

A NON-COMPENSATED CLARK–OCONE FORMULA FOR FUNCTIONALS OF COUNTING PROCESSES

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Abstract. In this paper, we develop a representation formula of Clark–Ocone type for any integrable Poisson functionals, which extends the Poisson imbedding for point processes. This representation formula differs from the classical Clark–Ocone formula on three accounts. First the representation holds with respect to the Poisson measure instead of the compensated one; second the representation holds true in L^1 and not in L^2 ; and finally contrary to the classical Clark–Ocone formula the integrand is defined as a pathwise operator and not as a L^2 -limiting object. We make use of Malliavin’s calculus and of a decomposition with uncompensated iterated integrals derived in [Hillairet and Réveillac, *Electron. J. Probab.* **29** (2024) 1–33] to establish this non-compensated Clark–Ocone representation formula and to characterize the integrand, which turns out to be a predictable integrable process.

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1. INTRODUCTION

Martingale representation formulas for Poisson functionals have been widely investigated in the literature (such as Løkka [1], Nualart and Schoutens [2]) in relation with stochastic analysis tools such as the chaotic expansion or the Malliavin calculus. At the crossroad of martingale representation and the Malliavin calculus lies the so called Clark–Ocone¹ formula which allows one to provide a description of the integrand in terms of the Malliavin derivative. More precisely, on the Poisson space, the Clark–Ocone formula gives the following representation of a square-integrable random variable F :

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+} (D_t F)^p \tilde{N}(dt)$$

where \tilde{N} is the compensated Poisson process and $(D_t F)^p$ is the predictable projection of the Malliavin derivative of F . Note that the Clark–Ocone formula is restricted to representation of random variables whereas martingale representation provides a dynamic representation of the integrand for a given martingale. Yet in a Poisson framework both representations are intricate. Various kinds of generalizations have been obtained. Using the

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¹Also referred as Clark–Ocone–Haussmann formula.

Malliavin integration by parts formula given in Picard [3], Zhang [4] establishes the following Clark–Ocone formula: for any bounded Poisson functional F

$$F = \mathbb{E}[F] + \int_{\mathbb{R}_+ \times \mathbb{X}} (\mathbf{D}_{(t,x)} F)^p \tilde{\mathbf{N}}(dt, dx)$$

where $\tilde{\mathbf{N}}$ is the compensated Poisson random measure, and $(\mathbf{D}_{(t,x)} F)^p$ is the predictable projection of the Malliavin derivative of F . Last and Penrose [5] also give a Clark–Ocone type martingale representation formula when the underlying filtration is generated by a Poisson process on a measurable space. Flint and Torrisi [6] provide a Clark–Ocone formula for point processes on a finite interval possessing a conditional intensity. Di Nunno and Vives [7] develop a Malliavin–Skorohod type calculus for additive processes and obtain a generalization of the Clark–Ocone formula for random variables in L^1 whose integrand operator is supposed to be in L^1 as well. At the same time, the so called Poisson imbedding (see Jacod [8], Chapter 4 or Brémaud and Massoulié [9]) provides also a representation of any point process with respect to an uncompensated Poisson measure. More precisely, if H is a point process on \mathbb{R}_+ with stochastic intensity λ , then these pair of processes (H, λ) can be represented on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random Poisson measure \mathbf{N} on $\mathbb{R}_+ \times \mathbb{R}_+$ and the following representation holds true:

$$H_T = \int_{(0,T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta); \quad \forall T > 0. \quad (1.1)$$

Yet Hillairet and Réveillac [10] investigate the so called pseudo-chaotic expansion which involves also non-compensated iterated integrals and which are related to the so called factorial Poisson measures associated to \mathbf{N} (whose definition will be recalled in Section 2.3). This recent contribution leads the way to the study of a Clark–Ocone representation formula with respect to the non-compensated Poisson measure \mathbf{N} , which in a sense will generalize the Poisson imbedding relation (1.1). The aim of this paper is to investigate which Poisson-functionals admit such non-compensated Clark–Ocone decomposition and to determine the integrand.

More precisely, we consider a Poisson random measure \mathbf{N} defined on $(\mathbf{X}, \mathfrak{X}) := ([0, T] \times \mathbb{X}; \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{X})$ equipped with a non-atomic σ -finite measure $\rho := dt \otimes \pi$, and where $(\mathbb{X}, \mathcal{X})$ is a complete separable metric space. The non-compensated Clark–Ocone formula is stated in Theorem 3.10 and gives for any F in $L^1(\Omega)$ the following representation (under the assumption that $\rho(\mathbf{X}) < +\infty$):

$$F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s. \quad (1.2)$$

with $F(\omega_\emptyset) := F \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}$. The operator $\mathcal{H}F$ is defined as follows:

$$\mathcal{H} : \begin{array}{l} L^0(\Omega) \rightarrow L^0(\Omega \times \mathbf{X}) \\ F \mapsto \mathcal{H}_{(t,x)} F := \mathbf{D}_{(t,x)} F \circ \tau_t \end{array}$$

where $\mathbf{D}_{(t,x)}$ is the Malliavin derivative operator and

$$\tau_t : \begin{array}{l} \Omega \rightarrow \Omega \\ \omega = \sum_{j=1}^n \delta_{(t_j, x_j)} \mapsto \omega_t := \sum_{j=1}^n \delta_{(t_j, x_j)} \mathbf{1}_{\{t_j < t\}}. \end{array}$$

Let us emphasize that this non-compensated Clark–Ocone formula holds for any F in $L^1(\Omega)$ and its integrand $\mathcal{H}F$ is well defined pathwise and is proved to be an integrable predictable process (see Thms. 3.10 and 3.12). This is a major difference with the standard Clark–Ocone formula that requires $F \in L^2(\Omega)$ and for which the integrand $(\mathbf{D}F)^p$ is only defined as a limit in $L^2(\Omega \times \mathbf{X})$ (see Rem. 3.4).

The paper is organized as follows. Notations and the description of the Poisson space and elements of Malliavin's calculus are presented in Section 2, as well as the operators and iterated integrals of the (pseudo)-chaotic expansions. The main contribution is stated in Section 3, which recalls the standard and pseudo-chaotic expansions and then derives the Clark–Ocone formula with respect to the Poisson measure. Section 4 provides some applications of this non-compensated Clark–Ocone formula. Finally Section 5 gathers some technical lemmata.

2. ELEMENTS OF MALLIAVIN'S CALCULUS ON THE POISSON SPACE

We introduce in this section some elements and notions of stochastic analysis on a general Poisson space. All the elements presented in this section are taken from [11, 12].

2.1. The Poisson space

$\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ denotes the set of positive integers and for any finite set S , $|S|$ denotes its cardinal. We fix $(\mathbb{X}, \mathcal{X})$ a complete separable metric space equipped with a non-atomic σ -finite measure π and we set $(\mathbf{X}, \mathfrak{X}) := ([0, T] \times \mathbb{X}; \mathcal{B}([0, T]) \otimes \mathcal{X})$ equipped with the non-atomic σ -finite measure $\rho := dt \otimes \pi$ with dt the Lebesgue measure on \mathbb{R}_+ and T is a fixed positive real number. Throughout this paper we will make use of the following notation:

Notation 2.1. We denote with bold letters \mathbf{x} elements in \mathbf{X} and for $\mathbf{x} \in \mathbf{X}$ we set $\mathbf{x} := (t, x)$ with $t \in \mathbb{R}_+$ and $x \in \mathbb{X}$.

We define Ω the space of configurations on \mathbf{X} as

$$\Omega := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{X}, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\}. \quad (2.1)$$

Notation 2.2. We also denote by $\omega_\emptyset \in \Omega$ the unique element ω of Ω such that $\omega(A \times B) = 0$ for any $A \times B \in \mathfrak{X}$.

Let \mathcal{F} be the σ -field associated to the vague topology on Ω . Let \mathbb{P} the Poisson measure on Ω under which the canonical evaluation \mathbf{N} defines a Poisson random measure with intensity measure ρ . To be more precise given any element $A \times B$ in \mathfrak{X} with $\rho(A \times B) > 0$, the random variable

$$(\mathbf{N}(\omega))(A \times B) := \omega(A \times B)$$

is a Poisson random variable with intensity $\rho(A \times B)$ and $\tilde{\mathbf{N}}$ defined as

$$(\tilde{\mathbf{N}}(\omega))(A \times B) := \omega(A \times B) - \rho(A \times B)$$

is the compensated Poisson measure. We define the natural history associated to \mathbf{N}

$$\mathcal{F}_t^{\mathbf{N}} := \sigma\{\mathbf{N}(A \times B); (A, B) \in \mathfrak{X}; A \subset [0, t]\}$$

and $\mathcal{F}^{\mathbf{N}} := \sigma\{\mathbf{N}(A \times B); (A, B) \in \mathfrak{X}\}$. For any $p \geq 0$, $L^p(\Omega; \mathbb{P}) := L^p(\Omega; \mathcal{F}^{\mathbf{N}}; \mathbb{P})$. For any $F \in L^1(\Omega; \mathbb{P})$ and any $t \geq 0$ we denote $\mathbb{E}_t[F] := \mathbb{E}[F | \mathcal{F}_t^{\mathbf{N}}]$. We also set for any $t > 0$,

$$\mathcal{F}_{t-}^{\mathbf{N}} := \bigvee_{0 < s < t} \mathcal{F}_s^{\mathbf{N}}.$$

We set $\mathcal{P}^{\mathbf{N}}$ the predictable σ -field associated to $\mathcal{F}^{\mathbf{N}}$ that is the σ -field on $[0, T] \times \Omega$ generated by left-continuous $\mathcal{F}^{\mathbf{N}}$ -adapted stochastic processes and we set $\mathcal{P} := \mathcal{P}^{\mathbf{N}} \otimes \mathcal{B}(\mathbb{X})$ the set of real-valued predictable processes $X : \Omega \times [0, T] \times \mathbb{X}$. Let

$$\mathbb{L}_{\mathcal{P}}^0 := \{X : \Omega \times [0, T] \times \mathbb{X} \text{ which are } \mathcal{P}\text{-measurable}\},$$

and for $r > 0$

$$\mathbb{L}_{\mathcal{P}}^r := \{X \in \mathbb{L}_{\mathcal{P}}^0, \|X\|_{\mathbb{L}_{\mathcal{P}}^r} < +\infty\}, \text{ with } \|X\|_{\mathbb{L}_{\mathcal{P}}^r} := \mathbb{E} \left[\int_{[0, T] \times \mathbb{X}} |X_{(t, x)}|^r dt \pi(dx) \right]^{1/r}.$$

According to Jacod [13] (see also Protter [14], Cor. 3), for any $X \in \mathbb{L}_{\mathcal{P}}^2$, the process $t \mapsto \int_{[0, t] \times \mathbb{X}} X_{(r, x)} \tilde{\mathbf{N}}(dr, dx)$ is a $\mathcal{F}^{\mathbf{N}}$ -martingale which satisfies the following L^2 -isometry

$$\mathbb{E} \left[\left| \int_{[0, t] \times \mathbb{X}} X_{(r, x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] = \mathbb{E} \left[\int_{[0, t] \times \mathbb{X}} |X_{(r, x)}|^2 dr \pi(dx) \right], \quad \forall t \in [0, T]. \quad (2.2)$$

We introduce some general notations in preparation of the definition of stochastic analysis tools on the Poisson space together with some pathwise measurable transformations. We set

$$L^0(\Omega) := \{F : \Omega \rightarrow \mathbb{R}, \mathcal{F}^{\mathbf{N}}\text{-measurable}\},$$

$$L^2(\Omega) := \{F \in L^0(\Omega), \mathbb{E}[|F|^2] < +\infty\}.$$

Similarly

$$L^0(\mathbf{X}^j; \boldsymbol{\rho}^{\otimes j}) := \{f : \mathbf{X}^j \rightarrow \mathbb{R}, \mathfrak{X}^{\otimes j}\text{-measurable}\}$$

and for $r \in \{1, 2\}$, for $j \in \mathbb{N}^*$

$$L^r(\mathbf{X}^j; \boldsymbol{\rho}^{\otimes j}) := \left\{ f \in L^0(\mathbf{X}^j; \boldsymbol{\rho}^{\otimes j}), \|f\|_{L^r(\mathbf{X}^j)}^r := \int_{\mathbf{X}^j} |f(\mathbf{x}_1, \dots, \mathbf{x}_j)|^r \boldsymbol{\rho}^{\otimes j}(d\mathbf{x}_1 \cdots d\mathbf{x}_j) < +\infty \right\}. \quad (2.3)$$

Besides,

$$L_s^r(\mathbf{X}^j; \boldsymbol{\rho}^{\otimes j}) := \{f \in L^r(\mathbf{X}^j) \text{ and } f \text{ is symmetric}\} \quad (2.4)$$

is the space of square-integrable symmetric mappings where we recall that $f : \mathbf{X}^j \rightarrow \mathbb{R}$ is said to be symmetric if for any element σ in \mathcal{S}_j (the set of all permutations of $\{1, \dots, j\}$),

$$f(\mathbf{x}_1, \dots, \mathbf{x}_j) = f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_j) \in \mathbf{X}^j.$$

Given $f : \mathbf{X}^j \rightarrow \mathbb{R}$, we write \tilde{f} its symmetrization defined as:

$$\tilde{f}(\mathbf{x}_1, \dots, \mathbf{x}_j) := \frac{1}{j!} \sum_{\sigma \in \mathcal{S}_j} f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(j)}).$$

The main ingredient we will make use of are the add-points operators on the Poisson space Ω .

