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SPLITTING FOR SOME CLASSES OF HOMEOMORPHIC AND COALESCING STOCHASTIC FLOWS

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Abstract. The splitting scheme (the Kato–Trotter formula) is applied to stochastic flows with common noise of the type introduced by Th.E. Harris. The case of possibly coalescing flows with continuous infinitesimal covariance is considered and the weak convergence of the corresponding finite-dimensional motions is established. As applications, results for the convergence of the associated pushforward measures and dual flows are given. Similarities between splitting and the Euler-Maruyama scheme yield estimates of the speed of the convergence under additional regularity assumptions.

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1. Introduction

Introduced (with zero drift) in [1], Harris flows are families of random transformation of the real line that represent the joint movement of one-dimensional interacting Brownian particles whose pairwise correlation depends on the distance between them and is given *via* so-called infinitesimal covariance φ. Since coalescence is possible, such transformations are not, in contrast to the case of diffeomorphic flows obtained as solutions to SDEs with sufficiently smooth coefficients [2], necessarily continuous. One natural and straightforward extension of the notion of the Harris flow is to add drift to affect the motion of particles in a way similar to the case of the Arratia flow in [3]. This brings us closer to biological and physical models that use potentials of different forms [4, 5] while introducing common noise as in [6, 7] including one that forces particles to collide.

The main goal of the paper is to apply the well-known method of splitting in the stochastic setting [8–16] to Harris flows, so that the actions of the semigroups generated by the corresponding driftless Harris flow and the ordinary ODE are separated.

The formal definition of a Harris flow with drift adopted in the paper is based of the definition of a driftless Harris flow from [17] (see also [1, 18, 19]), with a minor modification as in [20]. Let $D^{\uparrow}(\mathbb{R})$ be a separable metrizable topological space of non-decreasing càdlàg functions on $\mathbb{R}$ equipped with the Skorokhod J_1 topology [21, 22]. Since for any $f, g \in D^{\uparrow}(\mathbb{R})$ the composition $f \circ g \in D^{\uparrow}(\mathbb{R})$ [23], Lemma 13.2.4, and, for $D^{\uparrow}(\mathbb{R})-$valued random elements ξ, η, we have

$$\{\xi \circ \eta(t) \geq a\} = \left\{\eta(t) \geq \hat{\xi}(a)\right\},$$

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where $\hat{f}$ is the càglàd generalized inverse of f, the composition $\xi \circ \eta$ defines a random element in $D^{\uparrow}(\mathbb{R})$. The space of $\mathbb{R}^d-$valued càdlàg functions with non-decreasing coordinates endowed with the J_1-topology is denoted by $D^{\uparrow}(\mathbb{R}^d)$, and standard Skorokhod spaces of functions on $[0,T], T>0$, with values in $\mathbb{R}^d$ are denoted $D([0,T],\mathbb{R}^d)$.

Remark 1.1. Standard sources discuss spaces $D([0,\infty),\mathbb{R}^d)$ or $D([0,T),\mathbb{R}^d)$ but the extension of the parametric set to the whole $\mathbb{R}$ can be found in [24].

Definition 1.2. A Harris flow $X^{\varphi,a}$ with the infinitesimal covariance φ and drift a is a family of $D^{\uparrow}(\mathbb{R})-$valued random elements $\{X^{\varphi,a}_{s,t}(\cdot) \mid 0 \le s \le t\}$ such that

1. for any $s \le t \le r$, $\mathrm{P}\left(X^{\varphi,a}_{s,r} = X^{\varphi,a}_{t,r} \circ X^{\varphi,a}_{s,t}\right) = 1$; $X^{\varphi,a}_{s,s} = \mathrm{Id}$ a.s. (Id is the identity mapping);
2. for any $t_1 \le t_2 \le \ldots \le t_n$ random elements $X^{\varphi,a}_{t_1,t_2}, \ldots, X^{\varphi,a}_{t_{n-1},t_n}$ are independent;
3. for any $s,t,h > 0, s < t$, $\mathrm{Law}\left(X^{\varphi,a}_{s,t}\right) = \mathrm{Law}\left(X^{\varphi,a}_{s+h,t+h}\right)$;
4. as $h \to 0+$, $X^{\varphi,a}_{0,h} \to \mathrm{Id}$ in probability in $D^{\uparrow}(\mathbb{R})$;
5. for any $x \in \mathbb{R}, s \ge 0$, the process

$$t \mapsto w_t(x,s) \equiv X^{\varphi,a}_{s,t}(x) - x - \int_s^t a(X^{\varphi,a}_{s,r}(x))\mathrm{d}r, \quad t \ge s,$$

   is a $(\mathcal{F}^{X^{\varphi,a}}_{s,r})_{r\ge s}-$Wiener process started at 0, where

$$\mathcal{F}^{X^{\varphi,a}}_{s,t} = \sigma\left\{X^{\varphi,a}_{u_1,u_2}, s \le u_1 \le u_2 \le t\right\}, \quad 0 \le s \le t;$$

6. for any $x,y \in \mathbb{R}, s \ge 0$

$$\langle w(x,s), w(y,s)\rangle_t = \int_s^t \varphi\left(X^{\varphi,a}_{s,r}(x) - X^{\varphi,a}_{s,r}(y)\right)\mathrm{d}r, \quad t \ge s.$$

The following splitting scheme is used. Let $0 = t_0 < t_1 < \ldots < t_m = T$ for some T. Define recursively piecewise continuous processes (u,y) such that for any $k = \overline{0, m-1}$

$$\begin{aligned} u_t &= y_{t_k-} + \int_{t_k}^t a(u_s)\,\mathrm{d}s, \\ y_t &= X^{\varphi,a}_{t_k,t}\left(u_{t_{k+1}-}\right), \\ &\quad t \in [t_k, t_{k+1}), \end{aligned}$$

with $y_{0-} = x$.

We establish the weak convergence of finite-dimensional motions in Skorokhod spaces in the general case of continuous φ as the size of a partition tends to 0 (Thm. 5.1). This result is used to derive the convergence of the pushforward measures under the actions of the corresponding flows under an additional assumption that guarantees the initial flow to be a coalescing one (Thm. 6.1). As a second application, the convergence of the associated dual flows in the reversed time is established (Thm. 7.5).

If a Harris flow admits a representation as the unique strong solution of an SDE (see Sect. 3 for details), which corresponds to the additional assumption of $\sqrt{1-\varphi}$ being Holder continuous of order $\beta \ge \frac{1}{2}$, one can (almost) mechanically transfer proofs and conclusions for the Euler-Maruyama scheme [10, 12] into our setting (Thm. 8.1). We emphasize the highly derivative nature of the results in this case. To formulate the results, the Wasserstein distance on the space of distributions of random measures is chosen (Thm. 8.5).

2. Existence of Harris flows with non-trivial drift

Sufficient conditions for $X^{\varphi,a}$ to exist and be a coalescing flow are given in this section. Hereinafter $\mathbb{R}_+ = (0, \infty)$.

Definition 2.1. A continuous symmetric function $\varphi\colon \mathbb{R} \mapsto \mathbb{R}$ belongs to Φ_* if

1. φ is strictly positive definite;
2. $\varphi(0) = 1$;
3. φ is Lipschitz continuous outside any neighborhood of 0.

Definition 2.2. A function $\varphi \in \Phi_*$ belongs to Φ_α for some $\alpha \in (0, 2]$ if for some $C_\varphi, \widetilde{C}_\varphi > 0$

$$1 - \varphi(x) \geq C_\varphi |x|^\alpha, \quad x \in [-\widetilde{C}_\varphi, \widetilde{C}_\varphi]. \tag{2.1}$$

Definition 2.3. A measurable function $a\colon \mathbb{R} \mapsto \mathbb{R}$ belongs to A_β for some $\beta \in \mathbb{R}$ if there exists a non-negative function $\rho \in C(\mathbb{R}_+)$ such that for some $C_\rho, \widetilde{C}_\rho > 0$

$$\begin{aligned} |a(x+y) - a(x)| &\leq \rho(y), \quad x \in \mathbb{R}, y \in \mathbb{R}_+, \\ \rho(x) &\leq C_\rho x^\beta, \quad x \in (0, \widetilde{C}_\rho], \\ \rho(x) &\leq C_\rho(1 + |x|), \quad x \in \mathbb{R}_+. \end{aligned}$$

Remark 2.4. If $\varphi \in \Phi_*$, then $\varphi(x) < 1$ when $x \neq 0$ and φ is bounded. A function $a \in A_\beta$ does not have to be right continuous or bounded at 0 if $\beta < 0$.

Example 2.5. If $\varphi(x) = \mathrm{e}^{-|x|^\alpha}, x \in \mathbb{R}$, for $\alpha \in (0, 2]$, then $\varphi \in \Phi_\alpha$. Here $\alpha = 2$ corresponds to a homeomorphic non-coalescing flow.

Example 2.6. The Brownian web (the Arratia flow) [25] can be seen as an extreme example of the Harris flow with $\varphi(x) = 1\!\!\mathrm{I}[x = 0]$. In this case particles are independent before a collision.

Example 2.7. Consider the following example of $\varphi \in \Phi_{3/2}$ that does not satisfy the Holder condition of any positive order. Given positive C_1, C_2 with $C_1 + C_2 = 1$ set

$$\begin{aligned} \varphi(x) &= C_1 \mathrm{e}^{-\frac{x^2}{2}} + C_2 \varphi_1(x), \\ \varphi_1(x) &= \sum_{n \geq 1} \frac{\cos(\mathrm{e}^{n/2} n^{3/2} x)}{n^2}. \end{aligned}$$

Then for sufficiently large $m \in \mathbb{N}$ and $x \in (\mathrm{e}^{-2(m+1)}, \mathrm{e}^{-2m}]$

$$\varphi_1(0) - \varphi_1(x) \geq \frac{m\mathrm{e}^{-3m}}{3\mathrm{e}^4} \geq \frac{\left(\mathrm{e}^{-2m}\right)^{3/2}}{3\mathrm{e}^4} \geq \frac{x^{3/2}}{3\mathrm{e}^4}.$$

On the other hand, for any $k \in \mathbb{N}$ and $x_m = \mathrm{e}^{-km}, m \in \mathbb{N}$,

$$\varphi_1(0) - \varphi_1(x_m) \geq \frac{(2k-1) m \mathrm{e}^{(2k-1)m} x_m^2}{3} = \frac{2k-1}{3k} x_m^{\frac{1}{k}} \log x_m^{-1}.$$

Theorem 2.8. *Suppose $\varphi \in \Phi_*$ and a is measurable and of linear growth. Then the Harris flow $X^{\varphi,a}$ exists and is unique in distribution.*

Theorem 2.9. *Suppose that $\varphi \in \Phi_\alpha$ and $a \in A_\beta$ with $\beta - \alpha > -1, \alpha < 2$. Then for any $C, t \in \mathbb{R}_+$*

$$\mathrm{E}\,\sharp \left\{ X^{\varphi;a}_{0,t}(u) \mid u \in [-C, C] \right\} < \infty. \tag{2.2}$$

Either result is essentially an extension of the corresponding theorem in [1]. Still, original proofs need to be modified to accommodate the presence of non-zero drift, so we present a brief description of the necessary changes in Appendix A.

3. Harris flows as solutions to SDEs

This section describes the approach of [17] that provides the representation of $X^{\varphi,a}$ as a solution to a SDE w.r.t a cylindrical Wiener process.

Let H_φ be the separable Hilbert space obtained as the completion of

$$\operatorname{span}\Big\{ \sum_{k=\overline{1,n}} a_k \varphi(x_k - \cdot), a_k, x_k \in \mathbb{R}, k = \overline{1,n}, n \in \mathbb{N} \Big\}$$

w.r.t the inner product $(\varphi(x - \cdot), \varphi(y - \cdot))_{H_\varphi} = \varphi(x - y)$. Define

$$M_{s,t}(x) = X^{\varphi,a}_{s,t}(x) - \int_s^t a(X^{\varphi,a}_{s,r}(x))\mathrm{d}r, \quad 0 \le s \le t, x \in \mathbb{R}.$$

Proposition 3.1 ([17], p. 351 + p. 356)**.** *Assume that $\varphi \in C(\mathbb{R})$ and a is measurable and of linear growth. Then there exists a standard cylindrical Wiener process W on H_φ such that*

$$(W_t - W_s, \varphi(x - \cdot))_{H_\varphi} = L_2 - \lim_{n\to\infty} \sum_{k=\overline{0,n}} \left(M_{s+\frac{k}{n}(t-s), s+\frac{k+1}{n}(t-s)}(x) - x \right).$$

We denote such W as $\mathcal{W}(X^{\varphi,a})$.

Assume that $e_n, n \in \mathbb{N}$, is an orthonormal basis in H_φ. Then given $w^n_\cdot = (W_\cdot, e_n)_{H_\varphi}, n \in \mathbb{N}$,

$$\int_s^t \sigma(X^{\varphi;a}_{s,r}(x))\mathrm{d}W_r = \sum_{n\ge 1} \int_s^t e_n(X^{\varphi;a}_{s,r}(x))\mathrm{d}w^n_r = \int_s^t W\left(X^{\varphi;a}_{s,r}(x), \mathrm{d}r\right),$$

where the last integral is understood in the sense of [2] and $\sigma(x), x \in \mathbb{R}$, are Hilbert–Schmidt operators from H_φ to $\mathbb{R}$:

$$\sigma(x)(h) = \sum_{n\ge 1} (e_n, h)_{H_\varphi} e_n(x), \quad h \in H_\varphi.$$

Proposition 3.2 ([17], p. 356)**.** *Under the same assumptions as in Proposition 3.1 for all $x \in \mathbb{R}$ and $s \ge 0$ with probability 1*

$$X^{\varphi;a}_{s,t}(x) = x + \int_s^t a(X^{\varphi;a}_{s,r}(x))\mathrm{d}r + \int_s^t \sigma(X^{\varphi;a}_{s,r}(x))\mathrm{d}W_r, \quad t \ge s. \tag{3.1}$$

Here

$$\|\sigma(x)\|_{HS} = 1,$$
$$\|\sigma(x) - \sigma(y)\|^2_{HS} = 2(1 - \varphi(x - y)), \quad x, y \in \mathbb{R},$$

where $\|\cdot\|_{HS}$ is the corresponding Hilbert–Schmidt norm.

Both propositions are formulated in [17] without proofs and in the case of $a = 0$, so we need to justify them for nontrivial drift. Sketches of the corresponding proofs are presented in Appendix B.

We denote the space of Holder continuous functions of order β on $\mathbb{R}$ by $H_\beta(\mathbb{R})$ henceforth.

