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BRANCHING RANDOM WALKS WITH REGULARLY VARYING PERTURBATIONS

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Abstract. We consider a modification of classical branching random walk, where we add i.i.d. perturbations to the positions of the particles in each generation. In this model, which was introduced and studied by Bandyopadhyay and Ghosh (2023), perturbations take the form $\frac{1}{\theta}\log\frac{X}{E}$, where θ is a positive parameter, X has an arbitrary distribution μ supported on $\mathbb{R}_+^*$ and E is exponential with parameter 1, independent of X. Working under finite mean assumption for μ, they proved almost sure convergence of the rightmost position to a constant limit, and identified the weak centered asymptotics when θ does not exceed a certain critical parameter θ_0. This paper complements their work by providing weak centered asymptotics for the case when $\theta > \theta_0$ (this case was previously considered only in the case $\mu = \delta_0$) and extending the results to μ with regularly varying tails. We prove almost sure convergence of the rightmost position and identify the appropriate centering for the weak convergence, which is of the form $\alpha n + c\log n$, with constants α, c depending on the ratio of θ and θ_0. We describe the limiting distribution and provide explicitly the constants appearing in the centering.

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1. Introduction

Branching random walk on the real line, abbreviated as BRW, is constructed as follows. The process starts with a single particle placed at 0. Given a point process $\mathcal{Z} = \sum_{k=1}^{N} \delta_{\xi_k}$ on $\mathbb{R}$, where N, denoting the size of the offspring, is a random variable on $\mathbb{N}_0$, the original particle at time 1 dies and gives birth to N particles positioned according to $\mathcal{Z}$. These particles are called the first generation of the process. At time 2, each of these particles reproduces independently and has offspring with positions relative to their parents' position given by an independent copy of $\mathcal{Z}$.

The process continues infinitely. As a result we obtain a marked tree $(S, \mathbb{T})$, where the tree $\mathbb{T}$ is the set of all particles equipped with the natural tree structure, and S_v is the position of a given particle $v \in \mathbb{T}$. We write $|v|$ for the generation of v. For a BRW with displacements given by ξ, let $R_n = \sup_{|v|=n} S_v$ denote the position of the most right particle at time n. There is a rich literature concerning the asymptotic behaviour of R_n. In 1976, Biggins [1] proved under minimal assumptions the law of large numbers for R_n, *i.e.* $\frac{R_n}{n}$ converges almost surely to a constant α. The corresponding limit theorem was proved by Aïdékon [2] in 2013, who showed that

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$R_n - \alpha n + c \log n$ converges in distribution to a random shift of the Gumbel distribution. We refer to Shi [3] for an extensive description of recent results on branching random walks.

In this paper we consider a perturbed branching random walk S^*, which is a modification of S, in which we add a random perturbation to the position of every particle, *i.e.*

$$S_v^* = S_v + X_v,$$

where $\{X_v\}_{v\in\mathbb{T}}$ are i.i.d. random variables independent of S. Note that the perturbation added to the position of a vertex $v \in \mathbb{T}$ does not influence the positions of its offspring, which explains that the process is sometimes called *last progeny modified branching random walk*. In this paper we study the model introduced by Bandyopadhyay and Ghosh in [4], where the perturbations have the form

$$X_v(\theta) = \frac{1}{\theta} \log \frac{Y_v}{E_v}$$

for a given positive real number θ, and $\{Y_v\}_{v\in\mathbb{T}}$ which are independent positive random variables with distribution μ, and given $\mathbb{T}$ are independent of $\{E_v\}_{v\in\mathbb{T}}$, which are independent with distribution $\mathrm{Exp}(1)$. A more general situation, with X_n replaced by a general law, was considered in a recent paper [5]. The model introduced above was further studied in the context of large deviations [6], and inhomogeneous time BRW [7]. The main motivation for considering it comes from the connection between the supremum of the perturbed BRW $R_n^*(\theta) = \sup_{|v|=n} S_v^*$ and random weighted sums. More precisely, Corollary 3.6 in [4] states, that

$$\theta R_n^*(\theta) \stackrel{d}{=} \log Y_n(\theta) - \log E \tag{1.1}$$

where $Y_n(\theta) = \sum_{|v|=n} \mathrm{e}^{\theta S_v} Y_v$ and E is exponential with parameter 1, independent of $Y_n(\theta)$. We will sometimes write $R_n^* = R_n^*(\theta)$ and $Y_n = Y_n(\theta)$ if the parameter is clear from the context. The asymptotics of R_n^* will be related very closely to the behaviour of Y_n, which is well described in the literature, see *e.g.* [8] and [9]. It turns out that properties of R_n^* depend on the parameter θ and of course on a number of further parameters. It turns out that one needs to control its position with respect to the critical parameter θ_0 defined as

$$\theta_0 = \inf\{\theta > 0 : \nu(\theta) = \theta\nu'(\theta)\}$$

where

$$\nu(\theta) = \log \mathbb{E}\left[\sum_{i=1}^{N} \mathrm{e}^{\theta\xi_i}\right]$$

is the log-Laplace transform of $\mathcal{Z}$, and $\nu'(\theta) = \mathrm{e}^{-\nu(\theta)}\mathbb{E}\left[\sum_{i=1}^{N} \xi_i \mathrm{e}^{\theta\xi_i}\right]$. Note that ν does not have to be differentiable at θ for this quantity to exist, and that in general θ_0 may be infinite.

