], Theorem 1.7.1, we deduce from (5.1) that there exists a constant $c > 0$ such that

$$
\int_0^z \mathrm{e}^{\theta u}\mathbb{P}_{(0,y)}\left(X_T > \mathrm{e}^u\right)du \underset{z\to+\infty}{\sim} cz^m
$$

i.e., going back to the original variable

$$
\int_0^z t^{\theta-1}\mathbb{P}_{(0,y)}\left(X_T > t\right)dt \underset{z\to+\infty}{\sim} c\left(\ln(z)\right)^m.
$$

We now fix $\delta > 0$ and take $\varepsilon > 0$ small enough such that $c(1-\varepsilon)(1+\delta)^m - c(1+\varepsilon) > 0$. Then, for $z > 0$ large enough, we have

$$
\begin{aligned}
\frac{1}{\theta}z^{\theta(1+\delta)}\mathbb{P}_{(0,y)}\left(X_T > z\right) &\geq \int_z^{z^{1+\delta}} t^{\theta-1}\mathbb{P}_{(0,y)}\left(X_T > t\right)dt \\
&\geq c(1-\varepsilon)\left(\ln(z^{1+\delta})\right)^m - c(1+\varepsilon)\left(\ln(z)\right)^m \\
&\geq \left(c(1-\varepsilon)(1+\delta)^m - c(1+\varepsilon)\right)\left(\ln(z)\right)^m \xrightarrow[z\to+\infty]{} +\infty
\end{aligned}
$$

which yields the lower bound of Lemma 5.1. □

5.2. Proof of Corollary 1.4

Observe first that applying the Markov inequality and the maximal inequality (3.5), there exists a constant $K > 0$ such that for all $t \geq 2$,

$$
\mathbb{P}_{(0,y)}\left(T > t\right) \leq \frac{\mathbb{E}_{(0,y)}\left[T^{\theta-\frac{1}{\ln(t)}}\right]}{t^{\theta-\frac{1}{\ln(t)}}} \leq K\frac{\mathbb{E}_{(0,y)}\left[X_T^{\theta-\frac{1}{\ln(t)}}\right]}{t^{\theta-\frac{1}{\ln(t)}}} = \frac{Ke}{t^\theta}\mathbb{E}_{(0,y)}\left[X_T^{\theta-\frac{1}{\ln(t)}}\right]
$$

and the upper bound follows as above from (5.2). Then, taking $\varepsilon > 0$ small enough, we have

$$
\mathbb{P}_{(0,y)}(X_T > t) \leq \mathbb{P}_{(0,y)}\left(\sup_{[0,T]} X_s > t\right) \leq \mathbb{P}_{(0,y)}(T > t^{1-\varepsilon}) + \mathbb{P}_{(0,y)}\left(\sup_{[0,t^{1-\varepsilon}]} X_s > t\right).
$$

By scale invariance, the last term on the right-hand side equals $\mathbb{P}_{(0,y)}\left(\sup_{[0,1]} X_s > t^\varepsilon\right)$ which decreases exponentially since $\sup_{[0,1]} X_s$ admits exponential moments, see [18]. As a consequence, we deduce from Lemma 5.1 that there exists a constant $c_1 > 0$ such that

$$
\frac{c_1}{t^{\frac{\theta+\delta}{1-\varepsilon}}} \leq \mathbb{P}_{(0,y)}\left(T > t\right) \qquad \text{as } t \to +\infty.
$$

This is the lower bound of Corollary 1.4, after renaming the constants. □

6. The limit of $\theta(\alpha,\beta)$ as $\alpha\downarrow 0$

When $\alpha=0$, the process X is absorbed at 0. As a consequence, as observed in Figure 2, different situations occur, according as whether Y may or may not reach the level 0. Typically, when $\beta\geq 2$, the process Y cannot reach 0, so that $\mathbb{P}_{(x,y)}(T=+\infty)>0$, and $\lim_{\alpha\downarrow 0}\theta(\alpha,\beta)=0$. Conversely, when $\beta<2$, the process Y reaches 0 a.s., and thus the random variable X_T remains well-defined, but its distribution admits an atom at 0 (on the set $\{T\geq \tau_0^X\}$). We compute below the limit of $\theta(\alpha,\beta)$ as $\alpha\downarrow 0$ in these different regimes. Note that this limit necessarily exists since $\theta(\alpha,\beta)$ is non-decreasing in α.