Theorem 3.3 ([17], p. 356)**.** *If* $\sqrt{1-\varphi} \in H_\beta(\mathbb{R}), \beta \geq \frac{1}{2}, a \in Lip(\mathbb{R})$, *then for every* $x \in \mathbb{R}$ *and* $s \geq 0$ *the process* $X^{\varphi;a}_{s,\cdot}(x)$ *is the unique strong solution of* (3.1)*.*

4. The splitting scheme and the example of the Brownian web

Only $a \in Lip(\mathbb{R})$ is considered hereinafter unless stated otherwise explicitly. $T > 0$ is fixed. We define the composition of $m \in \mathbb{N}$ functions by

$$\underset{j=\overline{1,m}}{\circ} f_k = \Big(\underset{j=\overline{2,m}}{\circ} f_k \Big) \circ f_1.$$

Consider a sequence $\mathcal{T} = (\{t^n_j \mid j = \overline{0, N^n}\})_{n\in\mathbb{N}}$ of partitions of $[0, T]$:

$$0 = t^n_0 < \ldots t^n_{N^n} = T, \quad n \in \mathbb{N},$$

and set

$$\begin{aligned} I^n_k &= [t^n_k, t^n_{k+1}), \quad k = \overline{0, N^n - 2}, \\ I^n_{N^n-1} &= [t^n_{N^n-1}, T], \\ \delta^n_k &= t^n_{k+1} - t^n_k, \quad k = \overline{0, N^n - 1}, \\ \delta_n &= \max_{k=\overline{0,N^n-1}} \delta^n_k, \end{aligned}$$

with $\delta_n \to 0, n \to \infty$.

Let $F_t(x), t \geq 0$, be the solution to

$$\begin{aligned} \frac{\mathrm{d}F_t(x)}{\mathrm{d}t} &= a(F_t(x)), \\ F_0(x) &= x, \end{aligned}$$

for $x \in \mathbb{R}$.

For all $n \in \mathbb{N}$ set $y^n_{0-}(x) = x$ and define processes $(u^n, y^n) \in D([0, T], \mathbb{R}^2)$ such that for $k = \overline{0, N^n - 1}$ and $t \in I^n_k$ (*cf.* [26], equation (2.1) for the Brownian web)

$$\begin{aligned} u^n_t(x) &= F_{t-t^n_k}\left(y^n_{t^n_k-}(x)\right) = F_{t-t^n_k} \circ \Big[\underset{j=\overline{0,k-1}}{\circ} \Big(X^{\varphi,0}_{t^n_j, t^n_{j+1}} \circ F_{t^n_{j+1}-t^n_j} \Big) \Big](x), \\ y^n_t(x) &= X^{\varphi,0}_{t^n_k, t}\left(u^n_{t^n_{k+1}-}(x)\right) \\ &= X^{\varphi,0}_{t_{n_k}, t} \circ F_{t^n_{k+1}-t^n_k} \circ \Big[\underset{j=\overline{0,k-1}}{\circ} \Big(X^{\varphi,0}_{t^n_j, t^n_{j+1}} \circ F_{t^n_{j+1}-t^n_j} \Big) \Big](x). \end{aligned} \tag{4.1}$$

If $\varphi \in \Phi_*$ and $\sqrt{1-\varphi} \in H_\beta(\mathbb{R}), \beta \geq \frac{1}{2}$, (4.1) is equivalent in distribution to the following system. Let $W = \mathcal{W}(X^{\varphi,a})$. Formally define

$$\begin{aligned} u_t^n(x) &= y_{t_k^n-}^n(x) + \int_{t_k^n}^t a(u_s^n(x))\mathrm{d}s, \\ y_t^n(x) &= u_{t_{k+1}^n-}^n(x) + \int_{t_k^n}^t \sigma(y_s^n(x))\mathrm{d}W_s, \\ y_{0-}^n(x) &= x, \\ &t \in I_k^n, k = \overline{0, N^n-1}, n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Borrowing notation from [12] and setting

$$\begin{aligned} \overline{\mathrm{d}}_t^n &= \max\{t_k^n \mid t_k^n \leq t\}, \\ \mathrm{d}_t^n &= \min\{t_k^n \mid t_k^n > t\}, \quad t \in [0,T), \\ \mathrm{d}_T^n &= t_{N^n-1}^n,\ \overline{\mathrm{d}}_T^n = T, \end{aligned}$$

we can rewrite both (4.1) and (4.2) as

$$\begin{aligned} u_t^n(x) &= x + \int_0^t a(u_s^n(x))\mathrm{d}s + w_{\mathrm{d}_t^n}^n(x), \\ y_t^n(x) &= x + \int_0^{\overline{\mathrm{d}}_t^n} a(u_s^n(x))\mathrm{d}s + w_t^n(x), \\ &t \in [0,T], n \in \mathbb{N}, \end{aligned}$$

where $w^n(x)$ are standard Wiener processes. In the case of (4.2)

$$w_t^n(x) = \int_0^t \sigma(y_s^n(x))\mathrm{d}W_s, \quad t \in [0,T], n \in \mathbb{N}.$$

The collection $(u_t^n(\cdot), y_t^n(\cdot)), t \in [0,T]$, can be considered as a $D^{\uparrow}(\mathbb{R}^2)-$valued random process in either case.

Definition 4.1. We denote $((u^n, y^n))_{n\in\mathbb{N}}$ by $\mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ in the case of (4.1) and by $\widetilde{\mathrm{Spl}}(X^{\varphi,a}; a; \mathcal{T})$ in the case of (4.2), respectively.

We summarize the discussion of the well-posedness of (4.1) and (4.2) as follows.

Proposition 4.2. *1. If $\varphi \in \Phi_*, a \in Lip(\mathbb{R})$, then $\mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ is unique in distribution.*
2. If $\sqrt{1-\varphi} \in H_\beta(\mathbb{R}), \beta \geq \frac{1}{2}$, additionally, then for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ the pair $(y_\cdot^n(x), u_\cdot^n(x))$ from $\widetilde{\mathrm{Spl}}(X^{\varphi,a}; a; \mathcal{T})$ is the unique strong $(\mathcal{F}_{0,s}^{X^{\varphi,a}})_{s\in[0,T]}-$adapted solution to the system (4.2); moreover, $\mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ and $\widetilde{\mathrm{Spl}}(X^{\varphi,a}; a; \mathcal{T})$ are identically distributed.

The proof of the following lemma uses reasoning similar to that in [27], pp. 171–172 and is therefore omitted.

Lemma 4.3. *Assume that $((u^n, y^n))_{n\in\mathbb{N}} = \widetilde{\mathrm{Spl}}(X^{\varphi,a}; a; \mathcal{T})$, $W = \mathcal{W}(X^{\varphi,a})$ and $\varphi \in \Phi_*$, $\sqrt{1-\varphi} \in H_\beta(\mathbb{R})$ for some $\beta \geq \frac{1}{2}$. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists, possibly on an extension of the original probability space, a* *standard Wiener process $b^n(x)$ such that*

$$\int_0^t \left(\sigma\left(X_{0,s}^{\varphi;a}(x)\right) - \sigma\left(y_s^n(x)\right)\right) \mathrm{d}W_s = 2^{1/2} \int_0^t \left(1 - \varphi\left(X_{0,s}^{\varphi,a}(x) - y_s^n(x)\right)\right)^{1/2} db_s^n(x),$$
$$t \in [0,T].$$

Define

$$r_t^n(x) = -\int_t^{\overline{\mathrm{d}}_t^n} a(u_s^n(x))\mathrm{d}s,$$
$$l_t^n(x) = w_t^n(x) - w_{\mathrm{d}_t^n}^n(x),$$
$$t \in [0,T], x \in \mathbb{R}, n \in \mathbb{N},$$

so

$$y^n(x) - u^n(x) = l^n(x) + r^n(x). \tag{4.3}$$

Lemma 4.4. *Let $((u^n, y^n))_{n\in\mathbb{N}} = \mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$.*

1. *For any $x \in \mathbb{R}$ and $p \geq 1$ there exists $C = C(p,x,T)$ such that*

$$\sup_{t\in[0,T]} |u_t^n(x)|^p \leq C\Big(1 + \sup_{t\in[0,T]} |w_t^n(x)|\Big)^p, \quad n \in \mathbb{N},$$
$$\sup_{n\in\mathbb{N}} \mathrm{E} \sup_{t\in[0,T]} |u_t^n(x)|^p \leq C.$$

2. *For $p \geq 2$ there exists $C_1 = C_1(p)$ such that*

$$\mathrm{E} \sup_{t\in[0,T]} |r_t^n(x)|^p \leq TC_1 \,\mathrm{E}\Big(\int_0^T (1 + |u_s^n(x)|)^2 \,\mathrm{d}s\Big)^{p/2} (\delta_n)^{p/2-1}, \quad x \in \mathbb{R}, n \in \mathbb{N},$$
$$\sup_{x\in\mathbb{R}} \mathrm{E} \sup_{t\in[0,T]} |l_t^n(x)|^p \leq TC_1 (\delta_n)^{p/2-1}, \quad n \in \mathbb{N}.$$

3. *For any $x \in \mathbb{R}$ and $p \geq 2$ there exists $C_2 = C_2(x,T)$ such that*

$$\sup_{n\in\mathbb{N}} \mathrm{E} \sup_{t\in[0,T]} |r_t^n(x)|^p \leq C_2,$$
$$\sup_{n\in\mathbb{N}} \mathrm{E} \sup_{t\in[0,T]} |l_t^n(x)|^p \leq C_2.$$

4. *There exists $C_3 = C_3(T)$*

$$\sup_{x\in\mathbb{R}} \mathrm{E} \sup_{t\in[0,T]} \left(l_t^n(x)\right)^2 \leq C_3 \delta_n \log \delta_n^{-1}, \quad n \in \mathbb{N}.$$

5. *For any $x \in \mathbb{R}$ there exists $C_4 = C_4(x,T)$*

$$\mathrm{E} \sup_{t\in[0,T]} \left(r_t^n(x)\right)^2 \leq C_4 \delta_n, \quad n \in \mathbb{N}.$$

Proof. We drop the argument x which is assumed to be fixed.

(1) The inequality follows trivially by the Gronwall lemma.

(2) We have for some $\widetilde{C}$

$$\begin{aligned}
\mathrm{E}\sup_{t\in[0,T]}|r_t^n|^p &\le \sum_{k=\overline{0,N^n-1}} \mathrm{E}\sup_{t\in I_k^n}\Big|\int_t^{\overline{\mathrm{d}}_t^n} a\left(u_s^n\right)\mathrm{d}s\Big|^p \\
&\le \widetilde{C}\,\mathrm{E}\Big(\int_0^T \left(1+|u_s^n|\right)^2\mathrm{d}s\Big)^{p/2} \sum_{k=\overline{0,N^n-1}} (\delta_k^n)^{p/2}, \\
\mathrm{E}\sup_{t\in[0,T]}|l_t^n|^p &\le \sum_{k=\overline{0,N^n-1}} \mathrm{E}\sup_{t\in I_k^n}\Big|w_t^n - w_{\mathrm{d}_t^n}^n\Big|^p \\
&\le \widetilde{C}\sum_{k=\overline{0,N^n-1}} (\delta_k^n)^{p/2}.
\end{aligned}$$

(3) is a corollary of (1) and (2).

(4) Set

$$\xi_k^n = \sup_{s\in I_k^n}\Big(w_s^n - w_{t_k^n}^n\Big)^2, \quad k=\overline{0,N^n-1}.$$

Then we need to estimate

$$\mathrm{E}\sup_{t\in[0,T]}\Big(w_t^n - w_{\mathrm{d}_t^n}^n\Big)^2 = \int_0^\infty \Big[1-\mathrm{P}\Big(\max_{k=\overline{0,N^n-1}}\xi_k^n\le u\Big)\Big]\mathrm{d}u.$$

Since

$$\mathrm{P}\Big(\max_{k=\overline{0,N^n-1}}\xi_k^n\le u\Big) = \mathrm{E}\,\mathrm{P}\Big(\xi_{N^n-1}^n\le u|\mathcal{F}_{t_{N^n-1}^n}^{X^{\varphi,0}}\Big)\prod_{k=\overline{0,N^n-2}}1\!\!\mathrm{I}\left[\xi_k^n\le u\right]$$

and, for a standard Wiener process w and $k=\overline{0,N^n-1}$,

$$\begin{aligned}
\mathrm{P}\Big(\xi_k^n\le u|\mathcal{F}_{t_k^n}^{X^{\varphi,0}}\Big) &\ge 1-2\,\mathrm{P}\Big(\sup_{s\in I_k^n} w_s\ge u^{1/2}\Big) \\
&= 1-\frac{4}{(2\pi)^{1/2}}\int_{\sqrt{\frac{u}{\delta_k^n}}}^\infty \mathrm{e}^{-\frac{v^2}{2}}\mathrm{d}v,
\end{aligned}$$

we get, iterating conditioning, using the inequality

$$\prod_{k=\overline{1,m}}(1-x_k)\ge 1-\sum_{k=\overline{1,m}}x_k, \quad x_k\in[0,1], k=\overline{1,m}, m\in\mathbb{N},$$

and standard estimates for the Gaussian distribution, that given $\alpha = 2\delta_n\log\frac{1}{\delta_n}$ we have for sufficiently large n and some absolute constants K_1, K_2

$$
\begin{aligned}
\mathrm{E}\sup_{t\in[0,T]}\Big(w^n_t - w^n_{\mathrm{d}^n_t}\Big)^2 &\le \alpha + K_1\int_\alpha^\infty\Big(\sum_{k=\overline{0,N^n-1}}\int_{\sqrt{\frac{u}{\delta^n_n}}}^\infty \mathrm{e}^{-\frac{v^2}{2}}\mathrm{d}v\Big)\mathrm{d}u\\
&\le \alpha + K_1\sum_{k=\overline{0,N^n-1}}\int_\alpha^\infty \mathrm{e}^{-\frac{u}{2\delta^n_k}}\sqrt{\frac{\delta^n_k}{u}}\mathrm{d}u\\
&\le \alpha + \frac{K_2}{\sqrt{\alpha}}\sum_{k=\overline{0,N^n-1}}(\delta^n_k)^{3/2}\cdot\int_{\frac{\alpha}{2\delta^n}}^\infty \mathrm{e}^{-u}\mathrm{d}u\\
&= \alpha + K_2T\mathrm{e}^{-\frac{\alpha}{2\delta_n}}\sqrt{\frac{\delta_n}{\alpha}}\\
&= 2\delta_n\Big(\log\frac{1}{\delta_n} + \frac{K_2T}{\sqrt{2\log\frac{1}{\delta_n}}}\Big).
\end{aligned}
$$

(5) For some $\widetilde{C}$

$$
\begin{aligned}
\mathrm{E}\sup_{t\in[0,T]}(r^n_t)^2 &\le \widetilde{C}\,\mathrm{E}\sup_{k=\overline{0,N^n-1}}\delta^n_k\int_{I^n_k}\Big(1+(u^n_s)^2\Big)\,\mathrm{d}s\\
&\le \widetilde{C}\delta_n\,\mathrm{E}\int_0^T\Big(1+(u^n_s)^2\Big)\,\mathrm{d}s,
\end{aligned}
$$

so the application of (1) yields the desired estimate. □

Let $\mathcal{M}_p(\mathbb{R})$ be the metric space of probability measures on $\mathbb{R}$ with finite $p-$th moment and let W_p be the corresponding Wasserstein distance [28]. For fixed p, we define the Wasserstein distance between probability measures L_1, L_2 on $\mathcal{M}_p(\mathbb{R})$ as

$$W_{1,p}(L_1, L_2) = \inf \mathrm{E}\; W_p(\mu', \mu''),$$

where the infinum is taken over the set of pairs of $\mathcal{M}_p(\mathbb{R})-$valued random elements μ_1, μ_2 satisfying $\mathrm{Law}(\mu_k) = L_k, k = 1, 2$.

