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SOME RESULTS ON THE ODDS RATE ORDER ALONG WITH ITS STATISTICAL TESTING

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Abstract. Recently, Lando *et al.* [J. Statist. Plann. Inference **221** (2022) 313–325] define the odds rate order and show that it is stronger than the usual stochastic order, but weaker than the hazard rate order. The purpose of this paper is to investigate the other properties of this new stochastic order. Some characterization and preservation results and its relationship with other well-known stochastic orders are given. In addition, two relative ordering based on the odds rate functions are also defined and their relation with the odds rate order is investigated. Applications in statistical theory of reliability and a statistical test for the odds rate order are also included.

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1. Introduction

In the reliability theory and survival analysis, classifying the lifetime random variables and the corresponding distribution functions into different aging classes has always been taken a great deal of attention in the literature. The main aim of the classifying is to reveal how a lifetime variable (negatively or positively) is affected by age (*cf.* [1]). The exponential distribution is usually located at the middle of these classes and exhibits the "no aging" property. Most of the aging notions may be equivalently expressed as a stochastic order which is hold between the distribution in the corresponding aging class and the exponential distribution. For instance, a lifetime random variable X belongs to the increasing (decreasing) failure rate, abbreviated as IFR(DFR), distributions class if and only if its corresponding distribution is dominated by the exponential distribution in the sense of the convex order. Recalling that a random variable X with distribution function F is said to be smaller than the random variable Y with distribution function G in the convex stochastic order (denoted by $X \leq_c Y$) if $G^{-1} \circ F$ is convex, for other examples, we refer the reader to [2].

Reminding that the IFR family excludes several important models, such as bathtub or heavy tailed distributions, [3] were looking for alternative models of broader applicability and ended up with constructing a wider family called increasing odds rate (IOR) family. In order to do this, they replaced the exponential distribution in the above convex order with the log–logistic distribution, with shape parameter equal to 1 and distribution

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function

$$G_0(x) = \frac{x}{1+x},\ x \geq 0,$$

and quantile function

$$G_0^{-1}(u) = \frac{u}{1-u},\ 0 < u < 1.$$

Thus, according to [3], a random variable X with distribution function F and survival function $\bar{F} = 1 - F$ is said to be IOR if

$$\Lambda_F(x) = G_0^{-1}F(x) = \frac{F(x)}{\bar{F}(x)},$$

is convex, where $G_0^{-1}F(x) = G_0^{-1} \circ F(x) = G_0^{-1}(F(x))$. When the density function exists, the IOR condition is equivalent to that the corresponding odds rate function, defined as

$$\lambda_F(x) = \frac{\mathrm{d}\Lambda_F(x)}{\mathrm{d}x} = \frac{f(x)}{\bar{F}^2(x)} = \frac{h_F(x)}{\bar{F}(x)},$$

is increasing, where f and h_F are the corresponding density and failure rate functions, respectively. The family of distributions with decreasing odds rate (DOR), corresponding to the concavity of Λ_F (or a deceleration of the odds of failure, *i.e.* λ_F decreases) is also considered as a negative aging notion. Both the IFR(DFR) and IOR(DOR) properties represent models in which the lifetimes are affected negatively (positively) by age. As given in [3], if $X \in IFR(DOR)$, then $X \in IOR(DFR)$.

Studying the properties of the IOR family and giving its applications, [3] have also defined the following new stochastic order.

Definition 1.1. We say that X is smaller than Y in the odds rate order, and write $X \leq_{or} Y$, if $\lambda_F(x) \geq \lambda_G(x)$, for every $x \in \mathbb{R}$.

[3] have pointed out that the interpretation of the OR order is very similar to that of the hazard rate order, that is, X ages faster than Y, since having a higher OR means that the probability of failure over survival grows faster.

Recalling that (for more details, see [2])

Definition 1.2. **(i)** The random variable X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x > 0$.

(ii) The random variable X is said to be smaller than Y in the hazard rate ordering (denoted by $X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x. When the densities exist, this is equivalent to that $h_F(x) \geq h_G(x)$ for all $x > 0$,

they have also shown that the odds rate order is stronger than the usual stochastic order, but weaker than the failure rate order.

For practical purposes, in order to verify aging classes and to determine whether a stochastic ordering holds between two distributions of data, diverse testing procedures have been developed in literature. For example, [4] have suggested a test statistic of the null hypothesis of monotone nondecreasing failure rate. [5] have proposed a nonparametric test to detect violations of the increasing hazard rate property. [6] have developed an approach for testing equality of k-distributions against the alternative that they are uniformly stochastically ordered (or equivalently, ordered in hazard rate). Regarding the IOR notion, [7] has proposed a nonparametric test of the increasing log-odds rate null hypothesis. Furthermore, mentioning that the IOR condition is useful to improve nonparametric estimates based on shape assumptions, [8] have provided two different nonparametric tests of the IOR null hypothesis.

In this paper, we study some other properties of the new stochastic order $\leq_{or}$ and provide a test for the presence of this order among two random variables. Analogously to the relationship between the IFR and IFR in average notions, we also consider the increasing odds rate average (IORA) notion along with two related relative aging concepts and investigate their relationship with the odds rate order. The rest of the paper is organized as follows. In Section 2, we give some main properties of the order including the characterization results. Its applications in the reliability theory and comparing the epoch times of two nonhomogeneous Poisson processes are also given in this section. Section 3 is devoted to the relationship between the odds rate order and some well-known stochastic orders and two relative ordering based on the odds rate function. A statistical test for testing the odds rate order along with a simulation exercise for assessing the test and real data analysis are provided in Section 4. Finally, some conclusions are given in Section 5. Though most of the results are valid for the real valued random variables, throughout the paper we assume that the random variables are non-negative.

2. Some basic properties and characterizations

Before proceeding to give the results, we overview some preliminary concepts of ageing and stochastic orders. (For more details of these concepts see, for example, [1], and [2]). Let X and Y be two non-negative random variables with their density functions f and g, distribution functions F and G and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Furthermore, $m_X(t) = \frac{\int_t^\infty \bar{F}(x)\mathrm{d}x}{\bar{F}(t)}$ denotes the mean residual life function corresponding to random variable X (m_Y is defined analogously). Throughout the paper we assume that these functions all exist and increasing (decreasing) means non-decreasing (non-increasing).

Definition 2.1. **(i)** The random variable X is said to be new better (worse) than used, abbreviated as NBU(NWU), if $\bar{F}(x+t) \leq (\geq) \bar{F}(x)\bar{F}(t)$, for all $x, t > 0$.

(ii) The random variable X is said to be increasing (decreasing) mean residual life, abbreviated as IMRL(DMRL), if $m_X(t)$ is increasing (decreasing) in t.

(iii) The random variable X is said to be smaller than Y in the likelihood ratio ordering (denoted by $X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ is increasing in x.

(iv) The random variable X is said to be smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if $G^{-1}(u) - F^{-1}(u)$ is increasing in $u \in (0,1)$.

The usual stochastic and the hazard rate orders given in Definition 3.7 and the above stochastic orders are related as the following (see, [2]).

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y, \quad X \leq_{disp} Y \Rightarrow X \leq_{st} Y$$

Now, we first give some characterization results. Let $\Lambda_F(x) = G_0^{-1}F(x)$ and $\Lambda_G(x) = G_0^{-1}G(x)$ be the odd functions corresponding to F and G, respectively, and Y_0 be a random variable with distribution function G_0 and quantile function G_0^{-1}. The following theorem gives some equivalent conditions for the odds rate order.

