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STOCHASTIC SEWING WITH BESOV REGULARITY

Petr Čoupek* and František Hendrych

Abstract. Under various conditions, sewing lemmas provide convergence of the Riemann-type sum $\sum_{[s,t]\in\pi} \Xi_{s,t}$ for a given two-parametric map Ξ as the mesh sizes of the considered partitions π tend to zero. In this note, we prove a stochastic sewing lemma for two-parameter processes whose increments, when viewed as functions with values in $L^m(\Omega;\mathbb{V})$ for $m \geq 2$ and a real separable Banach space $\mathbb{V}$ with a non-trivial martingale type, are of Besov regularity. The contribution is two-fold: First, we generalize the stochastic sewing lemma of Lê [Electron. J. Probab. **25** (2020) 1–55] for processes whose increments belong to a Besov and not necessarily Hölder space. Second, we show here that the assumptions of the Besov sewing lemma of Friz *et al.* [J. Differ. Equ. **339** (2022) 152–231] can be relaxed if stochastics is incorporated in the sewing from the beginning. As an application, Besov regularity of the Itô integral of Brownian functionals is obtained.

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1. Introduction

Consider a two-parametric map

$$\Xi : \{(s,t) \in [0,T]^2 : s \leq t\} \to \mathbb{E},$$

where $\mathbb{E}$ is a real Banach space and $T > 0$. In general, sewing lemmas provide sufficient conditions for the convergence of the sum

$$\sum_{[\tau_{i-1},\tau_i]\in\pi_n} \Xi_{\tau_{i-1},\tau_i}$$

as $n \to \infty$ for a sequence of partitions $\{\pi_n\}$ of the interval $[0,T]$ whose mesh sizes tend to zero. Such results are of particular interest for abstract integration and, consequently, for the treatment of differential equations driven by irregular functions. For example, the choice $\Xi_{s,t} = Y_s(X_t - X_s)$ for $Y \in C^\alpha([0,T])$ and $X \in C^\beta([0,T])$ with $\alpha + \beta > 1$ leads to the Young (generalized Riemann) integral $\int_0^T Y_r \,\mathrm{d}X_r$ (see, *e.g.*, Sect. 4.1 of [1] or Sect. 6 of [2]); the choice $\Xi_{s,t} = Y_s(X_t - X_s) + Y'_s\mathbb{X}_{s,t}$ for a rough path $\boldsymbol{X} = (X,\mathbb{X})$ of Hölder regularity $\alpha \in (1/3, 1/2]$ and a path $\boldsymbol{Y} = (Y,Y')$ controlled by X of regularity 2α leads to the rough integral $\int_0^T \boldsymbol{Y}_r \,\mathrm{d}\boldsymbol{X}_r$ (see, *e.g.*, Sect. 4.3

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Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75, Prague 8, Czech Republic.

* Corresponding author: coupek@karlin.mff.cuni.cz

of [1] or Prop. 1 of [3]). Various other sewing lemmas and their applications are also given, for example, in [4–9].

An important contribution is made by Lê in [10] where a version of a sewing lemma in the case when $\mathbb{E} = L^m(\Omega)$ for $m \geq 2$ is given. This so-called *stochastic* sewing lemma shows that it is possible to relax the regularity assumptions in one part by half if stochastics is incorporated in the sewing from the beginning. The result has proved useful in a number of applications (see, *e.g.*, [11–14]) and some of its generalizations can be found in [15, 16].

Most of the classical results mentioned above are given in the Hölder setting, *i.e.* when the considered to-be-sewed map Ξ (as well as its second-order increments) is of Hölder regularity. Recently, however, Friz and Seeger [17] obtained a sewing lemma for maps that are of Besov regularity, see Theorems 3.1–3.3 therein. Such a result is important because many of the stochastic processes to which sewing lemmas are commonly applied are of Besov regularity—for example, while the paths of the Wiener process do not lie in the Hölder space $C^{1/2}([0,T])$ almost surely, they do lie in the Besov space $B^{1/2}_{p,\infty}([0,T])$ almost surely for any $p \in [1,\infty)$; see [18].

In this note, we prove a stochastic sewing lemma for processes Ξ that are of Besov regularity. Namely, we consider $\mathbb{E} = L^m(\Omega;\mathbb{V})$ for $m \geq 2$ and a real separable Banach space $\mathbb{V}$ of non-trivial martingale type. We assume that Ξ is $\mathscr{F}_t$-adapted and that $\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi$ and $\mathsf{E}^{\mathscr{F}}\delta\Xi$ belong to certain Besov-type spaces. (Here we denote $\mathsf{E}^{\mathscr{F}}\delta\Xi : (s,u,t) \mapsto \mathsf{E}^{\mathscr{F}_s}\delta\Xi_{s,u,t}$). We obtain the existence of a stochastic process $\mathcal{I}$ (which is of Besov regularity, provided that Ξ is) whose increments $(s,t) \mapsto \mathcal{I}_t - \mathcal{I}_s$ are the limit of

$$(s,t) \mapsto \sum_{[\tau_{i-1},\tau_i]\in\pi_n} \Xi_{s+\tau_{i-1}(t-s),s+\tau_i(t-s)}$$

as $n \to \infty$ in a Besov-type norm. (Here, $\{\pi_n\}$ is a sequence of partitions of interval $[0,1]$ whose mesh sizes tend to zero as $n \to \infty$). As explained in [17], such parametrization is done because functions from a Besov space are, a priori, defined only almost everywhere.

The main result described above is given in Theorem 3.4. It covers both the stochastic sewing lemma (Thm. 2.1) of [10] and the Besov sewing lemma (Thm. 3.1) of [17]. For the former, it allows more flexibility because also processes of lower Besov regularity, as opposed to Hölder regularity, can be considered. For the latter, it shows that if stochastics is incorporated in the sewing from the beginning, the (Besov) regularity requirement can be relaxed. A more detailed discussion is given in Remark 3.5. As an application, Besov regularity of the Itô integral of Brownian functionals is obtained in Example 3.6 and Example 3.7.

Organization of the article. The article is organized as follows. In Section 2, the multi-parameter Besov spaces are defined and their properties that are useful for our analysis are given. The main result in Theorem 3.4 is formulated in Section 3 and proved in Section 4. Appendix A and Appendix B contain some auxiliary lemmas.

2. Preliminaries: Multi-parameter Besov-type spaces

In this section, we define the multi-parameter Besov-type spaces that will be needed throughout the article and collect some of their properties. Let us begin by fixing some notation. For $d \in \mathbb{N}$ and $A \subseteq \mathbb{R}$, we denote $\triangle^d A = A$ if $d = 1$ and

$$\triangle^d A = \{(u_1,\dots,u_d) \in A^d \mid u_1 \leq \cdots \leq u_d\}$$

if $d \geq 2$. For a map $\xi : [0,T] \to \mathbb{E}$ (where $T > 0$ and $\mathbb{E}$ is a real Banach space), we define the (first-order) difference map by

$$\Delta\xi : \triangle^2[0,T] \to \mathbb{E} : (s,t) \mapsto \xi_t - \xi_s.$$

The multi-parameter Besov-type spaces are defined now.

Definition 2.1. Let $T > 0$ and $d \in \{1, 2, 3\}$, and let $\mathbb{E}$ be a real Banach space. Let also $\Xi : \triangle^d[0,T] \to \mathbb{E}$ be a Borel measurable function. For $p \in [1,\infty]$ and $\tau \in [0,T]$, set

$$\omega_p^d(\Xi,\tau) = \begin{cases} \operatorname*{ess\,sup}_{h\in[0,\tau]} \big\| \|\Delta\Xi_{r,r+h}\|_{\mathbb{E}} \big\|_{L^p([0,T-h];\,\mathrm{d}r)}, & d=1, \\ \operatorname*{ess\,sup}_{h\in[0,\tau]} \big\| \|\Xi_{r,r+h}\|_{\mathbb{E}} \big\|_{L^p([0,T-h];\,\mathrm{d}r)}, & d=2, \\ \sup_{\theta\in[0,1]} \operatorname*{ess\,sup}_{h\in[0,\tau]} \big\| \|\Xi_{r,r+\theta h,r+h}\|_{\mathbb{E}} \big\|_{L^p([0,T-h];\,\mathrm{d}r)}, & d=3. \end{cases} \tag{2.1}$$

Additionally, for $s \in (0,\infty)$ and $q \in [1,\infty]$, we define

$$\|\Xi\|_{\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})} = \left\| \frac{\omega_p^d(\Xi,\tau)}{\tau^s} \right\|_{L^q([0,T];\,\frac{\mathrm{d}\tau}{\tau})}$$

and we call the space

$$\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E}) = \{\Xi : \triangle^d[0,T] \to \mathbb{E} \text{ Borel measurable} \mid \|\Xi\|_{\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})} < \infty\}$$

equipped with the (Besov-type) norm $\|\cdot\|_{L^p([0,T];\mathbb{E})} + \|\cdot\|_{\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})}$ for $d = 1$ and with $\|\cdot\|_{\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})}$ for $d = 2, 3$ a (*d-parameter*) *Besov-type space*.

Remark 2.2. Let us make two brief remarks.

- For $d = 1$ and $s \in (0,1)$, the space $\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})$ from Definition 2.1 is the usual Besov space $B^s_{p,q}([0,T];\mathbb{E})$ defined either in terms of Littlewood–Paley blocks or in terms of the first-order difference; see Remark 2.3 in [17] and [19, 20]. For $d = 2, 3$, the space $\mathcal{B}^{s;d}_{p,q}([0,T];\mathbb{E})$ is not the usual Besov space $B^s_{p,q}(\triangle^d[0,T];\mathbb{E})$.
- It can be shown that the Besov-type space $\mathcal{B}^{s;2}_{p,q}([0,T];\mathbb{E})$ is a Banach space. This will be important in the proof of Theorem 3.4.

Remark 2.3. Definition 2.1 should be compared to Definition 2.1 and Definition 2.3 in [17]. In particular, here, the moduli of continuity for $d = 1$ and $d = 2$ in (2.1) are defined with the essential supremum while in [17], the supremum is used. If $d = 1$, this does not make a difference; however, if $d = 2$, then the use of the essential supremum leads to a Besov-type space in which functions that are equal λ^d-a.e. on $\triangle^d[0,T]$ have the same Besov-type norm while if the supremum is used, the norms of two functions that are equal λ^d-a.e. can be different (consider functions f and g on $\triangle^2[0,T]$ such that $g \equiv 0$ and $f = \mathbf{1}_{\{(u,v):\, v=u+\frac{T}{2}\}}$).

Two inequalities between the Besov-type norms that will be needed in the subsequent sections are given in the following proposition. The proofs are straightforward. (Here and in the rest of the article, we use the symbol $A \lesssim B$ for two expressions A and B if there is a constant $C \in (0,\infty)$ whose particular value is not important such that $A \le CB$. If we need to stress the dependence of the constant C on some parameter κ, we simply write $A \lesssim_\kappa B$).

Proposition 2.4. *Suppose that $\mathbb{E}$ is a real Banach space and let $d \in \{2,3\}$. Let, moreover, $s, s_1, s_2 \in (0,\infty)$, and let $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$. Then*

(i) $\|\cdot\|_{\mathcal{B}^{s_1;d}_{p,q}([0,T];\mathbb{E})} \lesssim_{s_1,s_2,T} \|\cdot\|_{\mathcal{B}^{s_2;d}_{p,q}([0,T];\mathbb{E})}$ *if* $s_1 \le s_2$,
(ii) $\|\cdot\|_{\mathcal{B}^{s;d}_{p_1,q}([0,T];\mathbb{E})} \lesssim_{p_1,p_2,T} \|\cdot\|_{\mathcal{B}^{s;d}_{p_2,q}([0,T];\mathbb{E})}$ *if* $p_1 \le p_2$.