Define random pushforward measures

$$
\begin{aligned}
\mu_t &= \mathrm{Leb}_{[0,1]}\circ\left(X^{\varphi,a}_t\right)^{-1},\\
\mu^n_t &= \mathrm{Leb}_{[0,1]}\circ\left(y^n_t\right)^{-1}, \quad t\in[0,T], n\in\mathbb{N},
\end{aligned}
\tag{4.4}
$$

and their distributions as measures on $\mathcal{M}_p(\mathbb{R})$

$$
\begin{aligned}
L_t &= \mathrm{Law}(\mu_t),\\
L^n_t &= \mathrm{Law}\left(\mu^n_t\right), \quad t\in[0,T], n\in\mathbb{N}.
\end{aligned}
$$

The rest of the section describes the example of splitting for the Brownian web. Let B be a Brownian web. The corresponding counterpart with drift B^a is defined and constructed in [3], Chapter 7 as a family $\{B^a_\cdot(x) \mid x\in\mathbb{R}\}$ of coalescing semimartingales. One defines the associated splitting $\mathrm{Spl}(B; a; \mathcal{T})$ *via* (4.1) by replacing $X^{\varphi,0}$ with B. It is worth noting that the limit in Proposition 3.1 does not exist due to [26], Proposition 1.5.

Theorem 4.5 ([26], Thm. 4.1)**.** *Assume $a\in L_\infty(\mathbb{R})$. For any $m\in\mathbb{N}$ and any $x_1,\dots,x_m\in\mathbb{R}$*

$$\left(y^n_\cdot(x_1),\dots,y^n_\cdot(x_m)\right)\Rightarrow\left(B^a_\cdot(x_1),\dots,B^a_\cdot(x_m)\right), \quad n\to\infty,$$

in $D([0,T],\mathbb{R}^m)$.

Set $L_t = \mathrm{Law}(\mathrm{Leb}_{[0,1]} \circ (B_t^a)^{-1})$.

Theorem 4.6 ([29], Thm. 2.1)**.** *Assume that the sequence $\{n\delta_n\}_{n\in\mathbb{N}}$ is bounded by K and $a \in L_\infty(\mathbb{R})$. Then for every $p \geq 2$ there exist $C = C(p, K, T) > 0$ such that*

$$W_{1,p}(L_t, L_t^n) \leq \frac{C}{\left(\log\log \delta_n^{-1}\right)^{1/p}}, \quad n \in \mathbb{N}.$$

Remark 4.7. The formulation of [29], Theorem 2.1 is erroneously missing the second logarithm due to a calculational error in the end of the proof.

5. Weak convergence

Let $((y^n, u^n))_{n\in\mathbb{N}} = \mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ for some $\mathcal{T}$. The main result of this section is the following theorem.

Theorem 5.1. *Assume that $\varphi \in \Phi_*$ and $a \in Lip(\mathbb{R})$. For any $m \in \mathbb{N}$ and any $x_1, \ldots, x_m \in \mathbb{R}$*

$$(y_\cdot^n(x_1), \ldots, y_\cdot^n(x_m)) \Rightarrow \left(X_{0,\cdot}^{\varphi,a}(x_1), \ldots, X_{0,\cdot}^{\varphi,a}(x_m)\right), \quad n \to \infty,$$

in $D([0,T], \mathbb{R}^m)$.

The proof of Theorem 5.1 is split into a series of lemmas.
Recall

$$w_t^n(x) = y_t^n(x) - x - \int_0^{\overline{\mathrm{d}}_t^n} a\left(u_s^n(x)\right) \mathrm{d}s, \quad t \in [0,T], x \in \mathbb{R}, n \in \mathbb{N},$$

to be standard Wiener processes.
We denote the modulus of continuity by ω and the Lipschitz constant for a by C_a.

Remark 5.2. Proceeding exactly as in the proof of [26], Proposition 2.2, one can show that for $u_1, u_2 \in \mathbb{R}$

$$\langle w^n(u_1), w^n(u_2)\rangle_t = \int_0^t \varphi\left(y_s^n(u_1) - y_s^n(u_2)\right) \mathrm{d}s, \quad t \in [0,T], n \in \mathbb{N}.$$

Denote

$$\begin{aligned} x &= (x_1, \ldots, x_m), \\ y^n(x) &= (y_\cdot^n(x_1), \ldots, y_\cdot^n(x_m)), \\ u^n(x) &= (u_\cdot^n(x_1), \ldots, u_\cdot^n(x_m)), \\ &n \in \mathbb{N}. \end{aligned}$$

Lemma 5.3. *The sequence $(u^n(x), y^n(x))_{n\in\mathbb{N}}$ is weakly relatively compact in $D([0,T], \mathbb{R}^{2m})$.*

Proof. By [21], Theorem 15.2, it is sufficient to prove that for any $\varepsilon > 0$ and any $k = \overline{1, m}$

$$\lim_{\kappa\to 0+} \limsup_{n\to\infty} \mathrm{P}\left(\max\left\{\widehat{\omega}(y^n(x_k), \kappa), \widehat{\omega}(u^n(x_k), \kappa)\right\} \geq \varepsilon\right) = 0, \tag{5.1}$$

where for $f \in D([0,T],\mathbb{R})$

$$\widehat{\omega}(f,\kappa) = \inf_{\substack{0=t_0<t_1<\ldots<t_r=T,\\ t_i-t_{i-1}>\kappa, i=\overline{1,r},\\ r\in\mathbb{N}}} \max_{i=\overline{1,r}} \sup_{s_1,s_2\in[t_{i-1},t_i)} (f_{s_1}-f_{s_2}).$$

For fixed k and ε consider s_1, s_2 such that $0 < s_2 - s_1 < \kappa$. We drop argument x to simplify notation. Since

$$\begin{aligned} y^n_{s_2} - y^n_{s_1} &= \int_{\overline{\mathrm{d}}^n_{s_1}}^{\overline{\mathrm{d}}^n_{s_2}} a(u^n_r)\mathrm{d}r + w^n_{s_2} - w^n_{s_1}, \\ u^n_{s_2} - u^n_{s_1} &= \int_{s_1}^{s_2} a(u^n_r)\mathrm{d}r + w^n_{\mathrm{d}^n_{s_2}} - w^n_{\mathrm{d}^n_{s_1}}, \end{aligned}$$

we have for some $C > 0$

$$\begin{aligned} \omega(y^n,\kappa) &\le \omega(w^n,\kappa) + C\Big(1+\sup_{t\in[0,T]}|u^n_t|\Big)(\kappa+2\delta_n), \\ \omega(u^n,\kappa) &\le \omega(w^n,\kappa+2\delta_n) + C\Big(1+\sup_{t\in[0,T]}|u^n_t|\Big)\kappa. \end{aligned}$$

Let w be a standard Wiener process. By Lemma 4.4, for some fixed constant K

$$\begin{aligned} &\mathrm{P}\left(\max\{\omega(y^n(x_k),\kappa), \omega(u^n(x_k),\kappa)\} \ge \varepsilon\right) \\ &\qquad \le \mathrm{P}\Big(\omega(w,\kappa+2\delta_n) \ge \frac{\varepsilon}{2}\Big) + \mathrm{P}\Big(\Big(1+\sup_{t\in[0,T]}|w_t|\Big)(\kappa+\delta_n) \ge \frac{\varepsilon}{K}\Big) \end{aligned} \tag{5.2}$$

Since $\widehat{\omega}(\cdot,\kappa) \le \omega(\cdot,2\kappa)$ for $\kappa \ll 1$, (5.1) follows. ☐

Lemma 5.4. *For any weak limit $\xi = (\xi_1,\ldots,\xi_m)$ of the sequence $(y^n(x))_{n\in\mathbb{N}}$ and for any pair $i, j \in \{1,\ldots,m\}, i \ne j$,*

$$\mathrm{P}\Big(\exists t \in [0;T]\ \xi_{it} = \xi_{jt} \text{ and } \sup_{s\in[t;T]} |\xi_{is} - \xi_{js}| > 0\Big) = 0.$$

Proof. We adopt the idea from [26], Proposition 3.8, referring to the aforementioned proof for those calculations that are shared between the proofs. Define

$$\begin{aligned} D^+([0,T],\mathbb{R}) &= \Big\{f \in D([0,T],\mathbb{R}) \mid \inf_{r\in[0,T]} f_r \ge 0\Big\}, \\ \Gamma^\kappa_\varepsilon &= \Big\{f \in D^+([0,T],\mathbb{R}) \mid \exists t\in[0,T]\colon\ f_t < \varepsilon, \int_t^T f_r \mathrm{d}r > \kappa\Big\}, \\ \Gamma^\kappa &= \Big\{f \in D^+([0,T],\mathbb{R}) \mid \exists t\in[0,T]\colon f_t = 0, \int_t^T f_r \mathrm{d}r > \kappa\Big\}, \\ &\kappa, \varepsilon > 0. \end{aligned}$$

Assume that i, j are fixed and that $x_j > x_i$. It is sufficient to show that for any κ

$$\liminf_{\varepsilon\to 0+} \liminf_{n\to\infty} \mathrm{P}\left(\Delta y^n \in \Gamma^\kappa_\varepsilon\right) = 0. \tag{5.3}$$

where $\Delta y^n = y^n(x_j) - y^n(x_i)$. Put

$$\omega^n = \omega(w^n(x_i), \delta_n) + \omega(w^n(x_j), \delta_n).$$

One can show that for $\varepsilon \ll \kappa$

$$\begin{aligned}
\mathrm{P}\left(\Delta y^n \in \Gamma_\varepsilon^\kappa\right) &= \mathrm{P}\left(\exists t\colon \Delta y_t^n < \varepsilon, \int_t^T \Delta y_r^n \mathrm{d}r > \kappa\right) \\
&\le \mathrm{P}\left(\omega^n \ge \varepsilon\right) + \sum_{l=\overline{0,N^n-1}} \mathrm{P}\Big(\inf_{r\in[0,t_{l-1}^n)} \Delta y_r^n \ge \varepsilon; \inf_{r\in[t_{l-1}^n,t_l^n)} \Delta y_r^n \le \varepsilon; \\
&\qquad \Delta y_{t_l^n-}^n \le 2\varepsilon; \int_{t_l^n}^T \Delta y_r^n \mathrm{d}r \ge \frac{\kappa}{2}\Big) \\
&= \mathrm{P}\left(\omega^n \ge \varepsilon\right) + I^n,
\end{aligned}$$

where $t_{-1}^n = 0$. Since

$$\lim_{n\to\infty} \mathrm{P}\left\{\omega^n \ge \varepsilon\right\} = 0,$$

we consider only the sum I^n. For $t \in [t_p^n, t_{p+1}^n), p \ge l$,

$$\begin{aligned}
\mathrm{E}\left(\Delta y_t^n | \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) &= \mathrm{E}\left(\mathrm{E}\left(\Delta y_t^n | \mathcal{F}_{t_p^n}^{X^{\varphi,0}}\right) | \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) \\
&= \mathrm{E}\left(\Delta y_{t_p^n}^n | \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) \\
&\le \mathrm{e}^{C_a \delta_p^n} \mathrm{E}\left(\Delta y_{t_p^n-}^n | \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) \\
&\le \dots \\
&\le \mathrm{e}^{C_a (T - t_l^n)} \Delta y_{t_l^n-}^n,
\end{aligned} \tag{5.4}$$

so

$$\begin{aligned}
\mathrm{P}\left(\int_{t_l^n}^T \Delta y_r^n \mathrm{d}r \ge \frac{\kappa}{2} \Big| \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) &\le \frac{2}{\kappa} \mathrm{E}\left(\int_{t_l^n}^T \Delta y_r^n \mathrm{d}r \Big| \mathcal{F}_{t_l^n}^{X^{\varphi,0}}\right) \\
&\le \frac{2\mathrm{e}^{C_a(T-t_l^n)}(T - t_l^n)}{\kappa} \Delta y_{t_l^n-}^n.
\end{aligned} \tag{5.5}$$

Thus

$$I^n \le \frac{4\mathrm{e}^{C_a T} T}{\kappa} \varepsilon,$$

which yields (5.3). □

Lemma 5.5. *Let $\mathcal{C}_m$ be the set of elements of $C(\mathbb{R}_+, \mathbb{R}^m)$ whose coordinates merge after a meeting. Any weak limit ξ of the sequence $(y^n(x))_{n\in\mathbb{N}}$ is a $\mathcal{C}_m$-solution in the sense of Definition A.1 in Appendix A to the* *martingale problem on* $\mathbb{R}^m$ *for the operator*

$$\mathcal{A}_m = \frac{1}{2} \sum_{k,j=\overline{1,m}} \varphi(x_k - x_j) \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=\overline{1,m}} a(x_k) \frac{\partial}{\partial x_k}.$$

Proof. W.l.o.g. we can suppose that $y^n(x) \Rightarrow \xi, n \to \infty$. Using [21], Theorem 15.5 and (5.2) we can check that ξ is continuous a.s.. Recalling (4.3), define

$$\begin{aligned} v^n(x_k) &= y^n(x_k) - r^n(x_k), \quad k = \overline{1,m}, \\ v^n(x) &= (v^n(x_1), \ldots, v^n(x_m)), \\ u^n(x) &= (u^n(x_1), \ldots, u^n(x_m)), \quad n \in \mathbb{N}. \end{aligned}$$

Lemma 4.4 implies

$$\begin{aligned} (r^n(x_1), \ldots, r^n(x_m)) &\to 0, \quad n \to \infty, \\ (l^n(x_1), \ldots, l^n(x_m)) &\to 0, \quad n \to \infty, \end{aligned}$$

in probability in $D([0,T], \mathbb{R}^m)$ in the uniform metric and therefore in the J_1 topology. Thus,

$$\begin{aligned} y^n(x) - u^n(x) &\to 0, \quad n \to \infty, \\ y^n(x) - v^n(x) &\to 0, \quad n \to \infty, \end{aligned} \tag{5.6}$$

in probability and therefore

$$\begin{aligned} v^n(x) &\Rightarrow \xi, \quad n \to \infty, \\ u^n(x) &\Rightarrow \xi, \quad n \to \infty, \end{aligned}$$

in $D([0,T], \mathbb{R}^m)$. Proceeding as in Lemma 5.3, we can check that the sequence

$$((y^n(x), u^n(x), v^n(x)))_{n \in \mathbb{N}}$$

is weakly relatively compact in $D([0,T], \mathbb{R}^{3m})$. Hence, by the Skorokhod representation theorem and (5.6) we can assume w.l.o.g that

$$(y^n(x), u^n(x), v^n(x)) \Rightarrow (\xi, \xi, \xi), \quad n \to \infty, \tag{5.7}$$

in $D([0,T], \mathbb{R}^{3m})$.

By the Itô lemma and Proposition 3.2, for any bounded $f \in C^2(\mathbb{R}^m)$

$$\begin{aligned} f(v^n_t(x)) = f(x) &+ \int_0^t \sum_{k=\overline{1,m}} a(u^n_s(x_k)) \frac{\partial}{\partial x_k} f(v^n_s(x)) \mathrm{d}s \\ &+ \frac{1}{2} \int_0^t \sum_{j,k=\overline{1,m}} \varphi\left(y^n_s(x_k) - y^n_s(x_j)\right) \frac{\partial^2}{\partial x_k \partial x_j} f(v^n_s(x)) \mathrm{d}s \\ &+ \int_0^t \sum_{k=\overline{1,m}} \sigma(y^n_s(x_k)) \frac{\partial}{\partial x_k} f(v^n_s(x)) \mathrm{d}W_s, \end{aligned}$$

where $W = \mathcal{W}(X^{\varphi,0})$.