Definition 2.3. [Add-points operators] Given $n \in \mathbb{N}^*$, and $J := \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbf{X}$ a subset of \mathbf{X} with $|J| = n$, we set the measurable mapping:

$$\begin{aligned} \varepsilon_J^{+,n} : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}}. \end{aligned}$$

Note that by definition

$$\omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}} = \omega + \sum_{j=1}^n \delta_{\mathbf{x}_j} \mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$$

that is we add the atoms \mathbf{x}_j to the path ω unless they already were part of it (which is the meaning of the term $\mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$). Note that since ρ is assumed to be atomless, given a set J as above, $\mathbb{P}[\mathbf{N}(J) = 0] = 1$ hence in what follows we will simply write $\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j}$ for $\varepsilon_{\mathbf{x}}^{+,n}(\omega)$.

2.2. Malliavin calculus on the Poisson space

This section gathers the main elements of the Malliavin calculus on the Poisson space, in particular the classical definition of the iterated integrals and the Malliavin derivative with respect to the compensated Poisson measure. We recall that according to Notation 2.1, any element $\mathbf{x} \in \mathbf{X}$ is written as $\mathbf{x} := (t, x)$, $t \in [0, T]$, $x \in \mathbb{X}$.

Notation 2.4. For $m \in \mathbb{N}^*$ we set

$$\mathbf{X}^{m,\neq} := \{(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbf{X}^m, \forall (i, j) \in \{1, \dots, m\}^2, [\mathbf{x}_i = \mathbf{x}_j] \Rightarrow [i = j]\}.$$

Definition 2.5. We consider the random measure $\tilde{\mathbf{N}}$ on $(\mathbf{X}, \mathfrak{X})$ defined as:

$$\tilde{\mathbf{N}}(A) := \mathbf{N}(A) - \rho(A), \quad A \in \mathfrak{X}.$$

Let $m \in \mathbb{N}^*$ and $f_m \in L^2(\mathbf{X}^m; \rho^{\otimes m})$. We define the m^{th} iterated integral of f_m with respect to $\tilde{\mathbf{N}}$ as

$$\mathbf{I}_m(f_m) := \int_{\mathbf{X}^m} f_m(\mathbf{x}_1, \dots, \mathbf{x}_m) \mathbf{1}_{\{\mathbf{X}^{m,\neq}\}}(\mathbf{x}_1, \dots, \mathbf{x}_m) \tilde{\mathbf{N}}(d\mathbf{x}_1) \cdots \tilde{\mathbf{N}}(d\mathbf{x}_m).$$

For $m = 0$ and $f_0 \in \mathbb{R}$ we set $\mathbf{I}_0(f_0) := f_0$.

According to [15] the operators \mathbf{I}_m are well defined as operators in L^2 , are orthogonal and satisfy the isometry: $\mathbb{E}[|\mathbf{I}_m(f_m)|^2] = \|f_m\|_{L^2(\mathbf{X}^{\otimes m}, \rho^{\otimes m})}^2$. In addition $\mathbf{I}_m(f_m) = \mathbf{I}_m(\tilde{f}_m)$ where \tilde{f}_m denotes the symmetrization of f_m . We now introduce the Malliavin derivative \mathbf{D} as follows.

Definition 2.6. For any element $F := \mathbf{I}_m(f_m)$, $m \in \mathbb{N}$, $f_m \in L_s^2(\mathbf{X}^m; \rho^{\otimes m})$ we define $\mathbf{D}F$ as the linear operator valued in $L^2(\Omega \times \mathbf{X}; \mathbb{P} \otimes \rho)$:

$$\mathbf{D}_{\mathbf{x}}F := m \mathbf{I}_{m-1}(f_m(\cdot, \mathbf{x})).$$

This definition extends naturally by linearity to $\mathcal{S} := \text{Span}\{\mathbf{I}_m(f_m), m \in \mathbb{N}, f_m \in L_s^2(\mathbf{X}^m; \rho^{\otimes m})\}$ and to a dense subset of $L^2(\Omega; \mathbb{P})$ denoted $\text{Dom}(\mathbf{D})$ (see [15, 16]).

The previous definition can be extended: for any $n \geq 1$ and for any F in \mathcal{S} , we set

$$\mathbf{D}^n F := \mathbf{D} \mathbf{D}^{n-1} F \tag{2.5}$$

which provides a linear operator on $L^2(\Omega \times \mathbf{X}^n; \mathbb{P} \otimes \boldsymbol{\rho}^n)$ (which is symmetric on \mathbf{X}^n by definition). As for the case $n = 1$, this definition can be extended to a larger set of random variables denoted by $\text{Dom}(\mathbf{D}^n)$.

2.3. Pathwise calculus

Coming back to the definition of the iterated integrals given in Definition 2.5, it appears that if an element $f_m \in L^2(\mathbf{X}^m; \boldsymbol{\rho}^{\otimes m})$ satisfies in addition that $f_m \in L^1(\mathbf{X}^m; \boldsymbol{\rho}^{\otimes m})$ then it is possible to give a pathwise interpretation to $\mathbf{I}_m(f_m)$. This remark finds a counterpart regarding the Malliavin derivative.

Proposition 2.7. *For any F in $\text{Dom}(\mathbf{D})$,*

$$(\mathbf{D}_{\mathbf{x}} F)(\omega) = F \circ \varepsilon_{\mathbf{x}}^+(\omega) - F(\omega), \mathbb{P} \otimes \boldsymbol{\rho} - a.s..$$

More generally for any $n \geq 1$ and any F in $\text{Dom}(\mathbf{D}^n)$,

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F = \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \circ \varepsilon_J^{+, |J|}, \mathbb{P} \otimes \boldsymbol{\rho}^{\otimes n} - a.s..$$

Recall that $J = \emptyset$ is an element of the sum.

Note that the right-hand side of the previous relations are defined for any F in $L^0(\Omega; \mathbb{P})$. We first introduce the so called factorial Poisson measures.

Proposition 2.8. *(Factorial Poisson measures; see e.g. [11], Prop. 1.) There exists a unique sequence of counting random measures $(\mathbf{N}^{(m)})_{m \in \mathbb{N}^*}$ where for any m , $\mathbf{N}^{(m)}$ is a counting random measure on $(\mathbf{X}^m, \mathfrak{X}^{\otimes m})$ with*

$$\begin{aligned} \mathbf{N}^{(1)} &:= \mathbf{N} \quad \text{and for } A \in \mathfrak{X}^{m+1}, \\ \mathbf{N}^{(m+1)}(A) &:= \int_{\mathfrak{X}^m} \left[\int_{\mathfrak{X}} \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_{m+1}) \in A\}} \mathbf{N}(d\mathbf{x}_{m+1}) - \sum_{j=1}^m \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}_j) \in A\}} \right] \mathbf{N}^{(m)}(d\mathbf{x}_1, \dots, d\mathbf{x}_m). \end{aligned}$$

With this definition at hand we introduce the notion of iterated integrals. In particular for $B \in \mathfrak{X}$,

$$\mathbf{N}^{(m)}(B^{\otimes m}) = \mathbf{N}(B)(\mathbf{N}(B) - 1) \times \dots \times (\mathbf{N}(B) - m + 1).$$

Note that by definition $\mathbf{N}^{(m)}(B) \mathbf{1}_{\{\mathbf{N}(B) < m\}} = 0$. To each factorial measure $\mathbf{N}^{(m)}$ is associated a so called factorial moment measure on $(\mathbf{X}^m, \mathfrak{X}^{\otimes m})$ given by the relation below (see e.g. [11], Relation (12)):

$$\mathbb{E}[\mathbf{N}^{(m)}(A)] = \boldsymbol{\rho}^{\otimes m}(A), \quad \forall A \in \mathfrak{X}^{\otimes m}. \tag{2.6}$$

We now introduce a counterpart of iterated integrals $\mathbf{I}_m(f_m)$ with respect to the non-compensated Poisson measure \mathbf{N} as follows.

Definition 2.9. Let $m \in \mathbb{N}^*$ and $f_m \in L^1(\mathbf{X}^m; \boldsymbol{\rho}^{\otimes m})$. $\mathcal{I}_m(f_m)$ the integral of f_m against the m^{th} factorial Poisson measure \mathbf{N} is defined as

$$\mathcal{I}_m(f_m) := \int_{\mathbf{X}^m} f_m(\mathbf{x}_1, \dots, \mathbf{x}_m) \mathbf{N}^{(m)}(d\mathbf{x}_1, \dots, d\mathbf{x}_m),$$

where each of the integrals above is well defined pathwise for \mathbb{P} -a.e. $\omega \in \Omega$.

As for iterated integrals $\mathbf{I}_m(f_m)$, $\mathcal{I}_m(f_m) = \mathcal{I}_m(\tilde{f}_m)$ where \tilde{f}_m denotes the symmetrization of f_m .

Remark 2.10. It is worth noting that in the Poisson framework the iterated integrals \mathbf{I}_m can be constructed in L^1 as integrals with respect to the factorial measures in a Lebesgue–Stieltjes fashion; or as the elements of the chaotic expansion which gives a construction of $L^2(\Omega; \mathbb{P})$ as an orthonormal sum of subsets named chaos.

According to [15], Proposition 4.1, iterated integrals $\mathbf{I}_m(f_m)$ rewrites in terms of the factorial measures as

$$\mathbf{I}_m(f_m) = \sum_{J \subset \{1, \dots, m\}} (-1)^{m-|J|} \int_{\mathbf{X}^{m-|J|}} \int_{\mathbf{X}^{|J|}} f_m(\mathbf{x}_1, \dots, \mathbf{x}_m) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \boldsymbol{\rho}^{\otimes(m-|J|)}(d\mathbf{x}_{J^c}),$$

where $J^c := \{1, \dots, m\} \setminus J$, $d\mathbf{x}_J := (d\mathbf{x}_j)_{j \in J}$ and where each of the integrals above is well defined pathwise for \mathbb{P} -a.e. $\omega \in \Omega$. Recall that $J = \emptyset$ is an element of the sum.

