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ITÔ-KRYLOV'S FORMULA FOR A FLOW OF MEASURES

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Abstract. In this article, we prove Itô's formula for the flow of measures associated with an Itô process having a bounded drift and a uniformly elliptic and bounded diffusion matrix, and for functions in an appropriate Sobolev-type space. This formula is the almost analogue, in the measure-dependent case, of the Itô-Krylov formula for functions in a Sobolev space on $\mathbf{R}^+ \times \mathbf{R}^d$.

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1. Introduction

We fix $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ a filtered probability space satisfying the usual conditions. Let $T > 0$ be a finite horizon of time, $d, d_1 \in \mathbf{N}^*$ with $d_1 \geq d$, and $(B_t)_{t\geq 0}$ a $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion of dimension d_1. We consider the Itô process on $\mathbf{R}^d$ defined, for $t \in [0, T]$, by

$$X_t := X_0 + \int_0^t b_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s, \tag{1.1}$$

where $X_0 \in L^2(\Omega, \mathcal{F}_0; \mathbf{R}^d)$, $b : [0, T] \times \Omega \to \mathbf{R}^d$ and $\sigma : [0, T] \times \Omega \to \mathbf{R}^{d\times d_1}$ are bounded progressively measurable processes. In the following, we will denote by μ_t the law of X_t at a given time $t \in [0, T]$ and by a the matrix $\sigma\sigma^*$.

Let us fix a real-valued function u defined on the 2-Wasserstein space $\mathcal{P}_2(\mathbf{R}^d)$, *i.e.* the space of probability measures on $\mathbf{R}^d$ having a finite moment of order 2. In this paper, we are interested in Itô's formula for u computed along the flow of probability measures $(\mu_t)_{t\in[0,T]}$. This formula describes the dynamics of $t \mapsto u(\mu_t)$, essentially by computing its derivative (see (1.2) below). It has a wide range of applications for example in Mean-Field Games, McKean-Vlasov's control problems, McKean-Vlasov Stochastic Differential Equations (SDEs) but also in the study of interacting particle systems and the propagation of chaos. These applications will be detailed below.

Itô's formula for a flow of measures naturally requires differential calculus on the space of measures $\mathcal{P}_2(\mathbf{R}^d)$. We will use the linear (functional) derivative, which is a standard notion of differentiability for functions of measures relying on the Polish structure of $\mathcal{P}_2(\mathbf{R}^d)$. The function u admits a linear derivative if there exists a real-valued and continuous function $\frac{\delta u}{\delta m}$ defined on $\mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d$, at most of quadratic growth with respect to

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the space variable uniformly on each compact set of $\mathcal{P}_2(\mathbf{R}^d)$, and such that for all $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^d)$

$$u(\mu) - u(\nu) = \int_0^1 \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(t\mu + (1-t)\nu)(v)\, d(\mu - \nu)(v)\, \mathrm{d}t.$$

The standard Itô formula for a flow of measures can be found in [1] (see Thm. 6.1) or in Section 3 of [2] and Chapter 5 of [3] (see Thm. 5.99) under less restrictive assumptions. It states that for all $t \in [0,T]$

$$u(\mu_t) = u(\mu_0) + \int_0^t \mathbf{E}\left(\partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s) \cdot b_s\right) \mathrm{d}s + \frac{1}{2}\int_0^t \mathbf{E}\left(\partial_v^2 \frac{\delta u}{\delta m}(\mu_s)(X_s) \cdot a_s\right) \mathrm{d}s, \tag{1.2}$$

where $x \cdot y$ denotes the usual scalar product of two vectors $x, y \in \mathbf{R}^d$ and $A \cdot B := \mathrm{Tr}(A^* B)$ the usual scalar product of two matrices $A, B \in \mathbf{R}^{d\times d}$. The common point between these results is that the function u has to be $\mathcal{C}^2$ in some sense. More precisely, it is always assumed that for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, the linear derivative $\frac{\delta u}{\delta m}(\mu)(\cdot)$ belongs to $\mathcal{C}^2(\mathbf{R}^d)$ or equivalently that the L-derivative $\partial_\mu u(\mu)(\cdot)$ (understood as the derivative $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$ of $v \in \mathbf{R}^d \mapsto \frac{\delta u}{\delta m}(\mu)(v)$) belongs to $\mathcal{C}^1(\mathbf{R}^d)$ (see below for the precise definition of the L-derivative). The present paper aims at proving Itô's formula (1.2) for functions u having a linear derivative $\frac{\delta u}{\delta m}$ that is not $\mathcal{C}^2$ with respect to the space variable.