Assume $g \in C(\mathbb{R}^{Mm}) \cap L_\infty(\mathbb{R}^{Mm})$ for some $M \in \mathbb{N}$. Then for arbitrary $s, t, s \le t$ and $s_1, \dots, s_M \le s$

$$\mathrm{E}\, g(y^n_{s_1}(x), \dots, y^n_{s_M}(x)) \int_s^t \sum_{k=\overline{1,m}} \sigma(y^n_s(x_k)) \frac{\partial}{\partial x_k} f(v^n_s(x)) \mathrm{d}W_s = 0,$$

so the process

$$\begin{aligned} m^n_t(x) = f(v^n_t(x)) &- \int_0^t \sum_{k=\overline{1,m}} a(u^n_s(x_k)) \frac{\partial}{\partial x_k} f(v^n_s(x)) \mathrm{d}s \\ &- \frac{1}{2} \int_0^t \sum_{j,k=\overline{1,m}} \varphi\left(y^n_s(x_k) - y^n_s(x_j)\right) \frac{\partial^2}{\partial x_k \partial x_j} f(v^n_s(x)) \mathrm{d}s, \\ t \in [0, T], \end{aligned}$$

is a martingale w.r.t. the filtration generated by $y^n(x)$.

Applying the Skorokhod representation theorem to (5.7), we can assume that

$$(y^n(x), u^n(x), v^n(x)) \to (\xi, \xi, \xi), \quad n \to \infty,$$

a.s. in $D([0,T], \mathbb{R}^{3m})$. Since the limit ξ is continuous, the convergence is uniform. In particular, for any $j, k = \overline{1,m}$

$$y^n(x_k) - y^n(x_j) \to \xi_k - \xi_j, \quad n \to \infty,$$

uniformly. Thus one can check that

$$\begin{aligned} &\mathrm{E}\, g(y^n_{s_1}(x), \dots, y^n_{s_M}(x)) \left(m^n_t(x) - m^n_s(x)\right) \to \\ &\quad \mathrm{E}\, g(\xi_{s_1}(x), \dots, \xi_{s_M}(x)) \Big(f(\xi_t) - f(\xi_s) - \int_s^t \mathcal{A}_m f(\xi_r) \mathrm{d}r \Big), \quad n \to \infty, \end{aligned}$$

so the process $t \mapsto f(\xi_t) - \int_0^t \mathcal{A}_m f(\xi_r) \mathrm{d}r$ is a martingale.

By Lemma 5.4

$$\mathrm{P}\left(\xi \in \mathcal{C}_m\right) = 1,$$

which concludes the proof. □

Lemma 5.6. *For any weak limit ξ of the sequence $(y^n(x))_{n \in \mathbb{N}}$*

$$\mathrm{Law}\,(\xi) = \mathrm{Law}\left(\left(X^{\varphi,a}_{0,\cdot}(x_1), \dots, X^{\varphi,a}_{0,\cdot}(x_m)\right)\right).$$

Proof. $\mathcal{C}_m-$solutions are unique by Proposition A.2. □

This finishes the proof of Theorem 5.1.

Remark 5.7. The splitting scheme and Theorem 5.1 can be extended to some classes of non-Lipschitz a as follows. Assume that a satisfies the one-sided Lipschitz condition: for some C

$$a(x) - a(y) \le C(x - y), \quad x \ge y.$$

Then the unique flow $X^{\varphi,a}$ exists and has finite moments of any order. For any $s \in [0,T)$ consider a SDE

$$\begin{aligned} \mathrm{d}F^n_{s,t}(x) &= a\left(F^n_{s,t}(x)\right)\mathrm{d}t + \varepsilon_n \mathrm{d}w_t, \\ F^n_{s,s}(x) &= x, \end{aligned}$$

where $\varepsilon_n \to 0+, n \to \infty$, and w is a Wiener process on $[0,T]$ independent of $X^{\varphi,0}$. Such SDEs have unique strong solutions. At each step of the splitting procedure, replace u^n on I^n_k in (4.1) with

$$u^n_t = F^n_{t^n_k,t}\left(y^n_{t^n_k-}\right).$$

Then the new splitting scheme $\mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ is well defined. Moreover, Lemmas 4.4 and 5.3 follow immediately. For Lemma 5.4, note that general comparison theorems for SDEs [30–32] imply that F^n_t is monotone mapping for any t and n so we can use the Gronwall lemma to establish analogs of (5.4) and (5.5) for conditional expectations w.r.t. the extended filtration

$$\sigma\left\{w_{u_1}, X^{\varphi,0}_{u_1,u_2}, 0 \le u_1 \le u_2 \le s\right\}, \quad 0 \le s \le T,$$

instead of $(\mathcal{F}^{X^{\varphi,0}}_{0,s})_{s\in[0,T]}$, which yields the conclusion of Lemma 5.4 for such a. Lemma 5.5 is also valid for such drift. This establishes the claim. See also [33] for the Euler-Maruyama scheme for SDEs with discontinuous coefficients and [13–16] for splitting schemes for SPDEs with non-Lipschitz coefficients.

6. Convergence of pushforward measures

$((u^n, y^n))_{n\in\mathbb{N}} = \mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ is considered in this section.

Let $\mathcal{R}(\mathbb{R})$ be the set of Radon measures on $\mathbb{R}$.

Theorem 6.1. *Assume that $\varphi \in \Phi_\alpha$ for some $\alpha < 2$ and $\nu_0 \in \mathcal{R}(\mathbb{R})$ is such that*

$$\forall \gamma > 0 \int_{\mathbb{R}} \mathrm{e}^{-\gamma u^2} \nu_0(\mathrm{d}u) < \infty.$$

Define

$$\begin{aligned} \nu_t &= \nu_0 \circ \left(X^{\varphi,a}_t\right)^{-1}, \\ \nu^n_t &= \nu_0 \circ \left(y^n_t\right)^{-1}, \quad t \in [0,T], n \in \mathbb{N}. \end{aligned}$$

Then for any $t \in [0,T]$ $\nu^n_t, \nu_t \in \mathcal{R}(\mathbb{R})$ a.s. and

$$\nu^n_t \Rightarrow \nu_t, \quad n \to \infty,$$

in $\mathcal{R}(\mathbb{R})$ under the vague topology.

For the pushforward measures defined in (4.4), the following conclusion holds.

Corollary 6.2. *Assume that $\varphi \in \Phi_\alpha$ for some $\alpha < 2$. Then for any $t \in [0,T]$*

$$\mu^n_t \Rightarrow \mu_t, \quad n \to \infty,$$

in $\mathcal{R}(\mathbb{R})$ under the weak topology.

We need the following lemma whose proof is postponed until Appendix A.

Lemma 6.3. *1. For all $n \in \mathbb{N}$*

$$\sup_{t\in[0,T]} \mathrm{E}\,|y_t^n(v) - y_t^n(w)| \le \mathrm{e}^{C_a T}|v-w|, \quad v,w \in \mathbb{R}.$$

2. Assume that $\varphi \in \Phi_\alpha$ for some $\alpha < 2$. Then for any $p \ge 2$ and $\varepsilon \in (0; \frac{1}{1+\frac{(2-\alpha)p}{2}})$ there exists $C = C(p,\varepsilon,T) > 0$ such that

$$\mathrm{E} \sup_{t\in[0,T]} \left| X_{0,t}^{\varphi,a}(v) - X_{0,t}^{\varphi,a}(w)\right|^p \le C|v-w|^\varepsilon, \quad |v-w| \le 1.$$

Proof of Theorem 6.1. Using Lemma 6.3, we can repeat the reasoning in [20], proof of Theorem 1, pp. 87–90 as soon as we have proved the following two claims: for arbitrary $\kappa > 0$ and compactly supported f there exists $M = M(\kappa, f)$ such that

$$\max\Big\{ \sup_{n\in\mathbb{N}} \mathrm{E}\Big| \int_{|v|\ge M} f\left(y_t^n(v)\right) d\nu_0(v)\Big|, \mathrm{E}\Big|\int_{|v|\ge M} f\left(X_{0,t}^{\varphi,a}(v)\right) d\nu_0(v)\Big|\Big\} \le \kappa,$$

and for arbitrary $M > 0$

$$\max\Big\{ \sup_{n\in\mathbb{N}} \mathrm{E}\,\nu_t^n\left((-M;M]\right); \mathrm{E}\,\nu_t\left((-M;M]\right)\Big\} < \infty.$$

For that, it is sufficient to show that for any $S > 0$

$$\lim_{M\to\infty} \int_{|x|\ge M} \mathrm{P}\left(\left|X_{0,t}^{\varphi,a}(x)\right| \le S\right) d\nu_0(x) = 0, \tag{6.1}$$

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\ge M} \mathrm{P}\left(|y_t^n(x)| \le S\right) d\nu_0(x) = 0. \tag{6.2}$$

Let C_a be such that $|a(x)| \le C_a(1+|x|)$ on $\mathbb{R}$. The solution to the ODE

$$\frac{dg_t}{\mathrm{d}t} = -C_a(1+g_t)$$

is

$$g_t = \mathrm{e}^{-C_a t} - 1 + \mathrm{e}^{-C_a t} g_0.$$

Assume $x \gg S$. Setting

$$\zeta_k^n(x) = w_{t_{k+1}^n}^n(x) - w_{t_k^n}^n(x), \quad k = \overline{0, N^n - 1}, n \in \mathbb{N}, x \in \mathbb{R},$$

define

$$\eta_t^n(x) = \mathrm{e}^{-C_a t_k^n} x + w_t^n(x) - w_{t_{k+1}^n}^n(x) + \sum_{j=\overline{0,k}} \mathrm{e}^{-C_a(t_{k+1}^n - t_{j+1}^n)} \left(\mathrm{e}^{-C_a \delta_j^n} - 1 + \zeta_j^n\right),$$
$$t \in I_k^n, k = \overline{0, N^n - 1}, n \in \mathbb{R}, x \in \mathbb{R}.$$

Then on $\{\inf_{t\in[0,T]}\eta_t^n(x)>0\}$ for $t\in I_k^n$ for some k

$$\begin{aligned} y_t^n(x) &\ge w_t^n(x)-w_{t_k^n}^n(x)+\mathrm{e}^{-C_a\delta_k^n}-1+\mathrm{e}^{-C_a\delta_k^n}y_{t_k^n-}^n \\ &\ge \eta_t^n(x). \end{aligned}$$

Thus

$$\mathrm{P}\Big(\inf_{s\in[0,T]}y_s^n(x)\le S\Big)\le \mathrm{P}\Big(\inf_{s\in[0,T]}\eta_s^n(x)\le S\Big).$$

Here

$$\eta_t^n(x)\ge\big(\mathrm{e}^{-C_aT}x-C_aT\big)+\widetilde{\eta}_t^n(x),$$

where

$$\begin{aligned} \widetilde{\eta}_t^n(x)&=w_t^n(x)-w_{t_k^n}^n(x)+\sum_{j=\overline{0,k-1}}\mathrm{e}^{-C_a(t_{k+1}^n-t_{j+1}^n)}\zeta_j^n, \\ &t\in I_k^n, k=\overline{0,N^n-1}, \end{aligned}$$

is a centered Gaussian process with

$$\sup_{t\in[0,T]}\mathrm{Var}\left(\widetilde{\eta}_t^n(x)\right)\le T.$$

The concentration inequality for a supremum of a Gaussian process [34], Theorem 2.1.1, implies

$$\mathrm{P}\Big(\inf_{s\in[0,T]}y_s^n(x)\le S\Big)\le\exp\Big\{-\frac{1}{2T}\Big(\mathrm{e}^{-C_aT}x-S-C_aT-\mathrm{E}\sup_{t\in[0,T]}\widetilde{\eta}_t^n(x)\Big)^2\Big\}. \tag{6.3}$$

Due to Dudley's entropy bound [34], Theorem 1.3.3, for some absolute K

$$\mathrm{E}\sup_{t\in[0,T]}\widetilde{\eta}_t^n(x)\le K\int_0^\infty\left(\log M_\varepsilon^n\right)^{\frac12}d\varepsilon, \tag{6.4}$$

where M_ε^n is the smallest number of balls of size ε that cover $[0,T]$ in the intrinsic metric

$$\rho^n(s_1,s_2)=\Big(\mathrm{E}\left(\widetilde{\eta}_{s_1}^n(0)-\widetilde{\eta}_{s_2}^n(0)\right)^2\Big)^{\frac12}$$

of the process $\widetilde{\eta}^n(0)$. Note that $\widetilde{\eta}^n(0)$ is a Wiener process on every I_k^n and for $s_1\in I_{k_1}^n, s_2\in I_{k_2}^n, k_1<k_2$ we have

$$\begin{aligned} \rho^n(s_1,s_2)^2 = &\sum_{j=\overline{0,k_1-1}}\mathrm{e}^{-2C_at_{j+1}^n}\left(\mathrm{e}^{-C_at_{k_1+1}^n}-\mathrm{e}^{-C_at_{k_2+1}^n}\right)^2\delta_j^n \\ &+\sum_{j=\overline{k_1+1,k_2-1}}\mathrm{e}^{-2C_a(t_{k_2+1}^n-t_{j+1}^n)}\delta_j^n \\ &+\left(1-\mathrm{e}^{-C_a(t_{k_2+1}^n-t_{k_1+1}^n)}\right)^2(s_1-t_{k_1}) \\ &+\mathrm{e}^{-2C_a(t_{k_2+1}^n-t_{k_1+1}^n)}(t_{k_1+1}^n-s_1)+s_2-t_{k_2}, \end{aligned}$$

so for a universal constant $C > 1$

$$\rho^n(s_1, s_2) \le C\Big((s_2 - s_1)^{\frac{1}{2}} + \delta^n_{k_2}\left(t^n_{k_1}\right)^{\frac{1}{2}}\Big).$$

We assume $T = 1$ for the rest of the proof. The diameter of $[0, T]$ in ρ^n does not exceed $2C$ so $M^n_\varepsilon = 1$ for $\varepsilon \ge C$ for all n.

