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A LÉVY–ITÔ DECOMPOSITION ON A CLASS OF TOPOLOGICAL MONOIDS

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Abstract. We investigate a Lévy–Itô formula on a class of topological monoids with a suitable family of characters. This formula generalizes the classical ones for subordinators. Stochastic processes such as classical subordinators, extremal processes, measured-valued processes and the random interlacements model fall into this category.

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1. Introduction

A subordinator, as defined in [1], Chapter 3, is a stochastic process $(X_t)_{t\geq 0}$ taking values in $[0,\infty)$ with independent and identically distributed increments. An important feature of subordinators is that they possess a *Lévy–Itô decomposition*, that is, if $(X_t)_{t\geq 0}$ is a driftless subordinator, then $(X_t)_{t\geq 0}$ can be written as

$$X_t = \sum_{t_i \leq t} x_i, \quad t \geq 0, \tag{1.1}$$

where $\mathcal{N} = \sum_i \delta_{(t_i,x_i)}$ is a Poisson point process on $(0,+\infty)\times(0,+\infty)$ with intensity measure $\mathrm{d}t\otimes\pi$, and π is a measure on $(0,+\infty)$ such that $\int_{(0,+\infty)} \min\{1,x\}\pi(\mathrm{d}x) < \infty$. Moreover, classical exponential formula ([1], p. 8) implies that

$$\mathbb{E}\left[\exp\left\{-\lambda \sum_{t_i\leq t} x_i\right\}\right] = \exp\left\{-t\int_{(0,+\infty)} \left(1-\mathrm{e}^{-\lambda x}\right)\pi(\mathrm{d}x)\right\}, \quad \lambda \geq 0. \tag{1.2}$$

From now on, for reasons that will become clear later on, we will refer to these subordinators as *additive subordinators.*

In the literature, one can find several kinds of stochastic processes that possess a Lévy-Khintchine type formula and a Lévy–Itô like decomposition. For instance, the extremal processes ([1], p. 8) and super-extremal processes [3], the random interlacements model [4], and the flow of squared Bessel processes of dimension zero

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([5], p. 508; [6], Section 4). An interesting fact is that all these processes take values on some topological monoids. Thus, a natural question is to find a class of topological monoids that admits an analogue formula to (1.2), which at the same time unifies and generalizes the formulae in the examples just mentioned. In this work we propose a class of topological monoids that possesses this property, which we call L-*topological monoid* (see Def. 2.1). Our main result states:

Theorem 1.1 (Lévy–Itô Decomposition)**.** *Let* $(\mathbb{M}, \widetilde{\mathbb{M}})$ *be an* L*-topological monoid. Suppose that* $\mathcal{N} = \sum_i \delta_{(t_i,x_i)}$ *is a Poisson point process on* $(0,\infty) \times \mathbb{M}$ *with intensity measure* $\mathrm{d}t \otimes \Pi$*, where* Π *is a Borel* σ*-finite measure such that* $\int_{\mathbb{M}} \left(1 - \chi(x)\right) \Pi(\mathrm{d}x) < +\infty$ *for each* $\chi \in \widetilde{\mathbb{M}}$*, and* $\mathrm{d}t$ *is the Lebesgue measure on* $(0, +\infty)$*. Then, for every character* $\chi \in \widetilde{\mathbb{M}}$*, we have*

$$\mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t} x_i\right)\right] = \exp\left\{-t \int_{\mathbb{M}} \left(1 - \chi(x)\right) \Pi(\mathrm{d}x)\right\}, \qquad t \ge 0. \tag{1.3}$$

Under the assumptions of this theorem, the law of the stochastic process $\left(\bigoplus_{t_i \le t} x_i\right)_{t \ge 0}$ induces a convolution semigroup of measures. To the best of our knowledge, although Lévy-Khintchine formulae have been widely studied for convolution semigroups of measures (see for example [7]), Lévy–Itô representations for convolution semigroups of measures, as in (1.3), have not been considered in the literature.

In some other directions, the classical pathwise notion of Lévy processes, as defined in [1], has been extended to more general groups (for instance, Lie groups [8] and some Banach spaces [9]). In the same vein, subordinators have been extended to some particular cones of Banach spaces (see [10] or [11]).

The paper is organized as follows. In Section 2, we introduce the definition of an L-topological monoid and prove some of its properties. Then we gather some results about the Laplace transform on a topological monoid (as in [12]). Afterwards, we define subordinators and their Laplace exponent, and we generalize some well-known results related to this exponent. In Section 3, we prove the Lévy-Itô like decomposition (1.3). Finally, in Section 4 we recover some known examples.

2. A Class of topological monoids

In this section we introduce the notion of L-topological monoid. We also prove some of its properties. Examples of L-topological monoids that appear in the literature are presented as well.

First, let us recall some definitions. An abelian topological monoid is a triplet $(\mathbb{M}, \tau, \oplus)$ such that $(\mathbb{M}, \tau)$ is a topological space, $(\mathbb{M}, \oplus)$ is a commutative monoid, and the operation $\oplus \colon \mathbb{M} \times \mathbb{M} \to \mathbb{M}$ is continuous. We denote the neutral element in $(\mathbb{M}, \oplus)$ by $\boldsymbol{e}$. We refer the reader to [13], Section 1.5 and the references therein for a complete exposition on topological monoids.

Let ∂_∞ be an element not in $\mathbb{M}$ and set $\mathbb{M}^* \coloneqq \mathbb{M} \cup \{\partial_\infty\}$. We denote by $(\mathbb{M}^*, \tau^*)$ the Alexandroff extension of $(\mathbb{M}, \tau)$. We recall that a sequence $(x_n)_{n \ge 1} \subset \mathbb{M}^*$ converges to ∂_∞ if for any compact subset K of $(\mathbb{M}, \tau)$ there exists $N \ge 1$ such that $x_n \notin K$ for all $n \ge N$. As usual, we denote this by $x_n \to \partial_\infty$.

We say that a function $\chi \colon \mathbb{M} \to [0, 1]$ is a character on $\mathbb{M}$ if

$$\chi(\boldsymbol{e}) = 1 \quad \text{and} \quad \chi(x \oplus y) = \chi(x)\chi(y) \quad \text{for all} \quad x, y \in \mathbb{M}.$$

Equivalently, a character on $\mathbb{M}$ is a monoid homomorphism from $(\mathbb{M}, \oplus)$ to $([0, 1], \cdot)$, where $\cdot$ denotes the usual product on $\mathbb{R}$. Let $\mathcal{B}(\mathbb{M})$ be the Borel σ-algebra of $(\mathbb{M}, \tau)$. We denote by $\mathrm{Ch}(\mathbb{M})$ the set of all $\mathcal{B}(\mathbb{M})$-measurable characters on $\mathbb{M}$. We say that a subset $\widetilde{\mathbb{M}}$ of $\mathrm{Ch}(\mathbb{M})$ is closed under pointwise multiplication if $\chi_1\chi_2 \in \widetilde{\mathbb{M}}$ for any $\chi_1, \chi_2 \in \widetilde{\mathbb{M}}$, where $(\chi_1\chi_2)(x) \coloneqq \chi_1(x)\chi_2(x)$, $x \in \mathbb{M}$. The set of all characters in $\mathrm{Ch}(\mathbb{M})$ that tend to zero at ∂_∞ will be denoted by $\mathrm{Ch}_0(\mathbb{M})$.