In [4] branching random walks with such perturbations where studied in the case when μ has finite mean. In particular, the authors proved that

$$\frac{R_n^*(\theta)}{n} \xrightarrow[n\to\infty]{a.s.} \begin{cases} \frac{\nu(\theta)}{\theta} & \theta < \theta_0 \\ \frac{\nu(\theta_0)}{\theta_0} & \theta \geq \theta_0 \end{cases}$$

and identified weak centered asymptotics for $\theta \leq \theta_0$. However, the result for $\theta > \theta_0$ was only obtained for the degenerated perturbations with $\mu = \delta_1$. The goal of this paper is to complete the results from [4] by providing the missing weak centered asymptotics for the so called above the boundary case, and to extend them to μ with infinite mean, with special focus on distributions with regularly varying tails.

2. Main results

Let $\gamma \in (0,1)$. Our main assumption for μ is that

$$x^{\gamma}\left(1-F(x)\right) \xrightarrow[x\to+\infty]{} c_{+} > 0, \tag{$\mathbf{H}$}$$

where F is the probability distribution function of μ.

This assumption tells us that μ belongs to the domain of attraction of a stable law with characteristic function

$$\tilde{g}\left(t\right) = \mathrm{e}^{-k|t|^{\gamma}\left(1-i\tan\left(\frac{\pi\gamma}{2}\right)\mathrm{sign}t\right)} \tag{2.1}$$

where

$$k = \frac{\pi c_{+}}{2\Gamma\left(\gamma\right)\sin\left(\pi\gamma/2\right)} > 0$$

and $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}\mathrm{e}^{-t}\mathrm{d}t$ is the Gamma function.

Furthermore, it yields that if Y has distribution μ, then $\mathbb{E}\left[Y^{\gamma}\right] = \infty$, but $\mathbb{E}\left[Y^{r}\right] < \infty$ for any $r \in (0,\gamma)$. ($\mathbf{H}$) will be assumed in most of the paper, however the result for the above the boundary case will be stated under more general assumption.

We assume for the rest of the paper that $\mathrm{supp}\,(\mu) \subset \mathbb{R}_{+}$, the system survives with probability 1 ($\mathbb{P}(N=0)=0$) and $\mathbb{E}N \in (1,\infty]$. The first assumption $\mathbb{P}(N=0)=0$ is only made to simplify the notation, it can be easily avoided through conditioning on the survival set, whereas the second one in particular entails that the branching mechanism is not degenerated ($\mathbb{P}(N>1)>0$). We also assume that ν is finite on some open interval I containing 0. Since θ_0 is finite, the last assumption guarantees, by convexity of ν, that ν is differentiable on $(-s,\theta_0)$ for some $s>0$, and has a left derivative at θ_0. One can also characterize θ_0 as the unique argument minimizing $\frac{\nu(\theta)}{\theta}$ over $\theta > 0$. Throughout this paper, existence of finite θ_0 will only be assumed when necessary.

As proved in [1], if θ_0 is finite, then

$$\frac{R_n}{n} \xrightarrow[n\to\infty]{} \frac{\nu(\theta_0)}{\theta_0}, \quad \text{a.s.} \tag{2.2}$$

For θ such that $\nu(\theta) < \infty$, let

$$W_n(\theta) = \mathrm{e}^{-n\nu(\theta)}\sum_{|v|=n}\mathrm{e}^{\theta S_v}. \tag{2.3}$$

$W_n(\theta)$ is called the additive martingale associated with S. We denote $W_n = W_n(\theta_0)$. Note that as a positive martingale $W_n(\theta)$ converges almost surely to some finite limit. If $\nu'(\theta) < \infty$, then Biggins martingale convergence theorem [10] states, that the almost sure limit of $W_n(\theta)$ is non-degenerate if and only if $\nu'(\theta) < \nu(\theta)/\theta$ and

$$\mathbb{E}\left[W_1(\theta)\log_{+}W_1(\theta)\right] < \infty. \tag{2.4}$$

Furthermore, the limit is then positive almost surely. The first condition is equivalent to $\theta < \theta_0$, thus

$$W_n(\theta) \xrightarrow[n\to\infty]{a.s.} \begin{cases} W_\theta^\infty & \text{if } \theta < \theta_0 \text{ and (2.4) is satisfied,} \\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

where W_θ^∞ is finite and positive almost surely. We also define the **derivative martingale** associated with S as

$$D_n = -\sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0)) \,\mathrm{e}^{\theta_0 S_v - n\nu(\theta_0)}$$