Proposition 6.1. *The following limits hold as $\alpha\downarrow 0$:*

$$\begin{aligned}
&\theta(\alpha,\beta)\ \sim \alpha/(2^{\frac{\beta}{2}}-2) && \text{if } \beta>2,\\
&\theta(\alpha,\beta)\ \sim \sqrt{\tfrac{\alpha}{2\ln(2)}} && \text{if } \beta=2,\\
&\theta(\alpha,\beta) = 1-\tfrac{\beta}{2}+\alpha\left(\tfrac{1}{2-2^{\beta/2}}-\tfrac{1}{2}\right)+\mathrm{o}(\alpha) && \text{if } 0\leq\beta<2.
\end{aligned}$$

Proof. By definition, $\theta(\alpha,\beta)$ is a solution of the equation:

$$\begin{aligned}
0=F_{\alpha,\beta}(\theta(\alpha,\beta)) &= 1+\sum_{n=1}^{+\infty}\frac{\left(\frac{\alpha+\beta}{2}-1+\theta(\alpha,\beta)\right)_n(-\theta(\alpha,\beta))(1-\theta(\alpha,\beta))_{n-1}}{\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)_{n-1}n!}\left(\frac{1}{2}\right)^n\\
&=1-2\frac{\theta(\alpha,\beta)}{\alpha}\sum_{n=1}^{+\infty}\frac{\left(\frac{\alpha+\beta}{2}-1+\theta(\alpha,\beta)\right)_n(1-\theta(\alpha,\beta))_{n-1}}{\left(\frac{\alpha}{2}+1\right)_{n-1}n!}\left(\frac{1}{2}\right)^n. \qquad (6.1)
\end{aligned}$$

Also, recall that $\theta(\alpha,\beta)\in(0,1)$ for $\alpha<\beta$. We deal with each case $\beta>2$, $\beta=2$ and $\beta<2$ separately.

1. When $\beta>2$, we have $\lim\limits_{\alpha\downarrow 0}\theta(\alpha,\beta)=0$. Passing to the limit in (6.1), we deduce that the sum converges towards

$$\sum_{n=1}^{+\infty}\frac{\left(\frac{\beta}{2}-1\right)_n}{n!}\left(\frac{1}{2}\right)^n=2^{\frac{\beta}{2}-1}-1$$

and thus

$$\theta(\alpha,\beta)\underset{\alpha\to 0}{\sim}\frac{\alpha}{2^{\frac{\beta}{2}}-2}.$$

Note that when $\beta=4$, we obtain $\theta(\alpha,4)\underset{\alpha\downarrow 0}{\sim}\dfrac{\alpha}{2}$ which is consistent with the explicit value $\theta(\alpha,4-\alpha)=\alpha/2$ obtained in the case $\alpha+\beta=4$.

2. When $\beta=2$, we still have $\lim\limits_{\alpha\downarrow 0}\theta(\alpha,\beta)=0$. Equation (6.1) then reads

$$0=1-2\frac{\theta(\alpha,2)\left(\frac{\alpha}{2}+\theta(\alpha,2)\right)}{\alpha}\sum_{n=1}^{+\infty}\frac{\left(\frac{\alpha}{2}+\theta(\alpha,2)+1\right)_{n-1}(1-\theta(\alpha,2))_{n-1}}{\left(\frac{\alpha}{2}+1\right)_{n-1}n!}\left(\frac{1}{2}\right)^n.$$