We are now ready to formulate our notion of L-topological monoids.

Definition 2.1 (L-Topological Monoid)**.** Let $\mathbb{M} \equiv (\mathbb{M}, \tau, \otimes)$ be an abelian topological monoid. We say that a pair $(\mathbb{M}, \widetilde{\mathbb{M}})$ is an L-topological monoid if $(\mathbb{M}, \tau)$ is a locally compact second countable Hausdorff space, and $\widetilde{\mathbb{M}}$ is a non-empty, countable and closed under pointwise multiplication subset of $\mathrm{Ch}(\mathbb{M})$ such that for any sequence $(x_n)_{n\geq 1} \subset \mathbb{M}$ the following conditions are satisfied:

(i) $\chi(x_n) \to 0$ for all $\chi \in \widetilde{\mathbb{M}}$ if and only if $x_n \to \partial_\infty$;
(ii) if $\chi(x_n) \to \psi(\chi)$ for every $\chi \in \widetilde{\mathbb{M}}$ and $\psi \not\equiv 0$, then there exists $x \in \mathbb{M}$ such that $x_n \to x$ and $\psi(\chi) = \chi(x)$ for each $\chi \in \widetilde{\mathbb{M}}$.

In this case, we say that $\widetilde{\mathbb{M}}$ is a determining class of characters on $\mathbb{M}$.

We present next some examples of L-topological monoids that appear in the literature.

Example 2.2. Let $\mathbb{R}_+ := [0, \infty)$ be endowed with the usual topology $\tau_{\mathbb{R}_+}$ and the usual sum $+$. Then $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, +)$ is an abelian topological monoid. For each $\lambda \in \mathbb{Q}^+ := \mathbb{Q} \cap (0, \infty)$, let $e_\lambda \colon \mathbb{R}_+ \to (0, 1]$ be the exponential function given by $e_\lambda(t) := \mathrm{e}^{-\lambda t}$ for each $t \in \mathbb{R}_+$. Set $\mathrm{Exp}(\mathbb{R}_+) := \{e_\lambda : \lambda \in \mathbb{Q}^+\}$. Notice that $\mathrm{Exp}(\mathbb{R}_+)$ is closed under pointwise multiplication. Let $(t_n)_{n\geq 1} \subset \mathbb{R}_+$ and fix $\lambda \in \mathbb{Q}^+$. Then $(e_\lambda(t_n))_{n\geq 1}$ converges to zero if and only if $t_n \to \infty$. On the other hand, $(e_\lambda(t_n))_{n\geq 1}$ converges to an element in $(0, \infty)$ if and only if $(t_n)_{n\geq 1}$ converges in $\mathbb{R}_+$. Thus, $(\mathbb{R}_+, \mathrm{Exp}(\mathbb{R}_+))$ is an L-topological monoid.

Example 2.3. For each $x, y \in \mathbb{R}_+$, we set $x \vee y := \max\{x, y\}$. Then $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, \vee)$ is an abelian topological monoid. For each $\lambda \in \mathbb{Q}^+$, we denote by $\mathbf{1}_{[0,\lambda]}$ the indicator function of the interval $[0, \lambda]$. Set $\mathrm{Ind}(\mathbb{R}_+) := \{\mathbf{1}_{[0,\lambda]} : \lambda \in \mathbb{Q}^+\}$. Notice that $\mathrm{Ind}(\mathbb{R}_+)$ is closed under pointwise multiplication. Let $(t_n)_{n\geq 1} \subset \mathbb{R}_+$. Observe that $\mathbf{1}_{[0,\lambda]}(t_n) \to 0$ for all $\lambda \in \mathbb{Q}^+$ if and only if $t_n \to \infty$. On the other hand, if $(t_n)_{n\geq 1}$ is bounded and $\mathbf{1}_{[0,\lambda]}(t_n)$ converges to $\psi(\lambda)$ for all $\lambda \in \mathbb{Q}^+$, then $(t_n)_{n\geq 1}$ converges to some t and $\psi(\lambda) = \mathbf{1}_{[0,\lambda]}(t)$. Thus, $(\mathbb{R}_+, \mathrm{Ind}(\mathbb{R}_+))$ is an L-topological monoid.

Example 2.4. For any integer $d \geq 1$, let $\mathbb{Z}^d_*$ be the space of proper subsets of $\mathbb{Z}^d$, equipped with the discrete topology $\tau_{\mathbb{Z}^d_*}$ and with the binary operation of the union $\cup$. Then $(\mathbb{Z}^d_*, \tau_{\mathbb{Z}^d_*}, \cup)$ is an abelian topological monoid. Here the neutral element $\boldsymbol{e}$ and the point at infinity ∂_∞ are the empty set $\emptyset$ and the set $\mathbb{Z}^d$, respectively. Recall that the discrete topology on $\mathbb{Z}^d_*$ coincides with the topology induced by the set-theoretic convergence, that is to say, a sequence $(J_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}_*$ converges to J if and only if $J = \limsup_{n\to\infty} J_n = \liminf_{n\to\infty} J_n$ where, as usual, $\limsup_{n\to\infty} J_n := \cap_{n\geq 1} \cup_{k\geq n} J_k$ and $\liminf_{n\to\infty} J_n := \cup_{n\geq 1} \cap_{k\geq n} J_k$. For a subset K of $\mathbb{Z}^d$, we write $K \subset\subset \mathbb{Z}^d$, if K has finite and non-zero cardinality. For each $K \subset\subset \mathbb{Z}^d$, we define the hitting functional $\chi_K : \mathbb{Z}^d_* \to \{0, 1\}$ by $\chi_K(J) := \mathbf{1}_{\{J\cap K=\emptyset\}}$. Set $\mathrm{Hit}(\mathbb{Z}^d) := \{\chi_K : K \subset\subset \mathbb{Z}^d\}$. Notice that $\mathrm{Hit}(\mathbb{Z}^d)$ is closed under pointwise multiplication. Let $(J_n)_{n\geq 1} \subset \mathbb{Z}^d_*$. Observe that $\chi_K(J_n) \to 0$ for all $K \subset\subset \mathbb{Z}^d$ if and only if $\lim_{n\to\infty} J_n = \mathbb{Z}^d$. On the other hand, $\limsup_{n\to\infty} J_n \neq \mathbb{Z}^d$ and $\chi_K(J_n)$ converges to $\psi(K)$ for all $K \subset\subset \mathbb{Z}^d$ if and only if $\lim_{n\to\infty} J_n = J$ for some $J \in \mathbb{Z}^d_*$ and $\psi(K) = \chi_K(J)$. Thus, $(\mathbb{Z}^d_*, \mathrm{Hit}(\mathbb{Z}^d))$ is an L-topological monoid.

The following topological monoid does not completely satisfy the conditions from Definition 2.1 since it is not locally compact, but still most of the results hold for this example.