As seen in Proposition A.3 from [2], under assumptions

(L1) $\theta_0 < \infty$ and

$$\mathbb{E}\left[\sum_{i=1}^{N} \mathrm{e}^{\theta_0 \xi_i} \xi_i^2\right] < \infty.$$

(L2) $\theta_0 < \infty$, and for $\tilde{X} = \sum_{i=1}^{N} \mathrm{e}^{\theta_0 \xi_i} \xi_i^+$, $X = \sum_{i=1}^{N} \mathrm{e}^{\theta_0 \xi_i}$

$$\mathbb{E}\left[\tilde{X} \log_+ \tilde{X}\right] < \infty,$$
$$\mathbb{E}\left[X \log_+^2 X\right] < \infty,$$

where $\log_+ x = \max\{0, \log x\}$, we have

$$D_n \xrightarrow[n\to\infty]{a.s.} D_\infty \tag{2.6}$$

for some random variable D_∞ that is finite and positive almost surely.

These two martingales are connected through Theorem 1.1 from [11], which states that under (**L1**) and (**L2**)

$$n^{\frac{1}{2}} W_n(\theta_0) \xrightarrow[n\to\infty]{\mathbb{P}} c_\infty D_\infty \tag{2.7}$$

where

$$c_\infty = \left(\frac{2}{\pi\sigma^2}\right)^{\frac{1}{2}} \quad \text{and} \quad \sigma^2 = \mathbb{E}\left[\sum_{i=1}^{N} (\theta_0 \xi_i - \nu(\theta_0))^2 \,\mathrm{e}^{\theta_0 \xi_i - \nu(\theta_0)}\right].$$

For more results on these martingales and their limits see for example Chapter 3 in [3].

We are now ready to present our results, starting with the almost sure convergence.

Theorem 2.1. *Assume that $\theta < \frac{\theta_0}{\gamma}$, condition (**H**) is satisfied and*

$$\mathbb{E}\left[W_1(\gamma\theta) \log_+ W_1(\gamma\theta)\right] < \infty.$$

Then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\gamma\theta)}{\gamma\theta}.$$

Theorem 2.2. *If $\theta_0 \le \theta$ and μ has finite r-th moments for all $r < \frac{\theta_0}{\theta}$, then*

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\theta_0)}{\theta_0}.$$

In particular, the conditions of Theorem 2.2 hold if μ satisfies (**H**) and $\theta \geq \frac{\theta_0}{\gamma}$.

The results concerning convergence in distribution we split into three cases.

Theorem 2.3 (Below the boundary case)**.** *Assume that $\theta < \frac{\theta_0}{\gamma}$, condition (**H**) is satisfied and*

$$\mathbb{E}\left[W_1(\gamma\theta)\log_+ W_1(\gamma\theta)\right] < \infty.$$

Then

$$R_n^* - n\frac{\nu(\gamma\theta)}{\gamma\theta} \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log H_\theta - \log E\right),$$

where H_θ is finite and positive almost surely, and E is exponential with intensity 1, independent of H_θ. Furthermore, H_θ has the characteristic function $\mathbb{E}\left[\tilde{g}\left(t\left(W_{\gamma\theta}^\infty\right)^\gamma\right)\right]$ where $W_{\gamma\theta}^\infty$ is the limit from (2.5) and $\tilde{g}$ is defined in (2.1).

Theorem 2.4 (The boundary case)**.** *Assume (**L1**) and (**L2**). If μ satisfies (**H**) and $\theta = \frac{\theta_0}{\gamma}$, then*

$$R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log H_{\theta_0} - \log E\right)$$

where H_{θ_0} is finite and positive almost surely and E is exponential with intensity 1, independent of H_{θ_0}. Furthermore, H_{θ_0} has the characteristic function $\mathbb{E}\left[\tilde{g}\left(t(c_\infty D_\infty)^\gamma\right)\right]$, where $\tilde{g}$ is defined in (2.1).

It is a natural question to ask whether assumption (**H**) in Theorems 2.1, 2.3 and 2.4 can be weakened by adding a slowly varying function. This is addressed in Remark 3.2.

Theorem 2.5 (Above the boundary case)**.** *Assume (**L1**), (**L2**) and that for all $s \in \mathbb{R}$,*

$$\mathbb{P}\left(\xi_1, \xi_2, \ldots, \in s\mathbb{Z}\right) < 1.$$

If $\theta > \theta_0$ and μ has a finite r-th moment for some $r > \frac{\theta_0}{\theta}$ and is not concentrated on a single point, then

$$R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log Z_{\frac{\theta_0}{\theta}} - \log E\right)$$

where $Z_{\frac{\theta_0}{\theta}}$ is finite and positive almost surely and E is exponential with intensity 1, independent of $Z_{\frac{\theta_0}{\theta}}$. Furthermore, $Z_{\frac{\theta_0}{\theta}} \stackrel{d}{=} (D_\infty)^{\frac{\theta}{\theta_0}} U_{\frac{\theta_0}{\theta}}$, where D_∞ is the almost sure limit of the derivative martingale defined in (2.6) and $U_{\frac{\theta_0}{\theta}}$ is strictly $\frac{\theta_0}{\theta}$-stable independent of D_∞.