As before, passing to the limit as $\alpha \downarrow 0$, the sum converges towards

$$\sum_{n=1}^{+\infty} \frac{1}{n} \left(\frac{1}{2}\right)^n = \ln(2)$$

and we obtain

$$0 = 1 - 2\ln(2) \lim_{\alpha\downarrow 0} \frac{\theta^2(\alpha, 2)}{\alpha}$$

i.e.

$$\theta(\alpha, 2) \underset{\alpha\downarrow 0}{\sim} \sqrt{\frac{\alpha}{2\ln(2)}}.$$

3. When $0 \leq \beta < 2$, then Y will hit 0 a.s. and we have $T \leq \tau_0^Y$. As explained in Subsection 4.1, this implies that $\theta(\alpha, \beta) > 1 - \frac{\beta}{2}$. Let us set $\theta(\alpha, \beta) = 1 - \frac{\beta}{2} + \rho(\alpha, \beta)$ so that Equation (6.1) becomes

$$0 = 1 - 2\left(\frac{\frac{\alpha}{2} + \rho(\alpha,\beta)}{\alpha}\right)\theta(\alpha,\beta) \sum_{n=1}^{+\infty} \frac{\left(\frac{\alpha}{2} + \rho(\alpha,\beta) + 1\right)_{n-1} (1-\theta(\alpha,\beta))_{n-1}}{\left(\frac{\alpha}{2}+1\right)_{n-1} n!} \left(\frac{1}{2}\right)^n. \tag{6.2}$$

Passing to the limit as $\alpha \downarrow 0$, the sum converges towards

$$\sum_{n=1}^{+\infty} \frac{(\rho(0^+,\beta)+1)_{n-1} (1-\theta(0^+,\beta))_{n-1}}{(n-1)!n!} \left(\frac{1}{2}\right)^n > 0.$$

As a consequence, going back to (6.2), we necessarily have $\rho(0^+, \beta) = 0$ and the sum equals

$$\sum_{n=1}^{+\infty} \frac{\left(\frac{\beta}{2}\right)_{n-1}}{n!} \left(\frac{1}{2}\right)^n = \frac{1}{\frac{\beta}{2}-1} \sum_{n=1}^{+\infty} \frac{\left(\frac{\beta}{2}-1\right)_n}{n!} \left(\frac{1}{2}\right)^n = \frac{2}{\beta-2}\left(2^{\frac{\beta}{2}-1} - 1\right)$$

so finally

$$\rho(\alpha, \beta) \underset{\alpha\downarrow 0}{\sim} \alpha \left(\frac{1}{2 - 2^{\beta/2}} - \frac{1}{2}\right).$$

□

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Data availability statement

No new data/codes were created or analyzed in this study.

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Appendix A.

We write down a formula which is used several times in the paper. This identity may also be recovered from [6], p. 329, Formula (3.259) (3).

Lemma A.1. *Assume that* $\lambda > 0$, $\mu > 0$ *and* $0 < \gamma < \lambda + \mu$. *Then:*

$$\int_0^{+\infty} \xi^{\gamma-1}(1-2\imath\xi)^{-\lambda}(1+2\imath\xi)^{-\mu}\mathrm{d}\xi$$
$$= \frac{\Gamma(\gamma)\Gamma(\mu+\lambda-\gamma)}{\Gamma(\lambda)\Gamma(\mu)}\left((2\imath)^{-\gamma}B(\lambda, 1-\gamma)\, 2^{-\lambda}{}_2F_1\left[\begin{matrix}1-\mu & \lambda\\ \lambda-\gamma+1\end{matrix}; \frac{1}{2}\right]\right.$$
$$\left. + (-2\imath)^{-\gamma}B(\mu, 1-\gamma)\, 2^{-\mu}{}_2F_1\left[\begin{matrix}1-\lambda & \mu\\ \mu-\gamma+1\end{matrix}; \frac{1}{2}\right]\right).$$