Example 2.5. For each locally compact, separable metric space E, let $\mathbb{M}^E$ be the space of Radon measures on E with the topology $\tau_{\mathbb{M}^E}$ of vague convergence. Then $(\mathbb{M}^E, \tau_{\mathbb{M}^E}, +)$ is an abelian topological monoid, where $+$ denotes the sum of measures in $\mathbb{M}^E$. For every h in the space $C^+_c(E)$ of non-negative, real-valued functions with compact support on E, we define the Laplace functional $\chi_h : \mathbb{M}^E \to [0, 1]$ by $\chi_h(\eta) := \exp\{-\int_E h(s)\eta(\mathrm{d}s)\}$. For a countable dense subset $\mathcal{C}$ of $C^+_c(E)$, we set $\mathrm{Exp}(\mathbb{M}^E) := \{\chi_h : h \in \mathcal{C}\}$. It follows from the Riesz-Markov theorem that $(\mathbb{M}^E, \mathrm{Exp}(\mathbb{M}^E))$ satisfies (i) and (ii) from Definition 2.1.

In the following lemma we prove some useful properties of L-topological monoids.

Lemma 2.6. *Let* $(\mathbb{M}, \widetilde{\mathbb{M}})$ *be an* L*-topological monoid. The following statements hold:*

(i) $\widetilde{\mathbb{M}}$ *is non-vanishing everywhere, i.e., for any* $x \in \mathbb{M}$ *there exists a character* $\chi \in \widetilde{\mathbb{M}}$ *such that* $\chi(x) \neq 0$.

(ii) $\widetilde{\mathbb{M}}$ *separates points, i.e., for any two distinct elements* $x, y \in \mathbb{M}$ *there exists* $\chi \in \widetilde{\mathbb{M}}$ *such that* $\chi(x) \neq \chi(y)$.

Proof. Since $\mathbb{M}$ is locally compact, part (i) follows directly from Definition 2.1 (i). Let us now proceed to prove part (ii). Let $x, y \in \mathbb{M}$, and suppose that $\chi(x) = \chi(y)$ for all $\chi \in \mathbb{M}$. Let $(z_n)_{n\geq 1} \subset \mathbb{M}$ be the sequence given by

$$z_n := \begin{cases} x & \text{if } n \text{ is even,} \\ y & \text{if } n \text{ is odd.} \end{cases}$$

Notice that $(\chi(z_n))_{n\geq 1}$ converges to $\chi(x)$ for all $\chi \in \widetilde{\mathbb{M}}$. On the other hand, from part (i) we know that there exists $\tilde{\chi} \in \widetilde{\mathbb{M}}$ such that $\tilde{\chi}(x) \neq 0$. We deduce from Definition 2.1 (ii) that $(z_n)_{n\geq 1}$ converges to some element in $\mathbb{M}$. Hence, $x = y$, *i.e.*, $\widetilde{\mathbb{M}}$ separates points. □

Our next task is to construct elements of the form $\bigoplus_{n\in\mathbb{N}} x_n$ for any sequence $(x_n)_{n\geq 1}$ in an L-topological monoid $(\mathbb{M}, \widetilde{\mathbb{M}})$.

Theorem 2.7. *Let* $(\mathbb{M}, \widetilde{\mathbb{M}})$ *be an* L*-topological monoid. The following statements hold:*

(i) *(Divergence criterion) If* $(x_n)_{n\geq 1} \subset \mathbb{M}$ *is a sequence such that* $\prod_{n\geq 1} \chi(x_n) = 0$ *for all characters* $\chi \in \widetilde{\mathbb{M}}$, *then* $\bigoplus_{i=1}^n x_i \to \partial_\infty$, *as* $n \to \infty$.
(ii) *(Convergence criterion) If* $(x_n)_{n\geq 1} \subset \mathbb{M}$ *is a sequence such that* $\prod_{n\geq 1} \chi(x_n) > 0$ *for some character* $\chi \in \widetilde{\mathbb{M}}$, *then there exists an element in* $\mathbb{M}$ *denoted by* $\bigoplus_{i=1}^\infty x_i$ *such that*

$$\lim_{n\to\infty} \bigoplus_{i=1}^n x_i = \bigoplus_{i=1}^\infty x_i.$$

Furthermore, the definition of $\bigoplus_{i=1}^\infty x_i$ *is independent of the order of the* $\oplus$*-addends, so we can unambiguously denote it as* $\bigoplus_{n\in\mathbb{N}} x_n$.
(iii) *For any sequence* $(x_n)_{n\geq 1} \subset \mathbb{M}$,

$$\chi\left(\bigoplus_{i=1}^\infty x_i\right) = \prod_{i=1}^\infty \chi(x_i), \quad \chi \in \widetilde{\mathbb{M}},$$

where we set $\chi(\partial_\infty) := 0$.
(iv) *For any partition* $\{\mathcal{P}, \mathcal{Q}\}$ *of* $\mathbb{N}$, *we have* $\left(\bigoplus_{n\in\mathcal{Q}} x_n\right) \oplus \left(\bigoplus_{n\in\mathcal{P}} x_n\right) = \left(\bigoplus_{n\in\mathbb{N}} x_n\right)$.

Proof. Observe that the limit

$$\prod_{n\geq 1} \chi(x_n) := \lim_{m\to\infty} \prod_{n=1}^m \chi(x_n)$$

exists because the sequence $\left(\prod_{n=1}^m \chi(x_n)\right)_{m\geq 1}$ is monotone decreasing for any $\chi \in \widetilde{\mathbb{M}}$. Moreover, if $(\tilde{x}_n)_{n\geq 1}$ is a reordering of $(x_n)_{n\geq 1}$, then

$$\prod_{n\geq 1} \chi(\tilde{x}_n) = \prod_{n\geq 1} \chi(x_n) \quad \text{for any} \quad \chi \in \widetilde{\mathbb{M}}. \tag{2.1}$$

For each $\chi \in \widetilde{\mathbb{M}}$, let

$$\psi(\chi) \coloneqq \lim_{n\to\infty} \chi\left(\bigoplus_{i=1}^{n} x_i\right) = \lim_{m\to\infty} \prod_{n=1}^{m} \chi(x_n).$$

If $\psi(\chi) = 0$ for all $\chi \in \widetilde{\mathbb{M}}$, then it follows from Definition 2.1 (i) that $\bigoplus_{i=1}^{n} x_i \to \partial_\infty$. This proves the divergence criterion in (i). On the other hand, if $\psi \not\equiv 0$, then we deduce from Definition 2.1 (ii) that $(\bigoplus_{i=1}^{n} x_i)_{n\geq 1}$ converges to some element in $\mathbb{M}$. We denote such element by $\bigoplus_{i=1}^{\infty} x_i$. Let $(\tilde{x}_n)_{n\geq 1}$ be a reordering of $(x_n)_{n\geq 1}$. By using (2.1) we obtain that $(\bigoplus_{i=1}^{n} \tilde{x}_i)_{n\geq 1}$ converges to the element $\bigoplus_{i=1}^{\infty} \tilde{x}_i$. Moreover, from Definition 2.1 (ii) and (2.1) we get that

$$\chi\left(\bigoplus_{i=1}^{\infty} x_i\right) = \chi\left(\bigoplus_{i=1}^{\infty} \tilde{x}_i\right) \quad \text{for all} \quad \chi \in \widetilde{\mathbb{M}}.$$

Hence, from Lemma 2.6 (ii) we conclude that $\bigoplus_{i=1}^{\infty} x_i = \bigoplus_{i=1}^{\infty} \tilde{x}_i$. This proves the convergence criterion in (ii). Part (iii) follows readily from parts (i) and (ii).