In particular, if μ has finite mean and $\theta > \theta_0$, or if μ satisfies (**H**) and $\theta > \frac{\theta_0}{\gamma}$, then the assumption in the last theorem is satisfied. This result is also more extensive than Theorem 2.6 in [4], where the asymptotics were only given for the case $\mu = \delta_1$. It is worth noting that the logarithmic correction term in Theorems 2.4 and 2.5 or its absence in Theorem 2.3 correspond to corrections in classical settings, see *e.g.* [2, 12])

3. Proofs of Theorems 2.3, 2.4, and 2.5

We start with a short proof of Equation (1.1). The equation is proven in [4] as Corollary 3.6, but we include it here for completeness.

Proof of (1.1). Take $f \in C_b(\mathbb{R})$ and let $\mathcal{F}_n = \sigma(S_v, Y_v : |v| = n)$ be the σ-algebra generated by $\{S_v\}_{|v|=n}$ and $\{Y_v\}_{|v|=n}$. Then

$$\mathbb{E}\left[f\left(\theta R_n^*(\theta)\right)\right] = \mathbb{E}\left[f\left(\sup_{|v|=n} \theta S_v + \log\frac{Y_v}{E_v}\right)\right] = \mathbb{E}\left[\mathbb{E}\left[f\left(-\log\inf_{|v|=n}\frac{E_v}{\mathrm{e}^{\theta S_v}Y_v}\right)\middle|\mathcal{F}_n\right]\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[f\left(-\log\frac{E_1}{\sum_{|v|=n}\mathrm{e}^{\theta S_v}Y_v}\right)\middle|\mathcal{F}_n\right]\right] = \mathbb{E}\left[f\left(\log Y_n - \log E_1\right)\right]$$

where the penultimate equality follows from the fact, that the minimum of independent exponential random variables with parameters λ_i, $i = 1 \ldots n$, is again an exponential random variable, with parameter $\sum_{i=1}^n \lambda_i$. ☐

Now we recall Lemma 4.1 from [8] (presented here in a slightly more accessible form for our use), that will be useful to understand behaviour of the asymptotics of Y_n.

Lemma 3.1. *Let $\{Y_v\}_{v\in\mathbb{T}}$ be i.i.d. random variables with distribution μ satisfying (**H**), and $\{A_v\}_{v\in\mathbb{T}}$ be a sequence of positive random variables, independent of $\{Y_v\}_{v\in\mathbb{T}}$, such that*

$$\sum_{|v|=n} A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A \quad \textit{and} \quad \sup_{|v|=n} A_v \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

for some positive random variable A. Then

$$\sum_{|v|=n} A_v Y_v \xrightarrow[n\to\infty]{d} H$$

where H has the characteristic function $\varphi_H(t) = \mathbb{E}\left[\tilde{g}(tA^{\frac{1}{\gamma}})\right]$.

Proof of Theorem 2.3. First we will prove that

$$Y_n(\theta)\mathrm{e}^{-n\frac{\nu(\gamma\theta)}{\gamma}} \xrightarrow[n\to\infty]{d} H_\theta \tag{3.1}$$

where H_θ has the characteristic function $\mathbb{E}\tilde{g}\big(t(W_{\gamma\theta}^\infty)^{\frac{1}{\gamma}}\big)$ and moreover H_θ is positive almost surely. For this purpose we will use Lemma 3.1 for $A_v = \mathrm{e}^{\theta S_v - |v|\frac{\nu(\theta\gamma)}{\gamma}}$. To check its hypotheses observe

$$\sup_{|v|=n} A_v = \sup_{|v|=n} \mathrm{e}^{n\theta\left(\frac{S_v}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right)} = \mathrm{e}^{n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right)}.$$

As $\frac{R_n}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\theta_0)}{\theta_0}$ and θ_0 is the unique argument minimizing $\frac{\nu(t)}{t}$, then $n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right) \xrightarrow[n\to\infty]{a.s.} -\infty$ and so $\sup_{|v|=n} A_v \xrightarrow[n\to\infty]{a.s.} 0$. Furthermore, in view of (2.5),

$$\sum_{|v|=n} A_v^\gamma = W_n(\theta\gamma)$$

converges to $W_{\gamma\theta}^\infty$, which is positive almost surely, because $\gamma\theta < \theta_0$. Summarizing, Lemma 3.1 entails (3.1).