Assume $\varepsilon > 2C(\delta_n)^{1/2}$. For a unit interval, consider a $\frac{\varepsilon^2}{4C^2}-$net A in the Euclidean metric. Then for any s

$$\rho^n(s, A) \le C\left(\frac{\varepsilon}{2C} + (\delta_n)^{\frac{1}{2}}\right) \le \varepsilon,$$

so A is an $\varepsilon-$net for $[0, 1]$ in the metric ρ^n and

$$M^n_\varepsilon \le \frac{4C}{\varepsilon^2} + 1.$$

Now assume that $\varepsilon \le 2C(\delta_n)^{1/2}$. We call an interval I^n_k large, if $\varepsilon \le 2C(\delta^n_k)^{1/2}$, and small, otherwise. We construct an $\varepsilon-$net as follows. Let B be a $\frac{\varepsilon^2}{C_1}-$net for $[0, 1]$ in the Euclidean metric where

$$C_1 = 8(C^2 + 1).$$

For each large I^n_k let B_k be a net of the same size such that $t^n_k \in B_k$, also in the Euclidean metric. Set

$$A_1 = B \cup \bigcup_{k=\overline{0,N^n-1}: I^n_k \text{ is large}} B_k.$$

Since for large intervals $4C^2\frac{\delta^n_k}{\varepsilon^2} \ge 1$, we have then

$$\begin{aligned}\sharp A_1 &\le \frac{C_1}{\varepsilon^2} + 1 + \sum_{k=\overline{0,N^n-1}: I^n_k \text{ is large}} \left(\frac{C_1\delta^n_k}{\varepsilon^2} + 1\right) \\ &\le \frac{2C_1 + 4C^2}{\varepsilon^2} + 1.\end{aligned}$$

We will show that A_1 is an $\varepsilon-$net for $[0, 1]$ in the metric ρ^n. If $t \in I^n_k$ and I^n_k is large, $\rho^n(t, A_1) \le \varepsilon$ immediately. Assume $t \in I^n_k$ and I^n_k is small. Then there exist numbers $0 \le m_1 \le k \le m_2 \le N^n - 1$ such that

1. the intervals $I^n_{m_1}, \ldots, I^n_{m_2}$ are small;
2. either $m_1 = 0$ or $I^n_{m_1-1}$ is large;
3. either $m_2 = N^n - 1$ or $I^n_{m_2+1}$ is large;
4. at least one of $I^n_{m_1-1}$ and $I^n_{m_2+1}$ is large.

Consider the case

$$L = t^n_{m_2+1} - t^n_{m_1} > \frac{2\varepsilon^2}{C_1}.$$

Then there exists $s \in B \cap \bigcup_{j=\overline{m_1,m_2}} I_j^n$ such that $|t-s| \le \frac{\varepsilon^2}{C_1}$ and for some $j, m_1 \le j \le m_2$

$$\rho^n(s,t) \le C\Big(\frac{\varepsilon}{C_1^{1/2}} + \Big(\delta^n_{\max\{k,j\}}\Big)^{1/2}\Big) \le \varepsilon.$$

If $L \le \frac{2\varepsilon^2}{C_1}$, we consider two possibilities. If $I^n_{m_1-1}$ is large, then

$$\rho^n(t, B_{m_1-1}) \le \lim_{s\to t^n_{m_1}} \rho^n(t,s) + \frac{\varepsilon}{C_1^{1/2}} \le C\left(L^{\frac{1}{2}} + \delta_k^n\right) + \frac{\varepsilon}{C_1^{1/2}} \le \varepsilon.$$

If $m_1 = 0$, then $t^n_{m_2+1} = L$, so

$$\rho^n(t, B_{m_2+1}) = \rho^n(t, t^n_{m_2+1}) \le C\left((t^n_{m_2+1} - t)^{\frac{1}{2}} + L^{\frac{1}{2}}\right) \le 2CL^{\frac{1}{2}} \le \varepsilon.$$

This proves the claim.

Estimating the integral in (6.4) we obtain for some K_1

$$\sup_{n\in\mathbb{N}} \mathrm{E} \sup_{t\in[0,T]} \widetilde{\eta}^n_t(x) \le K \int_0^C \left(\log\left(\frac{K_1}{\varepsilon^2}+1\right)\right)^{\frac{1}{2}} d\varepsilon < \infty,$$

so (6.3) yields

$$\sup_{x\ge x_0}\sup_{n\in\mathbb{N}} \mathrm{P}\left(\inf_{s\in[0,T]} \eta^n_s(x) \le S\right) \le C_0 \mathrm{e}^{-\frac{x^2}{C_0}}.$$

for sufficiently large absolute x_0 and C_0. Since the same estimate can be obtained for $\mathrm{P}(\sup_{t\in[0,T]} \eta^n_t(x) \ge S)$ for $n \in \mathbb{N}$ and negative x, (6.2) follows.

For the limit process, we get for $x \gg S$

$$\mathrm{P}\left(\inf_{s\in[0,T]} X^{\varphi,a}_{0,s}(x) \le S\right) \le \mathrm{P}\left(\inf_{s\in[0,T]} \xi_s(x) \le S\right),$$

where

$$d\xi_t(x) = -C_a(1+\xi_t(x))\mathrm{d}t + \mathrm{d}w_t, \quad \xi_0(x) = x,$$

w being a Wiener process. Since

$$\xi_t(x) = x\mathrm{e}^{-C_a t} + \mathrm{e}^{-C_a t} - 1 + \mathrm{e}^{-C_a t}\int_0^t \mathrm{e}^{C_a s}\mathrm{d}w_s, \quad t \ge 0,$$

where the last term is a continuous square integrable martingale with bounded quadratic variation, and $\mathrm{P}(X^{\varphi,a}_{0,t}(x) \ge -S)$ can be estimated similarly, (6.1) follows. □

Proof of Corollary 6.2. By Theorem 6.1 $\mu_t^n \Rightarrow \mu_t, n \to \infty$, vaguely. Since all measures are probabilistic, the convergence also holds in the weak topology [35], Theorem 4.9. □

7. Convergence of dual flows

We assume that $((u^n, y^n))_{n\in\mathbb{N}} = \mathrm{Spl}(X^{\varphi,0}; a; \mathcal{T})$ and establish the convergence of so-called dual flows [1, 19, 20, 36]. Here given a coalescing stochastic flow X the dual flow $\widehat{X}$ is defined *via*

$$\begin{aligned}\widehat{X}_{s,t}(x) &= \inf\{y \in \mathbb{R} \mid X_{T-t,T-s}(y) > x\}\\ &= \inf\big\{X_{r,T-t}(y) \mid X_{r,T-s}(y) > x, y \in \mathbb{R}, r \in [0, T-t]\big\},\\ & s,t \in [0,T], s \le t,\ x \in \mathbb{R},\end{aligned} \tag{7.1}$$

and is again a collection of $D^{\uparrow}(\mathbb{R})-$valued random elements.

To start, we need one extension of the splitting scheme (4.1). Let $l(s)$ equal the unique k such that $s \in I_k^n, s \in [0,T]$. Define

$$\begin{aligned} t_k^n(s) &= \max\{s, t_k^n\}, \quad k = \overline{l(s), N^n-1}, n \in \mathbb{N},\\ u_{s,t}^n(x) &= F_{t-t_k^n(s)}\left(y_{s,t_k^n-}^n(x)\right),\\ y_{s,t}^n(x) &= X_{t_k^n(s),t}^{\varphi,0}\left(u_{s,t_{k+1}^n-}^n(x)\right),\\ y_{s,s-}^n(x) &= x,\\ & t \in I_k^n, k = \overline{l(s), N^n-1}.\end{aligned}$$

For any $s \in [0,T)$ and any $f \in D([s,T],\mathbb{R})$ we extend f onto $[0,T]$ by setting

$$f_r^e = f_r 1\!\!\mathrm{I}\,[r \in [s,T]] + f_s 1\!\!\mathrm{I}\,[r \in [0,s)]\,.$$

For instance, $y_{s,\cdot}^{n,e}(x), u_{s,\cdot}^{n,e}(x)$ are random elements in $D([0,T],\mathbb{R})$.

Since constructing the families $\{y_{s,\cdot}^n(x) \mid x \in \mathbb{R}\}$ uses the single flow $X^{\varphi,0}$ for all s, the mappings $\{y_{s,t}^n \mid 0 \le s \le t \le T\}$ are consistent and form a coalescing flow, so using (7.1), one defines the corresponding dual flow $\widehat{y}^n = \{\widehat{y}_{s,t}^n \mid 0 \le s \le t \le T\}$.

We use the idea from [20], where a precise construction of the dual flow as a function that preserves the weak convergence is given.

Consider the set $\{(s_n, x_n) \in [0,T] \times \mathbb{R} \mid n \in \mathbb{N}\}$ containing all points whose coordinates are dyadic numbers. The corresponding version of Theorem 5.1 implies that for any $m \in \mathbb{N}$

$$Y^{n,m} = \left(y_{s_1,\cdot}^{n,e}(x_1), \ldots, y_{s_m,\cdot}^{n,e}(x_m)\right) \Rightarrow X^m = \left(X_{s_1,\cdot}^{\varphi,a,e}(x_1), \ldots, X_{s_m,\cdot}^{\varphi,a,e}(x_m)\right), \quad n \to \infty,$$

in $D([0,T],\mathbb{R}^m)$.

Denote by P_k the projector on the first k coordinates in the space $D([0,T],\mathbb{R})^\infty$ endowed with the product topology and by Q_k the projector on the $k-$th coordinate in the same space. Consider the $D([0,T],\mathbb{R})^\infty-$valued random elements $X, \widehat{X}, Y^n, \widehat{Y}^n, n \in \mathbb{N}$, defined *via*

$$\begin{aligned} P_m(Y^n) &= Y^{n,m},\\ P_m(X) &= X^m,\\ P_m(\widehat{Y}^n) &= \left(\widehat{y}_{s_1,\cdot}^{n,e}(x_1), \ldots, \widehat{y}_{s_m,\cdot}^{n,e}(x_m)\right),\\ P_m(\widehat{X}) &= \Big(\widehat{X}_{s_1,\cdot}^{\varphi,a,e}(x_1), \ldots, \widehat{X}_{s_m,\cdot}^{\varphi,a,e}(x_m)\Big),\\ & n, m \in \mathbb{N}.\end{aligned}$$

As in [20], p. 86, we obtain

$$Y^n \Rightarrow X, \quad n \to \infty,$$

in $D([0,T],\mathbb{R})^\infty$.

Consider a mapping $I\colon D([0,T],\mathbb{R})^\infty \mapsto D([0,T],\mathbb{R})^\infty$ defined *via*

$$\begin{aligned} &Q_j I(\psi)_r = \inf\{Q_i\psi_r \mid Q_i\psi_T \ge x_j, s_i \le r\}, \quad r \in [s_j, T],\\ &Q_j I(\psi)_r = Q_j I(\psi)(s_j), \quad r \in [0, s_j),\\ &\qquad j \in \mathbb{N},\ \psi \in D([0,T]^\infty). \end{aligned}$$

Then one can check that $I(Y^n) = \widehat{Y}^n$ a.s., $n \in \mathbb{N}$, and $I(X) = \widehat{X}$ a.s..

Definition 7.1. Let D_1 be a set of $\psi \in D([0,T],\mathbb{R})^\infty$ such that

1. $Q_k\psi_{s_k} = x_k,\ k \in \mathbb{N}$;
2. if for some $j_1, j_2 \in \mathbb{N}$ and some $s \in [0,T]$ $Q_{j_1}\psi_s \ge Q_{j_2}\psi_s$, then $Q_{j_1}\psi_t \ge Q_{j_2}\psi_t$ for $t \in [s;T]$.

Definition 7.2. Let D_2 be a subset of D_1 such that for any $\psi \in D_2$

1. $\forall k \in \mathbb{N}\ \exists \varepsilon_k > 0$

$$\begin{aligned} &\forall i \in \mathbb{N}\colon \big(x_i \ge Q_k I(\psi)_{s_i}\big) \Rightarrow \big(Q_i\psi_T - x_k \ge \varepsilon_k\big),\\ &\forall i \in \mathbb{N}\colon \big(x_i < Q_k I(\psi)_{s_i}\big) \Rightarrow \big(x_k - Q_i\psi_T \ge \varepsilon_k\big); \end{aligned}$$

2. $\forall \delta > 0\ \forall M > 0\ \exists L \in \mathbb{N}$

$$\sup_{l=\overline{0,\lceil T\rceil 2^L}}\ \sup_{j\colon |x_j|\le M, s_j = l2^{-L}}\ \sup_{\tau\in[0,2^{-L}]} \big|Q_j\psi_{s_j+\tau} - x_j\big| \le \delta.$$

Lemma 7.3 ([20], pp. 86–87)**.** *Assume that $\psi_n \to \psi, n \to \infty$, in $D([0,T],\mathbb{R})^\infty$, $\psi_n \in D_1, n \in \mathbb{N}, \psi \in D_2$, and $Q_j\psi \in C([0,T]), j \in \mathbb{N}$. Then $I(\psi_n) \to I(\psi), n \to \infty$, in $D([0,T],\mathbb{R})^\infty$.*

Lemma 7.4. *Assume that $\varphi \in \Phi_\alpha$ and $a \in A_\beta$ with $\beta - \alpha > -1, \alpha < 2$. Then $Y^n \in D_1$ a.s., $n \in \mathbb{N}$. $X \in D_2$ a.s..*

Proof. Proceeding as in [20] p. 87, one uses (A.1) in the proof of Proposition A.3 and the fact that the set $\{X^{\varphi,a}_{s,t}(x) \mid x \in \mathbb{R}\}$ is a.s. locally finite by Theorem 2.9 for any $s, t, s < t$. ☐

Combining Lemmas 7.3 and 7.4 yields the following result.

Theorem 7.5. *Assume that $\varphi \in \Phi_\alpha$ and $a \in A_\beta$ with $\beta - \alpha > -1, \alpha < 2$. Then $\widehat{Y}^n \Rightarrow \widehat{X}, n \to \infty$, in $D([0,T],\mathbb{R})^\infty$. In particular, for any $t_1, \ldots, t_m$, $v_1, \ldots, v_m$ and $m \in \mathbb{N}$*

$$\big(\widehat{y}^{n,e}_{t_1,\cdot}(v_1), \ldots, \widehat{y}^{n,e}_{t_m,\cdot}(v_m)\big) \Rightarrow \Big(\widehat{X}^{\varphi,a,e}_{t_1,\cdot}(v_1), \ldots, \widehat{X}^{\varphi,a,e}_{t_m,\cdot}(v_m)\Big), \quad n \to \infty,$$

in $D([0,T],\mathbb{R}^m)$.

8. Estimates for flows with Holder continuous $\sqrt{1-\varphi}$

$((u^n, y^n))_{n\in\mathbb{N}} = \widetilde{\mathrm{Spl}}(X^{\varphi,a}; a; \mathcal{T})$ and $W = \mathcal{W}(X^{\varphi,a})$ are considered in this section.

Assume that $\varphi \in C^2(\mathbb{R})$, which is equivalent to the finiteness of $\varphi''(0)$. As noted in [1], Section 3, $\sqrt{1-\varphi} \in Lip(\mathbb{R})$ then. Thus results of [12] are applicable. $X^{\varphi,a}$ is a flow of homeomorphisms by [2], Theorem 4.5.1.

Theorem 8.1 ([12], Cor. 4.2)**.** *For any $M \ge 0$ and some $C = C(M,T) > 0$*

$$\sup_{x\in[-M,M]} \mathrm{E} \sup_{t\in[0,T]} \left(y^n_t(x) - X^{\varphi,a}_{0,t}(x)\right)^2 \le C\delta_n,$$
$$\sup_{x\in[-M,M]} \sup_{t\in[0,T]} \mathrm{E} \left(u^n_t(x) - X^{\varphi,a}_{0,t}(x)\right)^2 \le C\delta_n.$$

It is possible to refine the order of convergence for u^n.