Note finally, that integrals of the form $\mathcal{I}_m(f_m)$ are also referred in the literature as U -Statistics with respect to the Poisson measure \mathbf{N} . To conclude this section, we recall a particular case of Mecke’s formula (see *e.g.* [11], Relation (11)).

Lemma 2.11 (A particular case of Mecke’s formula). *Let $F \in L^0(\Omega; \mathbb{P})$, $k \in \mathbb{N}$ and $h \in L^0(\mathbf{X}^k; \boldsymbol{\rho}^{\otimes k})$ with*

$$\int_{\mathbf{X}^k} |h(\mathbf{x}_1, \dots, \mathbf{x}_k)| \mathbb{E} [|F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_k}^{+,k}|] \boldsymbol{\rho}^{\otimes k}(d\mathbf{x}_1, \dots, d\mathbf{x}_k) < +\infty$$

then

$$\mathbb{E} \left[F \int_{\mathbf{X}^k} h d\mathbf{N}^{(k)} \right] = \int_{\mathbf{X}^k} h(\mathbf{x}_1, \dots, \mathbf{x}_k) \mathbb{E} [F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_k}^{+,k}] \boldsymbol{\rho}^{\otimes k}(d\mathbf{x}_1, \dots, d\mathbf{x}_k).$$

3. STANDARD/PSEUDO-CHAOTIC EXPANSION AND CLARK–OCONE FORMULA

In the literature, results on representations of Poisson functionals are mainly concentrated on the chaos expansion involving iterated compensated integrals operators \mathbf{I}_n . Similarly the classical Clark–Ocone formula is stated with respect to the compensated Poisson measure $\tilde{\mathbf{N}}$. Yet [10] investigates the so called pseudo-chaotic expansion which involves iterated non-compensated integrals operators \mathcal{I}_n . This recent contribution leads the way to the study of a Clark–Ocone representation formula with respect to the non-compensated Poisson measure \mathbf{N} .

3.1. Chaotic expansion and the Clark–Ocone formula

Let us first recall the classical chaotic expansion and the Clark–Ocone formula with respect to the compensated Poisson measure $\tilde{\mathbf{N}}$.

Theorem 3.1 (See *e.g.* Thm. 2 in [11]). *Let F in $L^2(\Omega)$. There exists a unique sequence $(f_n^F)_{n \geq 1}$ with $f_n^F \in L_s^2(\mathbf{X}^n; \boldsymbol{\rho}^{\otimes n})$ such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbf{I}_n(f_n^F),$$

where the convergence of the series holds in $L^2(\Omega)$. In addition coefficients $(f_n^F)_n$ are given as

$$f_n^F = \mathbb{E}[\mathbf{D}^n F], \quad n \geq 1.$$

Remark 3.2. As we mentioned, the domain of each operator \mathbf{D}^n is a subset of $L^2(\Omega; \mathbb{P})$. However, it is possible to prove that the operator $F \mapsto (f_n^F)_{n \geq 1}$ can be extended to $L^2(\Omega)$ in a similar fashion than the proof of Theorem 3.3 in Appendix A. We refer to [11, 12] for a complete treatment of this question. A similar question arises while stating the Clark–Ocone formula, see Remark 3.4 below.

The other classical relation on the Poisson space is the Clark–Ocone formula, for which one can find in the literature different variants with different conditions, such as *e.g.* Privault [16], Zhang [4], Di Nunno and Vives [7]. In our setting it takes the following form.

Theorem 3.3 (Clark–Ocone formula). *Let F in $L^2(\Omega; \mathbb{P})$. Then*

$$F = \mathbb{E}[F] + \int_{[0, T] \times \mathbb{X}} (\mathbf{D}_{(t,x)} F)^p \tilde{\mathbf{N}}(dt, dx)$$

where the process $(\mathbf{D}F)^p$ belongs to $\mathbb{L}_{\mathcal{P}}^2$ and is in this context the predictable projection of $\mathbf{D}F$ which is properly defined in the Appendix.

Remark 3.4. Contrary to what it suggests, the meaning of the integrand in the stochastic integral has to be made precise. Indeed the definition of the operator $(\mathbf{D}F)^p$ does not seem guaranteed only with F in $L^2(\Omega; \mathbb{P})$. For instance F in $L^2(\Omega \times \mathbf{X}; \otimes \rho)$ does not entail $\mathbf{D}F \in L^2(\Omega \times \mathbf{X}; \mathbb{P} \otimes \rho)$. Several sufficient conditions can be found in the literature (see Last [11]). In general as we will make it precise in the proof of Theorem 3.3 (see Appendix A), it is possible to define $(\mathbf{D}F)^p$ in $L^2(\Omega \times \mathbf{X})$ in a limiting procedure and based on the continuous feature of the mapping $F \mapsto (\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F])_{(t,x)}$ together with the fact that for fixed (t, x) , $\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] = (\mathbf{D}_{(t,x)} F)^p \mathbb{P} - a.s.$ (see [4]). For sake of completeness, we reproduce the proof of this result in the Appendix (Appendix A) following [16], Section 3.2. Note that even in the case of the Clark–Ocone formula obtained in [7] where the L^2 assumption on F is relaxed to L^1 , the integrand operator is assumed to belong to L^1 as well. In Theorem 3.10 below (non-compensated Clark–Ocone formula) the integrability of the integrand is a consequence of F in L^1 .

3.2. Pseudo-chaotic expansion and non-compensated Clark–Ocone formula

We recall here the pseudo-chaotic expansion obtained with respect to the non-compensated iterated integrals operators \mathcal{I}_n . To this end we introduce a purely deterministic operator that will be at the core of the pseudo-chaotic expansion.

Definition 3.5. For $F \in L^0(\Omega; \mathbb{P})$, we define the deterministic operators:

$$\mathcal{T}_0 F := F(\omega_\emptyset),$$

and for $n \in \mathbb{N}^*$, $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$,

$$\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n) := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \left(\sum_{\mathbf{y} \in J} \delta_{\mathbf{y}} \right).$$

We point out that $\mathcal{T}_n F$ is not an add-point operator but an evaluation operator. This is not the case of the Malliavin derivative $\mathbf{D}_{\mathbf{x}} F$ which is random as:

$$(\mathbf{D}_{\mathbf{x}} F)(\omega) = F(\omega + \delta_{\mathbf{x}}) - F(\omega), \quad \omega \in \Omega.$$

Besides, given the event $\{\mathbf{N}(\mathbf{X}) = 0\}$, $\mathcal{T}_n F(\mathbf{x}_1, \dots, \mathbf{x}_n)$ coincides with $\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F$ and $\mathcal{T}_0 F$ coincides with F .

Theorem 3.6 (See Thm. 3.6 in [10]). *Assume $\rho(\mathbf{X}) < +\infty$ and F in $L^2(\Omega; \mathbb{P})$. Then*

$$F = F(\omega_\emptyset) + \sum_{k=1}^{+\infty} \frac{1}{k!} \mathcal{I}_k(\mathcal{T}_k F); \quad \mathbb{P} - a.s. \quad (3.1)$$

As a consequence, the notion of pseudo-chaotic expansion makes reference to the expansion (3.1) and once again the elements $\mathcal{I}_k(\mathcal{T}_k F)$ are defined through the integrals against the factorial Poisson measure \mathbf{N} (see Def. 2.9). Inspired by this pseudo-chaotic expansion, we introduce below the operator that will allow us to write a Clark–Ocone formula with respect to the non-compensated Poisson measure (that we called non-compensated Clark–Ocone formula).

Definition 3.7. For fixed $t > 0$, we consider the measurable transformation τ_t

$$\tau_t : \begin{array}{l} \Omega \rightarrow \Omega \\ \omega \mapsto \omega_t \end{array}$$

where for ω of the form $\omega := \sum_{j=1}^n \delta_{(t_j, x_j)}$, $(t_j, x_j) \in \mathbf{X}$, $j = 1, \dots, n$, $n \in \mathbb{N} \cup \{+\infty\}$, we set

$$\omega_t := \sum_{j=1}^n \delta_{(t_j, x_j)} \mathbf{1}_{\{t_j < t\}}.$$

We also set τ_0 as

$$\tau_0(\omega) := \omega_\emptyset; \quad \omega \in \Omega$$

where we recall Notation 2.2.

Roughly speaking, τ_t plays the role of a $\mathcal{F}_t^{\mathbf{N}}$ -conditional expectation.

Definition 3.8. Fix (t, x) in \mathbf{X} . For $F \in L^0(\Omega)$, we define the operator \mathcal{H} as:

$$\mathcal{H} : \begin{array}{l} L^0(\Omega) \rightarrow L^0(\Omega \times \mathbf{X}) \\ F \mapsto \mathcal{H}_{(t,x)} F := \mathbf{D}_{(t,x)} F \circ \tau_t \end{array}$$

that is for $\omega \in \Omega$

$$\mathcal{H}_{(t,x)} F(\omega) := \mathbf{D}_{(t,x)} F(\omega_t) = F(\omega_t) \circ \varepsilon_{(t,x)}^+ - F(\omega_t).$$

As highlighted in the following lemma, it turns out that the operator \mathcal{H} can be written as the combination of the Malliavin derivative with a very specific Radon–Nikodym density $L^{t,T}$. This density is at the core of the pseudo-chaotic expansion (see [10]) and under which the Poisson measure \mathbf{N} (with intensity ρ on \mathbf{X}) becomes a Poisson measure with intensity 0 on $[t, T] \times \mathbb{X}$.

Lemma 3.9. *Assume $\rho(\mathbf{X}) < +\infty$ and $F \in L^0(\Omega)$. Fix $(t, x) \in (0, T) \times \mathbb{X}$. Then the operator $\mathcal{H}_{(t,x)}$ rewrites as*

$$\mathcal{H}_{(t,x)} F = \mathbb{E}_{t-}[L^{t,T} \mathbf{D}_{(t,x)} F], \quad \mathbb{P} - a.s.$$

$$\text{with } L^{t,T} := \exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}}.$$

Proof. Let $\omega \in \Omega$.

$$\begin{aligned} (L^{t,T} \mathbf{D}_{(t,x)} F)(\omega) &= \exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\omega_{([t,T] \times \mathbb{X})} = 0\}} \mathbf{D}_{(t,x)} F(\omega) \\ &= (\exp((T-t)\pi(\mathbb{X})) \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X}) = 0\}} \mathbf{D}_{(t,x)} F \circ \tau_t)(\omega). \end{aligned}$$

Hence since $\mathbf{D}_{(t,x)} F \circ \tau_t$ is $\mathcal{F}_{t-}^{\mathbf{N}}$ -measurable

$$\begin{aligned} \mathbb{E}_{t-}[L^{t,T} \mathbf{D}_{(t,x)} F] &= \mathbf{D}_{(t,x)} F \circ \tau_t \exp((T-t)\pi(\mathbb{X})) \mathbb{E}_{t-}[\mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X}) = 0\}}] \\ &= (F(\omega_t) \circ \varepsilon_{(t,x)}^+ - F(\omega_t)) \exp((T-t)\pi(\mathbb{X})) \mathbb{E}_{t-}[\mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X}) = 0\}}] \\ &= \mathcal{H}_{(t,x)} F. \end{aligned}$$

□

We have now all the elements to state the non-compensated Clark–Ocone formula for any F in $L^1(\Omega; \mathbb{P})$, first under the condition that $\rho(\mathbf{X}) < +\infty$.

Theorem 3.10. *Assume $\rho(\mathbf{X}) < +\infty$. Let F in $L^1(\Omega)$. Then $\mathcal{H}F \in \mathbb{L}_{\mathcal{P}}^1$ and*

$$F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s.. \quad (3.2)$$

In addition this decomposition is unique in the sense that if there exists $(c, Z) \in \mathbb{R} \times \mathbb{L}_{\mathcal{P}}^1$ such that

$$F = c + \int_{[0,T] \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s.. \quad (3.3)$$

then $c = F(\omega_\emptyset)$ and for a.-e. $(t, x) \in [0, T] \times \mathbb{X}$, $Z_{(t,x)} = \mathcal{H}_{(t,x)} F$, \mathbb{P} -a.s..