To see that H_θ is positive almost surely, choose any $\varepsilon > 0$. Then using Exercise 3.3.2 from [13]

$$\begin{aligned}
\mathbb{P}\left(H_\theta = 0\right) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left[\tilde{g}\big(t(W_{\gamma\theta}^\infty)^{\frac{1}{\gamma}}\big)\right] \mathrm{d}t \\
&\le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left[\mathrm{e}^{-k|t|^\gamma W_{\gamma\theta}^\infty}\right] \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\mathbb{E}\left[\mathrm{e}^{-k|t|^\gamma |W_{\gamma\theta}^\infty|}; W_{\gamma\theta}^\infty \le \varepsilon\right] + \mathbb{E}\left[\mathrm{e}^{-k|t|^\gamma W_{\gamma\theta}^\infty}; W_{\gamma\theta}^\infty > \varepsilon\right]\right) \mathrm{d}t \\
&\le \mathbb{P}\left(W_{\gamma\theta}^\infty \le \varepsilon\right) + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathrm{e}^{-k|t|^\gamma \varepsilon} \mathrm{d}t.
\end{aligned}$$

Now the function $t \to \mathrm{e}^{-k\varepsilon|t|^\gamma}$ is integrable, hence the limit of the second term is 0 for any ε. The first term can be made arbitrarily small through the choice of ε. Since we know that $W_{\gamma\theta}^\infty$ is positive almost surely for $\gamma\theta < \theta_0$, we conclude positivity of H_θ.

Next, recalling (1.1)

$$R_n^* - n\frac{\nu(\gamma\theta)}{\gamma\theta} \overset{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu(\gamma\theta)}{\gamma\theta} = \frac{1}{\theta}\left(\log Y_n(\theta)\mathrm{e}^{-n\frac{\nu(\gamma\theta)}{\gamma}} - \log E\right)$$

with $E \sim Exp(1)$ independent of Y_n. Finally, by (3.1) and the continuous mapping theorem

$$\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\nu(\gamma\theta)}{\gamma}}\right) \xrightarrow[n\to\infty]{d} \log H_\theta$$

where the distribution of H_θ is as specified in the statement. □

Proof of Theorem 2.4 (the boundary case). Define $A_v = \mathrm{e}^{\theta S_v - n\frac{\nu(\theta_0)\theta}{\theta_0} + \frac{\theta}{2\theta_0}\log n}$. Then, by (2.7), we have

$$\sum_{|v|=n} A_v^\gamma = n^{\frac{1}{2}} W_n \xrightarrow[n\to\infty]{\mathbb{P}} \left(\frac{2}{\pi\sigma^2}\right)^{\frac{1}{2}} D_\infty = c_\infty D_\infty$$

The second condition of Lemma 3.1, $\sup_{|v|=n} A_v \xrightarrow[n\to\infty]{\mathbb{P}} 0$, follows by applying Proposition A.3. in [14] to $V_u = \theta_0 S_u - |u|\nu(\theta_0)$. Then once again by Lemma 3.1

$$Y_n(\theta) n^{\frac{\theta}{2\theta_0}} \mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} = \sum_{|v|=n} A_v Y_v \xrightarrow[n\to\infty]{d} H_{\theta_0}. \tag{3.2}$$

Next, by (1.1)

$$\begin{aligned}
&R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n \overset{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n \\
&= \frac{1}{\theta}\left(\log\left(n^{\frac{\theta}{2\theta_0}} Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) - \log E\right)
\end{aligned}$$

and by (3.2) and the continuous mapping theorem

$$\log\left(n^{\frac{\theta}{2\theta_0}} Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) \xrightarrow[n\to\infty]{d} \log H_{\theta_0}$$

where distribution of H_{θ_0} is as specified in the statement. □

Remark 3.2. It is clear that proofs of Theorems 2.3 and 2.4 rely on Lemma 3.1. If one were to allow a slowly varying function $L(x)$ in the assumption (**H**), then a close examination of the proof available in [8] reveals that the assumption $\sum_{|v|=n} A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A$ needs to be replaced with $\sum_{|v|=n} L(A_v^{-1})^\gamma A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A$ and we have no tools to study convergence of such sequences without the martingale property.

Proof of Theorem 2.5 (above the boundary case). The proof relies on Proposition 3.2 in [9]. The assumptions for Theorem 2.5 with condition $\theta_0\nu'(\theta_0) = \nu(\theta_0)$ for S are equivalent to assumptions **(A1)** through **(A3)** from [9] for a BRW

$$V_u = -\theta\left(S_u - |u|\frac{\nu(\theta_0)}{\theta_0}\right)$$

with critical parameter $\vartheta = \frac{\theta_0}{\theta}$. Proposition 3.2 in [9] entails

$$n^{\frac{3}{2}\vartheta}\sum_{|u|=n} \mathrm{e}^{-V_u} Y_u \xrightarrow[n\to\infty]{d} Z_{\frac{\theta_0}{\theta}}$$