3. Main result: Stochastic sewing lemma with Besov regularity

Fix $T > 0$ and a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathsf{P})$. We assume completeness of both P and $\{\mathscr{F}_t\}_{t\in[0,T]}$, *i.e.* $\{N \in \mathscr{F} : \mathsf{P}(N) = 0\} \in \mathscr{F}_0$. Let us also fix $m \in \mathbb{N}$ such that $m \geq 2$ and a real separable Banach space $\mathbb{V}$. For simplicity of notation, we shall write $\boldsymbol{L}_m = L^m(\Omega;\mathbb{V})$ and $\mathcal{B}^{s;d}_{p,q} = \mathcal{B}^{s;d}_{p,q}([0,T];\boldsymbol{L}_m)$ whenever there is no risk of confusion. We say that a stochastic process $\Xi : \triangle^d[0,T] \to \boldsymbol{L}_m$ is $\mathscr{F}_t$-*adapted* if $\Xi_{u_1,\dots,u_d}$ is $\mathscr{F}_{u_d}$-measurable for λ^d-a.a. $(u_1,\dots,u_d) \in \triangle^d[0,T]$. For such a process, we also define the mapping

$$\mathsf{E}^{\mathscr{F}}\Xi : \triangle^d[0,T] \to \boldsymbol{L}_m : (u_1,\dots,u_d) \mapsto \mathsf{E}^{\mathscr{F}_{u_1}}\Xi_{u_1,\dots,u_d},$$

where $\mathsf{E}^{\mathscr{G}}$ on the right denotes the standard conditional expectation $\mathsf{E}[\,\cdot\,|\mathscr{G}]$ with respect to a σ-algebra $\mathscr{G} \subseteq \mathscr{F}$. In the proof of our main result in Theorem 3.4 below, we will need that an analogue of the Burkholder–Davis–Gundy inequality holds. Let us therefore recall the following definition that provides such an inequality.

Definition 3.1 (Martingale-type of a Banach space)**.** The Banach space $\mathbb{V}$ is said to *have martingale type* $\mathfrak{p} \in [1,2]$ if there exists $r \in (1,\infty)$ and a finite positive constant $C = C_{r,\mathfrak{p},\mathbb{V}}$ such that

$$\big\|\|M_N\|_{\mathbb{V}}\big\|_{L^r(\Omega)} \leq C \left\| \left(\|M_0\|^{\mathfrak{p}}_{\mathbb{V}} + \sum_{n=1}^{N} \|M_n - M_{n-1}\|^{\mathfrak{p}}_{\mathbb{V}} \right)^{\frac{1}{\mathfrak{p}}} \right\|_{L^r(\Omega)} \tag{3.1}$$

holds for every L_r-integrable $\mathbb{V}$-valued martingale $\{M_n\}_{n=0}^N$ where $N \in \mathbb{N}$.

Remark 3.2. Every Banach space has (the so-called *trivial*) martingale type $\mathfrak{p} = 1$ because inequality (3.1) in Definition 3.1 holds by the triangle inequality. If $\mathbb{V}$ has (the so-called *non-trivial*) martingale type $\mathfrak{p} \in (1,2]$, then it also has martingale type $\mathfrak{r}$ for every $\mathfrak{r} \in [1,\mathfrak{p}]$ and inequality (3.1) holds for all $r \in (1,\infty)$; see, *e.g.*, Proposition 3.5.27 and Corollary 3.5.28 in [21].

Example 3.3. Let us recall a few examples from Proposition 2.3 in [15]. As already mentioned, every Banach space has martingale type 1. Further, every Hilbert space has martingale type 2, the Lebesgue space $L^p([0,T])$ for $p \in (1,\infty)$ has martingale type $\min\{2,p\}$, and the Besov space $B^s_{p,q}([0,T];\mathbb{R})$ for $p,q \in (1,\infty)$ and $s \in (0,1)$ has martingale type $\min\{2,p,q\}$.

Finally, to avoid ambiguity, we note that the term *partition* of interval $[s,t] \subseteq \mathbb{R}$ means some finite strictly increasing sequence $\pi = \{t_i\}_{i=0}^N$, where $N \in \mathbb{N}$, with values in $[s,t]$ which starts at s and ends at t, and we denote its *mesh size* by $|\pi|$, *i.e.* $|\pi| = \max_{i=1,\dots,N}|t_i - t_{i-1}|$. Moreover, we recall that if $\mathbb{E}$ is a Banach space and $\xi : [0,T] \to \mathbb{E}$, we have defined the (first-order) difference map by

$$\Delta\xi : \triangle^2[0,T] \to \mathbb{E} : (s,t) \mapsto \xi_t - \xi_s.$$

Additionally, if $\Xi : \triangle^2[0,T] \to \mathbb{E}$, we define the (second-order) difference map by

$$\delta\Xi : \triangle^3[0,T] \to \mathbb{E} : (s,u,t) \mapsto \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t},$$

and, if $\pi = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ is a partition of interval $[0,1]$, we also define

$$I^{\pi} : \triangle^2[0,T] \to \mathbb{E} : (s,t) \mapsto \sum_{i=1}^{N} \Xi_{s+t_{i-1}(t-s),s+t_i(t-s)}. \tag{3.2}$$

The main result of the present article can now be formulated.

Theorem 3.4 (Stochastic sewing with Besov regularity)**.** *Assume that* $\mathbb{V}$ *has martingale type* $\mathfrak{p} \in (1,2]$ *and let* $\Xi : \triangle^2[0,T] \to \boldsymbol{L}_m$ *be a Borel measurable* $\mathscr{F}_t$*-adapted process such that the mapping* $\mathsf{E}^{\mathscr{F}}\Xi : \triangle^2[0,T] \to \boldsymbol{L}_m$ *is also Borel measurable and*

$$\|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_1;3}_{p_1,q}} < \infty, \tag{3.3}$$

$$\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}} < \infty \tag{3.4}$$

hold for some $p_1, p_2 \in [1,\infty]$, $s_1 \in (\frac{1}{\mathfrak{p}\wedge p_1}, \infty)$, $s_2 \in (1,\infty)$, *and* $q \in [\mathfrak{p}\wedge p_1, \infty]$. *Then there exists a Borel measurable stochastic process* $\mathcal{I} : [0,T] \to \boldsymbol{L}_m$ *that satisfies*

$$\lim_{N\to\infty} \|\Delta\mathcal{I} - I^{\pi_N}\|_{\mathcal{B}^{s_1\wedge s_2;2}_{p_1\wedge p_2,q}} = 0 \tag{3.5}$$

for any sequence $\{\pi_N\}_{N=1}^{\infty}$ *of partitions of interval* $[0,1]$ *satisfying* $\lim_{N\to\infty}|\pi_N| = 0$. *Moreover, it holds that*

$$\|\Delta\mathcal{I} - \Xi\|_{\mathcal{B}^{s_1\wedge s_2;2}_{p_1\wedge p_2,q}} \lesssim_{s_1,s_2,\mathfrak{p},p_1,q,T} \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_1;3}_{p_1,q}} + \|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}, \tag{3.6}$$

$$\|\mathsf{E}^{\mathscr{F}}(\Delta\mathcal{I} - \Xi)\|_{\mathcal{B}^{s_2;2}_{p_2,q}} \lesssim_{s_2,\mathfrak{p},q} \|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}, \tag{3.7}$$

and $\Delta\mathcal{I}$ *is* $\mathscr{F}_t$*-adapted. If, in addition,*

$$\|\Xi\|_{\mathcal{B}^{\tilde{s};2}_{\tilde{p},q}} < \infty$$

holds for some $\tilde{s} \in (0,\infty)$ *and* $\tilde{p} \in [1,\infty]$, *then*

$$\|\mathcal{I}\|_{\mathcal{B}^{\tilde{s}\wedge s_1\wedge s_2;1}_{\tilde{p}\wedge p_1\wedge p_2,q}} < \infty$$

holds as well.

Remark 3.5. Let us briefly discuss Theorem 3.4 in view of related results in the literature. At first, a comparison is made with the stochastic sewing lemma from Theorem 2.1 from [10].

- If $\mathbb{V} = \mathbb{R}^n$ (for some $n \in \mathbb{N}$) and $p_1 = p_2 = q = \infty$, then convergence (3.5) with estimates (3.6) and (3.7) from Theorem 3.4 reduce to the corresponding statements in Theorem 2.1 from [10] where a Hölder-type condition for $\delta\Xi$ is assumed. Therefore, Theorem 3.4 is a generalization of that result and it is also a generalization of Theorem 3.1 of [15] where Banach-space-valued stochastic processes are considered (although there are some small differences that make Theorem 3.1 of [15] slightly more general in the Hölder case; *e.g.*, the use of another conditional moment and a control function).

Let us also compare Theorem 3.4 to the Besov sewing lemma in Theorem 3.1 of [17].

- If Ξ is deterministic and if we set $s_1 = s_2$, $p_1 = \infty$, and $\mathfrak{p} = 1$, it is readily seen that our convergence result (3.5) and inequality (3.6) reduce to the convergence (3.1) and estimate (3.2) from Theorem 3.1 of [17], respectively.
- We also note the appearance of the bound $s_1 > \frac{1}{\mathfrak{p}\wedge p_1}$ which shows that the regularity requirement for $\delta\Xi$ in Theorem 3.1 of [17] may be relaxed if the target space for Ξ is the space $\boldsymbol{L}_m$ (as opposed to a general Banach space). This is similar to what happens in the Hölder setting, *cf.* [10] for further discussion.

Example 3.6. Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a complete probability space endowed with a complete filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$, let $B = (B_t, t \in [0,T])$ be a standard $\mathscr{F}_t$-Brownian motion and let $f(x) = \mathbf{1}_{(0,\infty)}(x)$, $x \in \mathbb{R}$. We wish to use

Theorem 3.4 to define the Itô integral $\int_0^\bullet f(B_r)\mathrm{d}B_r$ as the limit (in a suitable Besov space of stochastic processes) of the Riemann-type approximations

$$I_{s,t}^{D_n} = \sum_{i=1}^{2^n} f(B_{s+d_{i-1}^n(t-s)})(B_{s+d_i^n(t-s)} - B_{s+d_{i-1}^n(t-s)}), \quad (s,t) \in \triangle^2[0,T], \tag{3.8}$$

where $\{D_n\}_n$ denotes the sequence of dyadic partitions $D_n = \{i/2^n\}_{i=0}^{2^n}$ of the interval $[0,1]$. To this end, let

$$\Xi_{s,t} = f(B_s)(B_t - B_s), \quad (s,t) \in \triangle^2[0,T]. \tag{3.9}$$

Let also m be an integer such that $m \geq 2$. Note first that for $h \in (0,T)$ and $r \in (0, T-h)$, there is the estimate

$$\mathsf{E}|\Xi_{r,r+h}|^m = \mathsf{E}|f(B_r)(B_{r+h} - B_r)|^m \leq \mathsf{E}|B_{r+h} - B_r|^m = c_m h^{\frac{m}{2}} \tag{3.10}$$

where $c_m = \left(\frac{2^m}{\pi}\right)^{\frac{1}{2}} \Gamma\left(\frac{m+1}{2}\right)$ denotes the m^{th} absolute moment of a standard normal random variable. Let now $h \in (0,T)$, $r \in (0, T-h)$, and $\theta \in (0,1)$ be fixed and note that

$$\delta\Xi_{r,r+\theta h,r+h} = -(f(B_{r+\theta h}) - f(B_r))(B_{r+h} - B_{r+\theta h}) \tag{3.11}$$

holds so that we obtain the equality

$$\mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\theta h,r+h} = -\mathsf{E}^{\mathscr{F}_r}(f(B_{r+\theta h}) - f(B_r))\mathsf{E}^{\mathscr{F}_{r+\theta h}}(B_{r+h} - B_{r+\theta h}) = 0. \tag{3.12}$$