Proof. To highlight the main keys of the proof, some technical lemmata (namely Lems. 5.1 and 5.5) are postponed to Section 5.

Uniqueness: Assume Relation (3.3) holds. Then note that $F(\omega_\emptyset) = F \mathbf{1}_{\{\mathbf{N}(\mathbf{X}) = 0\}} = c$.

Fix $(t, x) \in \mathbf{X}$, $t > 0$. We have for \mathbb{P} -a.e. ω in Ω , since $Z \in \mathbb{L}_{\mathcal{P}}^1$

$$\begin{aligned} \mathcal{H}_{(t,x)} F(\omega) &= \mathbf{D}_{(t,x)} F(\omega_t) \\ &= \int_{[0,T] \times \mathbb{X}} (Z_{(s,y)}(\omega_t) + \delta_{(t,x)}) - Z_{(s,y)}(\omega_t) \omega_t(ds, dy) + Z_{(t,x)}(\omega_t) + \delta_{(t,x)} \\ &= Z_{(t,x)}(\omega_t) + \delta_{(t,x)} = Z_{(t,x)}(\omega_t) = Z_{(t,x)}(\omega). \end{aligned}$$

Existence: We proceed in several parts.

Step 1. Since $\rho(\mathbf{X}) < +\infty$, $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] > 0$. Let $F \in L^2(\Omega)$. Thus $\lim_{p \rightarrow +\infty} \mathbb{E}[|F - F^k|^2] = 0$ with $F^k := \mathbb{E}[F] + \sum_{n=1}^k \mathcal{I}_n(T_n F)$. By Lemma 5.5 and the uniqueness of the pseudo-chaotic expansion, for any k ,

$$F^k = F^k(\omega_\emptyset) + \sum_{n=1}^k \mathcal{I}_n(T_n F^k), \quad \text{and} \quad F^k = F^k(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^k \mathbf{N}(dt, dx),$$

and $\mathcal{H}_{(t,x)}F^k$ belongs to $\mathbb{L}_{\mathcal{P}}^1$. In addition by Lemma 5.1, $\mathcal{H}F$ is well defined in $\mathbb{L}_{\mathcal{P}}^1$ and

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[\left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F^k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx) \right| \right] = 0.$$

Hence

$$\begin{aligned} & \mathbb{E} \left[\left| F - F(\omega_\emptyset) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx) \right| \right] \\ & \leq \mathbb{E} \left[\left| F^k - F^k(\omega_\emptyset) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F^k \mathbf{N}(dt, dx) \right| \right] \\ & \quad + \mathbb{E} \left[\left| \int_{[0,T] \times \mathbb{X}} (\mathcal{H}_{(t,x)}F^k - \mathcal{H}_{(t,x)}F) \mathbf{N}(dt, dx) \right| \right] \\ & \quad + \mathbb{E} [|F - F^k|] + |F(\omega_\emptyset) - F^k(\omega_\emptyset)| \xrightarrow{k \rightarrow +\infty} 0 \end{aligned}$$

$$\text{as } F(\omega_\emptyset) - F^k(\omega_\emptyset) = (\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1} \mathbb{E}[|F - F^k| \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}] \xrightarrow{k \rightarrow +\infty} 0.$$

Step 2. Assume $F \in L^1(\Omega; \mathbb{P})$. Let for $k \geq 1$, $F^k := -k \vee F \wedge k$. Then by monotone convergence theorem $\lim_{k \rightarrow +\infty} \mathbb{E}[|F - F^k|] = 0$. For any k , as $F^k \in L^2(\Omega; \mathbb{P})$, by Step 1,

$$F^k = F^k(\omega_\emptyset) + \int_{\mathbf{X}} \mathcal{H}_{(t,x)}F^k \mathbf{N}(dt, dx).$$

By definition,

$$\begin{aligned} |F^k(\omega_\emptyset) - F(\omega_\emptyset)| &= ((\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1}) \mathbb{E}[|F^k(\omega_\emptyset) - F(\omega_\emptyset)|] \\ &= ((\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0])^{-1}) \mathbb{E}[|F(\omega_\emptyset)| \mathbf{1}_{\{|F(\omega_\emptyset)| > k\}}] \\ &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

Finally by Lemma 5.1, $\mathcal{H}F \in \mathbb{L}_{\mathcal{P}}^1$ and

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[\left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F^k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx) \right| \right] = 0$$

leading to $F = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx)$. □

We now extend the previous result to the case $\rho(\mathbf{X}) = +\infty$.

This extension by essence is partial as $F(\omega_\emptyset)$ which is part of the formula is defined on a set of probability zero when $\rho(\mathbf{X}) = +\infty$. For example consider $F := \mathbf{1}_{\{\mathbf{N}(\mathbf{X}) < +\infty\}}$, by definition

$$\begin{aligned} \mathcal{H}_{(t,x)}F(\omega) &:= F(\omega_t) \circ \varepsilon_{(t,x)}^+ - F(\omega_t) \\ &= \mathbf{1}_{\{\#(\omega_t)+1 < +\infty\}} - \mathbf{1}_{\{\#(\omega_t) < +\infty\}} \\ &= 0. \end{aligned}$$

In addition, $F(\omega_\emptyset) = 1$ leading to a contradiction if Relation (3.2) could be extended as $F = 0$, \mathbb{P} -a.s.. In the classical Clark–Ocone formulae, this contradiction does not appear as $(\mathbf{D}F)^p = 0$ and $\mathbb{E}[F] = 0$. This example also shows that this phenomenon is not related to an integrability condition as F is bounded. We present below a sufficient condition to extend our formulae to the case $\rho(\mathbf{X}) = +\infty$.

Notation 3.11. *As ρ is a σ -finite measure on $(\mathbf{X}, \mathfrak{X})$, there exists a family of sets $(\mathbf{R}_j)_j$ such that for any j , $\rho(\mathbf{R}_j) < +\infty$; $\mathbf{R}_j \subset \mathbf{R}_{j+1}$ and $\cup_j \mathbf{R}_j = \mathbf{X}$; we define the family of subsets of Ω*

$$\Omega^j := \left\{ \omega = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{R}_j, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\} \subset \Omega.$$

By construction $\Omega^j \subset \Omega^{j+1} \forall j$ and $\Omega_j \nearrow \Omega$ when j tends to $+\infty$.

Theorem 3.12. *Assume $\rho(\mathbf{X}) = +\infty$. Let F in $L^1(\Omega; \mathbb{P})$ such that*

- (i) $\lim_j \mathbb{E}[|F^j - F|] = 0$, where $F^j := F \mathbf{1}_{\{\Omega^j\}}$,
- (ii) $\mathcal{H}F \in \mathbb{L}_{\mathcal{P}}^1$.

Then

$$F = F(\omega_\emptyset) + \int_{\mathbb{R}_+ \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx), \mathbb{P} - a.s.$$

In addition, if there exists $Z \in \mathbb{L}_{\mathcal{P}}^0$ such that

$$\mathbb{E} \left[\left| \int_{\mathbb{R}_+ \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx) \right| \right] < +\infty$$

and

$$F = F(\omega_\emptyset) + \int_{\mathbb{R}_+ \times \mathbb{X}} Z_{(t,x)} \mathbf{N}(dt, dx), \mathbb{P} - a.s.$$

then for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{X}$, $Z_{(t,x)} = \mathcal{H}_{(t,x)} F$, \mathbb{P} -a.s..

Proof. The proof is divided in two parts.

Uniqueness: First note that contrary to Theorem 3.10 the value of the random variable F on the set $\mathbf{N}(\mathbb{R}_+ \times \mathbb{X}) = 0$ is not uniquely defined as $\mathbb{P}[\mathbf{N}(\mathbb{R}_+ \times \mathbb{X}) = 0] = 0$ (unless F is non-random). The same proof as in Theorem 3.10 gives that for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{X}$, $\mathcal{H}_{(t,x)} F = Z_{(t,x)}$, \mathbb{P} -a.s..

Existence: Let $F \in L^1(\Omega; \mathbb{P})$ satisfying (i) and (ii). Fix $j \geq 1$ and consider $F_j := F \mathbf{1}_{\{\Omega_j\}}$ so that $F_j \in L^1(\Omega; \mathbb{P})$. Writing \mathbf{N}^j the projection of \mathbf{N} on \mathbf{R}_j (that is $\mathbf{N}^j(A \times B) := \mathbf{N}((A \times B) \cap \mathbf{R}_j)$) places ourselves in the framework of Theorem 3.10 and thus

$$F_j = F \mathbf{1}_{\{\Omega_j\}} \mathbf{1}_{\{\mathbf{N}^j(\mathbf{X})=0\}} + \int_{\mathbf{R}_j} \mathcal{H}_{(t,x)} F_j \mathbf{N}^j(dt, dx), \mathbb{P} - a.s.$$

and $\mathcal{H}F_j \mathbf{1}_{\{\mathbf{R}_j\}} \in \mathbb{L}_{\mathcal{P}}^1$. By definition \mathbf{N}^j coincides with \mathbf{N} on \mathbf{R}_j and for (t, x) in \mathbf{R}_j we have for any $\omega \in \Omega_j$

$$\begin{aligned} (\mathcal{H}_{(t,x)} F_j)(\omega) &= (\mathbf{D}_{(t,x)} F_j \circ \tau_t)(\omega) = F_j(\omega_t) + \delta_{(t,x)} - F_j(\omega_t) \\ &= F(\omega_t) + \delta_{(t,x)} \mathbf{1}_{\{\omega_t + \delta_{(t,x)} \in \Omega_j\}} - F(\omega_t) \mathbf{1}_{\{\omega_t \in \Omega_j\}} \\ &= (\mathbf{D}_{(t,x)} F \circ \tau_t)(\omega) \mathbf{1}_{\{\omega_t \in \Omega_j\}} = (\mathcal{H}_{(t,x)} F)(\omega) \mathbf{1}_{\{\omega_t \in \Omega_j\}}. \end{aligned}$$

So $\mathbf{1}_{\{\Omega_j\}} \mathcal{H}_{(t,x)} F_j = \mathbf{1}_{\{\Omega_j\}} \mathcal{H}_{(t,x)} F$ and thus $F_j = F \mathbf{1}_{\{\Omega_j\}} \mathbf{1}_{\{\mathbf{N}^j(\mathbf{X})=0\}} + \int_{\mathbf{R}_j} \mathcal{H}_{(t,x)} F \mathbf{N}^j(dt, dx)$, $\mathbb{P} - a.s.$. In addition by monotone convergence and (ii),

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \mathbb{E} \left[\left| \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}^j(dt, dx) - \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] \\ &= \lim_{j \rightarrow +\infty} \mathbb{E} \left[\left| \int_{\mathbb{R}_+ \times (\mathbf{X} \setminus \mathbf{R}_j)} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] = 0. \end{aligned}$$

Regarding the constant term we have (noting that $\mathbb{P}[\mathbf{N}(\mathbf{X}) = 0] = 0$)

$$\begin{aligned} & \mathbb{E} [|F_j(\boldsymbol{\omega}_\emptyset) - F(\boldsymbol{\omega}_\emptyset)|] \\ &= \mathbb{E} [|F| \mathbf{1}_{\{\mathbf{N}(\mathbf{X} \cap \mathbf{R}_j)=0\}} \mathbf{1}_{\{\Omega_j\}} - \mathbf{1}_{\{\mathbf{N}(\mathbf{X})=0\}}|] \\ &= \mathbb{E} [|F| \mathbf{1}_{\{\mathbf{N}(\mathbf{X} \cap \mathbf{R}_j)=0\}} \mathbf{1}_{\{\Omega_j\}}] \\ &\leq E [|F| \mathbf{1}_{\{\mathbf{N}(\mathbf{X} \cap \mathbf{R}_j)=0\}}] \rightarrow_{j \rightarrow +\infty} 0. \end{aligned}$$