where $Z_{\frac{\theta_0}{\theta}}$ is positive almost surely. Furthermore, by equation (1.13) in [9], we have $Z_{\frac{\theta_0}{\theta}} \stackrel{d}{=} D^{\frac{1}{\vartheta}} U_\vartheta$, where D is the limit of the derivative martingale associated with $-\vartheta V$, and U_ϑ is strictly ϑ-stable independent of D. If we let ψ be the log-Laplace transform of $-\vartheta V$, then it satisfies the equation $\psi(1) = 0 = \psi'(1)$, so the derivative martingale associated with $-\vartheta V$ is

$$\sum_{|u|=n} \vartheta V_u \mathrm{e}^{-\vartheta V_u} = -\sum_{|v|=n}\left(\theta_0 S_v - n\nu(\theta_0)\right)\mathrm{e}^{\theta_0 S_v - n\nu(\theta_0)} = D_n$$

so $D^{\frac{1}{\vartheta}} = (D_\infty)^{\frac{\theta}{\theta_0}}$. Therefore

$$n^{\frac{3\theta}{2\theta_0}} Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} = n^{\frac{3}{2}\vartheta}\sum_{|u|=n}\mathrm{e}^{-V_u}Y_u \xrightarrow[n\to\infty]{d} Z_{\frac{\theta_0}{\theta}}$$

where the distribution of $Z_{\frac{\theta_0}{\theta}}$ is as in the statement. By (1.1) we have

$$\begin{aligned} R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} &\stackrel{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} \\ &= \frac{1}{\theta}\left(\log n^{\frac{3\theta}{2\theta_0}} Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} - \log E\right) \end{aligned}$$

and by the continuous mapping theorem

$$\log n^{\frac{3\theta}{2\theta_0}} Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} \xrightarrow[n\to\infty]{d} \log Z_{\frac{\theta_0}{\theta}}$$

which completes the proof. □

4. Proof of Theorems 2.1 and 2.2

We start with the following Lemma, which gives the convergence in probability. It is an essential step in the proof of Theorems 2.1 and 2.2, as it provides the bound in (4.5).

Lemma 4.1.
(a) If conditions of Theorem 2.1 hold, then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{\nu(\gamma\theta)}{\gamma\theta}$$

(b) If conditions of Theorem 2.2 hold, then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{\nu(\theta_0)}{\theta_0}$$

Proof. Let $\beta = \gamma\theta$ in case (a) and $\beta = \theta_0$ in case (b). We will prove first that

$$\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\beta)}{\beta}}\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{4.1}$$

We consider first the case (b). Fix an arbitrary $\varepsilon > 0$ and choose $\delta < \frac{\theta_0}{\theta}$ satisfying

$$\frac{\nu(\theta_0)}{\theta_0} - \frac{\nu(\delta\theta)}{\delta\theta} + \varepsilon > 0.$$

Such δ always exists, since ν is continuous and θ_0 is the unique argument minimizing $\frac{\nu(t)}{t}$ over $t > 0$. The Markov inequality yields

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > \varepsilon\right) &= \mathbb{P}\left(\delta\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > n\theta\delta\varepsilon\right)\\
&= \mathbb{P}\left(Y_n(\theta)^\delta\mathrm{e}^{-n\delta\frac{\theta\nu(\theta_0)}{\theta_0}} > \mathrm{e}^{n\theta\delta\varepsilon}\right)\\
&\le \mathbb{E}[Y_n(\theta)^\delta]\mathrm{e}^{-\delta n\theta\left(\frac{\nu(\theta_0)}{\theta_0}+\varepsilon\right)}.
\end{aligned}$$

Applying the well-known inequality $(a+b)^\delta \le a^\delta + b^\delta$, valid for any positive a, b and $\delta < 1$ and the fact that for any v the random variable Y_v is independent of S_v, we obtain

$$\mathbb{E}[Y_n(\theta)^\delta] = \mathbb{E}\left[\left(\sum_{|v|=n}\mathrm{e}^{\theta S_v}Y_v\right)^\delta\right] \le \mathbb{E}\left[\sum_{|v|=n}\mathrm{e}^{\theta\delta S_v}Y_v^\delta\right] = \mathrm{e}^{n\nu(\theta\delta)}\mathbb{E}[Y^\delta], \tag{4.2}$$

where the last expectation is finite. Summarizing

$$\mathbb{P}\left(\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > \varepsilon\right) \le \mathbb{E}[Y^\delta]\mathrm{e}^{-\delta n\theta\left(\frac{\nu(\theta_0)}{\theta_0}-\frac{\nu(\delta\theta)}{\delta\theta}+\varepsilon\right)}$$

and thanks to our choice of δ the above expression converges to 0 as n tends to $+\infty$. To prove the remaining bound, denote $v_n = \arg\max_{|v|=n} S_v$. Since

$$\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) \ge \frac{1}{n\theta}\log\left(\mathrm{e}^{n\theta\left(\frac{S_{v_n}}{n}-\frac{\nu(\theta_0)}{\theta_0}\right)}Y_{v_n}\right) = \frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \frac{1}{n\theta}\log Y_{v_n},$$

for any parameters $0 < \delta < \varepsilon$ we obtain

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) < -\varepsilon\right) &\leq \mathbb{P}\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \frac{1}{n\theta}\log Y_{v_n} < -\varepsilon\right)\\ &= \mathbb{P}\left(\mathrm{e}^{n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0}\right)} Y_{v_n} < \mathrm{e}^{-\varepsilon n\theta}\right)\\ &= \mathbb{P}\left(\mathrm{e}^{n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \varepsilon\right)} Y_{v_n} < 1\right)\\ &\leq \mathbb{P}\left(\mathrm{e}^{n\theta\delta} Y_{v_n} < 1\right) + \mathbb{P}\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \varepsilon < \delta\right).\end{aligned}$$