It follows that

$$\mathsf{E}|\delta\Xi_{r,r+\theta h,r+h} - \mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\theta h,r+h}|^m = \mathsf{E}|\delta\Xi_{r,r+\theta h,r+h}|^m = I_1 + I_2$$

where we denote

$$\begin{aligned} I_1 &= \mathsf{E}\mathbf{1}_{[B_{r+\theta h}>0]}\mathbf{1}_{[B_r\leq 0]}|B_{r+h} - B_{r+\theta h}|^m, \\ I_2 &= \mathsf{E}\mathbf{1}_{[B_{r+\theta h}\leq 0]}\mathbf{1}_{[B_r>0]}|B_{r+h} - B_{r+\theta h}|^m. \end{aligned}$$

To evaluate term I_1, note that

$$I_1 = \mathsf{E}\left(\mathbf{1}_{[B_r\leq 0]}\mathsf{E}\left[\mathbf{1}_{[B_{r+\theta h}-B_r>-B_r]}|B_{r+h} - B_{r+\theta h}|^m \mid B_r\right]\right)$$

holds and, as the random variables B_r, $B_{r+\theta h} - B_r$, and $B_{r+h} - B_{r+\theta h}$ are independent and normally distributed with mean zero and variance r, θh, and $(1-\theta)h$, respectively, we obtain

$$\mathsf{E}\left[\mathbf{1}_{[B_{r+\theta h}-B_r>-x]}|B_{r+h} - B_{r+\theta h}|^m \mid B_r = x\right] = c_m(1-\theta)^{\frac{m}{2}}h^{\frac{m}{2}}\left(1 - \Phi\left(\frac{-x}{\sqrt{\theta h}}\right)\right), \quad x \in \mathbb{R},$$

where Φ denotes the cumulative distribution function of the standard normal distribution. It then follows that

$$I_1 = c_m(1-\theta)^{\frac{m}{2}}h^{\frac{m}{2}}\left(\frac{1}{2} - \mathsf{E}\mathbf{1}_{[B_r\leq 0]}\Phi\left(\frac{-B_r}{\sqrt{\theta h}}\right)\right).$$

Since

$$\mathsf{E}\mathbf{1}_{[B_r\leq 0]}\Phi\left(-\frac{B_r}{\sqrt{\theta h}}\right)=\int_0^\infty \Phi\left(\sqrt{\frac{r}{\theta h}}x\right)\mathrm{d}\Phi(x)=\frac{1}{4}+\frac{1}{2\pi}\arctan\left(\sqrt{\frac{r}{\theta h}}\right)$$

holds by using the fact that

$$\int_0^\infty \Phi(ax)\mathrm{d}\Phi(x)=\frac{1}{4}+\frac{1}{2\pi}\arctan(a),\quad a\in\mathbb{R},$$

which can be obtained, for example, by substitution to polar coordinates after rewriting Φ as the integral of the density of the standard normal distribution, we finally obtain

$$I_1=c_m(1-\theta)^{\frac{m}{2}}h^{\frac{m}{2}}\left(\frac{1}{4}-\frac{1}{2\pi}\arctan\left(\sqrt{\frac{r}{\theta h}}\right)\right).$$

By similar arguments, we also obtain

$$I_2=c_m(1-\theta)^{\frac{m}{2}}h^{\frac{m}{2}}\left(\frac{1}{4}+\frac{1}{2\pi}\arctan\left(-\sqrt{\frac{r}{\theta h}}\right)\right)$$

and it follows that

$$\mathsf{E}|\delta\Xi_{r,r+\theta h,r+h}-\mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\theta h,r+h}|^m=\frac{c_m}{\pi}(1-\theta)^{\frac{m}{2}}h^{\frac{m}{2}}\arctan\left(\sqrt{\frac{\theta h}{r}}\right).\tag{3.13}$$

Now, it clearly follows from (3.10) that $\Xi\in\mathcal{B}^{\frac{1}{2};2}_{\infty,\infty}$. From (3.13) we have that $\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi\in\mathcal{B}^{\frac{1}{2};3}_{\infty,\infty}$ but $\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi\notin\mathcal{B}^{\frac{1}{2}+\alpha;3}_{\infty,\infty}$ for any $\alpha>0$ so that Theorem 2.1 of [10] cannot be used. On the other hand, if $1\leq p<2m$, we obtain from (3.13) that $\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi\in\mathcal{B}^{\frac{1}{2}+\frac{1}{2m};3}_{p,\infty}$. It follows from Theorem 3.4 that if $2<p<2m$, there exists $\mathcal{I}\in\mathcal{B}^{\frac{1}{2};1}_{p,\infty}\subseteq C^{\frac{1}{2}-\frac{1}{p}}([0,T];\boldsymbol{L}_m)$ for which there is the convergence

$$\lim_{n\to\infty}\|\Delta\mathcal{I}-I^{D_n}\|_{\mathcal{B}^{\frac{1}{2}+\frac{1}{2m};2}_{p,\infty}}=0$$

where I^{D_n} are the Riemann-type approximations given by (3.8).

Example 3.7. Consider the same setting as in Example 3.6 but instead of a particular function f, consider a general Borel measurable function $f:\mathbb{R}\to\mathbb{R}$. Assume that there exists $\alpha\in(0,1)$, $p\in[2,\infty]$, and $n\in(2,\infty)$ such that $f(B)\in\mathcal{B}^{\alpha;1}_{p,\infty}([0,T];\boldsymbol{L}_n)$. Then for every integer $2\leq m<n$, there is the estimate

$$\|\Xi\|_{\mathcal{B}^{\frac{1}{2};2}_{p,\infty}([0,T];\boldsymbol{L}_m)}+\|\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{\frac{1}{2}+\alpha;3}_{p,\infty}([0,T];\boldsymbol{L}_m)}\lesssim\|f(B)\|_{L^p([0,T];\boldsymbol{L}_n)}+\|f(B)\|_{\mathcal{B}^{\alpha;1}_{p,\infty}([0,T];\boldsymbol{L}_n)}<\infty$$

so that it follows by Theorem 3.4 that there exists $\mathcal{I}\in\mathcal{B}^{\frac{1}{2};2}_{p,\infty}([0,T];\boldsymbol{L}_m)$ for which there is the convergence

$$\lim_{N\to\infty}\|\Delta\mathcal{I}-I^{D_N}\|_{\mathcal{B}^{\frac{1}{2}+\alpha;2}_{p,\infty}([0,T];\boldsymbol{L}_m)}=0$$

where again I^{D_N} are the Riemann-type sums given by (3.8). Indeed, if Ξ is again defined by (3.9), we obtain by using the Hölder inequality and the equivalence of moments of Gaussian random variables the estimate

$$\mathsf{E}|\Xi_{r,r+h}|^m \lesssim (\mathsf{E}|f(B_r)|^n)^{\frac{m}{n}} h^{\frac{m}{2}}$$

from which we immediately obtain the estimate

$$\|\Xi\|_{\mathcal{B}^{\frac{1}{2};2}_{p,\infty}([0,T];\boldsymbol{L}_m)} \lesssim \|f(B)\|_{L^p([0,T];\boldsymbol{L}_n)}.$$

Moreover, as (3.11) and (3.12) also hold, we obtain by Hölder's inequality and the equivalence of moments of Gaussian random variables that

$$\mathsf{E}|\delta\Xi_{r,r+\theta h,r+h} - \mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\theta h,r+h}|^m = \mathsf{E}|\delta\Xi_{r,r+\theta h,r+h}|^m \lesssim (\mathsf{E}|f(B_{r+\theta h}) - f(B_r)|^n)^{\frac{m}{n}} (1-\theta)^{\frac{m}{2}} h^{\frac{m}{2}}$$

from which we immediately obtain the estimate

$$\|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{\frac{1}{2}+\alpha;3}_{p,\infty}([0,T];\boldsymbol{L}_m)} \lesssim \|f(B)\|_{\mathcal{B}^{\alpha;1}_{p,\infty}([0,T];\boldsymbol{L}_n)}.$$

Note that the above condition is easily verified with f being a Hölder continuous function of order $\eta \in (0,1)$, $\alpha = \frac{\eta}{2}$, $p = \infty$, and any $n \in (2,\infty)$ (by appealing to the equivalence of moments of Gaussian random variables again) so that this example extends Example 2.10 of [10].

4. Proof of Theorem 3.4

We shall split the proof of Theorem 3.4 into three separate steps. At first, the existence of the process $\mathcal{I}$ that satisfies convergence (3.5) is proved in Section 4.1. Subsequently, in Section 4.2, we will show that process $\mathcal{I}$ satisfies the bounds in (3.6) and (3.7). Finally, the $\mathscr{F}_t$-adaptedness of $\Delta\mathcal{I}$ is shown in Section 4.3. For the remainder of this section, let us therefore assume that $\mathbb{V}$ has (non-trivial) martingale type $\mathfrak{p} \in (1,2]$ and that $\Xi : \triangle^2[0,T] \to \boldsymbol{L}_m$ is a Borel measurable $\mathscr{F}_t$-adapted stochastic process that satisfies (3.3) and (3.4) for some fixed $p_1, p_2 \in [1,\infty]$, $s_1 \in (\frac{1}{\mathfrak{p}\wedge p_1}, \infty)$, $s_2 \in (1,\infty)$, and $q \in [\mathfrak{p}\wedge p_1, \infty]$. We set

$$s_0 = s_1 \wedge s_2 \quad \text{and} \quad p_0 = p_1 \wedge p_2.$$

4.1. Existence of $\mathcal{I}$ and the proof of convergence (3.5)

Let $\pi = \{\pi_N\}_{N=1}^{\infty}$ be a sequence of partitions of interval $[0,1]$ that satisfies $|\pi_N| \to 0$ as $N \to \infty$. In this subsection, we first prove that the sequence $\{I^{\pi_N} - \Xi\}_{N=1}^{\infty}$, where I^{π_N} is defined by (3.2), is Cauchy in the Besov-type space $\mathcal{B}^{s_0;2}_{p_0,q}$. As this space is complete, this yields the existence of the limit R^π. (This is done in Sect. 4.1.1). Subsequently, we will show that the limit R^π is independent of the sequence of partitions π. (This is done in Sect. 4.1.2). From this point onward, we will be justified to simply use the symbol R for the limit. Subsequently, we will show that there exists a process $\mathcal{I} : [0,T] \to \boldsymbol{L}_m$ such that $I_{s,t} := (\Xi + R)_{s,t} \overset{\text{P-a.s.}}{=} \Delta\mathcal{I}_{s,t}$ holds for λ^2-a.a. $(s,t) \in \triangle^2[0,T]$; *i.e.* that the abstract map I essentially comprises of the increments of another stochastic process (the so-called *additivity* of I) which will yield convergence (3.5). (This is done in Sect. 4.1.3).

4.1.1. Existence of the limit R^π

In this section, we prove that the sequence $\{I^{\pi_N} - \Xi\}_{N=1}^{\infty}$ is Cauchy in the space $\mathcal{B}^{s_0;2}_{p_0,q}$. To this end, we use the bound from the following key lemma. Its proof is rather technical and it is given in Appendix B for clarity of the exposition.