Hence combining the previous facts and (i)

$$\begin{aligned} & \mathbb{E} \left[\left| F - F(\boldsymbol{\omega}_\emptyset) - \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] \\ &\leq \mathbb{E} [|F - F_j|] + \mathbb{E} [|F_j(\boldsymbol{\omega}_\emptyset) - F(\boldsymbol{\omega}_\emptyset)|] \\ &\quad + \mathbb{E} \left[\left| \int_{\mathbb{R}_+ \times \mathbf{R}_j} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) - \int_{\mathbb{R}_+ \times \mathbf{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] \\ &\rightarrow_{j \rightarrow +\infty} 0. \end{aligned}$$

□

Remark 3.13. For the case of point processes, the non-compensated Clark–Ocone formula stated in Theorem 3.10 coincides with the Poisson imbedding representation (1.1). Indeed let H be a point process with bounded predictable intensity λ defined through the thinning procedure

$$H_T = \int_{(0,T] \times \mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t\}} \mathbf{N}(dt, d\theta); \quad \forall T > 0.$$

Then $H_T(\boldsymbol{\omega}_\emptyset) = 0$, and using the previous representation of H_T , the difference operator is

$$\begin{aligned} \mathbf{D}_{(t,x)} H_T &= H_T(\cdot + \delta_{(t,x)}) - H_T(\cdot) \\ &= \mathbf{1}_{\{x \leq \lambda_t\}} + \int_{(t,T] \times \mathbb{R}_+} (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta). \end{aligned}$$

Therefore $\mathbf{D}_{(t,x)} H_T \circ \tau_t = \mathbf{1}_{\{x \leq \lambda_t\}}$ which is \mathcal{F}_{t-} -measurable. Thus (1.1) rewrites as

$$H_T = H_T(\boldsymbol{\omega}_\emptyset) + \int_{[0,T] \times [0,\infty[} \mathcal{H}_{(t,x)} H_T \mathbf{N}(dt, dx), \quad \mathbb{P} - a.s..$$

Note that the standard Clark–Ocone formula is much more tricky to compute, since one has to compute the conditional expectation (given \mathcal{F}_{t-}) of the integral term

$$\begin{aligned} H_T &= \mathbb{E}(H_T) + \int_{[0,T] \times [0,\infty[} \left(\mathbb{E}_{t-} \left[\mathbf{1}_{\{x \leq \lambda_t\}} + \int_{(t,T] \times \mathbb{R}_+} (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta) \right] \right) \tilde{\mathbf{N}}(dt, dx) \\ &= \mathbb{E} \left(\int_{[0,T]} \lambda_t dt \right) + \int_{[0,T] \times [0,\infty[} \mathbf{1}_{\{x \leq \lambda_t\}} \tilde{\mathbf{N}}(dt, dx) \\ &\quad + \int_{[0,T] \times [0,\infty[} \left(\mathbb{E}_{t-} \left[\int_t^T (\mathbf{1}_{\{\theta \leq \lambda_r \circ \varepsilon_{(t,x)}^+\}} - \mathbf{1}_{\{\theta \leq \lambda_r\}}) \mathbf{N}(dr, d\theta) \right] \right) \tilde{\mathbf{N}}(dt, dx). \end{aligned}$$

4. APPLICATIONS

This section provides some direct applications of the non-compensated Clark–Ocone formula. These applications exploit different characteristics of this formula: the positivity of the Poisson measure, the predictability of the operators, the validity of the formula for $L^1(\Omega; \mathbb{P})$ -random variables. In particular we give non-compensated versions of Poincaré’s inequality and a logarithmic Sobolev inequality, which turn out to be more accurate (in fact equalities) in the non-compensated version. Indeed, the non-compensated Clark–Ocone formula involves only predictable operators, contrary to the standard Clark–Ocone one, and therefore it does not require the use of a Jensen’s inequality to eliminate the predictable projection. In this section, we assume $\rho(\mathbf{X}) < +\infty$.

4.1. Expectation inequality

Thanks to the positivity of the Poisson measure \mathbf{N} , the non-compensated Clark–Ocone formula allows us to compare the expectation of random variables, as soon as inequalities on their respective integrand and their respective value on ω_\emptyset are satisfied.

Proposition 4.1. *Let F, G such that F and G are in $L^1(\Omega; \mathbb{P})$.*

If $\mathcal{H}_{(t,x)}(F) \leq \mathcal{H}_{(t,x)}(G)$ $\mathbb{P} \otimes \rho$ -a.e. and $F(\omega_\emptyset) \leq G(\omega_\emptyset)$, then $\mathbb{E}[F] \leq \mathbb{E}[G]$.

4.2. Stop-loss on the counting process

Another application concerns the value of stop-loss contract on the counting process, which can be written in terms of the intensity process.

Proposition 4.2. *Let $K \in \mathbb{N}$ and $H := (H_t)_{t \geq 0}$ be a counting process on \mathbb{R}_+ with stochastic bounded intensity λ . Then*

$$\mathbb{E}[(H_T - K)_+] = \int_0^T \mathbb{E}[\lambda_t \mathbf{1}_{\{H_t \geq K\}}] dt.$$

Proof. Denoting M the bound of the intensity λ , one can represent H as

$$H_t = \int_{[0,t] \times [0,M]} \mathbf{1}_{\{x \leq \lambda_s\}} \mathbf{N}(ds, dx); \quad t \geq 0.$$

Then

$$\mathbb{E}[(H_T - K)_+] = \int_{[0,T] \times [0,M]} \mathbb{E}[\mathcal{H}_{(t,x)}((H_T - K)_+)] dt dx.$$

Let $(t, x) \in [0, T] \times [0, M]$. We have

$$\begin{aligned}
& \mathbb{E}[\mathcal{H}_{(t,x)}((H_T - K)_+)] \\
&= \mathbb{E}[(H_{t-} + \mathbf{1}_{\{x \leq \lambda_t\}} - K)_+ - (H_{t-} - K)_+] \\
&= \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} ((H_{t-} + 1 - K)_+ - (H_{t-} - K)_+)] \\
&= \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} \mathbf{1}_{\{H_{t-} - K > 0\}}] + \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} \mathbf{1}_{\{H_{t-} - K \leq 0\}} ((H_{t-} + 1 - K)_+)] \\
&= \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} \mathbf{1}_{\{H_{t-} - K > 0\}}] + \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} \mathbf{1}_{\{H_{t-} = K\}}] \\
&= \mathbb{E}[\mathbf{1}_{\{x \leq \lambda_t\}} \mathbf{1}_{\{H_{t-} - K \geq 0\}}].
\end{aligned}$$

Therefore

$$\mathbb{E}[(H_T - K)_+] = \int_0^T \mathbb{E}[\lambda_t \mathbf{1}_{\{H_t - K \geq 0\}}] dt.$$

□

4.3. Moments and Poincaré's inequality

The non-compensated Clark–Ocone formula can be useful to compute moments of a random variable. Namely, let $n \in \mathbb{N}^*$ and $F \in L^n(\Omega; \mathbb{P})$. By Theorem 3.10

$$F^n = F^n(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F^n \mathbf{N}(dt, dx),$$

leading to

$$\mathbb{E}[F^n] = F^n(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathbb{E}[\mathcal{H}_{(t,x)} F^n] dt dx. \quad (4.1)$$

Obviously it remains to compute the quantity $\mathbb{E}[\mathcal{H}_{(t,x)} F^n]$ which is equal to $\mathbb{E}[L^{t,T} \mathbf{D}_{t,x} F^n]$. Note however that those quantities are usually more tractable than the ones given in the standard Clark–Ocone formula, as highlighted in Remark 4.9 of [10].

Focusing on the moments of order one and two, that is applying (4.1) with $n = 2$ and $n = 1$ we get that:

$$\mathbb{E}[F^2] = F^2(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathbb{E}[\mathcal{H}_{(t,x)} F^2] dt dx;$$

and

$$\mathbb{E}[F] = F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathbb{E}[\mathcal{H}_{(t,x)} F] dt dx;$$

leading to

$$\mathbb{E}[(F - \mathbb{E}[F])^2] = F^2(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathbb{E}[\mathcal{H}_{(t,x)} F^2] dt dx - \left(F(\omega_\emptyset) + \int_{[0,T] \times \mathbb{X}} \mathbb{E}[\mathcal{H}_{(t,x)} F] dt dx \right)^2.$$

This has to be compared with the proof of Poincaré’s inequality using the Clark–Ocone formula. By Clark–Ocone formula,

$$F = \mathbb{E}[F] + \int_{[0,T] \times \mathbb{X}} (\mathbf{D}_{(t,x)} F)^p \tilde{\mathbf{N}}(dt, dx).$$

Hence using Jensen inequality, and since the predictable projection coincides with the conditional expectation (see Prop. 3.2.7 in [16])

$$\begin{aligned} \mathbb{E}[(F - \mathbb{E}[F])^2] &= \int_{[0,T] \times \mathbb{X}} \mathbb{E} [|(\mathbf{D}_{(t,x)} F)^p|^2] dt dx \\ &= \int_{[0,T] \times \mathbb{X}} \mathbb{E} \left[|\mathbb{E}_{t-}[\mathbf{D}_{(t,x)} F]|^2 \right] dt dx \\ &\leq \int_{[0,T] \times \mathbb{X}} \mathbb{E} \left[|\mathbf{D}_{(t,x)} F|^2 \right] dt dx \end{aligned}$$

which leads to Poincaré inequality.

Remark 4.3. Contrary to the proof of Poincaré’s inequality where we make an estimation by Jensen’s inequality, our operator $\mathcal{H}_{(t,x)} F^2$ has directly the right measurability.

Remark 4.4. By Relation (4.1) with $n = 2$ we have that if $F(\omega_\emptyset) = 0$

$$\int_{[0,T] \times \mathbb{X}} \mathbb{E}[L^{t,T} F^2(\cdot + \delta_{(t,x)})] dt dx \geq \int_{[0,T] \times \mathbb{X}} \mathbb{E}[L^{t,T} F^2] dt dx.$$

The following example with N_T^2 , where $N := (N_t)_{t \geq 0}$ a Poisson process with intensity $\lambda = 1$, gives an illustration of the gap between the two formulas (the upper bound given by Poincaré’s inequality, and the exact value). Using the non-compensated Clark–Ocone formula

$$\begin{aligned} \mathbb{E}[(N_T^2 - \mathbb{E}[N_T^2])^2] &= 0 + \int_0^T \mathbb{E}[\mathcal{H}_{(t)} N_T^4] dt - \left(\int_0^T \mathbb{E}[\mathcal{H}_{(t)} N_T^2] dt \right)^2 \\ &= \int_0^T \mathbb{E}[(N_{t-} + 1)^4 - N_{t-}^4] dt - \left(\int_0^T \mathbb{E}[(N_{t-} + 1)^2 - N_{t-}^2] dt \right)^2 \\ &= \int_0^T (4t^3 + 18t^2 + 14t + 1) dt - \left(\int_0^T (2t + 1) dt \right)^2 \\ &= 4T^3 + 6T^2 + T. \end{aligned}$$

While with Poincaré's inequality we only have

$$\begin{aligned} \int_{[0,T]} \mathbb{E} \left[|\mathbf{D}_{(t,x)} N_T^2|^2 \right] dt &= \int_0^T \mathbb{E} [|(N_{T-} + 1)^2 - N_{T-}^2|] dt \\ &= \int_{[0,T]} (4T(T+1) + 4T + 1) dt \\ &= 4T^3 + 8T^2 + T. \end{aligned}$$

4.4. Logarithmic Sobolev relation

In the same way, the non-compensated Clark–Ocone formula allows us to provide a new version of a logarithmic Sobolev relation, in the same line as the logarithmic Sobolev inequality (see [17]), for random variables F in $L^1(\Omega; \mathbb{P})$.