Theorem 2.2. *The random variable X is smaller than Y in the odds rate order if and only if*

(i) $\Lambda_F(x) - \Lambda_G(x)$ *increases in x;*

(ii) $\frac{1}{\bar{F}(x)} - \frac{1}{\bar{G}(x)}$ *increases in x;*

(iii) $G_0^{-1}FG^{-1}(u) - G_0^{-1}(u)$ *increases in $u \in (0,1)$;*

(iv) $G_0^{-1}G(X) \leq_{disp} Y_0$*;*

(v) $Y_0 \leq_{disp} G_0^{-1}F(Y)$*;*

(vi) $G_0^{-1}FG^{-1}G_0(x) - x$ *increases in x.*

Proof. We only give the proof of part (iv). The proof of other parts are simple or similar to this part. It follows from part (i) that

$$G_0^{-1}F(y) - G_0^{-1}G(y) \geq G_0^{-1}F(x) - G_0^{-1}G(x), \text{ for } \ y \geq x.$$

The result now follows from Theorem 3.B.5 in ([2], p. 151). □

The following theorem shows that the order $\leq_{or}$ is closed under increasing transformation.

Theorem 2.3. *If $X \leq_{or} Y$, then, for any increasing function ϕ, $\phi(X) \leq_{or} \phi(Y)$.*

Proof. Let $\bar{F}_\phi$ and $\bar{G}_\phi$ denote the survival functions of $\phi(X)$ and $\phi(Y)$, respectively. Then,

$$\bar{F}_\phi(x) = \bar{F}\phi^{-1}(x), \quad \bar{G}_\phi(x) = \bar{G}\phi^{-1}(x).$$

Using Theorem 2.2(ii), the assumption is equivalent to that

$$\psi(x) = \frac{1}{\bar{F}(x)} - \frac{1}{\bar{G}(x)},$$

is increasing. This along with the assumption that ϕ is increasing, follow that

$$\frac{1}{\bar{F}_\phi(x)} - \frac{1}{\bar{G}_\phi(x)} = \psi\phi^{-1}(x),$$

is increasing which completes the proof. □

The following theorem gives condition under which X and $X+Y$ are ordered in the sense of the odds rate.

Theorem 2.4. *If $X \in IFR(DFR)$, then for any random variable Y independent of X, $X \leq_{or} (\geq_{or}) X+Y$.*

Proof. The assumption $X \in IFR(DFR)$ implies that, $\frac{F(x-y)}{\bar{F}(x)}$ is increasing (decreasing) in x. Hence,

$$\frac{1}{\bar{F}(x)} - \frac{1}{\bar{F}_{X+Y}(x)} = \frac{\int [\frac{\bar{F}(x-y)}{\bar{F}(x)} - 1] dG(y)}{\bar{F}_{X+Y}(x)},$$

is increasing (decreasing) in x, where $\bar{F}_{X+Y}$ is the survival function of $X+Y$. Applying Theorem 2.2(ii) now completes the proof. □

Corollary 2.5. *If $X \in DFR$ and $Y \in IFR$, and $X \leq_{or} Y$, then for any random variable Z independent of X and Y, $X+Z \leq_{or} Y+Z$.*

Proof. The result follows from the previous theorem and the fact that, under the assumptions

$$\frac{1}{\bar{F}_{X+Z}(x)} - \frac{1}{\bar{F}_{Y+Z}(x)} = \frac{1}{\bar{F}_{X+Z}(x)} - \frac{1}{\bar{F}(x)} + \frac{1}{\bar{F}(x)} - \frac{1}{\bar{G}(x)} + \frac{1}{\bar{G}(x)} - \frac{1}{\bar{F}_{Y+Z}(x)},$$

is increasing in x. □

Analog to Theorem 1.B.8. in ([2], p. 20) the next theorem, gives a condition under which the order $\leq_{or}$ is closed under mixtures. The similar proof is omitted.

Theorem 2.6. *Let X, Y, and Θ be random variables such that $[X|\Theta = \theta] \leq_{or} [Y|\Theta = \theta']$ for all θ and θ' in the support of Θ. Then, $X \leq_{or} Y$.*

The following theorem gives results about the maximum and minimum of X and Y.

Theorem 2.7. *Let X and Y be two independent random variables. Then,*

(a) $\min\{X,Y\} \leq_{or} X$,
(b) $\min\{X,Y\} \leq_{or} \max\{X,Y\}$,
(c) $X \leq_{or} \max\{X,Y\}$, *if* $X \leq_{or} Y$.

Proof. Part (a) follows from the fact that

$$\frac{1}{\bar{F}(x)\bar{G}(x)} - \frac{1}{\bar{F}(x)} = \frac{G(x)}{\bar{F}(x)\bar{G}(x)},$$

is increasing.

(b) Let $\lambda_{\min}$ and $\lambda_{\max}$ be the odds rate functions corresponding to $\min\{X,Y\}$ and $\max\{X,Y\}$, respectively. Then, one can obtain that

$$\lambda_{\min}(x) = \frac{\lambda_F(x)G(x)}{\bar{G}^2(x)} + \frac{\lambda_G(x)F(x)}{\bar{F}^2(x)},$$

and

$$\lambda_{\max}(x) = \frac{\lambda_F(x)G(x)\bar{F}^2(x) + \lambda_G(x)F(x)\bar{G}^2(x)}{[\bar{F}(x) + F(x)\bar{G}(x)]^2}.$$

Hence,

$$\frac{\lambda_{\min}(x)}{\lambda_{\max}(x)} = \left(\frac{1}{\bar{G}(x)} + \frac{F(x)}{\bar{F}(x)}\right)^2 \geq 1.$$

Part (c) also follows from that

$$\frac{\lambda_{\max}(x)}{\lambda_F(x)} \leq \frac{G(x)\bar{F}^2(x) + F(x)\bar{G}^2(x)}{[\bar{G}(x) + G(x)\bar{F}(x)]^2} \leq 1,$$

where, the inequalities follow from $X \leq_{or} Y$ which implies $\bar{F}(x) \leq \bar{G}(x)$. The proof is now completed. □

In the following results we give some properties of the odds rate order in reliability theory especially its connection with aging notions.

Theorem 2.8. **(a)** *$X \in IFR(DFR)$ if and only if $X_t \geq_{or} (\leq_{or}) X_{t'}$, for all $t < t'$, where $X_t = [X - t | X > t]$ is the residual lifetime variable corresponding to random variable X.*
(b) *$X \in IOR(DOR)$ if and only if $X_t \leq_{or} (\geq_{or}) X$.*
(c) *If $X_t \leq_{or} (\geq_{or}) X$, then $X \in NBU(NWU)$.*
(d) *If $X \in IMRL(DMRL)$, then $X \leq_{or} (\geq_{or}) X_e$,*
where X_e is the equilibrium random variable corresponding to X with survival function $\bar{F}_e(x) = \frac{1}{\mu}\int_x^\infty \bar{F}(y)\mathrm{d}y$, $\mu = E(X)$.

Proof. (a) We know that $X_t \geq_{or} (\leq_{or}) X_{t'}$ implies that $X_t \geq_{st} (\leq_{st}) X_{t'}$ which, using Theorem 1.A.30 (a) in ([2], p. 15), is equivalent to that $X \in IFR(DFR)$.