Lemma 4.1. *Let π_N and $\pi'_{N'}$ be two partitions of interval $[0,1]$ such that $\pi'_{N'}$ refines π_N, i.e. $\pi_N \subseteq \pi'_{N'}$, and $|\pi_N| \leq \frac{1}{2}$. Then there is the following bound:*

$$\|I^{\pi_N} - I^{\pi'_{N'}}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim_{s_1,s_2,\mathfrak{p},p_1,q,T} |\pi_N|^{s_0-\frac{1}{\mathfrak{p}\wedge p_1}} \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + |\pi_N|^{s_2-1}\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}. \tag{4.1}$$

Remark 4.2. Note that if π_N and $\pi'_{N'}$ are two partitions of $[0,1]$ that are not necessarily included in one another but which satisfy $|\pi_N| \vee |\pi_{N'}| \leq \frac{1}{2}$, one can define $\tilde{\pi}_{\tilde{N}} = \pi_N \cup \pi'_{N'}$ for $\tilde{N} = N \vee N'$ and apply Lemma 4.1 twice to obtain the inequality

$$\begin{aligned}\|I^{\pi_N} - I^{\pi'_{N'}}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim_{s_1,s_2,\mathfrak{p},p_1,q,T} |\pi_N|^{s_0-\frac{1}{\mathfrak{p}\wedge p_1}} \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + |\pi'_{N'}|^{s_0-\frac{1}{\mathfrak{p}\wedge p_1}} \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} \\ + |\pi_N|^{s_2-1}\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}} + |\pi'_{N'}|^{s_2-1}\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}.\end{aligned}$$

The existence of the limit R^π can be proved now. To this end, let $\pi_1^{\mathcal{D}} = \{0, \frac{1}{2}, 1\}$ and note that

$$\|I^{\pi_1^{\mathcal{D}}} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \leq \|\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_0,q}} \leq \|\delta\Xi - \mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_0,q}} + \|\mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_0,q}} \lesssim \|\delta\Xi - \mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + \|\mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}} \tag{4.2}$$

holds by using Theorem 2.4 and that

$$\|I^{\pi_N} - I^{\pi_1^{\mathcal{D}}}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim \|\delta\Xi - \mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + \|\mathbb{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}} \tag{4.3}$$

holds by Remark 4.2. By combining (4.2) and (4.3), we immediately obtain, for fixed $N \in \mathbb{N}$, the estimate

$$\|I^{\pi_N} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \leq \|I^{\pi_N} - I^{\pi_1^{\mathcal{D}}}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|I^{\pi_1^{\mathcal{D}}} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + \|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}},$$

where the constant depends on $s_1, s_2, \mathfrak{p}, p_1, p_2, T$ and $|\pi_N|$. The right-hand side of the above inequality is finite by assumptions (3.3) and (3.4). This means that the sequence $\{I^{\pi_N} - \Xi\}_{N=1}^{\infty}$ belongs to the space $\mathcal{B}^{s_0;2}_{p_0,q}$.

Recall now that the sequence of partitions $\pi = \{\pi_N\}_{N=1}^{\infty}$ satisfies $|\pi_N| \to 0$ as $N \to \infty$. From this sequence, select π_n and π_m for some $m, n \in \mathbb{N}$. Without loss of generality, we may and do suppose that $|\pi_n| \vee |\pi_m| \leq \frac{1}{2}$. Define $\tilde{\pi} = \pi_n \cup \pi_m$. Similarly as above, adding and subtracting $I^{\tilde{\pi}} - \Xi$ inside the norm and the subsequent successive use of the triangle inequality, Theorem 2.4, Remark 4.2, and assumptions (3.3) and (3.4) yield

$$\|(I^{\pi_n} - \Xi) - (I^{\pi_m} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim |\pi_n|^{s_0-\frac{1}{\mathfrak{p}\wedge p_1}} + |\pi_m|^{s_0-\frac{1}{\mathfrak{p}\wedge p_1}} + |\pi_n|^{s_2-1} + |\pi_m|^{s_2-1}, \tag{4.4}$$

where the constant depends on $s_1, s_2, \mathfrak{p}, p_1, q, T, \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}}$, and $\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}$. Thus it is shown that the sequence $\{I^{\pi_N} - \Xi\}_{N=1}^{\infty}$ is Cauchy in $\mathcal{B}^{s_0;2}_{p_0,q}$ and, as this space is complete, the existence of the limit R^π follows, *i.e.*

$$\lim_{N\to\infty} \|R^\pi - (I^{\pi_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} = 0. \tag{4.5}$$

4.1.2. Independence of R^π of π

Let us now show that the limit R^π does not depend on the sequence of partitions $\pi = \{\pi_N\}_{N=1}^{\infty}$. Let $\tilde{\pi} = \{\tilde{\pi}_N\}_{N=1}^{\infty}$ be another sequence of partitions of interval $[0,1]$ satisfying $|\tilde{\pi}_N| \to \infty$ as $N \to \infty$ and denote the limit of $\{I^{\tilde{\pi}_N} - \Xi\}_{N=1}^{\infty}$ in the space $\mathcal{B}^{s_0;2}_{p_0,q}$ by $R^{\tilde{\pi}}$. Then it holds for any $N \in \mathbb{N}$ that

$$\|R^\pi - R^{\tilde{\pi}}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \leq \|R^\pi - (I^{\pi_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|R^{\tilde{\pi}} - (I^{\tilde{\pi}_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|I^{\pi_N} - I^{\tilde{\pi}_N}\|_{\mathcal{B}^{s_0;2}_{p_0,q}}.$$

Because of convergence (4.5), the first and the second term on the right-hand side of the above inequality tend to zero as $N \to \infty$. The third term is shown to tend to zero as $N \to \infty$ as well by obtaining a bound that is the same as the one in (4.4) in the same manner as in there. Thus we obtain that $R^\pi = R^{\tilde{\pi}}$ in $\mathcal{B}^{s_0;2}_{p_0,q}$. Hence, we have a unique limit $R = R^\pi$ that is independent of π and setting $I = \Xi + R$ yields

$$\lim_{N\to\infty} \|I - I^{\pi_N}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} = \lim_{N\to\infty} \|R - (I^{\pi_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} = 0. \tag{4.6}$$

4.1.3. Additivity of the limit I and convergence (3.5)

From now on, we denote $I = \Xi + R$. We will show here that there exists a Borel measurable stochastic process $\mathcal{I} : [0,T] \to \boldsymbol{L}_m$ such that $\Delta\mathcal{I}_{s,t} \overset{\text{P-a.s.}}{=} I_{s,t}$ holds for λ^2-a.a. $(s,t) \in \triangle^2[0,T]$, *i.e.* that the abstract map I essentially comprises of the increments of another stochastic process. This together with convergence (4.6) implies that (3.5) holds. Moreover, we immediately obtain that if $\|\Xi\|_{\mathcal{B}^{\tilde{s};2}_{\tilde{p},q}} < \infty$ for some $\tilde{s} \in (0,\infty)$ and $\tilde{p} \in [1,\infty]$, then

$$\|\mathcal{I}\|_{\mathcal{B}^{\tilde{s}\wedge s_1\wedge s_2;1}_{\tilde{p}\wedge p_1\wedge p_2,q}} = \|\Delta\mathcal{I}\|_{\mathcal{B}^{\tilde{s}\wedge s_0;1}_{\tilde{p}\wedge p_0,q}} = \|I\|_{\mathcal{B}^{\tilde{s}\wedge s_0;2}_{\tilde{p}\wedge p_0,q}} \lesssim_{\tilde{s},s_1,s_2,\tilde{p},p_1,p_2,T} \|\Xi\|_{\mathcal{B}^{\tilde{s};2}_{\tilde{p},q}} + \|R\|_{\mathcal{B}^{s_0;2}_{p_0,q}} < \infty.$$

A criterion for the existence of stochastic process $\mathcal{I}$ is given in Lemma 4.3 below. Its proof is similar to the proof of Lemma 2.9 in [17].

Lemma 4.3. *If $A : \triangle^2[0,T] \to \boldsymbol{L}_m$ is a Borel measurable stochastic process such that*

$$\operatorname*{ess\,sup}_{h\in[0,\tau]} \| \|\delta A_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m} \|_{L^p([0,T-h];\,\mathrm{d}r)} = 0 \tag{4.7}$$

holds for some $p \in [1,\infty]$, λ-a.a. $\theta \in [0,1]$, and λ-a.a. $\tau \in [0,T]$, then there exists a Borel measurable stochastic process $\mathcal{A} : [0,T] \to \boldsymbol{L}_m$ such that

$$\Delta\mathcal{A}_{s,t} \overset{\text{P-a.s.}}{=} A_{s,t}$$

holds for λ^2-a.a. $(s,t) \in \triangle^2[0,T]$.

Proof. We extend A, and therefore also δA, to the whole $[0,T]^2$ and $[0,T]^3$, respectively, by setting $A_{t,s} = -A_{s,t}$ for $s < t$. Assumption (4.7) then implies that $\|\delta A_{s,u,t}\|_{\boldsymbol{L}_m} = 0$ holds for λ^3-a.a. $(s,u,t) \in \triangle^3[0,T]$. Moreover, it is also seen that $\|\delta A_{s,u,t}\|_{\boldsymbol{L}_m} = 0$ holds for λ^3-a.a. $(s,u,t) \in [0,T]^3$ by observing that

$$\delta A_{\sigma(s,u,t)} = \begin{cases} \delta A_{s,u,t}, & \operatorname{sgn}\sigma = 1, \\ -\delta A_{s,u,t}, & \operatorname{sgn}\sigma = -1, \end{cases}$$

holds for any permutation σ on three elements with sign $\operatorname{sgn}\sigma$. Fubini's Theorem then yields $s \in [0,T]$ for which $\|\delta A_{s,u,t}\|_{\boldsymbol{L}_m} = 0$ holds for λ^2-a.a. $(u,t) \in [0,T]^2$ and we define $\mathcal{A}_t = A_{s,t}$ for $t \in [0,T]$. Then we have

$$\mathcal{A}_t - \mathcal{A}_u = A_{s,t} - A_{s,u} = A_{u,t} + \delta A_{s,u,t} \overset{\text{P-a.s.}}{=} A_{u,t}$$

for λ^2-a.a. $(u,t) \in [0,T]^2$. As the map $A : \triangle^2[0,T] \to \boldsymbol{L}_m$ is Borel measurable, the same holds also for the map $A : [0,T]^2 \to \boldsymbol{L}_m$ and, consequently, also for the map $\mathcal{A} : [0,T] \to \boldsymbol{L}_m : t \mapsto \mathcal{A}_t$. □

Let us now verify that I satisfies the assumptions of Lemma 4.3. To this end, we will show that

$$\operatorname*{ess\,sup}_{h\in[0,\tau]} \| \|\delta I_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m} \|_{L^{p_0}([0,T-h];\,\mathrm{d}r)} = 0$$

holds for λ-a.a. $\theta \in [0,1]$ and λ-a.a. $\tau \in [0,T]$. Indeed, pick $\theta \in (0,1)$ arbitrarily and define a sequence of partitions of $[0,1]$ as follows:

$$\begin{aligned}\pi_1^\theta &= \{0 < \theta < 1\},\\ \pi_2^\theta &= \{0 < \theta^2 < \theta < \theta + \theta(1-\theta) < 1\},\\ &\vdots\\ \pi_{N+1}^\theta &= \{0 = t_0^{\theta,N+1} < t_1^{\theta,N+1} < \cdots < t_{2^{N+1}}^{\theta,N+1} = 1\}\end{aligned}$$

where for $N \in \mathbb{N}$ we define

$$\begin{aligned}t_{2i}^{\theta,N+1} &= t_i^{\theta,N} \qquad \text{for} \quad i = 0,\dots,2^N,\\ t_{2i-1}^{\theta,N+1} &= t_{i-1}^{\theta,N} + \theta(t_i^{\theta,N} - t_{i-1}^{\theta,N}) \qquad \text{for} \quad i = 1,\dots,2^N.\end{aligned}$$

We observe two important things. The first observation is that there is the following convergence:

$$\lim_{N\to\infty} |\pi_N^\theta| = \begin{cases} \lim\limits_{N\to\infty} \theta^N = 0, & \theta \geq \frac{1}{2},\\ \lim\limits_{N\to\infty} (1-\theta)^N = 0, & \theta \leq \frac{1}{2}.\end{cases}$$

The second observation is that there is the inclusion $\pi_N^\theta \subseteq \pi_M^\theta$ whenever $N \leq M$. Now, fix $N \in \mathbb{N}$ and write

$$\left\| \frac{\operatorname{ess\,sup}_{h\in[0,\tau]} \| \|\delta I_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m}\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)}}{\tau^{s_0}} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})} \leq Z_1^N + Z_2^N \tag{4.8}$$

where

$$\begin{aligned} Z_1^N &= \left\| \frac{\operatorname{ess\,sup}_{h\in[0,\tau]} \| \|\delta I_{r,r+\theta h,r+h} - \delta I^{\pi_N^\theta}_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m}\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)}}{\tau^{s_0}} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})},\\ Z_2^N &= \left\| \frac{\operatorname{ess\,sup}_{h\in[0,\tau]} \| \|\delta I^{\pi_N^\theta}_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m}\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)}}{\tau^{s_0}} \right\|_{L^q([0,T];\frac{\mathrm{d}\tau}{\tau})}. \end{aligned}$$