Now, since $\frac{R_n}{n}$ converges almost surely to $\frac{\nu(\theta_0)}{\theta_0}$ and $\delta < \varepsilon$, the second term converges to 0. For the first term we have

$$\mathbb{P}\left(\mathrm{e}^{n\theta\delta} Y_{v_n} < 1\right) = \mathbb{P}\left(Y < \mathrm{e}^{-n\theta\delta}\right) \to 0.$$

Thus, we conclude the proof of (4.1) for case (b).

For case (a), by Theorem 2.3, $\log Y_n(\theta)\mathrm{e}^{-n\frac{\nu(\gamma\theta)}{\gamma}}$ converges in distribution to $\log H_\theta$ and this limit is finite almost surely. Therefore $\frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\nu(\gamma\theta)}{\gamma}}\right)$ converges in distribution to 0, and hence the convergence holds in probability as well. Thus, the proof of (4.1) is completed.

To prove Lemma 4.1 notice that using (1.1) we can write

$$\frac{R_n^*(\theta)}{n} \stackrel{d}{=} \frac{\log Y_n(\theta)}{n\theta} - \frac{\log E}{n\theta} = \frac{1}{n\theta}\log\left(Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\beta)}{\beta}}\right) + \frac{\nu(\beta)}{\beta} - \frac{\log E}{n\theta}.$$

Now $\frac{\log E}{n\theta}$ converges to 0 almost surely and by (4.1), $\frac{1}{n\theta}\log Y_n(\theta)\mathrm{e}^{-n\frac{\theta\nu(\beta)}{\beta}}$ converges to 0 in probability. That completes the proof of the Lemma.

□

Proof of Theorems 2.1 and 2.2 (almost sure convergence). To prove the almost sure convergence we utilize here the arguments given in the proof of Theorem 2.1 in [4]. For the sake of completeness, we present a complete proof. We note that the main difference is that, due to Lemma 4.1, there is no need to treat separately the cases below and above the boundary. Again, let $\beta = \gamma\theta$ if conditions of Theorem 2.1 are satisfied and $\beta = \theta_0$ if conditions of Theorem 2.2 are satisfied. We start with the upper bound

$$\limsup_{n\to\infty}\frac{R_n^*(\theta)}{n} \leq \frac{\nu(\beta)}{\beta} \qquad \text{a.s.} \tag{4.3}$$

Fix any $\varepsilon > 0$. By (1.1) and the Markov inequality we get that for any $\delta < \min(\frac{\theta_0}{\theta}, 1)$

$$\begin{aligned}\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} > \varepsilon\right) &= \mathbb{P}\left(\theta\delta R_n^*(\theta) - \frac{\theta\delta n\nu(\beta)}{\beta} > n\delta\theta\varepsilon\right)\\ &= \mathbb{P}\left(\log\frac{Y_n(\theta)^\delta}{\mathrm{e}^\delta} - \frac{\theta\delta n\nu(\beta)}{\beta} > n\delta\theta\varepsilon\right)\\ &\leq \mathrm{e}^{-\delta n\theta\left(\frac{\nu(\beta)}{\beta}+\varepsilon\right)}\mathbb{E}\left[\mathrm{e}^{-\delta}\right]\mathbb{E}\left[Y_n(\theta)^\delta\right]\\ &\leq \mathrm{e}^{-\delta n\theta\left(\frac{\nu(\beta)}{\beta} - \frac{\nu(\theta\delta)}{\theta\delta}+\varepsilon\right)}\Gamma(1-\delta)\,\mathbb{E}\left[Y^\delta\right],\end{aligned}$$

where the last inequality follows from (4.2).

Since ν is continuous, we can choose δ so that

$$\frac{\nu(\beta)}{\beta} - \frac{\nu(\delta\theta)}{\delta\theta} + \varepsilon > 0.$$

Therefore the series

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} > \varepsilon\right)$$

converges. The Borel–Cantelli lemma and arbitrariness of ε entails (4.3).