Let $\mathcal{P} \subset \mathbb{N}$, and let $(x_n)_{n\in\mathcal{P}} \subset \mathbb{M}$. From the preceding parts, we know that $\bigoplus_{n=1}^{|\mathcal{P}|} \tilde{x}_n = \bigoplus_{n=1}^{|\mathcal{P}|} \hat{x}_n$ for any orderings $(\tilde{x}_n)_{n=1}^{|\mathcal{P}|}$ and $(\hat{x}_n)_{n=1}^{|\mathcal{P}|}$ of $(x_n)_{n\in\mathcal{P}}$. Thus, we can define unambiguously the element $\bigoplus_{n\in\mathcal{P}} x_n \coloneqq \bigoplus_{n=1}^{|\mathcal{P}|} \tilde{x}_n$. Suppose now that $\{\mathcal{P}, \mathcal{Q}\}$ is a partition of $\mathbb{N}$, and let $(x_n)_{n\geq 1} \subset \mathbb{M}$. Let $(\tilde{x}_n)_{n=1}^{|\mathcal{P}|}$ and $(\hat{x}_n)_{n=1}^{|\mathcal{Q}|}$ be orderings of $(x_n)_{n\in\mathcal{P}}$ and $(x_n)_{n\in\mathcal{Q}}$, respectively. By using Definition 2.1 and (2.1) we obtain

$$\begin{aligned}
\chi\left(\left(\bigoplus_{n\in\mathcal{P}} x_n\right) \oplus \left(\bigoplus_{n\in\mathcal{Q}} x_n\right)\right) &= \chi\left(\bigoplus_{n\in\mathcal{P}} x_n\right) \chi\left(\bigoplus_{n\in\mathcal{Q}} x_n\right) \\
&= \left(\prod_{n=1}^{|\mathcal{P}|} \chi(\tilde{x}_n)\right) \left(\prod_{n=1}^{|\mathcal{Q}|} \chi(\hat{x}_n)\right) \\
&= \prod_{n=1}^{\infty} \chi(x_n) \\
&= \chi\left(\bigoplus_{n\in\mathbb{N}} x_n\right)
\end{aligned}$$

for all $\chi \in \widetilde{\mathbb{M}}$. Therefore, it follows from Lemma 2.6 (ii) that $\left(\bigoplus_{n\in\mathcal{Q}} x_n\right) \oplus \left(\bigoplus_{n\in\mathcal{P}} x_n\right) = \left(\bigoplus_{n\in\mathbb{N}} x_n\right)$. This proves part (iv). □

Let $(\mathbb{M}, \widetilde{\mathbb{M}})$ be an L-topological monoid. We can define a natural metric $\rho_{\widetilde{\mathbb{M}}}$ on $\mathbb{M}$ given by

$$\rho_{\widetilde{\mathbb{M}}}(x, y) = \sum_{n\geq 1} \frac{|\chi_n(x) - \chi_n(y)|}{2^n}, \quad x, y \in \mathbb{M}, \tag{2.2}$$

where $(\chi_n)_{n\geq 1}$ is some ordering of $\widetilde{\mathbb{M}}$.

Denote as $\tau_{\widetilde{\mathbb{M}}}$ the topology on $\mathbb{M}$ induced by the metric $\rho_{\widetilde{\mathbb{M}}}$, and let $\widetilde{C}(\mathbb{M})$ be the space of continuous functions from $\mathbb{M}$ into $\mathbb{R}$ with respect to the topology $\tau_{\widetilde{\mathbb{M}}}$. Setting $\chi_n(\partial_\infty) = 0$ for $n \geq 1$, the formula (2.2) makes sense for any $x, y \in \mathbb{M}^*$ and defines a metric for $\mathbb{M}^*$. By a slight abuse of notation we also denote this metric as $\rho_{\widetilde{\mathbb{M}}}$. Let $\mathcal{B}(\tau_{\mathbb{M}})$ be the Borel σ-algebra of $(\mathbb{M}, \tau_{\widetilde{\mathbb{M}}})$.

Proposition 2.8. *The followings statements hold:*

(i) If $\sigma\big(\widetilde{\mathbb{M}}\big)$ *is the* σ*-algebra generated by the characters in* $\widetilde{\mathbb{M}}$*, then*

$$\mathcal{B}(\mathbb{M}) = \sigma(\widetilde{\mathbb{M}}) = \mathcal{B}(\tau_{\widetilde{\mathbb{M}}}).$$

(ii) $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$ *is a topological monoid, and* $(\mathbb{M}^*, \rho_{\widetilde{\mathbb{M}}})$ *is a compact space.*

Proof. We first proceed to prove (i). Notice that $\mathcal{B}(\tau_{\widetilde{\mathbb{M}}}) \subset \sigma(\widetilde{\mathbb{M}}) \subset \mathcal{B}(\mathbb{M})$. On the other hand, since $(\mathbb{M}, \tau)$ is in particular a Polish space and $\widetilde{\mathbb{M}}$ separates points, we can apply Lemma 3 in [14] to obtain that the Borel σ-algebras of $(\mathbb{M}, \tau)$ and $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$ coincide. Hence, part (i) follows.

It only remains to prove the statement in (ii). Suppose that $x_m \to x$, $y_m \to y$, as $m \to \infty$, with respect to the metric $\rho_{\widetilde{\mathbb{M}}}$. This means that $\chi_n(x_m) \to \chi_n(x)$, $\chi_n(y_m) \to \chi_n(y)$, as $m \to \infty$, for $n \geq 1$. Then $\chi_n(x_m \oplus y_m) \to \chi_n(x \oplus y)$, as $m \to \infty$, for $n \geq 1$. Hence $x_m \oplus y_m \to x \oplus y$, as $m \to \infty$, with respect to the metric $\rho_{\widetilde{\mathbb{M}}}$, so that $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$ is a topological monoid.

Let us prove that $(\mathbb{M}^*, \rho_{\widetilde{\mathbb{M}}})$ is a compact space. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{M} \cup \partial_\infty$ and consider the array $(\chi_n(x_m))_{n\geq 1, m\geq 1}$. By Cantor's diagonal method there exists an increasing sequence $(m(k))_{k\geq 1}$ of integers such that $\lim_{k\to\infty} \chi_n(x_{m(k)})$ exists for $n \geq 1$. It follows from Definition 2.1 that there exists $x \in \mathbb{M}^*$ such that

$$\lim_{k\to\infty} \chi_n(x_{m(k)}) = \chi_n(x), \quad n \geq 1.$$

□

2.1. Laplace transform on L-topological monoids

We gather here some of the fundamental properties of the Laplace transform on an L-topological monoid $(\mathbb{M}, \widetilde{\mathbb{M}})$, following nearly the approach in [12], IX, §5, No. 7. For each finite measure μ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$, we define the Laplace transform of μ as the function $\mathscr{L}\mu$ from $\widetilde{\mathbb{M}}$ into $[0, \infty)$ given by

$$\mathscr{L}\mu(\chi) \coloneqq \int_{\mathbb{M}} \chi(s)\, \mathrm{d}\mu(s).$$

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{M}$-valued random element X on it, we define the Laplace transform of X, denoted by $\mathscr{L}_X$, as the Laplace transform of the probability measure $\mathbb{P}(X \in \cdot)$. Observe that $\mathscr{L}_X(\chi) = \mathbb{E}[\chi(X)]$ for every $\chi \in \widetilde{\mathbb{M}}$, where $\mathbb{E}$ is the expectation operator associated to $\mathbb{P}$.