We shall show that both Z_1^N and Z_2^N on the right-hand side of inequality (4.8) tend to zero as $N \to \infty$ for λ-a.a. $\theta \in [0,1]$. In order to prove that $\lim_{N\to\infty} Z_1^N = 0$ holds for λ-a.a. $\theta \in [0,1]$, let us write

$$\delta I_{r,r+\theta h,r+h} - \delta I^{\pi_N^\theta}_{r,r+\theta h,r+h} = I_{r,r+h} - I^{\pi_N^\theta}_{r,r+h} - (I_{r,r+\theta h} - I^{\pi_N^\theta}_{r,r+\theta h}) - (I_{r+\theta h,r+h} - I^{\pi_N^\theta}_{r+\theta h,r+h})$$

for $h \in [0,T]$, $r \in [0,T-h]$, and $\theta \in (0,1)$. It follows that

$$Z_1^N \leq 3\|I - I^{\pi_N^\theta}\|_{\mathcal{B}^{s_0;2}_{p_0,q}}$$

holds for λ-a.a. $\theta \in [0,1]$. By appealing to formula (4.6), we obtain that the right-hand side of the above inequality and, therefore also Z_1^N, tends to zero as $N \to \infty$ for λ-a.a. $\theta \in [0,1]$. To prove that also $\lim_{N\to\infty} Z_2^N = 0$ holds

for λ-a.a. $\theta \in [0,1]$, we can write

$$\delta I^{\pi_N^\theta}_{r,r+\theta h,r+h} = I^{\pi_N^\theta}_{r,r+h} - I^{\pi_N^\theta}_{r,r+\theta h} - I^{\pi_N^\theta}_{r+\theta h,r+h} = I^{\pi_N^\theta}_{r,r+h} - I^{\pi_{N+1}^\theta}_{r,r+h}$$

for $h \in [0,T]$, $r \in [0,T-h]$, and $\theta \in (0,1)$, and estimate

$$Z_2^N \le \|I^{\pi_N^\theta} - I^{\pi_{N+1}^\theta}\|_{\mathcal{B}^{s_0;2}_{p_0,q}}$$

for λ-a.a. $\theta \in [0,1]$. However, π^θ_{N+1} refines π^θ_N and hence, we can use bound (4.1) (at least for those N large enough to satisfy $|\pi_N| \le \frac{1}{2}$) from Lemma 4.1 which, combined with Theorem 2.4, yields the claim. It thus follows that the left-hand side of (4.8) equals to zero for λ-a.a. $\tau \in [0,T]$ and λ-a.a. $\theta \in [0,1]$ which means that

$$\operatorname*{ess\,sup}_{h\in[0,\tau]} \| \|\delta I_{r,r+\theta h,r+h}\|_{\boldsymbol{L}_m} \|_{L^{p_0}([0,T-h];\,\mathrm{d}r)} = 0$$

holds for λ-a.a. $\theta \in [0,1]$ and λ-a.a. $\tau \in [0,T]$. This concludes the proof of the existence of $\mathcal{I}$ satisfying (3.5).

4.2. Proofs of estimates (3.6) and (3.7)

In this section, estimates (3.6) and (3.7) are proved. Take the sequence $\{\pi_N\}_{N=1}^\infty$ of dyadic partitions $\pi_N = \{k/2^N\}_{k=0}^{2^N}$, $N \in \mathbb{N}$, of interval $[0,1]$. For $h \in [0,T]$, $r \in [0,T-h]$, and $N \in \mathbb{N}$, we rewrite

$$I^{\pi_N}_{r,r+h} - \Xi_{r,r+h} = I^{\pi_N}_{r,r+h} - I^{\pi_1}_{r,r+h} + \delta\Xi_{r,r+\frac{1}{2}h,r+h}. \tag{4.9}$$

4.2.1. Estimate (3.6)

We shall proceed through

$$\|\Delta\mathcal{I} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \le \|\Delta\mathcal{I} - I^{\pi_N}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|I^{\pi_N} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}}. \tag{4.10}$$

The first summand on the right-hand side of (4.10) tends to zero as $N \to \infty$ thanks to (3.5). Therefore, we only need to estimate the second summand appropriately and let $N \to \infty$. Formula (4.9) yields

$$\begin{aligned}\omega^2_{p_0}(I^{\pi_N} - \Xi, \tau) \le{}& \omega^2_{p_0}(I^{\pi_N} - I^{\pi_1}, \tau)\\ &+ \operatorname*{ess\,sup}_{h\in[0,\tau]} \left\| \left\| \delta\Xi_{r,r+\frac{1}{2}h,r+h} - \mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\frac{1}{2}h,r+h} \right\|_{\boldsymbol{L}_m} \right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}\\ &+ \operatorname*{ess\,sup}_{h\in[0,\tau]} \left\| \left\| \mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\frac{1}{2}h,r+h} \right\|_{\boldsymbol{L}_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)}\end{aligned}$$

for $\tau \in [0,T]$ from which the estimate

$$\omega^2_{p_0}(I^{\pi_N} - \Xi, \tau) \le \omega^2_{p_0}(I^{\pi_N} - I^{\pi_1}, \tau) + \omega^3_{p_1}(\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi, \tau) + \omega^3_{p_2}(\mathsf{E}^{\mathscr{F}}\delta\Xi, \tau)$$

as well as the estimate

$$\|I^{\pi_N} - \Xi\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \le \|I^{\pi_N} - I^{\pi_1}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + \|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}$$

are obtained immediately. We only need to estimate the first term of the above inequality. As π_N refines π_1 and $|\pi_1| = \frac{1}{2}$, the bound in (4.1) yields

$$\|I^{\pi_N} - I^{\pi_1}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \lesssim_{s_1,s_2,\mathfrak{p},p_1,q,T} 2^{-s_0+\frac{1}{\mathfrak{p}\wedge p_1}} \|\delta\Xi - \mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_0;3}_{p_1,q}} + 2^{-s_2+1}\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}.$$

Finally, bound (3.6) is obtained by combining the last two estimates with inequality (4.10) and letting $N \to \infty$.

4.2.2. Estimate (3.7)

At first, recall the contraction property of the conditional expectation, *i.e.*

$$\|\mathsf{E}^{\mathscr{G}_1}(\,\cdot\,)\|_{\boldsymbol{L}_m} \le \|\mathsf{E}^{\mathscr{G}_2}(\,\cdot\,)\|_{\boldsymbol{L}_m} \tag{4.11}$$

holds for two σ-algebras satisfying $\mathscr{G}_1 \subseteq \mathscr{G}_2 \subseteq \mathscr{F}$; see, *e.g.*, [22]. We shall proceed through

$$\|\mathsf{E}^{\mathscr{F}}(\Delta\mathcal{I} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \le \|\Delta\mathcal{I} - I^{\pi_N}\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \tag{4.12}$$

where we used the contraction property (4.11) in the first term on the right-hand side, and hence, this term tends to zero as $N \to \infty$ thanks to formula (3.5). It is therefore sufficient to only estimate the second summand appropriately and let $N \to \infty$ again.

First, we take $\mathsf{E}^{\mathscr{F}_r}$ of both sides of equation (4.9). We obtain that

$$\omega^2_{p_0}(\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - \Xi), \tau) \le \omega^2_{p_0}(\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - I^{\pi_1}), \tau) + \operatorname*{ess\,sup}_{h\in[0,\tau]} \left\| \left\| \mathsf{E}^{\mathscr{F}_r}\delta\Xi_{r,r+\frac{1}{2}h,r+h} \right\|_{\boldsymbol{L}_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)}$$

holds for $\tau \in [0,T]$. (We have also used formula (4.11) again). By the same argument as in the proof of estimate (3.6), the estimate

$$\|\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - \Xi)\|_{\mathcal{B}^{s_0;2}_{p_0,q}} \le \|\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - I^{\pi_1})\|_{\mathcal{B}^{s_0;2}_{p_0,q}} + \|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}} \tag{4.13}$$

is then shown to hold. It remains to estimate the first summand of the previous inequality. Let us choose $h \in [0,T]$ and $r \in [0, T-h]$. We can take the conditional expectation $\mathsf{E}^{\mathscr{F}_r}$ of both sides of equality (B.5) and use inequality (4.11) again to get

$$\begin{aligned}\|\mathsf{E}^{\mathscr{F}_r}(I^{\pi_N}_{r,r+h} - I^{\pi_1}_{r,r+h})\|_{\boldsymbol{L}_m} \le{}& \sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n} \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_2^{i,n,k},\hat{s}_3^{i,n,k}} \right\|_{\boldsymbol{L}_m} \\ &+ \sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n} \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_3^{i,n,k},\hat{s}_4^{i,n,k}} \right\|_{\boldsymbol{L}_m}\end{aligned}$$

which yields

$$\begin{aligned}\omega^2_{p_2}(\mathsf{E}^{\mathscr{F}}(I^{\pi_N} - I^{\pi_1}), h) \le{}& \sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n} \operatorname*{ess\,sup}_{h\in[0,H]} \left\| \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_2^{i,n,k},\hat{s}_3^{i,n,k}} \right\|_{\boldsymbol{L}_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)} \\ &+ \sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n} \operatorname*{ess\,sup}_{h\in[0,H]} \left\| \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_3^{i,n,k},\hat{s}_4^{i,n,k}} \right\|_{\boldsymbol{L}_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)},\end{aligned}$$

and we would proceed in the same manner as we would proceed from identity (B.7) to the use of Lemma B.3 to obtain

$$\left\| \frac{\omega_{p_2}^2(\mathsf{E}^{\mathscr{F}}(I^{\pi_N}-I^{\pi_1}),h)}{h^{s_2}} \right\|_{L^q([0,T];\frac{\mathrm{d}h}{h})} \leq \left\| \sum_{i=1}^{N}\sum_{n=1}^{n_0^i} \frac{\omega_{p_2}^3(\mathsf{E}^{\mathscr{F}}\delta\Xi, 2(t_i-t_{i-1})h2^{-n})}{2^{-n}h^{s_2}} \right\|_{L^q([0,T];\frac{\mathrm{d}h}{h})}.$$

Now, Lemma B.3 yields

$$\|\mathsf{E}^{\mathscr{F}}(I^{\pi_N}-I^{\pi_1})\|_{\mathcal{B}^{s_2;2}_{p_2,q}} \lesssim_{s_2,\mathfrak{p},q} 2^{-s_2+1}\|\mathsf{E}^{\mathscr{F}}\delta\Xi\|_{\mathcal{B}^{s_2;3}_{p_2,q}}.$$

We conclude the proof of the bound in estimate (3.7) by combining the last estimate with inequalities (4.12) and (4.13) and by letting $N\to\infty$.