If $\mathbb{E}[F] = 1$, then the entropy is given by (with the convention $\Phi(0) := 0$):

$$\text{Ent}[F] := \mathbb{E}[\Phi(F)]; \quad \Phi(x) := x \log(x)$$

Proposition 4.5. *Let $F \in L^1(\Omega; \mathbb{P})$ with $\mathbb{P}[F \geq 0] = 1$ and $\mathbb{E}[F] = 1$ (so that $\Phi(\mathbb{E}[F]) = 0$). Then*

$$\text{Ent}[F] = \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[(\Phi(F(\omega_t) + \delta_{(t,x)}) - \Phi(F(\omega_t)))] dt \pi(dx).$$

Proof.

$$\begin{aligned} \text{Ent}[F] &= \mathbb{E}[\Phi(F)] \\ &= \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[\mathcal{H}_{(t,x)}(\Phi(F))] dt \pi(dx) \\ &= \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[L^{t,T}(\Phi(F)(\omega_t) + \delta_{(t,x)}) - \Phi(F)(\omega_t)] dt \pi(dx) \\ &= \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[(\Phi(F(\omega_t) + \delta_{(t,x)}) - \Phi(F(\omega_t)))] dt \pi(dx). \end{aligned}$$

This also re-writes as

$$\text{Ent}[F] = \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[\beta(F(\omega_t), \mathcal{H}_{(t,x)}F)] dt \pi(dx),$$

with

$$\beta(u, v) := (u + v) \log(u + v) - u \log(u).$$

□

This exact formula could be compared with an equality relation given in Wu [17], from which the logarithmic Sobolev inequality is deduced for $F \in L^2(\Omega; \mathbb{P})$. More precisely

$$\mathbf{D}_{(t,x)} \Phi(F) - \Phi'(F) \mathbf{D}_{(t,x)} F = \Psi(F, \mathbf{D}_{(t,x)} F)$$

where Ψ is a convex function in u and v given by

$$\Psi(u, v) := (u + v) \log(u + v) - u \log(u) - (1 + \log(u))v; \quad u > 0; u + v > 0,$$

it is proved in [17], that

$$\begin{aligned} \text{Ent}[F] &= \int_{\mathbf{X}} \mathbb{E} [\Psi(\mathbb{E}_{t-}[F], (\mathbf{D}_{(t,x)}F)^p)] dt\pi(dx) \\ &\leq \int_{\mathbf{X}} \mathbb{E} [\Psi(F, \mathbf{D}_{(t,x)}F)] dt\pi(dx). \end{aligned}$$

A comparison of these two relations is left for future research.

Remark 4.6. As a by product, we obtain an exact formula between the function Φ and the predictable projection on Poisson space. Let $F \in L^2(\Omega; \mathbb{P})$ with $\mathbb{P}[F \geq 0] = 1$ and $\mathbb{E}[F] = 1$. Then

$$\mathbb{E}[\Phi(F)] = \int_{\mathbf{X}} \mathbb{E} [\Psi(\mathbb{E}_{t-}[F], \mathbb{E}_{t-}[\mathbf{D}_{(t,x)}F])] dt\pi(dx) = \Phi(F(\omega_\emptyset)) + \int_{\mathbf{X}} \mathbb{E}[L^{t,T}\beta(F, \mathbf{D}_{(t,x)}F)] dt\pi(dx).$$

5. TECHNICAL LEMMATA

The section gathers some technical lemmata. We first prove convergence results of the operator $\mathcal{H}_{(t,x)}$. More precisely:

Lemma 5.1. *Assume $\rho(\mathbf{X}) < +\infty$. Let $r \in [1, 2]$ and $F \in L^r(\Omega; \mathbb{P})$. Assume there exists $(G_k)_k \subset L^r(\Omega; \mathbb{P})$ converging in $L^r(\Omega; \mathbb{P})$ to F such that for any k , $\mathcal{H}G_k$ belongs to \mathbb{L}_P^r . Then*

$$\lim_{k \rightarrow +\infty} \left(\int_{[0,T] \times \mathbb{X}} \mathbb{E} [|\mathcal{H}_{(t,x)}G_k - \mathcal{H}_{(t,x)}F|^r] dt\pi(dx) \right)^{1/r} = 0.$$

In particular, the stochastic process $\mathcal{H}F$ belongs to \mathbb{L}_P^r and we have the following convergence in L^1

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[\left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}G_k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)}F \mathbf{N}(dt, dx) \right| \right] = 0.$$

Proof. We adapt the proof of [12], Lemma 2.3 and give only the main arguments. Using Malliavin derivative (see Prop. 2.7) and Lemma 3.9.

$$\begin{aligned} &\left(\int_{[0,T] \times \mathbb{X}} \mathbb{E} [|\mathcal{H}_{(t,x)}G_k - \mathcal{H}_{(t,x)}F|^r] dt\pi(dx) \right)^{1/r} \\ &= \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} \left| \mathbb{E}_{t-} [L^{t,T} \mathbf{D}_{(t,x)}G_k] - \mathbb{E}_{t-} [L^{t,T} \mathbf{D}_{(t,x)}F] \right|^r dt\pi(dx) \right]^{1/r} \\ &= \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} \left| \mathbb{E}_{t-} [L^{t,T} (\mathbf{D}_{(t,x)}G_k - \mathbf{D}_{(t,x)}F)] \right|^r dt\pi(dx) \right]^{1/r} \\ &\leq \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |L^{t,T} (\mathbf{D}_{(t,x)}G_k - \mathbf{D}_{(t,x)}F)|^r dt\pi(dx) \right]^{1/r} \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |\mathbf{D}_{(t,x)} G_k - \mathbf{D}_{(t,x)} F|^r dt \pi(dx) \right]^{1/r} \\
&\leq C_r \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |G_k \circ \varepsilon_{(t,x)}^+ - F \circ \varepsilon_{(t,x)}^+|^r dt \pi(dx) \right]^{1/r} \\
&\quad + C_r \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |G_k - F|^r dt \pi(dx) \right]^{1/r},
\end{aligned}$$

where $C_r > 0$ is a combinatorial constant depending only on r . Using Mecke's formula (see Lem. 2.11)

$$\begin{aligned}
&\mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |G_k \circ \varepsilon_{(t,x)}^+ - F \circ \varepsilon_{(t,x)}^+|^r dt \pi(dx) \right]^{1/r} \\
&= \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} \exp(r(T-t)\pi(\mathbb{X})) |F \circ \varepsilon_{(t,x)}^+ - G_k \circ \varepsilon_{(t,x)}^+|^r \mathbf{1}_{\{\mathbf{N}([t,T] \times \mathbb{X})=0\}} dt \pi(dx) \right]^{1/r} \\
&\leq \exp(T\pi(\mathbb{X})) \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |F \circ \varepsilon_{(t,x)}^+ - G_k \circ \varepsilon_{(t,x)}^+|^r \mathbf{1}_{\{\mathbf{N}((t,T] \times \mathbb{X})=0\}} dt \pi(dx) \right]^{1/r} \\
&= \exp(T\pi(\mathbb{X})) \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |F \circ \varepsilon_{(t,x)}^+ - G_k \circ \varepsilon_{(t,x)}^+|^r \mathbf{1}_{\{\mathbf{N}((t,T] \times \mathbb{X})=0\}} \circ \varepsilon_{(t,x)}^+ dt \pi(dx) \right]^{1/r} \\
&= \exp(T\pi(\mathbb{X})) \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |F - G_k|^r \mathbf{1}_{\{\mathbf{N}((t,T] \times \mathbb{X})=0\}} \mathbf{N}(dt, dx) \right]^{1/r} \\
&= \exp(T\pi(\mathbb{X})) \mathbb{E} \left[|F - G_k|^r \int_{[0,T] \times \mathbb{X}} \mathbf{1}_{\{\mathbf{N}((t,T] \times \mathbb{X})=0\}} \mathbf{N}(dt, dx) \right]^{1/r}.
\end{aligned}$$

Fix $\omega = \sum_{j=1}^n \delta_{(t_j, x_j)}$, $(t_j, x_j) \in [0, T] \times \mathbb{X}$ and $n \in \mathbb{N}$ (as $\pi(\mathbb{X}) < +\infty$, $\omega([0, T] \times \mathbb{X}) < +\infty$ \mathbb{P} -a.s.). We have

$$\begin{aligned}
&\int_{[0,T] \times \mathbb{X}} \mathbf{1}_{\{\mathbf{N}((t,T] \times \mathbb{X})=0\}} \mathbf{N}(dt, dx)(\omega) \\
&= \sum_{j=1}^n \mathbf{1}_{\{\omega((t_j, T] \times \mathbb{X})=0\}} \\
&= \mathbf{1}_{\{\omega((t_n, T] \times \mathbb{X})=0\}} = \mathbf{1}_{\{\omega((0, T] \times \mathbb{X})>0\}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |G_k \circ \varepsilon_{(t,x)}^+ - F \circ \varepsilon_{(t,x)}^+|^r dt \pi(dx) \right]^{1/r} \\
&\leq \exp(rT\pi(\mathbb{X})) \mathbb{E} [|F - G_k|^r \mathbf{1}_{\{\mathbf{N}((0, T] \times \mathbb{X})>0\}}]^{1/r} \\
&\leq \exp(rT\pi(\mathbb{X})) \mathbb{E} [|F - G_k|^r]^{1/r} \xrightarrow{k \rightarrow +\infty} 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
& \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} (L^{t,T})^r |G_k - F|^r dt \pi(dx) \right]^{1/r} \\
&= \mathbb{E} \left[|G_k - F|^r \int_{[0,T] \times \mathbb{X}} (L^{t,T})^r dt \pi(dx) \right]^{1/r} \\
&\leq \exp(T\pi(\mathbb{X})) (T\pi(\mathbb{X}))^{1/r} \mathbb{E} [|G_k - F|^r]^{1/r} \xrightarrow{k \rightarrow +\infty} 0.
\end{aligned}$$

In addition as $\mathcal{H}G_k$ converges in \mathbb{L}_p^r it converges pointwise and thus $\mathcal{H}F$ belongs to \mathbb{L}_p^r . Furthermore

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} G_k \mathbf{N}(dt, dx) - \int_{[0,T] \times \mathbb{X}} \mathcal{H}_{(t,x)} F \mathbf{N}(dt, dx) \right| \right] \\
&\leq \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |\mathcal{H}_{(t,x)} G_k - \mathcal{H}_{(t,x)} F| \mathbf{N}(dt, dx) \right] \\
&= \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |\mathcal{H}_{(t,x)} G_k - \mathcal{H}_{(t,x)} F| dt \pi(dx) \right] \\
&\leq (T\pi(\mathbb{X}))^{\frac{r-1}{r}} \mathbb{E} \left[\int_{[0,T] \times \mathbb{X}} |\mathcal{H}_{(t,x)} G_k - \mathcal{H}_{(t,x)} F|^r dt \pi(dx) \right]^{1/r} \xrightarrow{k \rightarrow +\infty} 0
\end{aligned}$$

using Hölder inequality. □

We continue our analysis by elaborating on the nested structure of factorial measures $\mathbf{N}^{(m)}$. To do so we introduce the notations below.