A *full submonoid* $\mathcal{S}$ is a subset of continuous characters such that $\mathcal{S} \cap \mathrm{Ch}_0(\mathbb{M})$ separates points, $\mathcal{S} \cap \mathrm{Ch}_0(\mathbb{M})$ is non-vanishing everywhere, and $\chi_{\mathbf{1}} \in \mathcal{S}$, where $\chi_{\mathbf{1}}$ is the constant character equals to 1.

Theorem 2.9. *Let* $\mathcal{S}$ *be a full submonoid of an abelian topological monoid* $(\mathbb{M}, \tau, \oplus)$*. If* μ_1 *and* μ_2 *are two finite measures on* $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$*, such that* $\mathscr{L}\mu_1$ *and* $\mathscr{L}\mu_2$ *have the same restriction to* $\mathcal{S} \cap \mathrm{Ch}_0(\mathbb{M})$*, then* $\mu_1 = \mu_2$ *on* $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$*.*

Corollary 2.10. *Let* $(\mathbb{M}, \widetilde{\mathbb{M}})$ *be an* L*-topological monoid. If* μ_1 *and* μ_2 *are two finite measures on* $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ *such that* $\mathscr{L}\mu_1$ *and* $\mathscr{L}\mu_2$ *have the same restriction to* $\widetilde{\mathbb{M}}$*, then* $\mu_1 = \mu_2$ *on* $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$*.*

Proof. By part (i) in Proposition 2.8, $\mathcal{B}(\mathbb{M}) = \mathcal{B}(\tau_{\widetilde{\mathbb{M}}})$. Therefore, μ_1 and μ_2 are two finite measures on $(\mathbb{M}, \mathcal{B}(\tau_{\widetilde{\mathbb{M}}}))$. By part (ii) from Proposition 2.8, $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$ is a topological monoid. Moreover, on $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$ the set of characters $\mathcal{S} \coloneqq \widetilde{\mathbb{M}} \cup \{\chi_{\mathbf{1}}\}$ is a full submonoid; indeed, by parts (i) and (ii) from Lemma 2.6, $\mathcal{S} \cap \mathrm{Ch}_0(\mathbb{M}) = \widetilde{\mathbb{M}}$ separates points and is non-vanishing everywhere, and clearly the characters in $\widetilde{\mathbb{M}}$ are continuous in $(\mathbb{M}, \rho_{\widetilde{\mathbb{M}}})$. The result follows immediately from the previous theorem. □

Corollary 2.11. *Let μ_1 and μ_2 be two measures on* $(\mathbb{M}\backslash\{\boldsymbol{e}\}, \mathcal{B}(\mathbb{M}\backslash\{\boldsymbol{e}\}))$ *such that*

$$\int_{\mathbb{M}\backslash\{\boldsymbol{e}\}} \big(1-\chi(x)\big)\mu_1(\mathrm{d}x) = \int_{\mathbb{M}\backslash\{\boldsymbol{e}\}} \big(1-\chi(x)\big)\mu_2(\mathrm{d}x) < +\infty, \quad \chi \in \widetilde{\mathbb{M}}, \tag{2.3}$$

then $\mu_1 = \mu_2$ *on* $(\mathbb{M}\backslash\{\boldsymbol{e}\}, \mathcal{B}(\mathbb{M}\backslash\{\boldsymbol{e}\}))$.

Proof. The arguments are adapted from those on Theorem 1.23 in [15]. Let us extend μ_1 and μ_2 by setting $\mu_1(\{\boldsymbol{e}\}) = \mu_2(\{\boldsymbol{e}\}) = 0$. For fixed $\chi' \in \widetilde{\mathbb{M}}$, consider the measures ν_i for $i = 1, 2$, defined by $\nu_i(\mathrm{d}x) = \big(1 - \chi'(x)\big)\mu_i(\mathrm{d}x)$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$. Since $\widetilde{\mathbb{M}}$ is closed under pointwise multiplication, then

$$\int_{\mathbb{M}} \big(1-(\chi\cdot\chi')\big)(x)\mu_1(\mathrm{d}x) = \int_{\mathbb{M}} \big(1-(\chi\cdot\chi')\big)(x)\mu_2(\mathrm{d}x) < +\infty, \quad \chi \in \widetilde{\mathbb{M}}.$$

By subtracting (2.3) from this last equation, we get that $\mathscr{L}\nu_1$ and $\mathscr{L}\nu_2$ coincide on $\mathbb{M}\cup\{\chi_{\mathbf{1}}\}$ and, by the preceding corollary, $\nu_1 = \nu_2$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ for every $\chi' \in \widetilde{\mathbb{M}}$. Therefore, $\mu_1 = \mu_2$ on

$$\bigcup_{\chi' \in \widetilde{\mathbb{M}}} \{x \in \mathbb{M} : 1 - \chi'(x) > 0\} = \mathbb{M}\backslash\{\boldsymbol{e}\},$$

where the last equality is due to the fact that $\widetilde{\mathbb{M}}$ is non-vanishing everywhere. □

2.2. Subordinators on L-topological monoids

Throughout this section, we fix an L-topological monoid $(\mathbb{M}, \widetilde{\mathbb{M}})$ and a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The stochastic processes below will be $\mathbb{M}$-valued and defined on $(\Omega, \mathscr{F}, \mathbb{P})$.

For now, we give below a definition of subordinator, which is enough for our immediate purposes.

Definition 2.12 (Subordinator)**.** We say that $(X_t)_{t\geq 0}$ is a subordinator if

1. $X_0 = \boldsymbol{e}$.
2. $(X_t)_{t\geq 0}$ is right-continuous.
3. Given $t \geq 0$, for every $s \geq 0$, there exists a random element $X_s^{(t)}$, which could be defined on an enlargement of $(\Omega, \mathscr{F}, \mathbb{P})$, such that $X_{s+t} \overset{d}{=} X_s^{(t)} \oplus X_t$ where $X_s^{(t)} \overset{d}{=} X_s$ and $X_s^{(t)}$ is independent of X_t.