4.3. Proof of $\mathscr{F}_t$-adaptedness of $\Delta\mathcal{I}$

In what follows, we prove that $\Delta\mathcal{I}$ is $\mathscr{F}_t$-adapted. To this end, pick any sequence $\{\pi_N\}_{N=1}^{\infty}$ of partitions of interval $[0,1]$ satisfying $\lim_{N\to\infty}|\pi_N|=0$. Then

$$\lim_{N\to\infty}\left\| \frac{\omega_{p_1\wedge p_2}^2(\Delta\mathcal{I}-I^{\pi_N})}{\tau^{s_1\wedge s_2}} \right\|_{L^q([0,T],\frac{\mathrm{d}\tau}{\tau})}=0$$

holds by (3.5). The convergence in the $L^q([0,T],\frac{\mathrm{d}\tau}{\tau})$-norm implies the convergence in the measure $\frac{\mathrm{d}\tau}{\tau}$ and therefore, we can select a subsequence $\{\pi_{N_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\omega_{p_1\wedge p_2}^2(\Delta\mathcal{I}-I^{\pi_{N_k}})=\lim_{k\to\infty}\operatorname*{ess\,sup}_{h\in[0,\tau]}\|\|\Delta\mathcal{I}_{r,r+h}-I^{\pi_{N_k}}_{r,r+h}\|_{\boldsymbol{L}_m}\|_{L^{p_1\wedge p_2}([0,T-h],\mathrm{d}r)}=0$$

holds for λ-a.a. $\tau\in[0,T]$. It follows that for λ-a.a. $h\in[0,T]$, we have

$$\lim_{k\to\infty}\|\|\Delta\mathcal{I}_{r,r+h}-I^{\pi_{N_k}}_{r,r+h}\|_{\boldsymbol{L}_m}\|_{L^{p_1\wedge p_2}([0,T-h],\mathrm{d}r)}=0$$

and we fix one such h arbitrarily. Repeating the argument, we can select a subsequence $\{\pi_{N_{k_j}}\}_{j=1}^{\infty}$, depending on h, such that

$$\lim_{j\to\infty}\|\Delta\mathcal{I}_{r,r+h}-I^{\pi_{N_{k_j}}}_{r,r+h}\|_{\boldsymbol{L}_m}=0$$

holds for λ-a.a. $r\in[0,T-h]$ and we fix one such r arbitrarily. Once more, we can select a subsequence $\{\pi_{N_{k_{j_l}}}\}_{l=1}^{\infty}$, depending on r (and h), such that it holds that

$$\Delta\mathcal{I}_{r,r+h}(\omega)=\lim_{l\to\infty}I^{\pi_{N_{k_{j_l}}}}_{r,r+h}(\omega)$$

for P-a.a. $\omega\in\Omega$. However, $I^{\pi_m}_{u,u+v}$ is $\mathscr{F}_{u+v}$-measurable for any $m\in\mathbb{N}$, $v\in[0,T]$, and $u\in[0,T-v]$. This means that also $\Delta\mathcal{I}_{r,r+h}$, being the pointwise limit of $\mathscr{F}_{r+h}$-measurable functions, is $\mathscr{F}_{r+h}$-measurable.

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The research data associated with this article are included in the article.

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Appendix A. Auxiliary lemma

Lemma A.1. *Suppose we have a non-decreasing function $f : [0,\infty) \to [0,\infty)$; $n, m \in \mathbb{N}$ such that $m \geq n$; $h \in (0,T]$; and $c \in (0,1]$. Moreover, let $p \in [0,\infty)$ and $\gamma \in (0,\infty)$. Then*

$$\sum_{k=n}^{m} \frac{f(ch2^{-k})}{2^{-kp}} \leq \frac{2^p h^p}{\log 2} \int_0^{h2^{-n+1}} \frac{f(c\tau)}{\tau^p} \frac{\mathrm{d}\tau}{\tau} \leq \frac{2^p c^p h^p}{\log 2} \left\| \frac{f(\tau)}{\tau^\gamma} \right\|_{L^\infty([0,cT];\frac{\mathrm{d}\tau}{\tau})} \int_0^{ch2^{-n+1}} y^{\gamma-p} \frac{\mathrm{d}y}{y}. \tag{A.1}$$

Proof. Because the function $x \mapsto f(ch2^{-x})$ is non-increasing, we can write

$$f(ch2^{-k}) \leq \int_{k-1}^{k} f(ch2^{-x})\,\mathrm{d}x$$

for any $k = n, n+1, \dots, m$. As $2^{xp} \geq 2^{(k-1)p}$ holds for $x \in [k-1,k]$, we obtain the inequality

$$\sum_{k=n}^{m} \frac{f(ch2^{-k})}{2^{-kp}} \leq 2^p h^p \int_{n-1}^{m} \frac{f(ch2^{-x})}{[h2^{-x}]^p}\,\mathrm{d}x.$$

The first inequality in (A.1) is then obtained by substituting $\tau = h2^{-x}$ and by decreasing the lower bound in the resulting integral from $h2^{-m}$ to 0. The second inequality in (A.1) is obtained from the first inequality by further substituting $y = c\tau$ and estimating

$$\frac{f(y)}{y^\gamma} \leq \left\| \frac{f(\tau)}{\tau^\gamma} \right\|_{L^\infty([0,cT];\,\mathrm{d}\tau)} = \left\| \frac{f(\tau)}{\tau^\gamma} \right\|_{L^\infty([0,cT];\frac{\mathrm{d}\tau}{\tau})}, \quad y \in \left(0, ch2^{-n+1}\right].$$

□

Appendix B. Proof of Lemma 4.1

Here, the proof of Lemma 4.1 is given. We recall the notation

$$s_0 = s_1 \wedge s_2 \quad \text{and} \quad p_0 = p_1 \wedge p_2.$$

Let us assume that

$$\pi_N = \{0 = t_0 < t_1 < \cdots < t_N = 1\} \subseteq \pi'_{N'} = \{0 = t'_0 < t'_1 < \cdots < t'_{N'} = 1\}$$

and let us take $H \in [0,T]$, $h \in [0,H]$, and $r \in [0, T-h]$. We shall assume that $N' > N$ holds. (The case $N' < N$ is not possible if we assume that $\pi_N \subseteq \pi'_{N'}$ and if $N' = N$, then the norm on the left-hand side of (4.1) is zero). We can write

$$I^{\pi_N}_{r,r+h} - I^{\pi'_{N'}}_{r,r+h} = \sum_{i=1}^{N} \left(\Xi_{r+t_{i-1}h, r+t_i h} - \sum_{j\,:\,t_{i-1} < t'_j \leq t_i} \Xi_{r+t'_{j-1}h, r+t'_j h} \right).$$

Now we can use the following Lemma B.2 which is a consequence of the non-trivial martingale type of $\mathbb{V}$ and which provides us with a Burkholder–Davis–Gundy-type inequality.

Lemma B.1. *Fix $K \in \mathbb{N}$ and $t > 0$. Assume that we are given Borel measurable stochastic processes $Z^i : [0,t] \to \boldsymbol{L}_m$ for $i = 1, \dots, K$. Suppose that there is a collection of σ-algebras $\{\mathscr{G}^i_r\}_{i=1,\dots,K; r\in[0,t]}$ satisfying $\mathscr{G}^1_r \subseteq \cdots \subseteq \mathscr{G}^K_r \subseteq \mathscr{F}$ for any $r \in [0,t]$. Moreover, assume that $\sigma(Z^1_r), \dots, \sigma(Z^n_r) \subseteq \mathscr{G}^{n+1}_r$ for any $n = 1, \dots, K-1$ and $r \in [0,t]$ (the symbol σ stands*

for the generated σ-algebra). Finally, suppose that $\mathsf{E}^{\mathscr{G}^n} Z^n$ is Borel measurable for $n = 1, \dots, K$. Then

$$\left\| \left\| \sum_{i=1}^{K} (Z_r^i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i) \right\|_{L_m} \right\|_{L^p([0,t];\,\mathrm{d}r)} \lesssim_{\mathfrak{p}} \left(\sum_{i=1}^{K} \left\| \|Z_r^i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i\|_{L_m} \right\|_{L^p([0,t];\,\mathrm{d}r)}^{\mathfrak{p}\wedge p} \right)^{\frac{1}{\mathfrak{p}\wedge p}} \tag{B.1}$$

holds for any $p \in [1, \infty]$.

Proof. Fix $r \in [0,t]$. Set $M_0^r = 0 \in \mathbb{V}$ and $M_n^r = \sum_{i=1}^{n} (Z_r^i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i)$ for $n = 1, \dots, K$. Then the sequence $\{M_n^r\}_{n=0}^K$ is a martingale and, consequently, we obtain

$$\left\| \sum_{i=1}^{K} (Z_r^i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i) \right\|_{L_m} \lesssim_{\mathfrak{p}} \left\| \sum_{i=1}^{K} \|Z_r^i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i\|_{\mathbb{V}}^{\mathfrak{p}} \right\|_{L^m(\Omega)}^{\frac{1}{\mathfrak{p}}} \leq \left(\sum_{i=1}^{K} \|Z_i - \mathsf{E}^{\mathscr{G}_r^i} Z_r^i\|_{L_m}^{\mathfrak{p}} \right)^{\frac{1}{\mathfrak{p}}} \tag{B.2}$$

by appealing to inequality (3.1) (as $\mathbb{V}$ is assumed to have non-trivial martingale type $\mathfrak{p}$) and by using Minkowski's inequality (as $m \geq 2 \geq \mathfrak{p}$). Now, the L^p-norm is applied to both sides of (B.2) and for the case $p \in [1, \mathfrak{p})$, inequality (B.1) follows by using the inequality $(a+b)^{\frac{p}{\mathfrak{p}}} \leq a^{\frac{p}{\mathfrak{p}}} + b^{\frac{p}{\mathfrak{p}}}$, $a, b > 0$ while for the case $p \in [\mathfrak{p}, \infty)$, we use the triangle inequality. For the case $p = \infty$, inequality (B.1) is immediate. □

Given $h \in [0, H]$, we set

$$Z_{r,h}^i = \Xi_{r+t_{i-1}h, r+t_i h} - \sum_{j\,:\, t_{i-1} < t_j' \leq t_i} \Xi_{r+t_{j-1}'h, r+t_j' h}, \quad r \in [0, T-h],$$

for the rest of the proof. We obtain

$$\begin{aligned} &\left\| \left\| I_{r,r+h}^{\pi_N} - I_{r,r+h}^{\pi_{N'}'} \right\|_{L_m} \right\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)} \\ &\qquad \lesssim_{\mathfrak{p}} \left(\sum_{i=1}^{N} \left\| \left\| Z_{r,h}^i - \mathsf{E}^{\mathscr{F}_{r+t_{i-1}h}} Z_{r,h}^i \right\|_{L_m} \right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}^{\mathfrak{p}\wedge p_1} \right)^{\frac{1}{\mathfrak{p}}} + \sum_{i=1}^{N} \left\| \left\| \mathsf{E}^{\mathscr{F}_{r+t_{i-1}h}} Z_{r,h}^i \right\|_{L_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)}, \end{aligned}$$

by using Lemma B.1 to $\tilde{Z}_r^i = Z_{r,h}^i$ and $\tilde{\mathscr{G}}_r^i = \mathscr{F}_{r+t_{i-1}h}$. From this estimate, the bound

$$\begin{aligned} &\left\| \left\| I_{r,r+h}^{\pi_N} - I_{r,r+h}^{\pi_{N'}'} \right\|_{L_m} \right\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)} \\ &\qquad \lesssim_{\mathfrak{p}} 2 \left(\sum_{i=1}^{N} \left\| \left\| Z_{r,h}^i \right\|_{L_m} \right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}^{\mathfrak{p}\wedge p_1} \right)^{\frac{1}{\mathfrak{p}}} + \sum_{i=1}^{N} \left\| \left\| \mathsf{E}^{\mathscr{F}_{r+t_{i-1}h}} Z_{r,h}^i \right\|_{L_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)}, \end{aligned} \tag{B.3}$$

easily follows by using inequality (4.11). The following Lemma B.2 can be used on every Z^i in the inequality above. The proof of Lemma B.2 is only a slight generalization of the proof of Lemma 2.14 in [10] (we only need to take into account the dependency of all the objects on $r, r+h$) and we therefore omit it here.