Finally our goal is to prove the lower bound

$$\liminf_{n\to\infty} \frac{R_n^*(\theta)}{n} \geq \frac{\nu(\beta)}{\beta} \qquad \text{a.s.} \tag{4.4}$$

For u such that $|u| = m \leq n$, we define

$$R_{n-m}^{*(u)}(\theta) := \max_{v>u,|v|=n} \left(S(v) + \frac{1}{\theta}\log(Y_v/E_v)\right) - S(u),$$

where $v > u$ means that v is a descendant of u. Note that, due to the branching property of S, $\{R_{n-m}^{*(u)}(\theta)\}_{|u|=m}$ are i.i.d. and have the same distribution as $R_{n-m}^*(\theta)$. Now,

$$\begin{aligned} R_n^*(\theta) &= \max_{|u|=m} \max_{v>u,|v|=n} \left(S(v) + \frac{1}{\theta}\log(Y_v/E_v)\right) \\ &= \max_{|u|=m}\left(S(u) + R_{n-m}^{*(u)}(\theta)\right) \\ &\geq S(\tilde{u}_m) + \max_{|u|=m}\left(R_{n-m}^{*(u)}(\theta)\right), \end{aligned}$$

where

$$\tilde{u}_m := \arg\max_{|u|=m}\left(R_{n-m}^{*(u)}(\theta)\right).$$

Now, for any $\varepsilon \in (0,1)$ and small s such that $\nu(-s/2)$ is finite,

$$\begin{aligned} &\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon\right) \\ \leq\ &\mathbb{P}\left(S\left(\tilde{u}_{[\sqrt{n}]}\right) + \max_{|u|=[\sqrt{n}]}\left(R_{n-[\sqrt{n}]}^{*(u)}(\theta)\right) < n\left(\frac{\nu(\beta)}{\beta} - \varepsilon\right)\right) \\ \leq\ &\mathbb{P}\left(\max_{|u|=[\sqrt{n}]}\left(R_{n-[\sqrt{n}]}^{*(u)}(\theta)\right) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2}\right)\right) + \mathbb{P}\left(S\left(\tilde{u}_{[\sqrt{n}]}\right) < -\frac{n\varepsilon}{2}\right) \\ \leq\ &\mathbb{E}\left[\mathbb{P}\left(R_{n-[\sqrt{n}]}^*(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2}\right)\right)^{N_{[\sqrt{n}]}}\right] + \mathrm{e}^{-n\varepsilon s/4}\cdot\mathbb{E}\left[\mathrm{e}^{-sS\left(\tilde{u}_{[\sqrt{n}]}\right)/2}\right], \end{aligned}$$

where N_k is the number of offspring in k-th generation. Recalling Lemma 4.1, for all large enough n,

$$\mathbb{P}\left(R^*_{n-[\sqrt{n}]}(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2}\right)\right) < \varepsilon. \tag{4.5}$$

We have

$$\mathbb{E}\left[\mathbb{P}\left(R^*_{n-[\sqrt{n}]}(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2}\right)\right)^{N_{[\sqrt{n}]}}\right] \leq \mathbb{E}[\varepsilon^{N_{[\sqrt{n}]}}] \leq \mathbb{E}\left[\varepsilon^n; N_{[\sqrt{n}]} \geq n\right] + \mathbb{P}\left(N_{[\sqrt{n}]} < n\right)$$
$$\leq \varepsilon^n + \mathbb{P}\left(N_{[\sqrt{n}]} < n\right)$$

If $\mathbb{P}(N=1)=0$, then $N_{[\sqrt{n}]} \geq 2^{[\sqrt{n}]}$, so $\mathbb{P}\left(N_{[\sqrt{n}]} < n\right)$ obviously disappears. Otherwise, if $\mathbb{P}[N=1]>0$, then as seen in [15] (Cor. 5 with Eqs. (29) and (4b)) , there are positive constants $C>0$ and $\alpha>0$, such that for all large enough $k \in \mathbb{N}$,

$$\mathbb{P}\left(N_k < k^2\right) \leq C m^{-\alpha k},$$

where $m = \mathbb{E}[N]$. Therefore

$$\varepsilon^n + \mathbb{P}\left(N_{[\sqrt{n}]} < n\right) \leq \varepsilon_1^{\lceil\sqrt{n}\rceil}$$

for some $\varepsilon_1 < 1$. To estimate the second term, we bound supremum by the sum and we have

$$\mathbb{E}\left[\mathrm{e}^{-sS\left(\tilde{u}_{[\sqrt{n}]}\right)/2}\right] \leq \mathbb{E}\left[\sum_{|v|=\lceil\sqrt{n}\rceil} \mathrm{e}^{-\frac{s}{2}S_v}\right] = \mathrm{e}^{[\sqrt{n}]\nu(-s/2)}.$$

Therefore we have for all large enough n,

$$\mathbb{P}\left(\frac{R^*_n(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon\right) \leq \varepsilon_1^{\lceil\sqrt{n}\rceil} + \mathrm{e}^{-n\varepsilon s/4 + [\sqrt{n}]\nu(-s/2)}.$$

Since for every $\varepsilon \in (0,1)$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R^*_n(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon\right) < \infty,$$

using the Borel–Cantelli Lemma once again we deduce (4.4), completing the proof. □

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