For the only if part, recalling that $X \in IFR(DFR)$ is equivalent to

$$\frac{\bar{F}(t)}{\bar{F}(x+t)} \leq (\geq) \frac{\bar{F}(t')}{\bar{F}(x+t')}, \quad \text{whenever,} \ \ t < t',$$

we have

$$\begin{aligned} \lambda_{F_t}(x) = \frac{\bar{F}(t)h(x+t)}{\bar{F}(x+t)} &\leq (\geq) \ \frac{\bar{F}(t)h(x+t')}{\bar{F}(x+t)} \\ &\leq (\geq) \ \frac{\bar{F}(t')h(x+t')}{\bar{F}(x+t')} = \lambda_{F_{t'}}(x), \end{aligned}$$

where λ_{F_t} is the odds rate function corresponding to the residual lifetime variable X_t.

Part (b) follows from the fact that if $X_t \leq_{or} X$, then, for $t > 0$,

$$\lambda_F(x+t) - \lambda_F(x) = \frac{\lambda_{F_t}(x)}{\bar{F}(t)} - \lambda_F(x) \geq \lambda_{F_t}(x) - \lambda_F(x) \geq 0,$$

that is, $X \in IOR$. Conversely, if $X \in IOR$, then,

$$0 \leq \lambda_F(x+t) - \lambda_F(x) = \frac{\lambda_{F_t}(x)}{\bar{F}(t)} - \lambda_F(x) \leq \frac{\lambda_{F_t}(x)}{\bar{F}(t)} - \frac{\lambda_F(x)}{\bar{F}(t)}.$$

Hence, $X_t \geq_{or} X$. Furthermore, if $X_t \geq_{or} X$, then,

$$\lambda_F(x+t) - \lambda_F(x) = \frac{\lambda_{F_t}(x)}{\bar{F}(t)} - \lambda_F(x) \leq \frac{\lambda_{F_t}(x)}{\bar{F}(t)} - \frac{\lambda_F(x)}{\bar{F}(t)} \leq 0,$$

which implies that $X \in DOR$. On the other hand, if $X \in DOR$, then,

$$\lambda_{F_t}(x) = \bar{F}(t)\lambda_F(x+t) \leq \lambda_F(x),$$

that is, $X_t \geq_{or} X$.

(c) $X_t \leq_{or} (\geq_{or})X$, implies that $X_t \leq_{st} (\geq_{st})X$. That is, $\bar{F}(x+t) \leq (\geq)\bar{F}(x)\bar{F}(t)$ and $X \in NBU(NWU)$.

Finally, part (d) follows from the fact that under the assumptions $X \in IMRL$ and $X \in DMRL$

$$\frac{1}{\bar{F}(x)} - \frac{1}{\bar{F}_e(x)} = \frac{1}{\bar{F}(x)}[1 - \frac{\mu}{m_X(x)}], \quad \text{and} \quad \frac{1}{\bar{F}_e(x)} - \frac{1}{\bar{F}(x)} = \frac{1}{\bar{F}(x)}[\frac{\mu}{m_X(x)} - 1],$$

are increasing, respectively. This completes the proof. □

Noting that $\lambda_{F_t}(0) = h_F(t)$, as a corollary of Theorem 2.8(b) we obtain the following bound for the failure rate function.

Corollary 2.9. *If $X \in IOR(DOR)$, then $h_F(x) \geq (\leq) f(0)$, for all $x \geq 0$.*

Remark 2.10. It follows from Theorem 2.8 (d) that if $X \in DMRL$ then,

$$\lambda_{F_e}(x) = \frac{E(X)}{m_X^2(x)\bar{F}(x)} \geq \frac{f(x)}{\bar{F}^2(x)} = \lambda_F(x),$$

or $m_X^2(x)f(x) \leq \bar{F}(x)E(X)$. Now, integrating over $(0,\infty)$, w.r.t. x, gives that $E[m_X^2(X)] \leq E^2(X)$. One can show that $E[m_X^2(X)] = Var(X)$, and hence, $E(X^2) \leq 2E^2(X)$.

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent copies of the nonnegative random variables X and Y, respectively. We denote their corresponding order statistics by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$, respectively. It is easy to verify that $X_{(1)} \leq_{hr} X \leq_{hr} X_{(n)}$ ([9], p. 75) and hence, $X_{(1)} \leq_{or} X \leq_{or} X_{(n)}$.

Proposition 1.B.35. in ([2], p. 34) gives that if $X_i \leq_{hr} Y_i$, $i = 1, 2, \ldots, n$, then $X_{(r)} \leq_{hr} Y_{(r)}$, for $1 \leq r \leq n$. In the following, we give a similar result based on the odds rate order. Recalling that the lifetime of a $n-r+1$-out-of-n system made up of components with lifetimes $X_1, X_2, \ldots, X_n$ is the order statistics $X_{(r)}$, the result can be considered as a result for the comparison of $n-r+1$-out-of-n systems. We recall the following lemma from [10] which will need for proving the result.

Lemma 2.11. *Let $\phi : R^+ \to R^+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly increasing function $\psi : R^+ \to R^+$ the function $\psi\phi\psi^{-1}(x) - x$ is increasing.*

Theorem 2.12. *If $X_i \leq_{or} Y_i$, $i = 1, 2, \ldots, n$, then $X_{(r)} \leq_{or} Y_{(r)}$, for $1 \leq r \leq n$.*

Proof. Let $F_{(r)}$ be the distribution function of $X_{(r)}$. It is well-known that (*cf.* [9], p. 9)

$$F_{(r)}(x) = F_\beta F(x),$$

where F_β is the distribution function of a beta distribution with parameters r and $n-r+1$.

Now, using Theorem 2.2(vi), the assumption $X_i \leq_{or} Y_i$ is equivalent to that

$$G_0^{-1} F G^{-1} G_0(x) - x,$$

is increasing. On the other hand,

$$\begin{aligned} G_0^{-1} F_{(r)} G_{(r)}^{-1} G_0(x) - x &= G_0^{-1} F_\beta F G^{-1} F_\beta^{-1} G_0(x) - x \\ &= G_0^{-1} F_\beta G_0 G_0^{-1} F G^{-1} G_0 G_0^{-1} F_\beta^{-1} G_0(x) - x. \end{aligned}$$

The result now follows from Lemma 2.11, by taking $\phi = G_0^{-1} F G^{-1} G_0$ and $\psi = G_0^{-1} F_\beta G_0$. We just need to show that ψ is convex and strictly increasing. This is true because the log-logistic distribution with distribution function G_0 belongs to the IOR family and Theorem 5 in [3] ensures that $F_\beta G_0$ is in the IOR family and hence $\psi = G_0^{-1} F_\beta G_0$ is convex and strictly increasing. □

The above comparison can be extended to the comparison of two coherent systems as well.

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be the lifetimes of the components of two coherent systems with common distribution functions F and G, respectively. Let F_T and G_T be the distribution functions of the lifetime variables of these two systems, say T_1 and T_2, respectively. [11] showed that F_T and G_T can be represented as

$$F_T(x) = H_1 F(x), \quad G_T(x) = H_2 G(x),$$

where, H_1 and H_2, called structure and dependence or domination functions, are increasing functions in $(0,1)$ from $H_1(0) = H_2(0) = 0$ to $H_1(1) = H_2(1) = 1$, and depend only on the structure of the systems and the survival copula of the components lifetimes. When, the components are independent, H_1 and H_2 are polynomials, called domination or reliability polynomials, and strictly increasing.