Lemma B.2. *Consider a $\mathbb{V}$-valued stochastic process $(\Xi_{u,v})_{(u,v)\in\triangle^2[s,t]}$ where $0 \leq s < t < \infty$. For $n \in \mathbb{N}_0$ and $k = 0, \dots, 2^n$, set $t_k^n = \frac{k}{2^n}$. Pick any partition $\{0 = t_0 < t_1 < \cdots < t_N = 1\}$ where $N \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ and random vectors $R_{r,r+h}^{n,k}$ for $n \in \mathbb{N}_0$, $k = 1, \dots, 2^n$, $h \in [0, t-s]$, and $r \in [s, t-h]$ such that for λ-a.a. $h \in [0, t-s]$ it holds that*

1. *$R_{r,r+h}^{n,k} \overset{\mathsf{P\text{-}a.s.}}{=} 0$ for any $n \geq n_0$, $k = 1, \dots, 2^n$, and for λ-a.a. $r \in [s, t-h]$;*

2. *for all* $n \in \mathbb{N}_0$ *and* $k = 1, \dots, 2^n$, *there is* $(s_1^{n,k}, s_2^{n,k}, s_3^{n,k}, s_4^{n,k}) \in \triangle^4[t_{k-1}^n, t_k^n]$ *such that*

$$R_{r,r+h}^{n,k} \overset{\text{P-a.s.}}{=} \Xi_{r+s_1^{n,k}h, r+s_4^{n,k}h} - \Xi_{r+s_1^{n,k}h, r+s_2^{n,k}h} - \Xi_{r+s_2^{n,k}h, r+s_3^{n,k}h} - \Xi_{r+s_3^{n,k}h, r+s_4^{n,k}h}$$

holds for λ*-a.a.* $r \in [s, t-h]$;

3. $\Xi_{r,r+h} - \sum_{k=1}^N \Xi_{r+t_{k-1}h, r+t_k h} \overset{\text{P-a.s.}}{=} \sum_{n=0}^{n_0-1} \sum_{k=1}^{2^n} R_{r,r+h}^{n,k}$ *for* λ*-a.a.* $r \in [s, t-h]$.

By using Lemma B.2 as indicated, we obtain that there exist $n_0^i \in \mathbb{N}$, $i = 1, \dots, N$, and $(s_1^{i,n,k}, s_2^{i,n,k}, s_3^{i,n,k}, s_4^{i,n,k}) \in \triangle^4[\frac{k-1}{2^n}, \frac{k}{2^n}]$, $k = 1, \dots, 2^n$, $n = 0, \dots, n_0^i - 1$, $i = 1, \dots, N$, such that for λ-a.a. $h \in [0, H]$, the equality

$$\begin{aligned} Z_{r,h}^i \overset{\text{P-a.s.}}{=} \sum_{n=0}^{n_0^i-1} \sum_{k=1}^{2^n} \Big[& \Xi_{r+t_{i-1}h+s_1^{i,n,k}(t_i-t_{i-1})h, r+t_{i-1}h+s_4^{i,n,k}(t_i-t_{i-1})h} \\ & - \Xi_{r+t_{i-1}h+s_1^{i,n,k}(t_i-t_{i-1})h, r+t_{i-1}h+s_2^{i,n,k}(t_i-t_{i-1})h} \\ & - \Xi_{r+t_{i-1}h+s_2^{i,n,k}(t_i-t_{i-1})h, r+t_{i-1}h+s_3^{i,n,k}(t_i-t_{i-1})h} \\ & - \Xi_{r+t_{i-1}h+s_3^{i,n,k}(t_i-t_{i-1})h, r+t_{i-1}h+s_4^{i,n,k}(t_i-t_{i-1})h} \Big] \end{aligned} \tag{B.4}$$

holds for λ-a.a. $r \in [0, T-h]$. Let us denote

$$\hat{s}_j^{i,n,k} = r + t_{i-1}h + s_j^{i,n,k}(t_i - t_{i-1})h, \qquad j \in \{1, 2, 3, 4\},$$

and let us add and subtract $\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_3^{i,n,k}}$ in the last sum of equality (B.4). Thus we obtain

$$Z_{r,h}^i \overset{\text{P-a.s.}}{=} \sum_{n=0}^{n_0^i-1} \sum_{k=1}^{2^n} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_2^{i,n,k}, \hat{s}_3^{i,n,k}} + \sum_{n=0}^{n_0^i-1} \sum_{k=1}^{2^n} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_3^{i,n,k}, \hat{s}_4^{i,n,k}}. \tag{B.5}$$

Therefore, we can estimate the first term on the right-hand side of inequality (B.3) as

$$\begin{aligned} & \left(\sum_{i=1}^N \left\| \left\| Z_{r,h}^i \right\|_{L_m} \right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}^{\mathfrak{p} \wedge p_1} \right)^{\frac{1}{\mathfrak{p} \wedge p_1}} \\ & \quad \le \sum_{j=2}^3 \Bigg(\sum_{i=1}^N \Bigg[\sum_{n=0}^{n_0^i-1} \left\| \left\| \sum_{k=1}^{2^n} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_j^{i,n,k}, \hat{s}_{j+1}^{i,n,k}} \right. \right. \\ & \qquad \left. \left. - \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_j^{i,n,k}, \hat{s}_{j+1}^{i,n,k}} \right\|_{L_m} \right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)} \Bigg]^{\mathfrak{p} \wedge p_1} \Bigg)^{\frac{1}{\mathfrak{p} \wedge p_1}} \\ & \quad + \sum_{j=2}^3 \sum_{i=1}^N \sum_{n=0}^{n_0^i-1} \sum_{k=1}^{2^n} \left\| \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_j^{i,n,k}, \hat{s}_{j+1}^{i,n,k}} \right\|_{L_m} \right\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)} \end{aligned}$$

and the second term, using inequality (4.11), as

$$\sum_{i=1}^N \left\| \left\| \mathsf{E}^{\mathscr{F}_{r+t_{i-1}h}} Z_r^i \right\|_{L_m} \right\|_{L^{p_2}([0,T-h];\,\mathrm{d}r)} \le \sum_{j=2}^3 \sum_{i=1}^N \sum_{n=0}^{n_0^i-1} \sum_{k=1}^{2^n} \left\| \left\| \mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}} \delta\Xi_{\hat{s}_1^{i,n,k}, \hat{s}_j^{i,n,k}, \hat{s}_{j+1}^{i,n,k}} \right\|_{L_m} \right\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)}.$$

If we insert these last two inequalities into inequality (B.3), we obtain

$$\begin{aligned}
&\left\|\left\|I^{\pi_N}_{r,r+h}-I^{\pi'_{N'}}_{r,r+h}\right\|_{L_m}\right\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)}\\
&\qquad\lesssim_{\mathfrak{p}}\sum_{j=2}^{3}\left(\sum_{i=1}^{N}\left[\sum_{n=0}^{n_0^i-1}\left\|\left\|\sum_{k=1}^{2^n}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right.\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.-\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right\|_{L_m}\right\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}\right]^{\mathfrak{p}\wedge p_1}\right)^{\frac{1}{\mathfrak{p}\wedge p_1}}\\
&\qquad+\sum_{j=2}^{3}\sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n}\left\|\left\|\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right\|_{L_m}\right\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)}
\end{aligned}$$

and the application of Lemma B.1 again with $\mathscr{G}^k_r=\mathscr{F}_{\hat{s}_1^{i,n,k+1}}$ for $k=0,\dots,2^n-1$ and $\mathscr{G}^{2^n}_r=\mathscr{F}_{r+t_ih}$ yields

$$\begin{aligned}
&\left\|\left\|I^{\pi_N}_{r,r+h}-I^{\pi'_{N'}}_{r,r+h}\right\|_{L_m}\right\|_{L^{p_0}([0,T-h];\,\mathrm{d}r)}\\
&\qquad\lesssim_{\mathfrak{p}}\sum_{j=2}^{3}\left(\sum_{i=1}^{N}\left[\sum_{n=0}^{n_0^i-1}\left(\sum_{k=1}^{2^n}\left\|\left\|\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right.\right.\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.\left.-\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right\|_{L_m}\right\|^{\mathfrak{p}\wedge p_1}_{L^{p_1}([0,T-h];\,\mathrm{d}r)}\right)^{\frac{1}{\mathfrak{p}\wedge p_1}}\right]^{\mathfrak{p}\wedge p_1}\right)^{\frac{1}{\mathfrak{p}\wedge p_1}}\\
&\qquad+\sum_{j=2}^{3}\sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n}\left\|\left\|\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right\|_{L_m}\right\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)}.
\end{aligned}\tag{B.6}$$

It follows from the above estimate (B.6) that

$$\begin{aligned}
&\omega^2_{p_0}\big(\mathsf{E}^{\mathscr{F}}(I^{\pi_N}-I^{\pi'_{N'}}),H\big)\\
&\qquad\lesssim_{\mathfrak{p}}\sum_{j=2}^{3}\left[\sum_{i=1}^{N}\left(\sum_{n=0}^{n_0^i-1}\left[\sum_{k=1}^{2^n}\Big(\operatorname*{ess\,sup}_{h\in[0,H]}\Big\|\big\|\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.-\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\big\|_{L_m}\Big\|_{L^{p_1}([0,T-h];\,\mathrm{d}r)}\Big)^{\mathfrak{p}\wedge p_1}\right]^{\frac{1}{\mathfrak{p}\wedge p_1}}\right)^{\mathfrak{p}\wedge p_1}\right]^{\frac{1}{\mathfrak{p}\wedge p_1}}\\
&\qquad+\sum_{j=2}^{3}\sum_{i=1}^{N}\sum_{n=0}^{n_0^i-1}\sum_{k=1}^{2^n}\operatorname*{ess\,sup}_{h\in[0,H]}\big\|\big\|\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\big\|_{L_m}\big\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)}
\end{aligned}\tag{B.7}$$

holds. Observe that for fixed $i=1,\dots,N$, $n=1,\dots,n_0^i-1$, $k=1,\dots,2^n$, and $j=2,3$ the inequalities

1. $\hat{s}_1^{i,n,k}\le\hat{s}_j^{i,n,k}\le\hat{s}_{j+1}^{i,n,k}$,
2. $\hat{s}_{j+1}^{i,n,k}-\hat{s}_1^{i,n,k}\le\frac{1}{2^n}(t_i-t_{i-1})H$

hold for any $h\in[0,H]$. These two inequalities lead to the estimates

$$\operatorname*{ess\,sup}_{h\in[0,H]}\left\|\big\|\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}-\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\big\|_{L_m}\right\|_{L^{p_1}([0,T-h],\,\mathrm{d}r)}\le\omega^3_{p_1}(\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi,(t_i-t_{i-1})H2^{-n})$$

and

$$\operatorname*{ess\,sup}_{h\in[0,H]}\left\|\left\|\mathsf{E}^{\mathscr{F}_{\hat{s}_1^{i,n,k}}}\delta\Xi_{\hat{s}_1^{i,n,k},\hat{s}_j^{i,n,k},\hat{s}_{j+1}^{i,n,k}}\right\|_{\boldsymbol{L}_m}\right\|_{L^{p_2}([0,T-h],\,\mathrm{d}r)}\leq\omega_{p_2}^3(\mathsf{E}^{\mathscr{F}}\delta\Xi,(t_i-t_{i-1})H2^{-n});$$

and these, after being inserted into estimate (B.7), yield

$$\omega_{p_0}^2(I^{\pi_N}-I^{\pi'_{N'}},h)\lesssim_{\mathfrak{p}}\left[\sum_{i=1}^N\left(\sum_{n=1}^{n_0^i}\left[\frac{\left(\omega_{p_1}^3(\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi,2(t_i-t_{i-1})h2^{-n})\right)^{\mathfrak{p}\wedge p_1}}{2^{-n}}\right]^{\frac{1}{\mathfrak{p}\wedge p_1}}\right)^{\mathfrak{p}\wedge p_1}\right]^{\frac{1}{\mathfrak{p}\wedge p_1}}+\sum_{i=1}^N\sum_{n=1}^{n_0^i}\frac{\omega_{p_2}^3(\mathsf{E}^{\mathscr{F}}\delta\Xi,2(t_i-t_{i-1})h2^{-n})}{2^{-n}}\tag{B.8}$$

(we have also switched from H to h because they need not be distinguished anymore). The above estimate (B.8) immediately yields the inequality

$$\begin{aligned}\left\|\frac{\omega_{p_0}^2(I^{\pi_N}-I^{\pi'_{N'}},h)}{h^{s_0}}\right\|_{L^q([0,T],\frac{\mathrm{d}h}{h})}&\lesssim_{\mathfrak{p}}T^{s_1-s_0}\left\|\left[\sum_{i=1}^N\left(\sum_{n=1}^{n_0^i}\frac{\omega_{p_1}^3(\delta\Xi-\mathsf{E}^{\mathscr{F}}\delta\Xi,2(t_i-t_{i-1})h2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge p_1}}h^{s_1}}\right)^{\mathfrak{p}\wedge p_1}\right]^{\frac{1}{\mathfrak{p}\wedge p_1}}\right\|_{L^q([0,T],\frac{\mathrm{d}h}{h})}\\&\quad+T^{s_2-s_0}\left\|\sum_{i=1}^N\sum_{n=1}^{n_0^i}\frac{\omega_{p_2}^3(\mathsf{E}^{\mathscr{F}}\delta\Xi,2(t_i-t_{i-1})h2^{-n})}{2^{-n}h^{s_2}}\right\|_{L^q([0,T],\frac{\mathrm{d}h}{h})}\end{aligned}\tag{B.9}$$

and it only remains to apply the following Lemma B.3, whose proof is similar to that of Theorem 3.1 in [17], to conclude the proof of bound (4.1).