Proposition 8.2. *For any $M \ge 0$ and some $C = C(M,T) > 0$*

$$\sup_{x\in[-M,M]} \mathrm{E} \sup_{t\in[0,T]} \left(u^n_t(x) - X^{\varphi,a}_{0,t}(x)\right)^2 \le C\delta_n \log \delta_n^{-1},$$

Proof. Dropping the x argument and setting

$$m_t = \mathrm{E} \sup_{s\in[0,T]} \max\Big\{ \left(X^{\varphi,a}_{0,s} - u^n_s\right)^2, \left(X^{\varphi,a}_{0,s} - y^n_s\right)^2 \Big\}.$$

we get by Lemma 4.4 for some C_1, C_2

$$\begin{aligned} m_t &\le C_1\Big(\int_0^t m_s \mathrm{d}s + \mathrm{E}\sup_{s\in[0,T]} \Big((r^n_s)^2 + (l^n_s)^2\Big)\Big) \\ &\le C_2\Big(\int_0^t m_s \mathrm{d}s + \delta_n \log \delta_n^{-1}\Big). \end{aligned}$$

□

The order of convergence in Proposition 8.2 cannot be improved, as shown by the following example.

Example 8.3. Let $a \equiv 0, T = 1$ and $t^n_k = \frac{k}{n}$. Then $u^n_t(x) = X^{\varphi,0}_{0,\frac{k}{n}}(x)$ on I^n_k, so

$$\mathrm{E} \sup_{k=\overline{0,n-1}} \sup_{t\in[\frac{k}{n},\frac{k+1}{n})} \Big(X^{\varphi,0}_{0,t}(x) - u^n_t(x)\Big)^2 = \frac{2}{n}\mathrm{E}\sup_{k=\overline{1,n}} \eta_k^2,$$

where $\eta_k, k = \overline{1,n}$, are independent $\mathcal{N}(0,1)$ random variables. It is well known that

$$\lim_{n\to\infty} \frac{\mathrm{E}\sup_{k=\overline{1,n}} \eta_k}{(2\log n)^{1/2}} = 1,$$

so

$$\liminf_{n\to\infty} \frac{\mathrm{E}\sup_{t\in[0,T]} \Big(X^{\varphi,0}_{0,t}(x) - u^n_t(x)\Big)^2}{4n^{-1}\log n} \ge 1.$$

The following fact is well known (*e.g.* [29, 37]).

Proposition 8.4. *Let f be càdlàg and non-decreasing on* $[0,1]$. *Set* $\kappa = \mathrm{Leb}|_{[0,1]} \circ f^{-1}$ *and* $F(\cdot) = \kappa([0,\cdot])$. *Then* $F^{-1} = f$ *on* $[0,1]$.

Theorem 8.5. *For some* $C = C(T) > 0$

$$\sup_{t\in[0,T]} W_{1,2}\left(L_t, L_t^n\right) \le C\delta_n, \quad n \in \mathbb{N}.$$

Proof. By [28], Remark 2.19

$$W_{1,2}\left(L_t, L_t^n\right) = \mathrm{E}\int_0^1 \left(F_t^{-1}(u) - F_{n,t}^{-1}(u)\right)^2 \mathrm{d}u,$$

where $F_t^{-1}, F_{n,t}^{-1}$ are generalized càdlàg inverses of

$$F_t(u) = \mu_t((-\infty,x]) = \mathrm{Leb}\left\{u \in [0,1] \mid X_{0,t}^{\varphi,a}(u) \le x\right\},$$
$$F_{n,t}(u) = \mu_t^n((-\infty,x]) = \mathrm{Leb}\left\{u \in [0,1] \mid y_t^n(u) \le x\right\},$$

respectively. Thus, by Proposition 8.4,

$$W_{1,2}\left(L_t, L_t^n\right) = \mathrm{E}\int_0^1 \left(X_{0,t}^{\varphi,a}(u) - y_t^n(u)\right)^2 \mathrm{d}u,$$

and the application of Theorem 8.1 establishes the claim. □

Now assume that $\sqrt{1-\varphi} \in H_\beta(\mathbb{R})$ for some $\beta \in [\frac{1}{2}, 1)$. The Euler-Maruyama scheme for an SDE with such coefficients was considered in [10], where a suitable modification of the Yamada-Watanabe method was developed. Essentially, recalling (4.3), using Lemmas 4.3 and 4.4 to estimate $\mathrm{E}\,|r_s^n(x)|^\beta$ and $\mathrm{E}\,|l_s^n(x)|^\beta$, one proceeds by repeating the reasoning for [10], Proposition 2.2 line by line to draw the following two conclusions.

Theorem 8.6. *For some* $C = C(x,\beta,T) > 0$

$$\mathrm{E}\sup_{t\in[0,T]} \left(X_{0,t}^{\varphi,a}(x) - y_t^n(x)\right)^2 \le \frac{C}{\log \delta_n^{-1}}, \quad \beta = \frac{1}{2},$$
$$\mathrm{E}\sup_{t\in[0,T]} \left(X_{0,t}^{\varphi,a}(x) - y_t^n(x)\right)^2 \le C\delta_n^{\beta-\frac{1}{2}}, \quad \beta \in \left(\frac{1}{2}, 1\right).$$

Theorem 8.7. *For some* $C = C(\beta,T) > 0$

$$\sup_{t\in[0,T]} W_{1,2}\left(L_t, L_t^n\right) \le \frac{C}{\log \delta_n^{-1}}, \quad \beta = \frac{1}{2},$$
$$\sup_{t\in[0,T]} W_{1,2}\left(L_t, L_t^n\right) \le C\delta_n^{\beta-\frac{1}{2}}, \quad \beta \in \left(\frac{1}{2}, 1\right).$$

Remark 8.8. Following assumptions in [10], consider $a = a_1 + a_2, a_1 \in Lip(\mathbb{R}), a_2 \in H_\alpha(\mathbb{R})$ for some $\alpha \in (0,1)$ and assume that a_2 is non-increasing. Theorems 8.6 and 8.7 can be extended to such a as follows. Consider the extension of the splitting scheme given in Remark 5.7 with $\varepsilon_n = \frac{1}{n^{1/3}\log\delta_n^{-1}}$ for $\beta = \frac{1}{2}$ and $\varepsilon_n = \frac{1}{n^{\frac{1}{2}+\min\{\frac{\alpha}{2},\beta-\frac{1}{2}\}}}$ for $\beta \in (\frac{1}{2}, 1)$. Then

$$\mathrm{E}\sup_{t\in[0,T]}\big(X^{\varphi,a}_{0,t}(x) - y^n_t(x)\big)^2 \le \frac{C}{\log\delta_n^{-1}}, \quad \beta = \frac{1}{2},$$

$$\mathrm{E}\sup_{t\in[0,T]}\big(X^{\varphi,a}_{0,t}(x) - y^n_t(x)\big)^2 \le C\delta_n^{\min\{\frac{\alpha}{2},\beta-\frac{1}{2}\}}, \quad \beta \in \Big(\frac{1}{2}, 1\Big).$$

Appendix A. Existence of Harris flows with drift

Assume that $\varphi \in \Phi_*$ and a measurable a with linear growth are fixed.

Consider the following operator acting on $C^2(\mathbb{R}^n)$

$$\mathcal{A}_n = \frac{1}{2}\sum_{k,j=\overline{1,n}}\varphi(x_k - x_j)\frac{\partial^2}{\partial x_k\partial x_j} + \sum_{k=\overline{1,n}} a(x_k)\frac{\partial}{\partial x_k}.$$

The degeneracy of the matrix $\|\varphi(x_k - x_j)\|_{k,j=\overline{1,n}}$ on the boundary of $D_n = \{x \mid x_1 < \ldots < x_n\}$ requires an extension of the results presented in [38], which was directly discussed by Harris. However, [39], Sections 1.9–13 later provided a rigorous framework in terms of generalized martingale problems in domains, and we adopt this approach.

Consider $C(\mathbb{R}_+, \mathbb{R}^n)$ endowed with the Borel $\sigma-$algebra and the topology of uniform convergence on compact sets. Define $\mathcal{C}_n$ be the set of elements of $C(\mathbb{R}_+, \mathbb{R}^n)$ whose coordinates merge after a meeting.

Definition A.1. ([1], Def. 2.1) A family $P_x, x \in \mathbb{R}^n$, is called a $\mathcal{C}_n-$solution if it solves the martingale problem for $\mathcal{A}_n$ in $\mathbb{R}^n$ (in the sense of [38]) and for each x $P_x(\mathcal{C}_n) = 1$ and $P_x(\omega \mid \omega_0 = x\}) = 1$.

The solution of the generalized martingale problem for $\mathcal{A}_n$ does not explode to infinity, so one can use solutions of generalized martingale problems to construct $\mathcal{C}_n-$solutions in [1], Lemma 3.2. The next proposition is essentially [1], Lemmas 2.2, 3.2, with the exception of the conclusion about the measurability of the mapping $x \mapsto P_x$, which follows from [22], Proof of Theorem 4.4.6, and the conclusion about the Feller property, which follows from [39], Theorem 1.13.1.

Proposition A.2. *There exists a unique $\mathcal{C}_n-$solution $P_x, x \in \mathbb{R}^n$. This solution is strong Markov, Feller and such that the mapping $x \mapsto P_x \in \mathcal{M}_0(C(\mathbb{R}_+, \mathbb{R}^n))$ is measurable, where $\mathcal{M}_0(C(\mathbb{R}_+, \mathbb{R}^n))$ is the space of probability measures on $C(\mathbb{R}_+, \mathbb{R}^n)$ endowed with the topology of weak convergence.*

To construct $X^{\varphi,a}$ in Theorem 2.8, one can repeat the reasoning in [1], Section 4, the only missing ingredient that has to be checked directly being the following extension of [1], Theorem 4.7.

Proposition A.3. *For any $t > 0$ the family $\{X^{\varphi,a}_{0,s} \mid s \in [0,T]\}$ is a.s. uniformly continuous in $D^{\uparrow}(\mathbb{R})$ endowed with the topology of uniform convergence on compact sets.*

The next lemma is a straightforward corollary of the Gronwall lemma.

Lemma A.4. *For C such that $|a(x)| \le C(1 + |x|), x \in \mathbb{R}$, and a standard Wiener process w*

$$\mathrm{E}\sup_{s\in[0,T]}\big|X^{\varphi,a}_{0,t}(x) - x\big|^p \le 2^{p-1}\mathrm{e}^{pCT}\Big(C^p(1+|x|)^p t^p + \mathrm{E}\sup_{s\in[0,T]}|w(s)|^p\Big), \quad p \ge 1.$$

Proof of Proposition A.3. We modify the original proof. Define, for fixed $T > 1$ and $N > 1$

$$U_{knN} = \sup_{t\in[k2^{-n}T,(k+1)2^{-n}T]} \sup_{|x|\le N} \left|X^{\varphi,a}_{0,t}(x) - X^{\varphi,a}_{0,k2^{-n}T}(x)\right|, \quad k,n \in \mathbb{N}.$$

It is sufficient to show for any $\varepsilon > 0$

$$\sum_{n\ge1} \mathrm{P}\Big(\bigcup_{k=\overline{0,2^n-1}} \{U_{knN} \ge \varepsilon\}\Big) \le \sum_{n\ge1} \sum_{k=\overline{0,2^n-1}} \mathrm{P}\left(U_{knN} \ge \varepsilon\right) < \infty. \tag{A.1}$$

Proceeding as in [1], Proof of Lemma 4.5 and using Lemma A.4, we have, for any $m \ge 1$, $a, b \in \mathbb{R}, b - a > 1$, and some $C > 0$

$$\begin{aligned}
&\mathrm{P}\Big(\sup_{s\in[0,T],x\in[a,b]} \left|X^{\varphi,a}_{0,s}(x) - x\right| \ge \varepsilon\Big)\\
&\qquad \le 2^{m+1}(b-a) \sup_{x\in[a,b]} \mathrm{P}\Big(\sup_{s\in[0,T]} \left|X^{\varphi,a}_{0,s}(x) - x\right| \ge \varepsilon - 2^{-m}\Big)\\
&\qquad \le \frac{2^{m+1}(b-a)}{(\varepsilon-2^{-m})^4} \sup_{x\in[a,b]} \mathrm{E} \sup_{s\in[0,T]} \left(X^{\varphi,a}_{0,s}(x) - x\right)^4\\
&\qquad \le C2^m \frac{(b-a)^5t^4 + (b-a)t^2}{(\varepsilon-2^{-m})^4}.
\end{aligned}$$

Thus, for any fixed m such that $2^{-m} \le \frac{\varepsilon}{2}$ and some $C_1 = C_1(m, N, T) > 0$,

$$\begin{aligned}
\mathrm{P}\left(U_{knN} \ge \varepsilon\right) &\le \mathrm{E}\,\mathrm{P}\Big(\sup_{t\in[0,2^{-n}T]} \sup_{y\in A} \left|X^{\varphi,a}_{0,t}(x) - x\right| \ge \varepsilon\Big)\Big|_{A=\left[X^{\varphi,a}_{0,k2^{-n}T}(-N),X^{\varphi,a}_{0,k2^{-n}T}(N)\right]}\\
&\le C\varepsilon^{-4}2^{-2n},
\end{aligned}$$

which (A.1) follows from. □

This concludes the proof of Theorem 2.8.

Assume $\varphi \in \Phi_\alpha$, $a \in A_\beta$, $\beta - \alpha > -1$ and $\alpha < 2$. Theorem 2.9 corresponds to [1], Theorem 7.4, the crucial point being the estimate

$$\mathrm{E}\,\sharp\left\{X^{\varphi,0}_{0,t}(u) \mid u \in [0, M]\right\} \le 1 + \limsup_{n\to\infty} \sum_{k=\overline{0,n-1}} \mathrm{P}\left(X^{\varphi,0}_{0,t}\Big(\frac{k+1}{n}M\Big) > X^{\varphi,0}_{0,t}\Big(\frac{k}{n}M\Big)\right),$$
$$M \in \mathbb{R}_+, n \in \mathbb{N}, \tag{A.2}$$

where for some $C = C(t)$ and $0 < y - x$

$$\mathrm{P}\left(X^{\varphi,0}_{0,t}(y) > X^{\varphi,0}_{0,t}(x)\right) \le C(y - x), \tag{A.3}$$

so then

$$\mathrm{E}\,\sharp\left\{X^{\varphi,0}_{0,t}(u) \mid u \in [0, M]\right\} \le 1 + CM.$$

Since (A.2) holds for $X^{\varphi,a}$, too, it is sufficient to prove (A.3) in the case of non-zero drift.

In what follows we refer to [27], Section 5.5 and [40], Chapter 16 for the theory of Feller's one-dimensional diffusions and a classification of boundaries and to [41] for basic facts about Bessel processes including those with negative dimension.

Let ξ be a diffusion on $\mathbb{R}_+$ with generator

$$(1-\varphi(x))\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

and an absorbing boundary at 0. The scale function can be defined as

$$p(x)=\int_0^x \mathrm{e}^{-\int_0^y \frac{\rho(z)}{1-\varphi(z)}\mathrm{d}z}\mathrm{d}y,$$

so the speed measure is

$$m(\mathrm{d}y)=\frac{\mathrm{e}^{\int_0^y \frac{\rho(z)}{1-\varphi(z)}\mathrm{d}z}}{1-\varphi(y)}\mathrm{d}y.$$

Set $\widetilde{C}=\min\{\widetilde{C}_\rho,\widetilde{C}_\varphi\}$ and define

$$v(x)=\int_{\widetilde{C}}^x (p(x)-p(y))m(\mathrm{d}y)\geq 0,\quad x\in\mathbb{R}_+.$$

Since

$$\lim_{\varepsilon\to 0+} v(\varepsilon)\leq C_\varphi^{-1}\mathrm{e}^{\int_0^{\widetilde{C}} \frac{\rho(y)}{1-\varphi(y)}\mathrm{d}y}\limsup_{\varepsilon\to 0+}\int_\varepsilon^{\widetilde{C}}\frac{y-\varepsilon}{y^\alpha}\mathrm{d}y<\infty,$$

the boundary 0 is accessible in finite time with positive probability p_0. It is worth noting that $p_0<1$ in general, contrary to the case $\rho\equiv 0$, as shown below.