We are interested in the Laplace transform of these subordinators, *i.e.*

$$\mathscr{L}_{X_t}(\chi) = \mathbb{E}[\chi(X_t)], \quad \chi \in \widetilde{\mathbb{M}},\ t \geq 0.$$

From (3) in Definition 2.12 and since $\chi(x) \in [0, 1]$, $x \in \mathbb{M}$, it follows that $t \mapsto \mathscr{L}_{X_t}(\chi)$ is a non-increasing function that satisfies the Cauchy equation

$$\mathscr{L}_{X_{s+t}}(\chi) = \mathscr{L}_{X_s}(\chi)\cdot\mathscr{L}_{X_t}(\chi), \quad s, t \geq 0.$$

Therefore, there exists a mapping Ψ from $\widetilde{\mathbb{M}}$ into $[0, +\infty]$ such that

$$\mathscr{L}_{X_t}(\chi) = \exp\{-t\Psi(\chi)\}, \quad \chi \in \widetilde{\mathbb{M}},\ t \geq 0,$$

which gives rise to the *Laplace exponent* Ψ *of* $(X_t)_{t\geq 0}$.

For any integer $n \geq 1$, let $\big(X_t^{(1)}\big)_{t\geq 0}, \cdots, \big(X_t^{(n)}\big)_{t\geq 0}$ be independent subordinators having the same law as $(X_t)_{t\geq 0}$, which could be defined on an enlargement of $(\Omega, \mathscr{F}, \mathbb{P})$. We then have for $0 \leq t_1 < \cdots < t_n$,

$$\begin{aligned}\mathbb{E}[\chi_1(X_{t_1})\cdots\chi_n(X_{t_n})] &= \mathbb{E}\left[\chi_1(X_{t_1}^{(1)})\cdot\chi_2\left(X_{t_1}^{(1)}\oplus X_{t_2-t_1}^{(2)}\right)\cdots\chi_n\left(X_{t_1}^{(1)}\oplus\cdots\oplus X_{t_n-t_{n-1}}^{(n)}\right)\right]\\ &= \mathrm{e}^{-t_1\Psi(\chi_1\cdots\chi_n)}\mathrm{e}^{-(t_2-t_1)\Psi(\chi_2\cdots\chi_n)}\cdots\mathrm{e}^{-(t_n-t_{n-1})\Psi(\chi_n)}.\end{aligned} \tag{2.4}$$

In particular, when $\chi_1 = \cdots = \chi_n = \chi$,

$$\mathbb{E}[\chi(X_{t_1})\cdots\chi(X_{t_n})] = \mathrm{e}^{-t_1\Psi(\chi^n)}\mathrm{e}^{-(t_2-t_1)\Psi(\chi^{n-1})}\cdots\mathrm{e}^{-(t_n-t_{n-1})\Psi(\chi)}, \tag{2.5}$$

where χ^n is the product of χ with itself n-times.

Remark 2.13. Reciprocally, if $(X_t)_{t\geq 0}$ is a stochastic process which satisfies (2.4) for every $\chi \in \widetilde{\mathbb{M}}$ and $n \in \mathbb{N}$, then it is a subordinator. Indeed, the result from the previous subsection ensures that $(X_t)_{t\geq 0}$ satisfies conditions (1) and (3) in Definition 2.12. Since $Y_t := \chi(X_t)\exp\{t\Psi(\chi)\}$, $t \geq 0$, is a martingale for every $\chi \in \widetilde{\mathbb{M}}$, by the regularization theorem, $t \mapsto \chi(X_t)$ has a càdlàg modification.

3. Proof of Theorem 1.1

Let $(\mathbb{M}, \widetilde{\mathbb{M}})$ be an L-topological monoid. We will denote as $\mathrm{d}t$ the Lebesgue measure on $(0,\infty)$ and by $\mu\otimes\nu$ the product measure of two measures μ and ν.

Proof of Theorem 1.1. Let us first prove that (1.3) holds when Π is a finite measure. Define $\Psi(\chi) := \int_{\mathbb{M}}\big(1 - \chi(x)\big)\,\Pi(\mathrm{d}x)$. Since Π is a finite measure on $\mathbb{M}$, we can write $\mathcal{N} = \sum_{i=1}^{\infty}\delta_{(t_i,x_i)}$, where $(t_i)_{i=1}^{\infty}$ is an increasing sequence. Now we consider, for every $t > 0$, the Laplace transform of $\bigoplus_{t_i\leq t,\, i=1,\cdots,n} x_i$. By the memoryless property of marked Poisson processes [16], Theorem 7.4, we get

$$\begin{aligned}&\mathbb{E}\left[\chi\left(\bigoplus_{t_i\leq t,\, i=1,\cdots,n} x_i\right)\right]\\ &= \mathbb{E}\left[\prod_{i=1}^{n}\Big(1-\big(1-\chi(x_i)\big)1_{\{t_i\leq t\}}\Big)\right]\\ &= \int_{((0,\infty)\times\mathbb{M})^n}\mathbf{1}_{\{t_1<\cdots<t_n\}}\prod_{i=1}^{n}\Big(1-\big(1-\chi(y_i)\big)1_{\{t_i\leq t\}}\Big)\mathrm{e}^{-t_n\Pi(\mathbb{M})}\mathrm{d}t_1\Pi(\mathrm{d}y_1)\cdots\mathrm{d}t_n\Pi(\mathrm{d}y_n)\\ &= \sum_{i=0}^{n-1}\int_{((0,\infty)\times\mathbb{M})^n}\mathbf{1}_{\{t_1<\cdots<t_i<t<t_{i+1}<\cdots<t_n\}}\prod_{j=1}^{i}\big(1-(1-\chi(y_j))\big)\mathrm{e}^{-t_n\Pi(\mathbb{M})}\mathrm{d}t_1\Pi(\mathrm{d}y_1)\cdots\mathrm{d}t_n\Pi(\mathrm{d}y_n)\\ &\quad+\int_{((0,\infty)\times\mathbb{M})^n}\mathbf{1}_{\{t_1<\cdots<t_n<t\}}\prod_{i=1}^{n}\Big(1-\big(1-\chi(y_i)\big)\Big)\mathrm{e}^{-t_n\Pi(\mathbb{M})}\mathrm{d}t_1\Pi(\mathrm{d}y_1)\cdots\mathrm{d}t_n\Pi(\mathrm{d}y_n)\\ &= \sum_{i=0}^{n-1}\frac{\big(\Pi(\mathbb{M})-\Psi(\chi)\big)^i}{\Pi(\mathbb{M})^{i-n}}\int_{(0,+\infty)^n}\mathbf{1}_{\{t_1<\cdots<t_i<t<t_{i+1}<\cdots<t_n\}}\mathrm{e}^{-t_n\Pi(\mathbb{M})}\mathrm{d}t_1\cdots\mathrm{d}t_n\\ &\quad+\big(\Pi(\mathbb{M})-\Psi(\chi)\big)^n\int_{(0,+\infty)^n}\mathbf{1}_{\{t_1<\cdots<t_n<t\}}\mathrm{e}^{-t_n\Pi(\mathbb{M})}\mathrm{d}t_1\cdots\mathrm{d}t_n.\end{aligned}$$