Theorem 2.13. *Let $\psi_1 = G_0^{-1} H_1 G_0$ and $\psi_2 = G_0^{-1} H_2 G_0$ and assume that H_1 and H_2 are strictly increasing. If $X_i \leq_{or} Y_i$, $i = 1, 2, \ldots, n$ and $\psi_1(x) - \psi_2(x)$ is increasing, then $T_1 \leq_{or} T_2$.*

Proof. Using Theorem 2.2(vi), it is enough to show that

$$\begin{aligned} G_0^{-1}F_TG_T^{-1}G_0(x) - x &= G_0^{-1}H_1FG^{-1}H_2^{-1}G_0(x) - x \\ &= G_0^{-1}H_1G_0G_0^{-1}FG^{-1}G_0G_0^{-1}H_2^{-1}G_0(x) - x \\ &= \psi_1\phi\psi_2^{-1}(x) - x, \end{aligned}$$

or equivalently (since, ψ_2^{-1} is increasing and exist)

$$\psi_1\phi(x) - \psi_2(x),$$

is increasing, where $\phi = G_0^{-1}FG^{-1}G_0$. Under the assumptions, Lemma 2.11 implies that $\psi_1\phi(x) - \psi_1(x)$ is increasing (since, ψ_1^{-1} is increasing). Therefore,

$$\psi_1\phi(x) - \psi_2(x) = \psi_1\phi(x) - \psi_1(x) + \psi_1(x) - \psi_2(x),$$

is increasing. This completes the proof. □

We end this section by giving an application of the odds rate order in comparing the epoch times of two nonhomogeneous Poisson processes (NHPP). Let $T_{1,n}$ and $T_{2,n}$, $n \geq 1$ be the epoch points of two NHPPs with intensity functions h_1 and h_2, respectively, such that $\int_x^\infty h_i(y)\mathrm{d}y = \infty$, $i = 1, 2$, for all $x \geq 0$. Let also X and Y be two nonnegative random variables with hazard rates h_1 and h_2 and with distribution functions F and G, respectively. The distribution functions of $T_{1,n}$ and $T_{2,n}$ can be expressed as (see [12])

$$F_n(t) = \Phi_n F(x), \quad \text{and} \quad G_n(x) = \Phi_n G(x), \quad x \geq 0,$$

respectively, where $\Phi_n(p) = \Gamma_n(-\ln(1-p))$ for $p \in (0, 1)$, and Γ_n is the distribution function of a gamma distribution with scale parameter 1 and shape parameter n.

Theorem 2.14. *If $X \leq_{or} Y$, then $T_{1,n} \leq_{or} T_{2,n}$ for all $n \geq 1$.*

Proof. Under the assumption, Lemma 2.11 follows that

$$\begin{aligned} G_0^{-1}F_nG_n^{-1}G_0(x) - x &= G_0^{-1}\Phi_nFG^{-1}\Phi_n^{-1}G_0(x) - x \\ &= G_0^{-1}\Phi_nG_0G_0^{-1}FG^{-1}G_0G_0^{-1}\Phi_n^{-1}G_0(x) - x, \end{aligned}$$

is increasing, by taking $\phi = G_0^{-1}FG^{-1}G_0$ and $\psi = G_0^{-1}\Phi_nG_0$, which is convex and strictly increasing, since the gamma distribution belongs to the IOR family, for $n \geq 1$. Applying Theorem 2.2(vi), now completes the proof. □

3. Relationship with other stochastic orders

First note that, as shown in [3], the odds rate order implies the usual stochastic order. However, using the distributions in Theorem 8.7 in ([21], p. 289), for instance, one can see that the reverse is not necessarily true. The following counterexample also indicates that the odds rate order does not follow the hazard rate order.

Example 3.1. Let X be distributed as beta with parameters 0.1 and 0.75 and Y be a random variable with density function $g(x) = \ln(2)2^x, 0 < x < 1$. Figure 1 depicts that $X \leq_{or} Y$ but the hazard rate order does not hold.

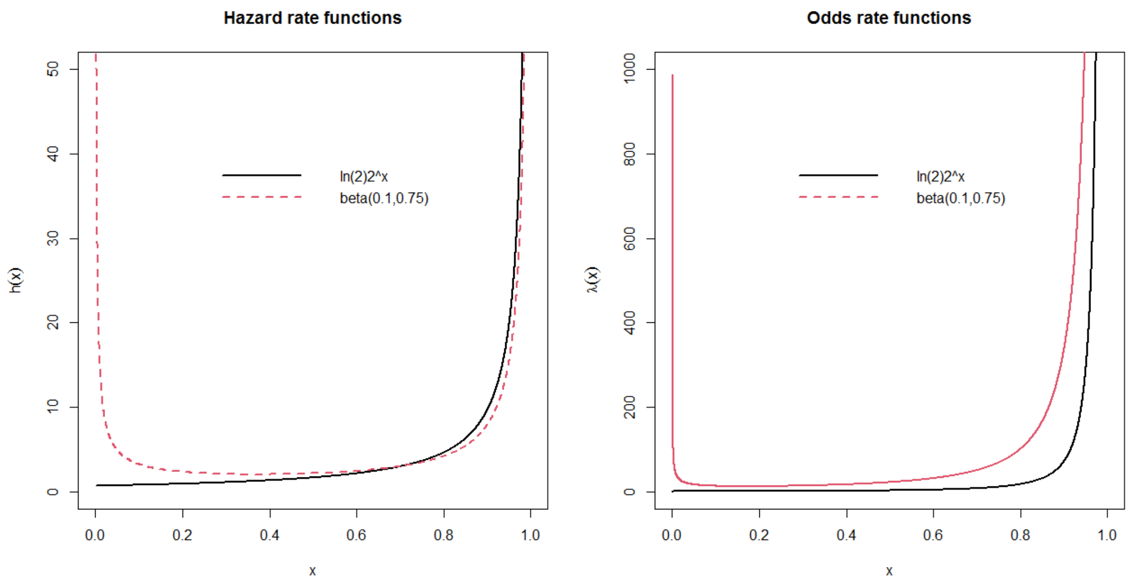

FIGURE 1. Plots of $h(x)$ and $\lambda(x)$ for distributions given in Example 3.1.

Let X and Y be two nonnegative random variables with finite means and suppose that $X \leq_{st} Y$ and that $E(X) < E(Y)$. Then,

$$f_Z(z) = \frac{\bar{G}(z) - \bar{F}(z)}{E(Y) - E(X)}, \ z \geq 0,$$

is a probability density function with corresponding random variable Z. The following theorem describes that the odds rate order can be characterized by means of the likelihood ratio order and an appropriate equilibrium distribution at the presence of the usual stochastic order.

Theorem 3.2. *Let X and Y be two independent nonnegative random variables with finite means such that $X \leq_{st} Y$ and such that $E(Y) > E(X) > 0$. Then $X \leq_{or} Y$ if and only if $U_e \leq_{lr} Z$, where U_e is the equilibrium random variable corresponding to the random variable $U = \min\{X, Y\}$.*

Proof. Using Theorem 2.2 (ii) the order $\leq_{or}$ is equivalent to that, for $x \geq t$,

$$\frac{\bar{F}(x)\bar{G}(x)}{\bar{F}(t)\bar{G}(t)} \leq \frac{\bar{G}(x) - \bar{F}(x)}{\bar{G}(t) - \bar{F}(t)}$$

or

$$\frac{f_{U_e}(x)}{f_Z(x)} \leq \frac{f_{U_e}(t)}{f_Z(t)},$$

which completes the proof.