Notation 5.2. For $m \in \mathbb{N}^*$ we set

$$\mathbf{X}^{m, t^\neq} := \{(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbf{X}^m, \forall (i, j) \in \{1, \dots, m\}^2, [t_i = t_j] \Rightarrow [i = j]\}.$$

Notation 5.3. For $a, n, m \in \mathbb{N}^*$ we set $\llbracket a, n \rrbracket := \{a, \dots, n\}$;

$$\llbracket a, n \rrbracket^{m, \neq} := \{(i_1, \dots, i_m) \in \llbracket a, n \rrbracket^m, \forall (j, k) \in \llbracket a, m \rrbracket^2, [i_j = i_k] \Rightarrow [j = k]\};$$

and

$$\llbracket a, n \rrbracket^{m, <} := \{(i_1, \dots, i_m) \in \llbracket a, n \rrbracket^m, i_1 < \dots < i_m\}.$$

Lemma 5.4. Assume $\rho(\mathbf{X}) < +\infty$. Let $m \geq 1$ and $f_m \in L_s^1(\mathbf{X}^m; \rho^{\otimes m})$. Then

$$\mathcal{I}_m(f_m) = m \int_{\mathbf{X}} \mathcal{I}_{m-1} \left(f_m(\mathbf{x}, \cdot \mathbf{I}_{\{[0,t] \times \mathbb{X}\}}^{\otimes (m-1)}) \right) \mathbf{N}(d\mathbf{x}), \quad (5.1)$$

in particular $\mathcal{H}_{(t,x)} \mathcal{I}_m(f_m) = m \mathcal{I}_{m-1} \left(f_m(\mathbf{x}, \cdot \mathbf{I}_{\{[0,t] \times \mathbb{X}\}}^{\otimes (m-1)}) \right)$ for any $(t, x) \in \mathbf{X}$.

Proof. First we prove that the random measure $\mathbf{N}^{(m)}$ has support \mathbf{X}^{m,t^\neq} . Indeed let $A \in \mathfrak{X}^{\otimes m}$ and let $\overline{\mathbf{X}^{m,t^\neq}} := \mathbf{X}^m \setminus \mathbf{X}^{m,t^\neq}$. By Relation (2.6)

$$\mathbb{E} \left[\mathbf{N}^{(m)}(A \cap \overline{\mathbf{X}^{m,t^\neq}}) \right] = \boldsymbol{\rho}^{\otimes m}(A \cap \overline{\mathbf{X}^{m,t^\neq}}) = 0$$

as any diagonal of dimension $r \in \{2, \dots, m\}$ (of the form $\mathbf{1}_{\{t_{i_1} = \dots = t_{i_r}\}}$ for pairwise distinct indices i_1, \dots, i_r in $\{1, \dots, m\}$) is of measure 0 for the Lebesgue measure $\lambda^{\otimes m}$. This is a simple refinement of the fact that factorial measures $\mathbf{N}^{(m)}$ are supported on pairwise distinct elements of \mathbf{X}^m ; since the choice of the measure $\boldsymbol{\rho}$ imposes that also the t components of the points have to be pairwise disjoint. This leads us to the proof of the statement. Let $\boldsymbol{\omega} = \sum_{j=1}^n \delta_{\mathbf{x}_j} \in \Omega$ with $n \geq m$ (note that as $\boldsymbol{\rho}(\mathbf{X}) < +\infty$, $\mathbf{N}(\mathbf{X}) < +\infty$, \mathbb{P} -a.s.). Without loss of generality, we can assume (from the first part of the proof) that $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^{m,t^\neq}$. Hence as the t components are pairwise disjoint, we also assume that $t_1 < t_2 < \dots < t_n$ (the definition of $\boldsymbol{\omega}$ is unchanged by relabelling the atoms \mathbf{x}_j). We have

$$\begin{aligned} (\mathcal{I}_m(f_m))(\boldsymbol{\omega}) &= \sum_{(i_1, \dots, i_m) \in \llbracket 1, n \rrbracket^{m, \neq}} f_m(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_m}) \\ &= m! \sum_{(i_1, \dots, i_m) \in \llbracket 1, n \rrbracket^{m, <}} f_m(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_m}) \\ &= m! \sum_{i_m=m}^n \sum_{(i_1, \dots, i_{m-1}) \in \llbracket 1, i_m \rrbracket^{m-1, <}} f_m(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_m}) \\ &= m \sum_{i_m=m}^n (m-1)! \sum_{(i_1, \dots, i_{m-1}) \in \llbracket 1, i_m \rrbracket^{m-1, \neq}} f_m(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_m}) \\ &= m \sum_{i_m=m}^n \mathcal{I}_{m-1} \left(f_m(\mathbf{x}_{i_m}, \cdot \mathbf{1}_{\{0, t_{i_m}\} \times \mathbb{X}}^{\otimes(m-1)}) \right) (\boldsymbol{\omega}) \\ &= m \left(\int_{\mathbf{X}} \mathcal{I}_{m-1} \left(f_m(\mathbf{x}, \cdot \mathbf{1}_{\{0, t\} \times \mathbb{X}}^{\otimes(m-1)}) \right) \mathbf{N}(d\mathbf{x}) \right) (\boldsymbol{\omega}) \end{aligned}$$

which proves Relation (5.1). The identification of the quantity $\mathcal{H}_{(t,x)} \mathcal{I}_m(f_m)$ follows from this relation and the definition of the operator. \square

Note that Relation (5.1) is the counterpart of a well-known similar relation for the iterated integrals $\mathbf{I}_n(f_n)$ (see *e.g.* [16], Prop. 2.7.1). Finally, we prove that for any $n \in \mathbb{N}$ the space generated by the iterated integrals $\mathbf{I}_n(f_n)$ and $\mathcal{I}_n(f_n)$ (for $f_n \in L^r(\mathbf{X}^n)$) are identical.

Lemma 5.5. *Assume $\boldsymbol{\rho}(\mathbf{X}) < +\infty$ and let $r \geq 1$.*

$$\text{Span} \{ \mathbf{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \} = \text{Span} \{ \mathcal{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \}$$

where $\mathbf{I}_0(f) := f$ and $\mathcal{I}_0(f) := f$ for $f \in \mathbb{R}$. In addition for any $n \in \mathbb{N}^*$ and $f_n \in L^r(\mathbf{X}^n)$,

$$\mathbf{I}_n(f_n) = \int_{[0, T] \times \mathbb{X}} \mathbb{E}_{t-} [\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n)] \tilde{\mathbf{N}}(dt, dx), \quad (5.2)$$

$$\mathcal{I}_n(f_n) = \int_{[0, T] \times \mathbb{X}} \mathcal{H}_{(t,x)} \mathcal{I}_n(f_n) \mathbf{N}(dt, dx), \quad (5.3)$$

and $\mathcal{H}\mathcal{I}_n(f_n)$ belongs to $\mathbb{L}_{\mathcal{P}}^1$. Finally if $r = 2$, then $\mathbb{E}_-[\mathbf{D}\mathbf{I}_n(f_n)] = (\mathbf{D}\mathbf{I}_n(f_n))^p$ and it belongs to $\mathbb{L}_{\mathcal{P}}^2$.

Proof. Assume $\rho(\mathbf{X}) < +\infty$. Recalling that $\mathbf{I}_n(f_n) = \mathbf{I}_n(\tilde{f}_n)$ and $\mathcal{I}_n(f_n) = \mathcal{I}_n(\tilde{f}_n)$ where \tilde{f}_n denotes the symmetrization of f_n , without loss of generality we assume that all the mappings below are symmetric.

Step 1: Let $n \in \mathbb{N}^*$ and $F := \mathcal{I}_n(f_n)$ with $f_n \in L^r_s(\mathbf{X}^n)$, $r \geq 1$. Then according to Proposition 12.11 in [12], $\overline{F} = \sum_{j=0}^n \mathbf{I}_j(g_j)$ with

$$g_j(\mathbf{x}_1, \dots, \mathbf{x}_j) := \frac{n!}{j!(n-j)!} \int_{\mathbf{X}^{n-j}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \mathbf{x}_1, \dots, \mathbf{x}_j) \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \in L^r(\mathbf{X}^{n-j})$$

as

$$\begin{aligned} & \left\| \int_{\mathbf{X}^{n-j}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \cdot) \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \right\|_{L^r(\mathbf{X}^{n-j})} \\ & \leq \int_{\mathbf{X}^{n-j}} \|f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \cdot)\|_{L^r(\mathbf{X}^{n-j})} \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \\ & = \int_{\mathbf{X}^{n-j}} \left(\int_{\mathbf{X}^j} |f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-j}, \mathbf{x}_1, \dots, \mathbf{x}_j)|^r \rho^{\otimes(n-j)}(d\mathbf{x}_1, \dots, d\mathbf{x}_j) \right)^{1/r} \rho^{\otimes(n-j)}(d\mathbf{y}_1, \dots, d\mathbf{y}_{n-j}) \\ & \leq (T\pi(\mathbb{X}))^{\frac{r}{r-1}} \|f_n\|_{L^r(\mathbf{X}^n)}. \end{aligned}$$

Let $n \in \mathbb{N}^*$ and $F := \mathbf{I}_n(f_n)$ with $f_n \in L^r(\mathbf{X}^n; \rho^{\otimes n})$, $r \geq 1$. Then

$$\begin{aligned} F &= \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{\otimes(n-|J|)}(d\mathbf{x}_{J^c}) \\ &= (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \rho^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &\quad + \sum_{J \subset \{1, \dots, n\}; J \neq \emptyset} (-1)^{n-|J|} \int_{\mathbf{X}^{n-|J|}} \int_{\mathbf{X}^{|J|}} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(|J|)}(d\mathbf{x}_J) \rho^{\otimes(n-|J|)}(d\mathbf{x}_{J^c}) \\ &= (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \rho^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &\quad + \sum_{k=1}^n \int_{\mathbf{X}^k} \frac{n!}{k!(n-k)!} (-1)^{n-k} \int_{\mathbf{X}^{n-k}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-k}, \mathbf{x}_1, \dots, \mathbf{x}_k) \rho^{\otimes(n-k)}(d\mathbf{y}) \mathbf{N}^{(k)}(d\mathbf{x}) \\ &= \mathcal{I}_0(g_0) + \sum_{k=1}^n \mathcal{I}_k(g_k) \end{aligned}$$

with $g_0 := (-1)^n \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \rho^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n)$ and as above

$$g_k := \frac{n!}{k!(n-k)!} (-1)^{n-k} \int_{\mathbf{X}^{n-k}} f_n(\mathbf{y}_1, \dots, \mathbf{y}_{n-k}, \cdot) \rho^{\otimes(n-k)}(d\mathbf{y}) \in L^r(\mathbf{X}^k; \rho^{\otimes k})$$

Then step 1 together with step 2 implies that

$$\text{Span} \{ \mathbf{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \} = \text{Span} \{ \mathcal{I}_k(f_k); k \in \llbracket 0, n \rrbracket, f_k \in L^r(\mathbf{X}^k) \}.$$

Step 3: Let $n \in \mathbb{N}^*$ and $f_n \in L_s^r(\mathbf{X}^n)$. We have that for any $(t, x) \in \mathbf{X}$,

$$\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n) = n \mathbf{I}_{n-1}(f_n((t, x), \cdot))$$

and thus

$$\mathbb{E}_{t-}[\mathbf{D}_{(t,x)} \mathbf{I}_n(f_n)] = n \mathbf{I}_{n-1}\left(f_n((t, x), \cdot \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)})\right),$$

with $\mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)}((t_2, x_2), \dots, (t_n, x_n)) := \prod_{i=2}^n \mathbf{1}_{\{t_i < t\}}$. The Clark–Ocone representation (5.2) follows from classical Malliavin’s calculus. Representation (5.3) follows from Lemma 5.4. Thus uniqueness part of the proof gives that (5.3) is in force. The last thing to be proved is that both integrands in (5.2) and (5.3) are predictable.