The first term in the above line equals

$$\sum_{i=0}^{n-1} \frac{\big(\Pi(\mathbb{M}) - \Psi(\chi)\big)^i}{\Pi(\mathbb{M})^{i-n}} \int_0^t \mathrm{d}t_i \cdots \int_0^{t_2} \mathrm{d}t_1 \int_t^\infty \mathrm{d}t_{i+1} \int_{t_{i+1}}^\infty \mathrm{d}t_{i+2} \cdots \int_{t_{n-2}}^\infty \mathrm{d}t_{n-1} \int_{t_{n-1}}^\infty \mathrm{d}t_n \mathrm{e}^{-t_n \Pi(\mathbb{M})}$$
$$= \sum_{i=0}^{n-1} \big(\Pi(\mathbb{M}) - \Psi(\chi)\big)^i \int_0^t \mathrm{d}t_i \cdots \int_0^{t_2} \mathrm{d}t_1 \, \mathrm{e}^{-t\Pi(\mathbb{M})}$$
$$= \mathrm{e}^{-t\Pi(\mathbb{M})} \sum_{i=0}^{n-1} \frac{\Big(t\big(\Pi(\mathbb{M}) - \Psi(\chi)\big)\Big)^i}{i!}.$$

Therefore,

$$\mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t,\, i=1,\cdots,n} x_i\right)\right] = \mathrm{e}^{-t\Pi(\mathbb{M})} \sum_{i=0}^{n-1} \frac{\Big(t\big(\Pi(\mathbb{M}) - \Psi(\chi)\big)\Big)^i}{i!} + \Delta_n, \quad n \ge 1, \tag{3.1}$$

where $\Delta_n \coloneqq \big(\Pi(\mathbb{M}) - \Psi(\chi)\big)^n \int 1_{\{t_1 < \cdots < t_n < t\}} \mathrm{e}^{-t_n \Pi(\mathbb{M})} \mathrm{d}t_1 \cdots \mathrm{d}t_n$. Since Π is a finite measure on $\mathbb{M}$, it follows that the (random) set $\{x_i \in \mathbb{M} : t_i \le t\}$ is finite a.s. for every $t > 0$, and therefore

$$\lim_{n\to\infty} \mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t,\, i=1,\cdots,n} x_i\right)\right] = \mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t} x_i\right)\right]. \tag{3.2}$$

Letting n tend to infinity in (3.1) and using, on one hand $|\Delta_n| \le |\Pi(\mathbb{M}) - \Psi(\chi)|^n t^n/n!$ for every $n \ge 1$, and on the other hand (3.2), we get

$$\mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t} x_i\right)\right] = \exp\{-t\Psi(\chi)\} = \exp\left\{-t \int_{\mathbb{M}} \big(1 - \chi(x)\big)\, \Pi(\mathrm{d}x)\right\}, \qquad \chi \in \widetilde{\mathbb{M}},$$

for every finite measure Π on $\mathbb{M}$.

Let now Π be a σ-finite measure on $\mathbb{M}$. We can then choose an increasing sequence $(K_n)_{n\ge1}$ of Borel sets of $\mathbb{M}$ such that $\cup_{n\ge1} K_n = \mathbb{M}$ and $\Pi(K_n) < \infty$ for $n \ge 1$. Denote as $\mathcal{N}^{(n)}$ the Poisson point process $\mathcal{N}$ restricted to $(0,\infty) \times K_n \subseteq (0,\infty) \times \mathbb{M}$. By applying the result just proved to $\mathcal{N}^{(n)}$, we get

$$\mathbb{E}\left[\chi\left(\bigoplus_{t_i \le t} x_i^{(n)}\right)\right] = \exp\left\{-t \int_{K_n} \big(1 - \chi(x)\big)\, \Pi(\mathrm{d}x)\right\}, \quad n \ge 1, \tag{3.3}$$

where $\left\{x_i^{(n)}\right\}$ are the marks of $\mathcal{N} = \sum_i \delta_{(t_i, x_i)}$ that belong to K_n. On the other hand, since $\cup_{n\ge1} K_n = \mathbb{M}$, we have $\bigoplus_{t_i \le t} x_i = \lim_{n\to\infty} \bigoplus_{t_i \le t} x_i^{(n)}$, and moreover, $\lim_{n\to\infty} \chi\left(\bigoplus_{t_i \le t} x_i^{(n)}\right) = \chi\left(\bigoplus_{t_i \le t} x_i\right)$. Finally, letting n tend to infinity in (3.3), we obtain the desired result. □

Theorem 3.1. *Consider $\mathcal{N} = \sum_i \delta_{(t_i, x_i)}$ a Poisson point process on $(0,\infty) \times \mathbb{M}$ with intensity $\mathrm{d}t \times \Pi$, where Π is a σ-finite measure such that $\int_{\mathbb{M}} \big(1 - \chi(x)\big)\Pi(\mathrm{d}x) < +\infty$ for every χ. Then, the process defined by $X_t \coloneqq \bigoplus_{t_i \le t} x_i$ is a subordinator in the* L*-topological monoid $\mathbb{M}$.*

Proof. First observe that, for $s, t \geq 0$,

$$X_{s+t} = \left(\bigoplus_{t_i \leq t} x_i \right) \oplus \left(\bigoplus_{t < t_i \leq s+t} x_i \right) =: X_t \oplus X_s^{(t)},$$

X_t is independent of $X_s^{(t)}$, and $X_s \stackrel{d}{=} X_s^{(t)}$. Recall that the process $Y_t := \chi(X_t) \exp\{t\Psi(\chi)\},\ t \geq 0$, is a martingale for every $\chi \in \widetilde{\mathbb{M}}$. By the regularization theorem, for every $\chi \in \widetilde{\mathbb{M}}$, $t \mapsto \chi(X_t)$ has a càdlàg modification. Then $\lim_{t \downarrow t_0} \chi(X_t) = \chi(X_{t_0})$ for every $\chi \in \widetilde{\mathbb{M}}$. This implies $\lim_{t \downarrow t_0} X_t = X_{t_0}$ since $\mathbb{M}$ is an L-topological monoid. ☐

4. Examples

We provide here some illustrative examples of subordinators on L-topological monoids that have appeared in the literature and heavily motivated this work.

4.1. Additive subordinators

Classical (or additive) subordinators (see [1], Chap. 3) are the subordinators in the L-topological monoid $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, +)$. From Theorem 1.1 it follows that the additive subordinators have the classical Lévy–Itô decomposition (1.1).

4.2. Extremal processes

Let us recall the classical definition of an extremal process that can be found in Chapter 4, Section 3 in Resnick [2]. An F-extremal process $(M_t)_{t \geq 0}$ on $\mathbb{R}_+ := [0, \infty)$ is a stochastic process that satisfies

$$\begin{aligned} \mathbb{P}(M_{t_1} \leq x_1, M_{t_2} \leq x_2, \ldots, M_{t_m} \leq x_m) \\ = F^{t_1} \left(\bigwedge_{i=1}^m x_i \right) F^{t_2 - t_1} \left(\bigwedge_{i=2}^m x_i \right) \cdots F^{t_m - t_{m-1}} (x_m), \end{aligned} \tag{4.1}$$

where $F^t(x) := \big(F(x)\big)^t$ for some distribution function F concentrated in $[0, \infty)$, that is, $F : [0, \infty) \to [0, 1]$ is right-continuous and non-decreasing, $\lim_{x \to +\infty} F(x) = 1$ and $0 = \inf\{x > 0 : F(x) > 0\}$.