Proof. We start by writing the left-hand side under the form

$$\int_0^{+\infty} \xi^{\gamma-1}(1-2\imath\xi)^{-\lambda}(1+2\imath\xi)^{-\mu}\mathrm{d}\xi$$
$$= \frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^{+\infty}\xi^{\gamma-1}\left(\int_0^{+\infty}a^{\lambda-1}\mathrm{e}^{-a(1-2\imath\xi)}\mathrm{d}a\right)\left(\int_0^{+\infty}b^{\mu-1}\mathrm{e}^{-b(1+2\imath\xi)}\mathrm{d}b\right)\mathrm{d}\xi.$$

Assume first that $a < b$:

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^{+\infty}\xi^{\gamma-1}\int_0^{+\infty}b^{\mu-1}\mathrm{e}^{-b(1+2\imath\xi)}\int_0^{b}a^{\lambda-1}\mathrm{e}^{-a(1-2\imath\xi)}\mathrm{d}b\mathrm{d}a\mathrm{d}\xi\\
&\quad= \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\mu)}\int_0^{+\infty}(2\imath(b-a))^{-\gamma}b^{\mu-1}\mathrm{e}^{-b}\int_0^{b}a^{\lambda-1}\mathrm{e}^{-a}\mathrm{d}a\mathrm{d}b\\
&\quad= \frac{\Gamma(\gamma)(2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\int_0^{+\infty}(1-a)^{-\gamma}b^{\mu+\lambda-\gamma-1}\mathrm{e}^{-b}\int_0^{1}a^{\lambda-1}\mathrm{e}^{-ab}\mathrm{d}a\mathrm{d}b\\
&\quad= \frac{\Gamma(\gamma)(2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)\int_0^{1}(1-a)^{-\gamma}(1+a)^{\gamma-\lambda-\mu}a^{\lambda-1}\mathrm{d}a\\
&\quad= \frac{\Gamma(\gamma)(2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)B(\lambda,1-\gamma)\,{}_2F_1\left[\begin{matrix}\mu+\lambda-\gamma & \lambda\\ \lambda-\gamma+1\end{matrix}; -1\right]
\end{aligned}$$

Recalling the Pfaff transformation

$${}_2F_1\left[\begin{matrix}\mu+\lambda-\gamma & \lambda\\ \lambda-\gamma+1\end{matrix}; -1\right] = 2^{-\lambda}{}_2F_1\left[\begin{matrix}1-\mu & \lambda\\ \lambda-\gamma+1\end{matrix}; \frac{1}{2}\right]$$

yields the first term on the right-hand side. Similarly, when $b < a$, we obtain:

$$\begin{aligned}
&\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^{+\infty}\xi^{\gamma-1}\int_0^{+\infty}b^{\mu-1}\mathrm{e}^{-b(1+2\imath\xi)}\int_b^{+\infty}a^{\lambda-1}\mathrm{e}^{-a(1-2\imath\xi)}\mathrm{d}b\mathrm{d}a\mathrm{d}\xi\\
&\quad= \frac{\Gamma(\gamma)(-2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)\int_1^{+\infty}(a-1)^{-\gamma}(1+a)^{\gamma-\lambda-\mu}a^{\lambda-1}\mathrm{d}a\\
&\quad= \frac{\Gamma(\gamma)(-2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)\int_0^{1}b^{\mu-1}(1-b)^{-\gamma}(1+b)^{\gamma-\lambda-\mu}\mathrm{d}b\\
&\quad= \frac{\Gamma(\gamma)(-2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)B(\mu,1-\gamma)\,{}_2F_1\left[\begin{matrix}\mu+\lambda-\gamma & \mu\\ \mu-\gamma+1\end{matrix}; -1\right]\\
&\quad= \frac{\Gamma(\gamma)(-2\imath)^{-\gamma}}{\Gamma(\lambda)\Gamma(\mu)}\Gamma(\mu+\lambda-\gamma)B(\mu,1-\gamma)\,2^{-\mu}{}_2F_1\left[\begin{matrix}1-\lambda & \mu\\ \mu-\gamma+1\end{matrix}; \frac{1}{2}\right]
\end{aligned}$$

and Lemma A.1 follows by summing both terms. □