It is clear form part (iii) and (iv) of Theorem 2.2 that if $X \leq_{or} Y$, then $G_0^{-1}G(X) \leq_{disp} G_0^{-1}F(Y)$. The following two results give another relation between the orders $\leq_{or}$ and $\leq_{disp}$. □

Theorem 3.3. **(a)** *If $X \leq_{or} Y$ and X or $Y \in DOR$, then $X \leq_{disp} Y$.*
(b) *If $X \leq_{disp} Y$ and X or $Y \in IOR$, then $X \leq_{or} Y$.*

Proof. For proving part (a), take $\phi = G_0^{-1}FG^{-1}G_0$ and $\psi = F^{-1}G_0$ or $\psi = G^{-1}G_0$. Under the assumptions, $\phi(x) - x$ is increasing and ψ is convex and strictly increasing. Therefore, it follows from Lemma 2.11 that

$$\psi\phi\psi^{-1}(x) - x = G^{-1}F(x) - x,$$

is increasing, that is $X \leq_{disp} Y$.

To prove part (b), taking $\phi = G^{-1}F$ and $\psi = G_0^{-1}F$ *or* $\psi = G_0^{-1}G$, similarly, implies that

$$G_0^{-1}FG^{-1}G_0(x) - x = G_0^{-1}FG^{-1}FF^{-1}G_0(x) - x = \psi\phi\psi^{-1}(x) - x$$

or

$$G_0^{-1}FG^{-1}G_0(x) - x = G_0^{-1}GG^{-1}FG^{-1}G_0(x) - x = \psi\phi\psi^{-1}(x) - x,$$

is increasing. Applying Theorem 2.2(vi) now completes the proof. □

Remark 3.4. It is easily seen that $X \leq_{or} Y_0$ if and only if $X \leq_{disp} Y_0$, or equivalently, $\lambda_F(x) \geq 1$ where Y_0 is a random variable with distribution function G_0.

Let X be absolutely continuous nonnegative random variable with distribution function F. Denoting, $I_F(x) = \int_0^x \bar{F}(y)\mathrm{d}y$, it follows that $T_F = FI_F^{-1}$ is an absolutely continuous distribution function on $[0, E(X)]$ (I_G and T_G are defined analogously). Note that T_F^{-1} is called the TTT transform corresponding to the distribution function F. Let X_T and Y_T be the corresponding random variables with T_F and T_G, respectively. The following result gives the connection between these distributions.

Theorem 3.5. *Let $X \in IOR$ and $Y \in DOR$. $X \leq_{or} Y$ if and only if $X_T \leq_{hr} Y_T$.*

Proof. First note that, the hazard rate functions corresponding to the random variables X_T and Y_T are $h_{T_F} = \lambda_F I_F^{-1}$ and $h_{T_G} = \lambda_G I_G^{-1}$, respectively. For the only if part, from $X \leq_{or} Y$ it follows that $I_G^{-1}(u) \leq I_F^{-1}(u)$, for all $0 < u < 1$. Thus, if $X \in IOR$, then

$$h_{T_F}(x) = \lambda_F I_F^{-1}(x) \geq \lambda_F I_G^{-1}(x) \geq \lambda_G I_G^{-1}(x) = h_{T_G}(x).$$

For the if part, it follows from $X_T \leq_{hr} Y_T$ that $I_F F^{-1}(u) \leq I_G G^{-1}(u)$ for all $0 < u < 1$ or $I_F(x) \leq I_G G^{-1}F(x)$ for all $x \geq 0$. Also, using Proposition 2 in [3], $Y \in DOR$ is equivalent to that $Y_T \in DFR$. Thus,

$$\lambda_F(x) = h_{T_F} I_F(x) \geq h_{T_G} I_F(x) \geq h_{T_G} I_G G^{-1}F(x) = \lambda_G G^{-1}F(x).$$

or equivalently, $\lambda_F F^{-1}(u) \geq \lambda_G G^{-1}(u)$ which is also equivalent to that $X \leq_{disp} Y$ (*cf.* Eq. (3.B.12) in [2], p. 149). Using Theorem 3.3(b) now yields $X \leq_{or} Y$. This completes the proof. □

Let F and G be absolutely continuous distribution functions with 0 as the left endpoint of their supports and the corresponding densities f and g, respectively. The next result gives the relation between the ordered densities and the $\leq_{or}$ order.

Theorem 3.6. (a) *If $f(x) \geq g(x)$, for all $x \geq 0$, then $X \leq_{or} Y$.*
(b) *If $f(x) - g(x)$ is increasing and $X \leq_{or} Y$, then $f(x) \geq g(x)$.*

Proof. (a) It is easy to see that $f(x) \geq g(x)$ is equivalent to that $F(x) - G(x)$ is increasing which implies that

$$\frac{1}{\bar{F}(x)} - \frac{1}{\bar{G}(x)} = \frac{F(x) - G(x)}{\bar{F}(x)\bar{G}(x)},$$

is increasing and therefore, $X \leq_{or} Y$.

(b) First note that from $X \leq_{or} Y$ it follows that $f(0) \geq g(0)$. On the other hand, $f(x) - g(x) \geq f(0) - g(0) \geq 0$, when $f(x) - g(x)$ is increasing. The proof is now completed. □

3.1. Two relative orderings based on the odds rate

In this subsection, we consider two relative ageing concepts and investigate their relationship with the odds rate order. First, we note that in similar with the increasing (decreasing) failure rare average, abbreviated as IFRA (DFRA), notions (*cf.* [13], p. 84), one may also consider the following notions corresponding to the odds function.

Definition 3.7. The random variable X is said to be increasing (decreasing) odds rate average, abbreviated as IORA (DORA), if $\Lambda_F(x)/x$ is increasing (decreasing) in x.

It is easy to see that, if $X \in IOR(DOR)$, then $X \in IORA(DORA)$.

The relative ageing is a comparison method between lifetime distributions which has been studied in the literature. For example, [14] have studied the comparison of life distributions through a partial ordering which is found to be useful in reliability theory. Indeed, they call a random variable X is aging faster than Y (written as $X \prec_c Y$) if the random variable $H_G(X)$ has an increasing failure rate (IFR) distribution, where $H_G(x) = -\ln \bar{G}(x)$ is the cumulative failure rate function corresponding to G. It has been shown that this is equivalent to that $H_F \circ H_G^{-1}$ is convex or $h_F(x)/h_G(x)$ is increasing, when the failure rates exist and $h_G \neq 0$. [15] have considered two other related orderings which include the increasing cumulative failure ratio property. Motivated by these and considering the IOR notion, we consider the following relative orderings.

Definition 3.8. The random variable X is said to be aging faster than Y in the odds rate (written as $X \prec_c^{OR} Y$) if $\Lambda_G(X)$ has an increasing odds rate (IOR) distribution.

Definition 3.9. The random variable X is said to be aging faster than Y in the odds rate average (written as $X \prec_*^{OR} Y$) if $\Lambda_G(X)$ has an increasing odds rate average (IORA) distribution.