We now fix the assumptions on the Itô process $(X_t)_{t\in[0,T]}$. In this paper, we always assume that the drift b and the diffusion matrix σ in (1.1) satisfy the following properties.

(A) There exists $K > 0$ such that almost surely

$$\forall t \in [0,T],\ |b_t| + |\sigma_t| \leq K.$$

(B) There exists $\delta > 0$ such that almost surely

$$\forall t \in [0,T],\ \forall \lambda \in \mathbf{R}^d,\ a_t\lambda \cdot \lambda \geq \delta|\lambda|^2.$$

Assumptions **(A)** and **(B)** stem from Section 2.10 of [4]. Therein, Krylov deals with controlled diffusion processes and needs to apply the standard Itô formula for a pay-off function which is not $\mathcal{C}^2$. That is why he proves an extension of the classical Itô formula for the Itô process $(X_t)_{t\in[0,T]}$ satisfying Assumptions **(A)** and **(B)**, and for a function $g : \mathbf{R}^d \to \mathbf{R}$ belonging to an appropriate Sobolev space. The crucial point is that $(X_t)_t$ satisfies the non-degeneracy Assumption **(B)**. It ensures that the noise does not degenerate and allows to produce a regularizing effect. Let us explain how. The non-degeneracy assumption leads to Krylov's estimate (see Thm. 4.1 below taken from Sect. 2.3 of [4]). This inequality, in turn, implies that for almost all $t \in [0,T]$, μ_t, the law of X_t, has a density $p(t,\cdot)$ with respect to the Lebesgue measure (see Prop. 4.3). Moreover, this density belongs to $L^{(d+1)'}([0,T]\times\mathbf{R}^d)$, where $(d+1)'$ denotes the conjugate exponent of $d+1$ defined in Section 2. The existence of densities together with the integrability property permit to assume Sobolev regularity for the function g. More precisely, Itô-Krylov's formula is established under the assumption that g is continuous on $\mathbf{R}^d$ and that ∇g belongs to the Sobolev space $W^{1,k}_{\mathrm{loc}}(\mathbf{R}^d)$, for $k \geq d+1$, *i.e.* that ∇g and $\nabla^2 g$ are in $L^k_{\mathrm{loc}}(\mathbf{R}^d)$ (see Sect. 2.10 of [4]).

Our goal here is to take advantage of the regularizing effect of the noise, stemming from the existence of the densities $p(t,\cdot)$ and their integrability property, to establish an analogue of Itô-Krylov's formula in the measure-dependent case. If we consider the precise expression (1.2) of Itô's formula for a flow of measures, the regularizing effect comes from the presence of expectations which average, with respect to the space variable, the derivatives of $\frac{\delta u}{\delta m}$ on all the trajectories of $(X_t)_t$. Indeed, the regularization by noise will only appear through the space variable of the linear derivative but not through its measure variable. This is not surprising since the space of measures $\mathcal{P}_2(\mathbf{R}^d)$ is somehow infinite dimensional while the noise is of finite dimension. Thus, we cannot expect a true regularization in the measure variable of $\frac{\delta u}{\delta m}$. The fact that a finite dimensional noise cannot have a complete regularizing effect in the space $\mathcal{P}_2(\mathbf{R}^d)$ is explained in [5] in the context of McKean-Vlasov SDEs.