The extremal processes turn out to be subordinators on the L-topological monoid $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, \vee)$. Indeed, let us observe that equation (4.1) is a particular case of the identity (2.5), with $\chi_i = \mathbf{1}_{[0,x_i]}$ and $\Psi(\chi_i) = \Psi(\mathbf{1}_{[0,x_i]}) = -\log(F(x_i))$, and by Remark 2.13 every F-extremal process is a subordinator on the L-topological monoid $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, \vee)$. Moreover, for each distribution function F concentrated on $\mathbb{R}_+$, we can associate a σ-finite measure π on $(0, +\infty)$ such that $\pi\big((a, b]\big) = Q(b) - Q(a)$, where $Q(x) = -\log(F(x))$, $x > 0$. By Theorem 3.1, it follows that if $\mathcal{N} = \sum_i \delta_{(t_i, x_i)}$ is a Poisson point process on $(0, \infty) \times \mathbb{R}_+$ with intensity $\mathrm{d}t \otimes \pi$, then $M_t := \bigvee_{t_i \leq t} x_i$ is a subordinator on the L-topological monoid $(\mathbb{R}_+, \tau_{\mathbb{R}_+}, \vee)$ such that

$$\mathbb{P}[M_t \leq x] = \mathbb{E}[\mathbf{1}_{[0,x_i]}(M_t)] = \exp\left\{ -t \int_{(0,+\infty)} \big(1 - \mathbf{1}_{[0,x_i]}(y)\big)\, \pi(\mathrm{d}y) \right\} = F^t(x),$$

that is, $(M_t)_{t \geq 0}$ is an F-extremal process.

4.3. Random interlacements model

In his seminal paper [4], Sznitman introduced the model of random interlacements $(\mathscr{I}_t)_{t \geq 0}$ on $\mathbb{Z}^d$ for $d \geq 3$ (abbreviated RI_d) to study the local structure of a simple random walk covering a fixed proportion of the volume of a torus $(\mathbb{Z}/N\mathbb{Z})^d$. This $\mathbb{Z}^d_*$-valued stochastic process is characterized by the formula

$$\mathbb{P}[\mathscr{J}_t \cap K = \emptyset] = \mathrm{e}^{-t\,\mathrm{cap}(K)}, \quad K \subset\subset \mathbb{Z}^d,$$

where cap denotes the capacity of the symmetric random walk on $\mathbb{Z}^d$. The existence of a probability measure $\mathbb{P}$ satisfying the preceding relation follows immediately from the Choquet-Kendall-Matheron theorem (see Thm. 1.34 in [17]) and an extension of Proposition 6.5.3 from [18]. It follows from Theorem 2.3.3 in [19] that there exists a σ-finite measure Π such that

$$\mathrm{cap}(K) = \int_{\mathbb{Z}^d_*\setminus\{\emptyset\}} \big(1 - \chi_K(x)\big)\Pi(\mathrm{d}x), \quad K \subset\subset \mathbb{Z}^d.$$

As a consequence of Theorem 1.1, the random interlacements model has the Lévy–Itô decomposition $\mathscr{J}_t = \bigcup_{t_i \le t} J_i$, where $\mathcal{N} = \sum_i \delta_{(t_i, J_i)}$ is a Poisson point process on $(0, +\infty) \times \mathbb{Z}^d_*$ with intensity measure $\mathrm{d}t \otimes \Pi$. Thus, the random interlacements model is a subordinator on the L-topological monoid $(\mathbb{Z}^d_*, \mathrm{Hit}(\mathbb{Z}^d))$ with Laplace exponent given by $\Psi(\chi_K) = \mathrm{cap}(K),\ K \subset\subset \mathbb{Z}^d$.

4.4. Flow of squared Bessel processes of dimension zero

Let $((X_t(x))_{t\ge0}, x \in \mathbb{R}_+)$ be a flow of squared Bessel processes of dimension zero (abbreviated BESQ_0 in the sequel), this is, $(X_t(x))_{t\ge0}$ satisfies $X_t(x) = x + 2\int_0^t \sqrt{|X_s(x)|}\mathrm{d}B_s,\ t \ge 0$, for all $x \in \mathbb{R}_+$, where $(B_t)_{t\ge0}$ is a Brownian motion. We can define an $\mathbb{M}^{\mathbb{R}_+}$-valued process $(M_x)_{x\ge0}$ by setting $M_x(\mathrm{d}s) = X_s(x)\mathrm{d}s$. Then

$$\mathbb{E}[\chi_h(M_x)] = \exp\left\{-x\int \big(1 - \chi_h(m)\big)\eta(\mathrm{d}m)\right\},$$

where η is a measure concentrated on $C_c^+(\mathbb{R}_+) \subset \mathbb{M}^{\mathbb{R}_+}$. Here, we have identified each function $f \in C_c^+(\mathbb{R}_+)$ with the Radon measure $f(s)\mathrm{d}s \in \mathbb{M}^{\mathbb{R}_+}$. Let $(w_t)_{t\ge0}$ be the coordinate process on $C_c^+(\mathbb{R}_+)$. The measure η is characterized by the following two properties: i) $\eta(w_0 \ne 0) = \eta(w_t = 0 \text{ for all } t \ge 0) = 0$, and ii) under η the process $(w_t)_{t>0}$ is Markovian on $(0, +\infty)$ with the transition probabilities $P_t(x, \mathrm{d}y)$ of the BESQ_0 restricted to $(0, +\infty)$ and with entrance law on $(0, +\infty)$ given by $\lambda_t(\mathrm{d}x) = \frac{\mathrm{e}^{-x/2t}}{4t^2}$, that is,

$$\eta\left(w_{t_1} \in \mathrm{d}x_1, \cdots, w_{t_n} \in \mathrm{d}x_n\right) = \lambda_{t_1}\left(\mathrm{d}x_1\right) P_{t_2-t_1}\left(x_1, \mathrm{d}x_2\right) \cdots P_{t_n-t_{n-1}}\left(x_{n-1}, \mathrm{d}x_n\right),$$

where $0 < t_1 < \cdots < t_n$. The measure η is known as the *excursion law* of BESQ_0. We refer the reader to [6] for details.

From this identity and Theorem 1.1, we can deduce that the flow of BESQ_0 has the Lévy–Itô decomposition $M_x = \sum_{x_i \le x} M_{x_i}$, where $\mathcal{M} = \sum_i \delta_{(x_i, M_{x_i})}$ is a Poisson point process on $(0, +\infty) \times \mathbb{M}^{\mathbb{R}_+}$ with intensity measure $\mathrm{d}t \otimes \eta$.

4.5. Subordinators on some other monoids

Although the most natural extension of real-valued Lévy processes is to consider Lévy processes taking values in some topological group, the development of the theory has shown the existence of interesting infinitely divisible processes that take values on some topological monoid instead of a topological group.

In the literature, there are subordinators taking values in some non-abelian monoids, and consequently, they are not in the class of processes considered here. A concrete example of a subordinator on a non-abelian monoid is given by the Λ-coalescent, which takes values in the monoid of partitions of $\mathbb{N}$ (see [20]). It would be nice to have a unified approach that covers all these motivating examples.

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