Lemma B.3. *Let $A:\triangle^3[0,T]\to\boldsymbol{L}_m$ be such that there exists $p\in[1,\infty]$ for which $\omega_p^3(A,\tau)<\infty$ for λ-a.a. $\tau\in[0,T]$. Moreover, take any partition $\tilde{\pi}_K=\{0=s_0<s_1<\cdots<s_K=1\}$ of interval $[0,1]$, where $K\in\mathbb{N}$, satisfying $|\tilde{\pi}_K|\leq\frac{1}{2}$ and any sequence $\{l_i\}_{i=1}^K$ where $l_i\in\mathbb{N}$ for all $i=1,\dots,K$. Finally, take $\alpha\in[1,\infty]$, $q\in[1,\infty]$, and $\gamma\in(0,\infty)$. If $\gamma>\frac{1}{\mathfrak{p}\wedge\alpha}$ and $q\geq\mathfrak{p}\wedge\alpha$, then*

$$\left\|\left(\sum_{i=1}^K\left(\sum_{n=1}^{l_i}\frac{\omega_p^3(A,2(s_i-s_{i-1})h2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}}h^\gamma}\right)^{\mathfrak{p}\wedge\alpha}\right)^{\frac{1}{\mathfrak{p}\wedge\alpha}}\right\|_{L^q([0,T];\frac{\mathrm{d}h}{h})}\lesssim_{\gamma,\mathfrak{p},\alpha,q}|\tilde{\pi}_K|^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}}\|A\|_{\mathcal{B}_{p,q}^{\gamma;3}([0,T];\boldsymbol{L}_m)}.$$

Proof. Let us set $\varrho_i=2(s_i-s_{i-1})$ for $i=1,\dots,K$ to simplify notation and let us choose $h\in(0,T]$ arbitrarily.

Case $q<\infty$: Lemma A.1 gives for every $i=1,\dots,K$ the estimate

$$\sum_{n=1}^{l_i}\frac{\omega_p^3(A,\varrho_ih2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}}h^\gamma}\leq\frac{2^{\frac{1}{\mathfrak{p}\wedge\alpha}}h^{\frac{1}{\mathfrak{p}\wedge\alpha}}}{\log 2}\int_0^h\frac{\omega_p^3(A,\varrho_i\tau)}{\tau^{\frac{1}{\mathfrak{p}\wedge\alpha}}h^\gamma}\frac{\mathrm{d}\tau}{\tau}.$$

from which we immediately obtain the estimate

$$\left\| \left(\sum_{i=1}^{K} \left(\sum_{n=1}^{l_i} \frac{\omega_p^3(A, \varrho_i h 2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}} h^\gamma} \right)^{\mathfrak{p}\wedge\alpha} \right)^{\frac{1}{\mathfrak{p}\wedge\alpha}} \right\|_{L^q([0,T],\frac{\mathrm{d}h}{h})}$$
$$\lesssim_{\mathfrak{p},\alpha} \left(\sum_{i=1}^{K} \left\| \left(h^{\frac{1}{\mathfrak{p}\wedge\alpha}} \int_0^h \frac{\omega_p^3(A, \varrho_i \tau)}{\tau^{\frac{1}{\mathfrak{p}\wedge\alpha}} h^\gamma} \frac{\mathrm{d}\tau}{\tau} \right)^{\mathfrak{p}\wedge\alpha} \right\|_{L^{\frac{q}{\mathfrak{p}\wedge\alpha}}([0,T],\frac{\mathrm{d}h}{h})} \right)^{\frac{1}{\mathfrak{p}\wedge\alpha}}. \quad \text{(B.10)}$$

Let us proceed from the interior of the sum on the right-hand side of (B.10). Fix any $i \in \{1, \dots, K\}$. As γ is assumed to satisfy $\gamma > \frac{1}{\mathfrak{p}\wedge\alpha}$, we obtain

$$\int_0^h \tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \frac{\mathrm{d}\tau}{\tau} = \frac{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}}{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}$$

and we can simply normalize the (finite) measure $\mathbb{I}_{[0,h]}(\tau)\tau^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}}\frac{\mathrm{d}\tau}{\tau}$ to a probability measure. Then, it is possible to use Jensen's inequality with

$$X(\tau) = \frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \quad \text{and} \quad \varphi(x) = x^q$$

because the Monotone Convergence Theorem ensures that Jensen's inequality still holds for a, possibly not integrable, non-negative random variable and an increasing convex function which is zero at zero provided that we define its value at infinity as its limit there. We obtain

$$\left\| \left(h^{\frac{1}{\mathfrak{p}\wedge\alpha}} \int_0^h \frac{\omega_p^3(A, \varrho_i \tau)}{\tau^{\frac{1}{\mathfrak{p}\wedge\alpha}} h^\gamma} \frac{\mathrm{d}\tau}{\tau} \right)^{\mathfrak{p}\wedge\alpha} \right\|^{\frac{q}{\mathfrak{p}\wedge\alpha}}_{L^{\frac{q}{\mathfrak{p}\wedge\alpha}}([0,T],\frac{\mathrm{d}h}{h})}$$
$$= \frac{1}{(\gamma - \frac{1}{\mathfrak{p}\wedge\alpha})^q} \int_0^T \left(\frac{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}} \int_0^h \tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \frac{\mathrm{d}\tau}{\tau} \right)^q \frac{\mathrm{d}h}{h}$$
$$\leq \frac{1}{(\gamma - \frac{1}{\mathfrak{p}\wedge\alpha})^q} \int_0^T \frac{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}} \int_0^h \tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \left[\frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \right]^q \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}h}{h}.$$

In the last term, we have

$$\int_0^T \int_0^h \frac{1}{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}} \tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \left[\frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \right]^q \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}h}{h} = \int_0^T \tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \left[\int_\tau^T \frac{1}{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}} \frac{\mathrm{d}h}{h} \right] \left[\frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \right]^q \frac{\mathrm{d}\tau}{\tau}$$

by Fubini's theorem and, moreover, we can estimate

$$\tau^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}} \int_\tau^T \frac{1}{h^{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}} \frac{\mathrm{d}h}{h} \leq \frac{1}{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}.$$

Thus, the following inequality is obtained:

$$\left\| \left(h^{\frac{1}{\mathfrak{p}\wedge\alpha}} \int_0^h \frac{\omega_p^3(A, \varrho_i \tau)}{\tau^{\frac{1}{\mathfrak{p}\wedge\alpha}} h^\gamma} \frac{\mathrm{d}\tau}{\tau} \right)^{\mathfrak{p}\wedge\alpha} \right\|^{\frac{q}{\mathfrak{p}\wedge\alpha}}_{L^{\frac{q}{\mathfrak{p}\wedge\alpha}}([0,T],\frac{\mathrm{d}h}{h})} \lesssim_{\gamma,\mathfrak{p},\alpha,q} \int_0^T \left[\frac{\omega_p^3(A, \varrho_i \tau)}{\tau^\gamma} \right]^q \frac{\mathrm{d}\tau}{\tau}.$$

The substitution $y = \varrho_i \tau$ in the integral on the right-hand side of the above inequality yields the estimate

$$\left\| \left(h^{\frac{1}{\mathfrak{p}\wedge\alpha}} \int_0^h \frac{\omega_p^3(A,\varrho_i\tau)}{\tau^{\frac{1}{\mathfrak{p}\wedge\alpha}} h^\gamma} \frac{\mathrm{d}\tau}{\tau} \right)^{\mathfrak{p}\wedge\alpha} \right\|^{\frac{q}{\mathfrak{p}\wedge\alpha}}_{L^{\frac{q}{\mathfrak{p}\wedge\alpha}}([0,T],\frac{\mathrm{d}h}{h})} \lesssim_{\gamma,\mathfrak{p},\alpha,q} \varrho_i^{\gamma q} \int_0^{\varrho_i T} \left[\frac{\omega_p^3(A,y)}{y^\gamma} \right]^q \frac{\mathrm{d}y}{y} \leq \varrho_i^{\gamma q} \|A\|^q_{\mathcal{B}^{\gamma;3}_{p,q}([0,T];\boldsymbol{L}_m)}$$

which, after being inserted into (B.10), yields

$$\left\| \left(\sum_{i=1}^K \left(\sum_{n=1}^{l_i} \frac{\omega_p^3(A,\varrho_i h 2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}} h^\gamma} \right)^{\mathfrak{p}\wedge\alpha} \right)^{\frac{1}{\mathfrak{p}\wedge\alpha}} \right\|_{L^q([0,T],\frac{\mathrm{d}h}{h})} \lesssim_{\gamma,\mathfrak{p},\alpha,q} |\tilde{\pi}_K|^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}} \|A\|_{\mathcal{B}^{\gamma;3}_{p,q}([0,T];\boldsymbol{L}_m)} \sum_{i=1}^K (s_i - s_{i-1}).$$

Case $q = \infty$: Lemma A.1 leads to the inequality

$$\sum_{n=1}^{l_i} \frac{\omega_p^3(A,\varrho_i h 2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}} h^\gamma} \lesssim_{\mathfrak{p},\alpha} h^{\frac{1}{\mathfrak{p}\wedge\alpha}-\gamma} \varrho_i^{\frac{1}{\mathfrak{p}\wedge\alpha}} \|A\|_{\mathcal{B}^{\gamma;3}_{p,\infty}([0,T],\boldsymbol{L}_m)} \int_0^{\varrho_i h} y^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}} \frac{\mathrm{d}y}{y}.$$

Again, as we suppose $\gamma > \frac{1}{\mathfrak{p}\wedge\alpha}$, the integral on the right-hand side can be rewritten as

$$\int_0^{\varrho_i h} y^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}} \frac{\mathrm{d}y}{y} = \frac{(\varrho_i h)^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}}}{\gamma - \frac{1}{\mathfrak{p}\wedge\alpha}}.$$

Combining the last two expressions yields

$$\left\| \left(\sum_{i=1}^K \left(\sum_{n=1}^{l_i} \frac{\omega_p^3(A,\varrho_i h 2^{-n})}{2^{-\frac{n}{\mathfrak{p}\wedge\alpha}} h^\gamma} \right)^{\mathfrak{p}\wedge\alpha} \right)^{\frac{1}{\mathfrak{p}\wedge\alpha}} \right\|_{L^\infty([0,T],\frac{\mathrm{d}h}{h})} \lesssim_{\gamma,\mathfrak{p},\alpha} |\tilde{\pi}_K|^{\gamma-\frac{1}{\mathfrak{p}\wedge\alpha}} \|A\|_{\mathcal{B}^{\gamma;3}_{p,\infty}([0,T],\boldsymbol{L}_m)}.$$

□

The proof of Lemma 4.1 is now concluded by the application of Lemma B.3 to inequality (B.9).