In order to prove Itô's formula (1.2) for u, it is clear that u needs to admit a linear derivative with at least distributional derivatives, in the Sobolev sense, of order 1 and 2 with respect to the space variable in $L^k(\mathbf{R}^d)$ for some k, as for the standard Itô-Krylov formula. Let us describe more precisely our assumptions on u. As said before, for almost all $t \in [0,T]$, the law μ_t has a density $p(t,\cdot)$ such that p belongs to $L^{(d+1)'}([0,T] \times \mathbf{R}^d)$. Denoting by $\Pi(\mathbf{R}^d)$ the space of measures $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ having a density with respect to the Lebesgue measure in $L^{(d+1)'}(\mathbf{R}^d)$, our assumptions on the derivatives of $\frac{\delta u}{\delta m}(\mu)(\cdot)$ are only made for measures μ belonging to $\Pi(\mathbf{R}^d)$. This is natural since for almost all $t \in [0,T]$, μ_t belongs to $\Pi(\mathbf{R}^d)$, and the derivatives of $\frac{\delta u}{\delta m}$ are evaluated along the flow $(\mu_t)_{t\in[0,T]}$ and integrated in time. Moreover, because of the integrability property of the densities $p(t,\cdot)$, the derivatives of $\frac{\delta u}{\delta m}(\mu)(\cdot)$ do not need to be defined and continuous on the whole space $\mathbf{R}^d$ because they are somehow integrated against the densities $p(t,\cdot)$ (see (1.2)). We say "somehow" because it is not completely the case since b and a are random. But as they are bounded, we can omit them in some sense. More precisely, the integrability property of the densities leads us to assume that u admits a linear derivative such that for all $\mu \in \Pi(\mathbf{R}^d)$, $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$ belongs to the Sobolev space $W^{1,k}(\mathbf{R}^d)$ defined in Section 2, with $k \geq d+1$. This is exactly the same condition as in the standard Itô-Krylov formula, except that we replace $W^{1,k}_{\text{loc}}(\mathbf{R}^d)$ by $W^{1,k}(\mathbf{R}^d)$. This is essentially explained by the expectations in Itô's formula (1.2). Indeed, the process $(X_t)_t$ cannot be localized by stopping times. Moreover, we assume that the map $\mu \in \Pi(\mathbf{R}^d) \mapsto \partial_v \frac{\delta u}{\delta m}(\mu)(\cdot) \in W^{1,k}(\mathbf{R}^d)$ is continuous for a distance on $\Pi(\mathbf{R}^d)$ satisfying the assumptions of Definition 2.4. This continuity assumption can be interpreted as the fact that the noise has no regularizing effect in the measure variable of the linear derivative, as explained above. The precise assumptions of our Itô-Krylov's formula are given in Definition 3.1 and Theorem 3.3. Eventually, we extend in Theorem 3.7 our formula to functions depending also on time and space variables. We refer to Definition 3.5 for the regularity required on these arguments.

We now focus on some applications of Itô's formula for a flow of measures. This one has been developed with the increasing interest for Mean-Field Games and McKean-Vlasov SDEs over the last decade. Mean-Field Games were initiated independently by Caines, Huang and Malhame in [6] and by Lasry and Lions in [7]. A notion of Master equation for Mean-Field Games has been introduced by Lions in his lectures at Collège de France [8] in order to describe in a single infinite dimensional equation the characteristic Hamilton-Jacobi-Bellman equation of a Mean-Field Game system. These Master equations, which are Partial Differential Equations (PDEs) on the space of probability measures, can be usually derived from Itô's formula. Indeed, Itô's formula appears to be the natural way to connect a McKean-Vlasov SDE (more precisely the associated semigroup $(P_t)_t$ acting on the space of functions of measures) to a PDE on the space of probability measures, in the same manner as for classical SDEs. We refer to Lions' lectures [8], the notes written by Cardaliaguet [9], and the books of Carmona and Delarue [3, 10] for a more detailed presentation of Mean-Field Games and Master equations. We also mention Bensoussan, Frehse and Yam [11] and Carmona, Delarue [12] where Master equations are derived, with the help of Itô's formula in [12]. The question of existence and uniqueness of solutions to Master equations was addressed for example in [1, 2, 13, 14]. The link between McKean-Vlasov SDEs and PDEs on the space of measures is at the heart of the work of Buckdahn, Li, Peng and Rainer [1] where the authors prove that the PDE admits a unique classical solution expressed with the flow of measures associated with the McKean-Vlasov SDE. Moreover, in the parallel work [2], Chassagneux, Crisan and Delarue adopt a similar approach and study the flow generated by a forward-backward stochastic system of McKean-Vlasov type under weaker assumptions on the coefficients of the equation. In both works, Itô's formula plays a key role. From a different point of view, Mou and Zhang deal with the well-posedness of Master equations in some weaker senses in [15]. Itô's formula for a flow of measures is also important to deal with McKean-Vlasov control problems because it allows to derive a dynamic programming principle describing the value function of the problem as presented in Chapter 6 of [3]. Finally, the problem of propagation of chaos for the interacting particles system associated with a McKean-Vlasov SDE can also be addressed with the help of the associated PDE on the space of measures (see Chap. 5 of [3]). It allows to obtain quantitative weak propagation of chaos estimates between the law of the solution to the McKean-Vlasov SDE and the empirical measure of the associated particle system. This approach was adopted for example by Chaudru de Raynal and Frikha in [16, 17], by Delarue and Tse in [18] and by Chassagneux, Szpruch and Tse in [19]. Let us also mention that the Master equation satisfied by the semigroup has been recently used by Jourdain and Tse in [20] to study the mean-field fluctuation (CLT) of an interacting particle system.