The following characterization result is hold for the new ordering $\prec_c^{OR}$.

Theorem 3.10. *$X \prec_c^{OR} Y$ if and only the following equivalent conditions hold:*

(i) *$\Lambda_F \circ \Lambda_G^{-1}$ is convex;*
(ii) *$\Lambda_F(Y)$ has a decreasing odds rate (DOR) distribution;*
(iii) *$\lambda_F(x)/\lambda_G(x)$ is increasing, when the densities exist and $\lambda_G \neq 0$.*

Proof. Let $Z = \Lambda_G(X)$ and H be the distribution function of Z. Then, $H(x) = F(\Lambda_G^{-1}(x))$ and it follows from the definition that

$$\Lambda_H(x) = G_0^{-1}H(x) = G_0^{-1}(FG^{-1}G_0(x)) = \Lambda_F \circ \Lambda_G^{-1}(x)$$

is convex. This gives part (i). To prove part (ii), let $W = \Lambda_F(Y)$ and K be the distribution function of W. The DOR property of W is equivalent to that $\Lambda_K(x)$ is concave or $\Lambda_K^{-1}(x)$ is convex. This is true because it follows from part(i) that

$$\begin{aligned}\Lambda_K^{-1}(x) = \{G_0^{-1}K(x)\}^{-1} &= \{G_0^{-1}G\Lambda_F^{-1}(x)\}^{-1} \\ &= \{G_0^{-1}GF^{-1}G_0(x)\}^{-1} = \{\Lambda_G\Lambda_F^{-1}(x)\}^{-1} \\ &= \Lambda_F \circ \Lambda_G^{-1}(x),\end{aligned}$$

is convex. Now, part (ii) follows that the derivative of $\Lambda_F \circ \Lambda_G^{-1}(x)$ is increasing. That is $\lambda_F(\Lambda_G^{-1}(x))/\lambda_G(\Lambda_G^{-1}(x))$ is increasing which implies part (iii). Finally, part (iii) easily follows part (i). □

The following theorem gives the analogous characterization result for the relative ordering $\prec_*^{OR}$. The similar proof is omitted.

Theorem 3.11. *$X \prec_*^{OR} Y$ if and only the following equivalent conditions hold:*

(i) *$\Lambda_F \circ \Lambda_G^{-1}(x)/x$ is increasing (i.e., $\Lambda_F \circ \Lambda_G^{-1}$ is star-shaped);*
(ii) *$\Lambda_F(Y)$ has a decreasing odds rate average (DORA) distribution;*
(iii) *$\Lambda_F(x)/\Lambda_G(x)$ is increasing.*

[16] have developed a procedure for testing the proportional odds assumption against that the odds ratio is increasing or in terms of the above relative orderings the corresponding random variables are ordered in $\prec_*^{OR}$. The following are examples of two random variables that are ordered in the above relative orderings.

Example 3.12. Let X and Y be two nonnegative random variables with survival functions

$$\bar{F}(x) = \exp\{a(x + \frac{1}{2}b_1 x^2)\} \text{ and } \bar{G}(x) = \exp\{a(x + \frac{1}{2}b_2 x^2)\},$$

respectively. We have

$$\frac{\lambda_F(x)}{\lambda_G(x)} = \frac{1 + b_1 x}{1 + b_2 x} \exp\{\frac{1}{2}a(b_1 - b_2)x^2\}.$$

Now, using Theorem 3.10(iii), one can see that $X \prec_c^{OR} Y$ if $b_1 \geq b_2$.

Example 3.13. Let X and Y be two nonnegative random variables with survival functions

$$\bar{F}(x) = \frac{1}{1 + (x/\theta_1)^{\alpha_1}} \text{ and } \bar{G}(x) = \frac{1}{1 + (x/\theta_2)^{\alpha_2}},\ x, \theta_i, \alpha_i > 0, i = 1, 2,$$

respectively. Then,

$$\frac{\Lambda_F(x)}{\Lambda_G(x)} = \frac{\theta_2^{\alpha_2}}{\theta_1^{\alpha_1}} x^{\alpha_1 - \alpha_2},$$

which is increasing whenever $\alpha_1 \geq \alpha_2$, irrespective of θ_1 and θ_2. This, using Theorem 3.11(iii), means that $X \prec_*^{OR} Y$.

It is well known that a convex function passing through the origin is star-shaped (*cf.* [13], p. 90). Thus, if $X \prec_c^{OR} Y$, then $X \prec_*^{OR} Y$.

The following corollary is immediately followed from Theorem 3.10.

Corollary 3.14. *If $X \prec_c^{OR} Y$ and $Y \in IOR(X \in DOR)$, then $X \in IOR(Y \in DOR)$.*

Remark 3.15. It is clear from Theorem 3.10 (iii) that when $\lambda_F(0) \geq \lambda_G(0)$, the $\prec_c^{OR}$ order implies the odds rate order. Furthermore, if $X \prec_*^{OR} Y$, then from Theorem 3.11 (iii)

$$\Lambda_F(x) - \Lambda_G(x) = \Lambda_G(x)[\Lambda_F(x)/\Lambda_G(x) - 1],$$

is increasing. Thus, the $\prec_*^{OR}$ order also implies the odds rate order.

The following theorem also shows that the $\prec_*^{OR}$ order implies the hazard rate order.

Theorem 3.16. *If $X \prec_*^{OR} Y$, then $X \leq_{hr} Y$*

Proof. First, from Theorem 3.11(iii), the assumption is equivalent to that

$$\frac{\lambda_F(x)}{\Lambda_F(x)} \geq \frac{\lambda_G(x)}{\Lambda_G(x)}, \tag{3.1}$$

which by integrating on $(0, y)$ w.r.t. x, follows that $\Lambda_F(x) \geq \Lambda_G(x)$ or equivalently $X \leq_{st} Y$ or $F(x) \geq G(x)$. Repeatedly, from 3.1 we have

$$h_F(x) = \lambda_F(x)\bar{F}(x) = \frac{\lambda_F(x)}{\Lambda_F(x)}F(x) \geq \frac{\lambda_G(x)}{\Lambda_G(x)}G(x) = \lambda_G(x)\bar{G}(x) = h_G(x),$$

from which the result is obtained. □

4. Testing the odds rate order

In the present section, we consider the testing hypothesis $H_0 : \lambda_F(x) = \lambda_G(x)$ against the alternative hypothesis $H_1 : \lambda_F(x) > \lambda_G(x)$ or equivalently,

$$H_0 : X \overset{st}{=} Y \text{ against } H_1 : X \leq_{or} Y \text{ and } X \overset{st}{\neq} Y.$$

First note that, $X \leq_{or} Y$ is equivalent to that $f(x)\bar{G}^2(x) \geq g(x)\bar{F}^2(x)$ which by integrating from 0 to ∞ w.r.t x gives that $E[\bar{G}^2(X)] \geq E[\bar{F}^2(Y)]$ or equivalently,

$$E[I(\min\{Y_1, Y_2\} > X_1)] \geq E[I(\min\{X_1, X_2\} > Y_1)],$$

where X_1, X_2 and Y_1, Y_2 are independent copies of X and Y, respectively, and $I(.)$ is the indicator function. Thus, we propose the measure of departure from H_0 by the following functional

$$\Delta(F, G) = E[I(\min\{Y_1, Y_2\} > X_1)] - E[I(\min\{X_1, X_2\} > Y_1)].$$

One can see that $\Delta(F, G)$ is zero under H_0 and positive under H_1. An estimator of $\Delta(F, G)$, now can be used for formulating a test of H_0 versus H_1.