Example A.5. Let $\rho(x)=0$. Then $\lim_{x\to\infty}p(x)=\infty$, $m([1,\infty))=\infty$, so ∞ is a natural boundary and the diffusion ξ hits 0 a.s. in finite time τ, though $\mathrm{E}\,\tau=\infty$. Indeed, $\xi=\xi_0+w_{\langle\xi\rangle}$ for some Wiener process w, where $\langle\xi\rangle_t\leq 2t, t\geq 0$, so, if $\tau_y=\inf\{s\mid \xi_s=y\}, y\in\mathbb{R}_+$, we have

$$\begin{aligned}\mathrm{E}\,\tau &> \mathrm{P}\left(\tau_{\xi_0+1}<\infty\right)\mathrm{E}_{\xi_0+1}\,\tau_{\xi_0}\\ &=\mathrm{P}\left(\tau_{\xi_0+1}<\infty\right)\int_0^\infty \mathrm{P}\left(\xi_0+1+\inf_{s\in[0,t]}w_{\langle\xi\rangle_s}>\xi_0\right)\mathrm{d}t\\ &\geq \mathrm{P}\left(\tau_{\xi_0+1}<\infty\right)\int_0^\infty \mathrm{P}\left(\inf_{s\in[0,2t]}w_s>-1\right)\mathrm{d}t\\ &=+\infty.\end{aligned}$$

In fact, 0 is an exit for $\alpha\in[1;2)$ and a regular boundary for $\alpha\in(0;1)$.

Example A.6. Let $\rho(x)=x$. Since $-\frac{z}{1-\varphi(z)}\leq -z$ on $\mathbb{R}_+$,

$$\lim_{x\to\infty}p(x)\leq\int_0^\infty \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y<\infty,$$

so ∞ is accessible. Then

$$p_0 \le \frac{\lim_{x\to\infty} p(x) - p(\xi_0)}{\lim_{x\to\infty} p(x)} < 1.$$

Example A.7. Let $\rho(x) = 1$ and $\alpha < 1$. Then the situation of Example A.6 repeats. In particular, given $y > x$ the difference $X^{\varphi,a}_{0,\cdot}(y) - X^{\varphi,a}_{0,\cdot}(x)$ goes to infinity with a positive probability in either example.

The next lemma is proved with a standard localization argument.

Lemma A.8. *For any* $(u_1, u_2) \in D_2$ *and any* $t > 0$

$$\mathrm{P}\left(X^{\varphi,a}_{0,t}(u_2) > X^{\varphi,a}_{0,t}(u_1)\right) \le \mathrm{P}_{u_2-u_1}\left(\xi_t > 0\right).$$

Remark A.9. By the localization argument, one can prove a stronger statement: given $\xi_0 = u_2 - u_1 > 0$ we have $\xi \ge X^{\varphi,a}_{0,\cdot}(u_2) - X^{\varphi,a}_{0,\cdot}(u_1)$ a.s. if ξ solves, up to the moment of hitting 0,

$$\mathrm{d}\xi_t = \rho(\xi_t)\mathrm{d}t + \left(2(1-\varphi(\xi_t))\right)^{1/2}\mathrm{d}w_t,$$

where the Wiener process w is such that

$$X^{\varphi,a}_{0,t}(u_2) - X^{\varphi,a}_{0,t}(u_1) \\ = \int_0^t \left[a\left(X^{\varphi,a}_{0,s}(u_2)\right) - a\left(X^{\varphi,a}_{0,s}(u_1)\right)\right]\mathrm{d}s + 2^{1/2}\int_0^t \left[1-\varphi\left(X^{\varphi,a}_{0,s}(u_2) - X^{\varphi,a}_{0,s}(u_1)\right)\right]^{1/2}\mathrm{d}w_s.$$

Proposition A.10. *Suppose* $\varphi \in \Phi_\alpha$, $a \in A_\beta$, $\beta - \alpha > -1$ *and* $\alpha < 2$. *Then for any* $t > 0$ *and any* $x, y \in \mathbb{R}, y > x$,

$$\mathrm{P}\left(X^{\varphi,a}_{0,t}(y) > X^{\varphi,a}_{0,t}(x)\right) \le C(y-x), \tag{A.4}$$

for some $C = C(t)$.

Proof. By Lemma A.8 it is sufficient to prove (A.4) for the diffusion ξ. We follow the idea of [1] of switching to a squared Bessel process. Since ξ is not in the natural scale, one step is added and the coefficients are distorted, so we provide necessary details.

Set $\sigma = (2(1-\varphi))^{1/2}, p(\infty) = \lim_{x\to\infty} p(x)$. The diffusion $\widetilde{\xi} = p(\xi)$ on $(0, p(\infty))$ with absorption at 0 has generator $\frac{1}{2}\widetilde{\sigma}^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, where

$$\widetilde{\sigma}(y) = \sigma(p^{-1}(y))p'(p^{-1}(y)) = \sigma(p^{-1}(y))\mathrm{e}^{-\int_0^{p^{-1}(y)} \frac{\rho(z)\mathrm{d}z}{1-\varphi(z)}},$$
$$p^{-1}(y) = \int_0^y \frac{\mathrm{d}u}{p'(p^{-1}(u))} = \int_0^y \mathrm{e}^{\int_0^{p^{-1}(u)} \frac{\rho(z)\mathrm{d}z}{1-\varphi(z)}}\mathrm{d}u,$$
$$y \in [0, p(\infty)).$$

Set $\widetilde{\theta} = \inf\{s \mid \widetilde{\xi}_s = 0\}$. Define

$$\psi(y) = \frac{\widetilde{\sigma}^2(y)}{y^\alpha}, \quad y \in [0, p(\infty)),$$

$$\tau(t) = \inf\Big\{s \mid \int_0^s \psi(\widetilde{\xi}_r)\mathrm{d}r = t\Big\}, \quad t \le \int_0^{\widetilde{\theta}} \psi(\widetilde{\xi}_s)\mathrm{d}s.$$

Then $\eta = \widetilde{\xi}_\tau$ is a diffusion on $(0, p(\infty))$ with generator $\frac{1}{2}y^\alpha \frac{\mathrm{d}^2}{\mathrm{d}y^2}$ and absorption at 0, and $\widetilde{\eta} = (\frac{2}{2-\alpha})^2\eta^{2-\alpha}$ is a diffusion on $(0, (\frac{2}{2-\alpha})^2 p(\infty)^{2-\alpha})$ with generator $2y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \delta\frac{\mathrm{d}}{\mathrm{d}y}$ and absorption at 0. Here $\delta = \frac{2(1-\alpha)}{2-\alpha} \in (-\infty, 1)$. Note that the diffusion $\widetilde{\widetilde{\eta}}$ on $[0,\infty)$ with generator $2y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \delta\frac{\mathrm{d}}{\mathrm{d}y}$ is a squared Bessel process with dimension δ and always hits 0. It is known [42], Example 6.4 that

$$\mathrm{P}_{\widetilde{\widetilde{\eta}}_0}\left(\widetilde{\widetilde{\eta}}_t > 0\right) = \frac{1}{\Gamma(\frac{1}{2-\alpha})}\int_0^{\widetilde{\widetilde{\eta}}_0/(2t)} s^{-\delta/2}\mathrm{e}^{-s}\mathrm{d}s \le \frac{2-\alpha}{\Gamma(\frac{1}{2-\alpha})}\left(\frac{\widetilde{\widetilde{\eta}}_0}{2t}\right)^{\frac{1}{2-\alpha}}. \tag{A.5}$$

If $p(\infty) = \infty$ we have, for any $\eta_0 > 0$,

$$\mathrm{P}_{\eta_0}(\eta_t > 0) = \mathrm{P}_{(\frac{2}{2-\alpha})^2\eta_0^{2-\alpha}}\left(\widetilde{\widetilde{\eta}}_t > 0\right), \tag{A.6}$$

but if $p(\infty) < \infty$

$$\left\{\widetilde{\eta} \text{ hits } \left(\frac{2}{2-\alpha}\right)^2 p(\infty)^{2-\alpha} \text{ before } 0\right\} = \left\{\liminf_{t\to\infty} \xi_t = \infty\right\},$$

so, for $A = (\frac{2}{2-\alpha})^2 p(\infty)^{2-\alpha}$,

$$\begin{aligned}\mathrm{P}_{\widetilde{\eta}_0}(\widetilde{\eta}_t > 0) &< \mathrm{P}_{\widetilde{\eta}_0}\left(\widetilde{\widetilde{\eta}}_t > 0\right) + \mathrm{P}_{\widetilde{\eta}_0}\left(\widetilde{\eta} \text{ hits } A \text{ before } 0\right) \\ &= \mathrm{P}_{\widetilde{\eta}_0}\left(\widetilde{\widetilde{\eta}}_t > 0\right) + \frac{q(\widetilde{\eta}_0) - q(0)}{q(A) - q(0)},\end{aligned} \tag{A.7}$$

where the scale function q for $\widetilde{\widetilde{\eta}}$ equals, for fixed $\varepsilon_0 > 0$,

$$q(x) = \int_{\varepsilon_0}^x \mathrm{e}^{-\frac{\delta}{2}\int_{\varepsilon_0}^y \frac{\mathrm{d}z}{z}}\mathrm{d}y = \frac{\varepsilon_0^{\frac{\delta}{2}}}{1-\frac{\delta}{2}}\left(x^{1-\frac{\delta}{2}} - \varepsilon_0^{1-\frac{\delta}{2}}\right).$$

Since $1 - \frac{\delta}{2} = \frac{1}{2-\alpha}$ and

$$q(A) - q(0) = \frac{\varepsilon_0^{\frac{\delta}{2}}}{1-\frac{\delta}{2}}A^{1-\frac{\delta}{2}} = \frac{4\varepsilon_0^{\frac{\delta}{2}}}{(1-\frac{\delta}{2})(2-\alpha)^2}p(\widetilde{C})^{2-\alpha} > 0,$$

we get by combining (A.5) and (A.7) that for some $C = C(t, \alpha, \varepsilon_0, \widetilde{C}) > 0$,

$$\mathrm{P}_{\widetilde{\eta}_0}(\widetilde{\eta}_t > 0) < C\widetilde{\eta}_0^{\frac{1}{2-\alpha}},$$

which implies together with (A.6) that for some $C_1 = C_1(t, \alpha, \widetilde{C}) > 0$

$$\mathrm{P}_x(\eta_t > 0) \le C_1 x, \quad x > 0, \tag{A.8}$$

regardless of whether $p(\infty)$ is finite or not.

Define $\widetilde{\theta}(a) = \inf\{s \mid \widetilde{\xi}_s = a\}, a \geq 0$. For the diffusion ξ we have

$$\begin{aligned}\mathrm{P}_x(\xi_t > 0) &= \mathrm{P}_{p(x)}\left(\widetilde{\xi}_t > 0\right)\\ &\leq \mathrm{P}_{p(x)}\left(\widetilde{\xi}_t > 0, \widetilde{\theta}(p(\widetilde{C})) = \infty\right) + \mathrm{P}_{p(x)}\left(\widetilde{\theta}(p(\widetilde{C})) < \widetilde{\theta}(0)\right)\\ &= \mathrm{P}_{p(x)}\left(\widetilde{\xi}_t > 0, \widetilde{\theta}(p(\widetilde{C})) = \infty\right) + \frac{p(x)}{p(\widetilde{C})}.\end{aligned}$$

Since $p(x) \leq x, p^{-1}(y) \geq y$, so we have for $y \in (0, p(\widetilde{C}))$

$$\psi(y) \geq C_\varphi \frac{p^{-1}(y)^\alpha}{y^\alpha} \mathrm{e}^{-\int_0^{p^{-1}(y)} \frac{\rho(z)}{1-\varphi(z)}\mathrm{d}z} \geq C_\varphi \mathrm{e}^{-\int_0^{\widetilde{C}} \frac{\rho(z)}{1-\varphi(z)}\mathrm{d}z} = \gamma > 0,$$

and on $\{\theta(p(\widetilde{C})) = \infty\}$

$$\tau(t) = \int_0^t \frac{\mathrm{d}s}{\psi(\widetilde{\xi}_{\tau(s)})} \leq \gamma^{-1} t.$$

Thus we get, using (A.8),

$$\begin{aligned}\mathrm{P}_x(\xi_t > 0) &\leq \mathrm{P}_{p(x)}\left(\widetilde{\xi}_t > 0, \tau(s) \leq \gamma^{-1}s, s \leq \int_0^{\widetilde{\theta}} \psi(\widetilde{\xi}_r)\mathrm{d}r\right) + \frac{x}{p(\widetilde{C})}\\ &\leq \mathrm{P}_{p(x)}\left(\widetilde{\xi}_t = \eta_{\tau^{-1}(t)} > 0, \tau^{-1}(t) \geq \gamma t\right) + \frac{x}{p(\widetilde{C})}\\ &\leq \mathrm{P}_{p(x)}(\eta_{\gamma t} > 0) + \frac{x}{p(\widetilde{C})}\\ &\leq \left(C_1 + \frac{1}{p(\widetilde{C})}\right)x, \end{aligned} \tag{A.9}$$

which concludes the proof. □

Remark A.11. [42], Example 6.4 uses [43], Exercise XI.1.22 that is stated for only positive dimensions. However, the claim of [43], Exercise XI.1.22 is known to hold for Bessel processes with arbitrary dimensions (after restricting to trajectories that do not hit 0).

Remark A.12. In [18], the coalescing property is established and estimates for the number of surviving particles are obtained under weaker assumptions on φ and for zero drift by studying eigenfunction expansions of the corresponding transitional densities.

Proof of Lemma 6.3. (1) follows from taking expectation in (5.4).