Step 4: Let $n \geq 1$ and $f_n \in L^r(\mathbf{X}^n; \rho^{\otimes n})$. Using classical approximations (as $\rho(\mathbf{X}) < +\infty$) there exists a sequence $(f_n^\ell)_{\ell \geq 1} \subset C_b(\mathbf{X})$ (the set of bounded and continuous functions on \mathbf{X}) such that $f_n^\ell \rightarrow_{\ell \rightarrow +\infty} f_n$ in $L^r(\mathbf{X}^n)$. Set $F^\ell := \mathcal{I}_n(f_n^\ell)$ where by abuse of notation we make use of the same notation of f_n^ℓ for its symmetrization. We also set $F := \mathcal{I}_n(f_n)$. We have that

$$\mathbb{E}[|F^\ell - F|] \leq \int_0^T \int_{\mathbb{X}} \mathbb{E}[|n \mathcal{I}_{n-1}(f_n^\ell((t, x), \cdot)) - n \mathcal{I}_{n-1}(f_n((t, x), \cdot))|] \pi(dx) dt \leq \|f_n^\ell - f_n\|_{\mathbb{L}_{\mathcal{P}}^1} \rightarrow 0.$$

Thus,

$$\mathcal{H}_{(t,x)} F^\ell = n \mathcal{I}_{n-1}(f_n^\ell((t, x), \cdot \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)}))$$

is left continuous in t for any x , since f_n^ℓ is a continuous function. Hence we have that $\mathcal{H}F^\ell$ belongs to $\mathbb{L}_{\mathcal{P}}^1$ and so is $\mathcal{H}F$ by Lemma 5.1.

Similarly if $r = 2$, thanks to the isometry property of the operator \mathbf{I}_n (see the comment at the end of Def. 2.5)

$$\begin{aligned} & \int_0^T \int_{\mathbb{X}} \mathbb{E}\left[|\mathbb{E}_{t-}[\mathbf{I}_n(f_n)] - \mathbb{E}_{t-}[\mathbf{I}_n(f_n^\ell)]|^2\right] \pi(dx) dt \\ &= \int_0^T \int_{\mathbb{X}} \mathbb{E}\left[|n \mathbf{I}_{n-1}\left(f_n^\ell((t, x), \cdot \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)})\right) - n \mathbf{I}_{n-1}\left(f_n^\ell((t, x), \cdot \mathbf{1}_{\{[0,t] \times \mathbb{X}\}}^{\otimes(n-1)})\right)|^2\right] \pi(dx) dt \\ &= n^2 \|f_n - f_n^\ell\|_{L^2(\mathbf{X}^n)}^2 \rightarrow_{\ell \rightarrow +\infty} 0. \end{aligned}$$

Once again the continuity of the maps f_n^ℓ gives that $\mathbb{E}_{t-}[\mathbf{I}_n(f_n)]$ is the predictable projection of $\mathbf{D}\mathbf{I}_n(f_n)$ and thus that it belongs to $\mathbb{L}_{\mathcal{P}}^2$. \square

Conclusion

This paper extends to any integrable Poisson-functional the Poisson imbedding that provides a representation of any point process (with intensity $(\lambda_t)_t$) as the integral of $\mathbf{1}_{\{\theta \leq \lambda_t\}}$ with respect to an uncompensated Poisson measure $\mathbf{N}(dt, d\theta)$. More precisely, we provide for any $F \in L^1(\Omega; \mathbb{P})$ a non-compensated Clark–Ocone representation formula with respect to the uncompensated Poisson measure and whose integrand is fully characterized as $\mathcal{H}_{(t,x)} F = \mathbf{D}_{(t,x)} F \circ \tau_t$. Besides $\mathcal{H}F$ is well defined in $\mathbb{L}_{\mathcal{P}}^1$, contrary to the standard Clark–Ocone formula which requires $F \in L^2(\Omega)$ and whose integrand $(\mathbf{D}_{(t,x)} F)^p$ is only defined as a limit in $L^2(\Omega \times \mathbf{X})$.

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APPENDIX A.

For sake of completeness, we provide below some technical elements on the standard Clark–Ocone formula.

Lemma A.1. Consider X a $\mathcal{F}^{\mathbf{N}} \otimes \mathfrak{X}$ -measurable process with $\int_{[0,T] \times \mathfrak{X}} \mathbb{E}[|X_{(t,x)}|] dt \pi(dx) < +\infty$. Then there exists a unique element $X^p \in \mathbb{L}_{\mathcal{P}}^1$ such that for any predictable stopping time τ

$$\mathbb{E} \left[X_{\tau,x} | \mathcal{F}_{\tau-}^{\mathbf{N}} \right] \mathbf{1}_{\{\tau < \infty\}} = X_{\tau,x}^p \mathbf{1}_{\{\tau < \infty\}}.$$

Proof. The proof essentially follows Lemma 3.3 in [4], in which X is assumed to be bounded. Let X such that $\int_{[0,T] \times \mathfrak{X}} \mathbb{E}[|X_{(t,x)}|] dt \pi(dx) < +\infty$ and we set $X^k := -k \vee X \wedge k$ for $k \in \mathbb{N}^*$. For any $k \geq 1$, there exists a unique \mathcal{P} -measurable process Y^k (which is the predictable projection of X^k) such that for any predictable stopping τ and $x \in \mathfrak{X}$

$$\mathbb{E} \left[X_{\tau,x}^k | \mathcal{F}_{\tau-}^{\mathbf{N}} \right] \mathbf{1}_{\{\tau < \infty\}} = Y_{\tau,x}^k \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - a.s..$$

By monotone convergence for any $x \in \mathfrak{X}$,

$$\mathbb{E} \left[X_{\tau,x} | \mathcal{F}_{\tau-}^{\mathbf{N}} \right] \mathbf{1}_{\{\tau < \infty\}} = Y_{\tau,x} \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - a.s.,$$

where $Y := \lim_{k \rightarrow +\infty} Y^k$ belong to $\mathbb{L}_{\mathcal{P}}^1$. □

We now give a concise proof of the standard Clark–Ocone formula for any $F \in L^2(\Omega; \mathbb{P})$ (Thm. 3.3). In particular, following [16], Section 3.2, we make precise the definition of the integrand $(\mathbf{D}F)^p$ in $L^2(\Omega \times \mathbf{X}; \mathbb{P} \otimes \rho)$ in a limiting procedure and based on the continuous feature of the mapping $F \mapsto (\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F])_{(t,x)}$ together with the fact that for fixed (t, x) , $\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] = (\mathbf{D}_{(t,x)} F)^p \quad \mathbb{P} - a.s..$

Proof of Theorem 3.3

The chaotic expansion (Thm. 3.1) entails that $V := \text{Span} \{ \mathbf{I}_n(f_n); \quad n \in \mathbb{N}, f_n \in L^2(\mathbf{X}) \}$ is dense in $L^2(\Omega; \mathbb{P})$. Let $F := \mathbf{I}_n(f_n) \in V$ with $n > 0$ and without loss of generality f_n is assumed to be symmetric. We have that

$$\mathbf{D}_{(t,x)} F = n \mathbf{I}_{n-1}(f_n((t, x), \cdot))$$

and by Lemma 5.5

$$(\mathbf{D}_{(t,x)} F)^p = \mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] = n \mathbf{I}_{n-1}(f_n((t, x), \cdot) \mathbf{1}_{\{([0,t] \times \mathfrak{X})^{\otimes (n-1)}\}}(\cdot))$$

so that

$$F = \int_{[0,T] \times \mathfrak{X}} \mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] \tilde{\mathbf{N}}(dt, dx), \tag{A.1}$$

and

$$\begin{aligned} & \mathbb{E} \left[\int_{[0,T] \times \mathfrak{X}} |\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F]|^2 dt \pi(dx) \right] \\ &= n^2 \int_{[0,T] \times \mathfrak{X}} \mathbb{E} \left[\left| \mathbf{I}_{n-1}(f_n((t, x), \cdot) \mathbf{1}_{\{([0,t] \times \mathfrak{X})^{\otimes (n-1)}\}}(\cdot)) \right|^2 \right] dt \pi(dx) \\ &= n(n-1)! \int_{([0,T] \times \mathfrak{X})^n} |f_n|^2 (dt \pi(dx))^{\otimes n} \\ &= \mathbb{E}[|F|^2] < +\infty. \end{aligned}$$

Note that by orthogonality of the operators \mathbf{I}_n the previous result extends to any element F on V with $\mathbb{E}[F] = 0$, that is for any $F \in V$ with $\mathbb{E}[F] = 0$,

$$\mathbb{E} \left[\int_{[0, T] \times \mathbb{X}} |\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F]|^2 dt \pi(dx) \right] = \mathbb{E}[|F|^2] < +\infty.$$

Let the operators U and \tilde{U} defined as

$$U : \begin{array}{l} V \rightarrow \mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+ \\ F \mapsto \left((\mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F])_{(t,x)} ; \mathbb{E}[F]^2 \right). \end{array}$$

$$\tilde{U} : \begin{array}{l} V \rightarrow L^2(\Omega) \\ F \mapsto \int_{[0, T] \times \mathbb{X}} \mathbb{E}_{t-} [\mathbf{D}_{(t,x)} F] \tilde{\mathbf{N}}(dt, dx). \end{array}$$

For any $F \in V$, we have proved that F enjoys Representation (A.1) and Relation (2.2) implies that

$$\|U(F)\|_{L^2(\mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+)}^2 = \|F\|_{L^2(\Omega)}^2; \quad \|\tilde{U}(F)\|_{L^2(\Omega; \mathbb{P})}^2 = \mathbb{E}[F^2] - \mathbb{E}[F]^2 \leq \|F\|_{L^2(\Omega)}^2.$$

Let $F \in L^2(\Omega)$. There exists $(F_n)_n \subset V$ such that $\lim_{n \rightarrow +\infty} \mathbb{E}[|F_n - F|^2] = 0$. Using the previous isometries the sequences $(U(F_n))_n$ and $(\tilde{U}(F_n))_n$ are Cauchy and thus converging respectively in $L^2(\mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+)$ and $L^2(\Omega)$. Let $U(F) := (U^1(F), U^2(F))$ and $\tilde{U}(F)$ their limits where U^1 and U^2 denote the two components of the limits. In addition, the $L^2(\Omega \times [0, T] \times \mathbb{X})$ -convergence of the process $U^1(F_n)$ implies that $U^1(F)$ is \mathcal{P} -measurable. Hence the process $t \mapsto \int_{[0, t] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx)$ is a martingale and

$$\begin{aligned} \mathbb{E} \left[\left| \int_{[0, T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] &= \mathbb{E} \left[\int_{[0, T] \times \mathbb{X}} |U^1(F)_{(r,x)}|^2 dr \pi(dx) \right] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\int_{[0, T] \times \mathbb{X}} |U^1(F_n)_{(r,x)}|^2 dr \pi(dx) \right] \\ &= \lim_{n \rightarrow +\infty} (\mathbb{E}[F_n^2] - \mathbb{E}[F_n]^2) = \mathbb{E}[F^2] - \mathbb{E}[F]^2. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbb{E} \left[\left| F - \mathbb{E}[F] - \int_{[0, T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] \\ &\leq \lim_{n \rightarrow +\infty} (\mathbb{E}[|F - F_n|^2] + |\mathbb{E}[F] - \mathbb{E}[F_n]|) \\ &\quad + \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left| \int_{[0, T] \times \mathbb{X}} U^1(F_n)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) - \int_{[0, T] \times \mathbb{X}} U^1(F)_{(r,x)} \tilde{\mathbf{N}}(dr, dx) \right|^2 \right] \\ &\leq \lim_{n \rightarrow +\infty} \mathbb{E} \left[\int_{[0, T] \times \mathbb{X}} |U^1(F_n)_{(r,x)} - U^1(F)|^2 dr \pi(dx) \right] \\ &\leq \lim_{n \rightarrow +\infty} \|U^1(F_n)_{(r,x)} - U^1(F)\|_{L^2(\mathbb{L}_{\mathcal{P}}^2 \times \mathbb{R}_+)}^2 = 0. \end{aligned}$$

The proof is complete.