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be samples from the population of X and Y, respectively. A U-statistic estimator of $\Delta(F, G)$ is given by (see, for example [17], p. 370)

$$U = \binom{m}{2}^{-1}\binom{n}{2}^{-1}\sum\sum_{1\leq i<j\leq m}\sum\sum_{1\leq k<l\leq n}\phi(X_i, X_j, Y_k, Y_l), \tag{4.1}$$

where,

$$\begin{aligned}\phi(X_1, X_2, Y_1, Y_2) = \frac{1}{4}[&I(\min\{Y_1, Y_2\} > X_1) - I(\min\{X_1, X_2\} > Y_1)\\ &+ I(\min\{Y_1, Y_2\} > X_2) - I(\min\{X_1, X_2\} > Y_1)\\ &+ I(\min\{Y_1, Y_2\} > X_1) - I(\min\{X_1, X_2\} > Y_2)\\ &+ I(\min\{Y_1, Y_2\} > X_2) - I(\min\{X_1, X_2\} > Y_2)].\end{aligned}$$

It is clear that $E(U) = \Delta(F, G)$, which is zero under H_0 and greater that it under H_1. This suggests rejecting H_0 in favor of H_1 when the statistic U is large enough. The asymptotic distribution of the above test statistic U is given by the following theorem.

Theorem 4.1. *Assume that* $\frac{m}{m+n} \to \rho \in (0,1)$ *as* $m, n \to \infty$. *Then*

$$\sqrt{m+n}U \xrightarrow{d} N(0, \sigma^2),$$

($\xrightarrow{d}$ *stands for convergence in distribution.)* $N(0, \sigma^2)$ *represents the normal random variable with mean* 0 *and variance* $\sigma^2 = \frac{4}{\rho}\sigma_{10}^2 + \frac{4}{1-\rho}\sigma_{01}^2$ *where*

$$\sigma_{10}^2 = Cov[\phi(X_1, X_2, Y_1, Y_2), \phi(X_1, X_2', Y_1', Y_2')]), \sigma_{01}^2 = Cov[\phi(X_1, X_2, Y_1, Y_2), \phi(X_1', X_2', Y_1, Y_2')]),$$

and the X*'s and* Y*'s are independently distributed according to* F *and* G*, respectively.*

Proof. The result immediately follows from Theorem 6.1.4 in ([17], p. 375).

After some algebraic manipulations one can obtain that

$$\begin{aligned}\sigma_{10}^2 &= 0.25(E[\bar{G}^4(X)] - E^2[\bar{G}^2(X)]) - [P(X > Y)E[\bar{G}^2(X)G(X)] - E[\bar{G}^2(X)]E[\bar{F}^2(Y)]] \\ &\quad + P^2(X > Y)E[G^2(X)] - E^2[\bar{F}^2(Y)],\end{aligned}$$

$$\begin{aligned}\sigma_{01}^2 &= 0.25(E[\bar{F}^4(Y)] - E^2[\bar{F}^2(Y)]) - [P(Y > X)E[\bar{F}^2(Y)F(Y)] - E[\bar{F}^2(Y)]E[\bar{G}^2(X)]] \\ &\quad + P^2(Y > X)E[F^2(Y)] - E^2[\bar{G}^2(X)].\end{aligned}$$

Hence, under H_0 one can get that $\sigma_{10}^2 = \sigma_{01}^2 = \frac{23}{360}$.

□

4.1. Simulation study

We have carried out a simulation exercise to assess the performance of the test statistics U. In our simulation, we considered the following situations and the empirical powers of the test statistic was compared for each of the situations.

(a) Pareto model: In this case, X has always Pareto distribution with survival function $\bar{F}(x) = (\frac{1}{x+1})^2, x \geq 0$ whereas Y has a Pareto distribution with survival function $\bar{G}(y) = (\frac{\theta}{y+\theta})^2, y \geq 0$. We have considered $\theta = 1, 1.5, 2, 3, 5$. The null hypothesis corresponds to $\theta = 1$.

(b) Gamma model: In this model, X has always the Gamma distribution with shape parameter 1 and scale parameter 1 whereas Y has a Gamma distribution with shape parameter θ and scale parameter 1. We have considered $\theta = 1, 1.5, 2, 4, 8$. The null hypothesis corresponds to $\theta = 1$.

(c) Beta model: In this model, X has always the Beta distribution with parameters 1 and 1 whereas Y has a Beta distribution with parameters θ and 1. We have considered $\theta = 1, 1.25, 1.5, 2, 4$. The null hypothesis corresponds to $\theta = 1$.

(d) Log Normal model: In this situation, X has always a Log normal distribution with local parameter 0 and scale parameter 1 and Y has a log normal distribution with local parameter θ and scale parameter 1. We have considered $\theta = 0, 0.5, 1, 2, 4$. The null hypothesis corresponds to $\theta = 0$.

In each model, as θ increases, X and Y departures from the null hypothesis being identically distributed and are ordered as $X \leq_{or} Y$.

To assess the performance of the test statistic U, we have compared its empirical power with that of two sample Kolmogorov–Smirnov test (KS). As used in [18], one may compute the critical values of U *via* bootstrap method. In our simulation, for comparison purposes, we have also used the bootstrap version of U (labeled as U^B) as another test. The empirical power was computed for each statistic under a total of 2000 generated couples of independent samples of sizes $n = m = 15, 30, 50, 100, 150$. The power was taken as the fractional number of times, out of 2000, the corresponding statistic exceeded the relevant threshold. For each sample we

TABLE 1. Power comparison between the tests U, U^B, KS, at the significance level $\alpha = 0.05$, under the Pareto model.

$n = m$	Test	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 5$
	U	0.0745	0.2685	0.4150	0.6975	0.9140
15	U^B	0.0370	0.1410	0.2620	0.5330	0.8165
	KS	0.0285	0.0630	0.1475	0.3525	0.6910
	U	0.0815	0.3385	0.6145	0.8965	0.9935
30	U^B	0.0360	0.2205	0.4705	0.8110	0.9820
	KS	0.0385	0.1365	0.3115	0.7145	0.9550
	U	0.0805	0.4465	0.7665	0.9775	0.9995
50	U^B	0.0415	0.3265	0.6555	0.9585	0.9980
	KS	0.0395	0.1980	0.5305	0.9070	0.9990
	U	0.0805	0.6720	0.9620	1.0000	1
100	U^B	0.0385	0.5620	0.9255	0.9995	1
	KS	0.0335	0.3935	0.8345	0.9965	1
	U	0.0730	0.7965	0.9935	1	1
150	U^B	0.0425	0.7070	0.9845	1	1
	KS	0.0385	0.5630	0.9660	1	1

have applied the tests for H_0 with nominal significance level $\alpha = 0.05$. It should be noted that the KS test does not take into account any information about the stochastic dominance under the alternative, so it is not fair to compare its power with that of the tests we proposed here. Still, the KS test is informative to understand better the departures of the alternatives from the null hypothesis.