(2) Let $p \geq 2, u_2, u_1 \in \mathbb{R}, |u_2 - u_1| \leq 1$ be fixed. The flow $X^{\varphi,a}$ is a coalescing flow by Theorem 2.9. Define

$$v_t = \sup_{s\in[0,T]} \left|X^{\varphi,a}_{0,s}(u_2) - X^{\varphi,a}_{0,s}(u_1)\right|, \quad t \in [0,T],$$

and

$$\theta = \inf\left\{T; t \in [0,T] \mid X^{\varphi,a}_{0,t}(u_2) = X^{\varphi,a}_{0,t}(u_1)\right\}.$$

Note that $v_t = v_{\min\{\theta,t\}}, t \in [0,T]$. For some C_p

$$v_t^p \le C_p\Big(|u_2-u_1|^p + \int_0^t v_s^p \mathrm{d}s + \sup_{s\in[0,\theta]} |m_s|^p\Big),$$

where m is a continuous martingale with

$$\langle m\rangle_t = 2\int_0^t \left(1-\varphi\left(X_{0,s}^{\varphi,a}(u_2) - X_{0,s}^{\varphi,a}(u_1)\right)\right) \mathrm{d}s \le 2t, \quad t\in[0,T].$$

Thus

$$\mathrm{E}\, v_\theta^p \le \mathrm{e}^{C_pT}\Big(|u_2-u_1|^p + \mathrm{E}\sup_{s\in[0,\theta]}|m_s|^p\Big).$$

Let w be a standard Wiener process. For any $a\in[0,T]$ and any $\varepsilon\in(0;1)$

$$\begin{aligned}\mathrm{E}\sup_{s\in[0,\theta]}|m_s|^p &\le \mathrm{E}\sup_{s\in[0,\theta]}|m_s|^p\, 1\!\!\mathrm{I}\,[\theta\ge a] + \mathrm{E}\sup_{s\in[0,a]}|m_s|^p \\ &\le \Big(\mathrm{E}\sup_{t\in[0,2T]}|w_t|^{\frac{p}{\varepsilon}}\Big)^{\varepsilon} \mathrm{P}\,(\theta\ge a)^{1-\varepsilon} + \mathrm{E}\sup_{t\in[0,2a]}|w_t|^p.\end{aligned}$$

Here by (A.9) and (A.5) for some C

$$\mathrm{P}\,(\theta\ge a) \le C\frac{|u_2-u_1|}{a^{\frac{1}{2-\alpha}}},$$

while

$$\begin{aligned}\mathrm{E}\sup_{t\in[0,2a]}|w_t|^p &= p\int_0^\infty u^{p-1}\,\mathrm{P}\Big(\sup_{t\in[0,2a]}|w_t|\ge u\Big)\mathrm{d}u \\ &\le \frac{2pa^{1/2}}{\pi^{1/2}}\int_0^\infty u^{p-2}\mathrm{e}^{-\frac{u^2}{4a}}\,\mathrm{d}u \\ &= \frac{2p}{\pi^{1/2}}a^{p/2}\int_0^\infty u^{p-2}\mathrm{e}^{-\frac{u^2}{4}}\,\mathrm{d}u.\end{aligned}$$

Thus setting $a=|u_2-u_1|^\kappa$, where $\kappa=\gamma(2-\alpha), \gamma\in(0,1)$, gives, for a new constant C_1

$$\begin{aligned}\mathrm{E}\,v_\theta^p &\le C_1\Big(a^{p/2} + \frac{|u_2-u_1|^{1-\varepsilon}}{a^{\frac{1-\varepsilon}{2-\alpha}}}\Big) \\ &= C_1\Big(|u_2-u_1|^{\frac{\gamma(2-\alpha)p}{2}} + |u_2-u_1|^{(1-\varepsilon)(1-\gamma)}\Big).\end{aligned}$$

The optimal γ is

$$\frac{1-\varepsilon}{1-\varepsilon+\frac{(2-\alpha)p}{2}}.$$

□

Appendix B. Proofs of Propositions 3.1 and 3.2

Note that since the flow $X^{\varphi,a}$ does not necessarily have the inverse flow one cannot refer to results in [2], Chapter 4 directly.

Put

$$\overline{M}_{s,t}(x) = M_{s,t}(x) - x = X^{\varphi,a}_{s,t}(x) - x - \int_s^t a(X^{\varphi,a}_{s,r}(x))\mathrm{d}r,$$
$$\overline{X}^{\varphi,a}_{s,t}(x) = X^{\varphi,a}_{s,t}(x) - x,$$
$$0 \le s \le t, x \in \mathbb{R}.$$

Each $\overline{M}_{s,\cdot}$ is a standard Wiener process started at 0.

The following lemma which follows from the definition of a Harris flow is repeatedly used in the sequel. We also skip straightforward calculations.

Lemma B.1. *For all $s, r, t \ge 0, s \le r \le t$, and $u, v \in \mathbb{R}$*

$$\mathrm{E}\,\overline{M}_{s,t}(u)\overline{M}_{s,t}(v) = \int_s^t \varphi\left(X^{\varphi,a}_{s,r}(u) - X^{\varphi,a}_{s,r}(v)\right)\mathrm{d}r,$$
$$\mathrm{E}\left(\overline{M}_{s,t}(u) - \overline{M}_{s,t}(v)\right)^2 = 2\int_s^t \left(1 - \varphi\left(X^{\varphi,a}_{s,r}(u) - X^{\varphi,a}_{s,r}(v)\right)\right)\mathrm{d}t,$$
$$\overline{M}_{s,t}(u) = \overline{M}_{s,r}(x) + \overline{M}_{r,t}\left(X^{\varphi,a}_{s,r}(x)\right).$$

Proof of Proposition 3.1. To simplify notation consider $s = 0, t = 1, t_{n,k} = \frac{k}{2^n}, k = \overline{0, 2^n}$,

$$\xi_n = \sum_{k=\overline{0,2^n-1}} \overline{M}_{t_{n,k},t_{n,k+1}}(x), \quad n \in \mathbb{N}.$$

For the sequence $(\xi_n)_{n\in\mathbb{N}}$ to be Cauchy in $L_2(\Omega)$, the space of square integrable random variables, it is sufficient to prove

$$\alpha_{n,m} = \mathrm{E}\,\xi_n\xi_m \to t, \quad n, m \to \infty. \tag{B.1}$$

Define

$$A_{n,k,m} = \{j \in \{0, \dots, 2^m - 1\} \mid t_{m,j} \in [t_{n,k}, t_{n,k+1}]\},$$
$$n, m, k \in \mathbb{N}, m \ge n, k = \overline{0, 2^n - 1}.$$

Assume $m \ge n$ throughout the proof. We have

$$\alpha_{n,m} = \sum_{k=\overline{0,2^n-1}} \sum_{j\in A_{n,k,m}} \mathrm{E}\,\overline{M}_{t_{n,k},t_{m,j+1}}(x)\overline{M}_{t_{m,j},t_{m,j+1}}(x)$$
$$= \sum_{k=\overline{0,2^n-1}} \sum_{j\in A_{n,k,m}} \mathrm{E}\int_{t_{m,j}}^{t_{m,j+1}} \varphi\left(X^{\varphi,a}_{t_{m,j},r}(u) - X^{\varphi,a}_{t_{m,j},r}(x)\right)\Big|_{u=X^{\varphi,a}_{t_{n,k},t_{m,j}}(x)}\mathrm{d}r$$
$$= t - \sum_{k=\overline{0,2^n-1}} \sum_{j\in A_{n,k,m}} \mathrm{E}\,g_m(x, X^{\varphi,a}_{t_{n,k},t_{m,j}}(x)),$$

where

$$g_m(u,v) = \mathrm{E}\int_0^{\frac{t}{2^m}} \left(1-\varphi\left(X^{\varphi,a}_{0,r}(u) - X^{\varphi,a}_{0,r}(v)\right)\right)\mathrm{d}t,$$
$$u,v\in\mathbb{R}, m\in\mathbb{N}.$$

Assume $\varepsilon > 0$ is fixed. Then

$$\begin{aligned} g_m(u,v) \le 2^{-m}\Big[&\mathrm{P}\left(\sup_{r\in[0,2^{-m}]}\max\left\{\left|\overline{X}_{0,r}(u)\right|, \left|\overline{X}_{0,r}(v)\right|\right\}\ge\varepsilon\right)\\ &+\sup_{y\in\overline{B}(u-v,2\varepsilon)}(1-\varphi(y))\Big]\\ \le 2^{-m}\Big[&2\sup_{y\in\{u,v\}}\mathrm{P}\left(\sup_{r\in[0,2^{-m}]}\left|\overline{X}_{0,r}(y)\right|\ge\varepsilon\right)+\sup_{y\in\overline{B}(u-v,2\varepsilon)}(1-\varphi(y))\Big],\end{aligned}$$

so

$$\begin{aligned} 0\le t-\alpha_{n,m}\le 2^{-m+1}&\sum_{k=\overline{0,2^n-1}}\sum_{j\in A_{n,k,m}}\sup_{y\in\left\{x,X^{\varphi,a}_{t_{n,k},t_{m,j}}(x)\right\}}\mathrm{P}\left(\sup_{r\in[0,2^{-m}]}\left|\overline{X}_{0,r}(y)\right|\ge\varepsilon\right)\\ &+2^{-m+1}\sum_{k=\overline{0,2^n-1}}\sum_{j\in A_{n,k,m}}\mathrm{P}\left(\left|\overline{X}^{\varphi,a}_{t_{n,k},t_{m,j}}(x)\right|\ge\varepsilon\right)\\ &+\sup_{|y|\le 3\varepsilon}(1-\varphi(y)). \end{aligned}\tag{B.2}$$

Since for some absolute $C>0$

$$\sup_{r\in[r_1,r_2]}\left|\overline{X}^{\varphi,a}_{r_1,r}(y)\right|\le C\Big(\sup_{r\in[r_1,r_2]}\left|\overline{M}_{r_1,r}(y)\right|+(1+|y|)(r_2-r_1)\Big),$$
$$0\le r_1\le r_2\le 1, y\in\mathbb{R},\tag{B.3}$$

we have, for independent standard Wiener processes w_1, w_2 and $j\in A_{n,k,m}$

$$\begin{aligned}\mathrm{P}\left(\left|\overline{X}^{\varphi,a}_{t_{n,k},t_{m,j}}(x)\right|\ge\varepsilon\right)&+\sup_{y\in\left\{x,X^{\varphi,a}_{t_{n,k},t_{m,j}}(x)\right\}}\mathrm{P}\left(\sup_{r\in[0,2^{-m}]}\left|\overline{X}_{0,r}(y)\right|\ge\varepsilon\right)\\ &\le 2\,\mathrm{P}\left(\sup_{s\in[0,2^{-n}]}|w_{1s}|+C2^{-m}\sup_{s\in[0,2^{-n}]}|w_{2s}|\ge\frac{\varepsilon}{C}-2^{-m}-C2^{-n}(1+|x|)\right).\end{aligned}$$

Substituting the last estimate into (B.2) implies (B.1).

It is trivial to check that

$$s\mapsto V(s,x) = (W_s - W_0, \varphi(x-\cdot))_{H_\varphi},$$

is a martingale w.r.t the filtration of the flow and

$$\mathrm{E}\,V(r_1,x)V(r_2,y) = \min\{r_1,r_2\}\varphi(x-y),$$

so it is left to prove that $\{V(t,x)\mid t\ge 0, x\in\mathbb{R}\}$ is a jointly Gaussian system.

Let $u_k = V(r, x_k), x_k \in \mathbb{R}, r > 0, k = \overline{1, n}, n \in \mathbb{N}$. The Gram–Schmidt orthogonalization produces $v_k \in L_2(\Omega)$ such that

$$\mathrm{E}\, v_k v_j = 1\!\!\mathrm{I}\left[k \neq j\right],$$
$$v_k = \sum_{i=\overline{1,k}} b_{k,i} u_i,$$
$$u_k = \sum_{i=\overline{1,k}} c_{k,i} v_i \quad k, j = \overline{1, n},$$

where the coefficients $\{b_{k,i}\}$ and $\{c_{k,i}\}$ are deterministic and do not depend on r. Define martingales

$$z_k = \sum_{j=\overline{1,k}} b_{k,j} V(\cdot, x_j), \quad k = \overline{1, n}.$$

Then

$$\langle z_k, z_j \rangle_r = r\, \mathrm{E}\, v_k v_j = r 1\!\!\mathrm{I}\left[k \neq j\right], \quad k, j = \overline{1, n}.$$

Since uncorrelated continuous martingales with independent increments are independent, the proposition is proved. □

Proof of Proposition 3.2. By definition of the stochastic integral,

$$\int_s^t W\left(X^{\varphi,a}_{s,r}(x), \mathrm{d}r\right) = L_2 - \lim_{n\to\infty} \sum_{k=\overline{0,n-1}} \Big(W(t_{n,k+1}, X^{\varphi,a}_{s,t_{n,k}}(x)) - W(t_{n,k}, X^{\varphi,a}_{s,t_{n,k}}(x)\Big),$$

where $t_{n,k} = s + \frac{k}{n}(t - s), k = \overline{0, n}$, so

$$\begin{aligned} I &= \mathrm{E}\Big(\int_s^t W\left(X^{\varphi,a}_{s,r}(x), \mathrm{d}r\right) - \overline{M}_{s,t}(x)\Big)^2 \\ &= \lim_{n\to\infty} \sum_{k=\overline{0,n-1}} \mathrm{E}\Big[W\Big(t_{n,k+1}, X^{\varphi,a}_{s,t_{n,k}}(x)\Big) - W\Big(t_{n,k}, X^{\varphi,a}_{s,t_{n,k}}(x)\Big) - \overline{M}_{t_{n,k},t_{n,k+1}}\Big(X^{\varphi,a}_{s,t_{n,k}}(x)\Big)\Big]^2. \end{aligned} \tag{B.4}$$

Consider uniform partitions of the closed intervals $[t_{n,k}, t_{n,k+1}]$:

$$s^{n,k}_{m,j} = t_{n,k} + \frac{j}{m}(t_{n,k+1} - t_{n,k}), \quad j = \overline{1, m}, k = \overline{0, n-1}.$$

Then

$$\overline{M}_{t_{n,k},t_{n,k+1}}\Big(X^{\varphi,a}_{s,t_{n,k}}(x)\Big) = \sum_{j=\overline{0,m-1}} \overline{M}_{s^{n,k}_{m,j}, s^{n,k}_{m,j+1}}\Big(X^{\varphi,a}_{s,s^{n,k}_{m,j}}(x)\Big),$$
$$W\Big(t_{n,k+1}, X^{\varphi,a}_{s,t_{n,k}}\Big) - W\Big(t_{n,k}, X^{\varphi,a}_{s,t_{n,k}}\Big) = L_2 - \lim_{m\to\infty} \sum_{j=\overline{0,m-1}} \overline{M}_{s^{n,k}_{m,j}, s^{n,k}_{m,j+1}}\Big(X^{\varphi,a}_{s,t_{n,k}}(x)\Big),$$

so returning to (B.4) gives

$$\begin{aligned} I &\leq \lim_{n\to\infty} \sum_{k=\overline{0,n-1}} \lim_{m\to\infty} \mathrm{E}\Bigg[\sum_{j=\overline{0,m-1}} \Big(\overline{M}_{s^{n,k}_{m,j}, s^{n,k}_{m,j+1}}\Big(X^{\varphi,a}_{s,t_{n,k}}(x)\Big) - \overline{M}_{s^{n,k}_{m,j}, s^{n,k}_{m,j+1}}\Big(X^{\varphi,a}_{s,s^{n,k}_{m,j}}(x)\Big)\Big)\Bigg]^2 \\ &= \lim_{n\to\infty} \sum_{k=\overline{0,n-1}} \lim_{m\to\infty} \mathrm{E} \sum_{j=0,m-1} f_{n,m}\Big(X^{\varphi,a}_{s,t_{n,k}}(x), X^{\varphi,a}_{s,s^{n,k}_{m,j}}(x)\Big), \end{aligned}$$

where

$$f_{n,m}(u,v) = 2\,\mathrm{E}\int_0^{\frac{t-s}{nm}} \left(1 - \varphi\left(X_{0,r}^{\varphi,a}(u) - X_{0,r}^{\varphi,a}(v)\right)\right) \mathrm{d}r,$$
$$u, v \in \mathbb{R}, n, m \in \mathbb{N}.$$

To obtain the final estimate for (B.4), one proceeds as in the proof of Proposition 3.1. □

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