Tables 1–4 summarize the results of the simulation for each situation. One can see from the tables that the power of all tests against any alternative show an increasing pattern with respect to sample size. This reveals the consistency of the test statistics. The results show that the bootstrap based test (U^B) performances better than the normal based test (U) in maintaining the type I error level specially for small sample sizes. The test U tends to have bigger power and type I error. This may be comes from the fact that the asymptotic variance takes small values under H_0. Over all, the proposed test statistics performance well in discriminating between the null and alternative hypothesis.

For further investigation of the performance of the proposed test statistic, we also considered situations where the distributions of the two random variables belong to different families. For this purpose, we considered the following cases in which the corresponding random variables are ordered as $\leq_{or}$ (except for case (V) in which $X \nleq_{or} Y$) and the empirical powers of the test statistics were compared in similar with the above situations.

- **(I)** X is distributed as beta with parameters 0.1 and 0.75 and Y is a random variable with density function $g(y) = \ln(2)2^y, 0 < y < 1$.
- **(II)** X is distributed as exponential with scale parameter 1 and Y has a log normal distribution with local parameter 2 and scale parameter 1.
- **(III)** X is distributed as exponential with scale parameter 1 and Y has a log normal distribution with local parameter 0 and scale parameter 1.
- **(IV)** X has Weibull distribution with shape parameter 0.5 and scale parameter 1 and Y has a the Birnbaum–Saunders (BS) distribution with distribution function $G(y) = \Phi(0.5(\sqrt{y} - \frac{1}{\sqrt{y}})), y > 0$ (Φ denotes the standard normal distribution function).
- **(V)** X is distributed as Weibull distribution with shape parameter 2 and scale parameter 1 and Y has a log normal distribution with local parameter 0 and scale parameter 1.

TABLE 2. Power comparison between the tests U and KS, at the significance level $\alpha = 0.05$, under the Gamma model.

$n = m$	Test	$\theta = 1$	$\theta = 1.5$	$\theta = 2$	$\theta = 4$	$\theta = 8$
15	U	0.0820	0.5385	0.8825	1.000	1
	U^B	0.0350	0.3560	0.7640	1.000	1
	KS	0.0265	0.1650	0.4765	0.998	1
30	U	0.0805	0.7545	0.9865	1	1
	U^B	0.0380	0.6060	0.9630	1	1
	KS	0.0365	0.3595	0.8490	1	1
50	U	0.0790	0.8910	0.9990	1	1
	U^B	0.0415	0.8150	0.9975	1	1
	KS	0.0370	0.6075	0.9830	1	1
100	U	0.0725	0.9910	1	1	1
	U^B	0.0365	0.9785	1	1	1
	KS	0.0350	0.8925	1	1	1
150	U	0.0845	0.9995	1	1	1
	U^B	0.0530	0.9980	1	1	1
	KS	0.0460	0.9795	1	1	1

TABLE 3. Power comparison between the tests U and KS, at the significance level $\alpha = 0.05$, under the Beta model.

$n = m$	Test	$\theta = 1$	$\theta = 1.25$	$\theta = 1.5$	$\theta = 2$	$\theta = 4$
15	U	0.0930	0.2045	0.3685	0.636	0.9775
	U^B	0.0425	0.1115	0.2045	0.465	0.9370
	KS	0.0345	0.0380	0.0840	0.243	0.7365
30	U	0.0825	0.2845	0.5320	0.8545	0.9995
	U^B	0.0390	0.1610	0.3720	0.7505	0.9995
	KS	0.0350	0.0790	0.1805	0.4705	0.9805
50	U	0.0815	0.3595	0.708	0.9570	1
	U^B	0.0460	0.2365	0.568	0.9205	1
	KS	0.0365	0.1015	0.318	0.7275	1
100	U	0.0815	0.5405	0.8985	1.0000	1
	U^B	0.0435	0.4090	0.8340	1.0000	1
	KS	0.0345	0.1985	0.5485	0.9665	1
150	U	0.0745	0.6405	0.968	1.000	1
	U^B	0.0430	0.5170	0.938	1.000	1
	KS	0.0445	0.2950	0.765	0.997	1

The results are summarized in Table 5. It is clear from the table that the proposed test still perform well when the random variables belong to the different family of distributions even for small sample sizes.

It should be mentioned that when none of the null and the alternative hypothesis is true, the behaviour of the proposed test is not known. This happens for any test of this kind in which the null and alternative hypothesis are not the complement of each other. For example, for the case (V) in Table 5 in which $X \nleq_{or} Y$, the test tends to not rejecting null hypothesis which is a confusing result. A behaviour controllable test can be provided by dealing with testing $H_0 : X \leq_{or} Y$ against $H_1 : X \nleq_{or} Y$ which is still under our investigation.

TABLE 4. Power comparison between the tests U and KS, at the significance level $\alpha = 0.05$, under the Log normal model.

$n=m$	Test	$\theta=0$	$\theta=0.5$	$\theta=1$	$\theta=2$	$\theta=4$
15	U	0.0850	0.4700	0.8865	1.0000	1
	U^B	0.0345	0.2975	0.7795	0.9985	1
	KS	0.0345	0.1405	0.5295	0.9920	1
30	U	0.0775	0.6790	0.9865	1	1
	U^B	0.0390	0.5235	0.9635	1	1
	KS	0.0350	0.3190	0.8800	1	1
50	U	0.0790	0.8385	0.9985	1	1
	U^B	0.0395	0.7340	0.9975	1	1
	KS	0.0365	0.5295	0.9865	1	1
100	U	0.0855	0.9750	1	1	1
	U^B	0.0475	0.9450	1	1	1
	KS	0.0345	0.8325	1	1	1
150	U	0.0765	0.9980	1	1	1
	U^B	0.0500	0.9905	1	1	1
	KS	0.0445	0.9575	1	1	1

TABLE 5. Power comparison between the tests U and KS, at the significance level $\alpha = 0.05$, under the cases $(I) - (V)$.

$n=m$	Test	I	II	III	IV	V
15	U	0.9995	1	0.4185	0.4905	0.1785
	KS	0.9870	1	0.0890	0.1075	0.1110
50	U	1	1	0.7640	0.8555	0.2410
	KS	1	1	0.3375	0.4440	0.5805

4.2. Real data analysis

Example 4.2. [19] used mice cancer data to illustrate a representation of mortality data by competing risks. These data consist of the number of days until developing thymic lymphomia after 300 rads of radiation was given to a group of male mice. A group of 29 mice was kept in a germ-free environment, while the control group consisted of the other 22. [20] have analyzed these data and pointed out to faster aging of the group exposed to 300 rads of radiation. The P-values of the test statistics KS, U^B and U are $0.141, 0.370, 0.304$, respectively. That is, all the test statistics accept the equality of the distribution of the two groups.

5. Conclusion

Recently, pointing out that the IFR family excludes several important models, such as bathtub or heavy tailed distributions, [3] introduced an alternative wider family of broader applicability called increasing odds rate (IOR) family. Studying the properties of the IOR family, they have also defined the new odds rate stochastic order. In this paper, we obtained some new properties on this order including the characterization results and the relationship with other well-known stochastic orders such as the hazard rate, dispersive order and two new relative orderings based on the odds rate functions. We were also able to provide a statistical test for testing the odds rate order alternative. A simulation study for assessing the performance of the test and a real lifetime data analysis were given. In similar with its own kind, the behaviour of the proposed test when none of the null and alternative hypothesis are true is unknown. Providing a test for the ordered in the odds rate null hypothesis against not ordered in the odds rate is a more thorough approach which is under our investigation.

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