Recently, Itô's formula has been extended to flows of measures generated by càdlàg semi-martingales. It was achieved independently by Guo, Pham and Wei in [21], who studied McKean-Vlasov control problems with jumps and by Talbi, Touzi and Zhang in [22] who worked on mean-field optimal stopping problems. In both works, dynamic programming principles are established thanks to Itô's formula for a flow of measures. Finally, we also mention that several Itô-Wentzell-Lions formulae for functional random fields of Itô type depending on measure flows have been established by dos Reis and Platonov in [23].

Let us explain our choice to work with the linear derivative. Indeed, the L-derivative, which was introduced by Lions in his lectures at Collège de France [8], is also well-adapted to establish Itô's formula for a flow of measures. We say that u is L-differentiable if its lifting defined by

$$\tilde{u} : X \in L^2(\Omega; \mathbf{R}^d) \mapsto u(\mathcal{L}(X)) \in \mathbf{R},$$

where $\mathcal{L}(X)$ denotes the law of X, is Fréchet differentiable on $L^2(\Omega; \mathbf{R}^d)$. Moreover, there exists a $\mathbf{R}^d$-valued function $\partial_\mu u$ defined on $\mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d$ such that the gradient of $\tilde{u}$ at $X \in L^2(\Omega; \mathbf{R}^d)$ is given by the random variable $\partial_\mu u(\mathcal{L}(X))(X)$. The function $\partial_\mu u$ is called the L-derivative of u. One of the primary advantage of the L-derivative is that it permits to use stochastic calculus on a Hilbertian basis for functional originally defined on the space $\mathcal{P}_2(\mathbf{R}^d)$ (see [8] for more information on this origin and motivation of the L-derivative). In fact, there is a link between the L-derivative and the linear derivative of u. Indeed, in general, the L-derivative $\partial_\mu u(\mu)(\cdot)$ is equal to the gradient of the linear derivative $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$ (see Props. 5.48 and 5.51 in [3] for the precise assumptions). Under our assumptions presented above,the Sobolev embedding theorem ensures that for all $\mu \in \Pi(\mathbf{R}^d)$, $\frac{\delta u}{\delta m}(\mu)(\cdot)$ belongs to $\mathcal{C}^1(\mathbf{R}^d; \mathbf{R})$, and that $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$ is continuous and bounded on $\mathbf{R}^d$. We would be tempted to deduce that u admits a L-derivative given, as recalled above, by $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$. However, this term is assumed to exist only for measures $\mu \in \Pi(\mathbf{R}^d)$ and not for $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. This is the case in Example 3.11, where this term is not well-defined for any $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ (see Rem. 3.12). It seems therefore more restrictive to work with the L-derivative and thus justifies our choice to work with the linear derivative.

The paper is organized as follows. Section 2 gathers some notations and definitions used throughout the paper. In Section 3, more precisely in Definitions 3.1 and 3.5, we define the spaces of functions for which we will establish Itô-Krylov's formula. These formulas are given in Theorem 3.3 for functions defined on $\mathcal{P}_2(\mathbf{R}^d)$ and in Theorem 3.7 for functions depending also on the time and space variables. Moreover, we give examples of functions for which our formulas hold and we discuss our assumptions through them. The proofs of these examples are postponed to Appendix A for ease of reading. In Section 4, we give some preliminary results. We start by recalling Krylov's estimate and its consequences on the existence of densities for the flow of measures $(\mu_t)_{t\in[0,T]}$ in Proposition 4.3. Then we recall some classical results on convolution and regularization procedures. Finally, Sections 5 and 6 are respectively dedicated to the proofs of Theorems 3.3 and 3.7.

2. Notations and definitions

2.1. General notations

Let us introduce some notations used several times in the article.

- B_R is the open ball centered at 0 and of radius R in $\mathbf{R}^d$ for the euclidean norm.
- p' is the conjugate exponent of $p \in [1, +\infty]$, defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
- $L^p_{\text{loc}}(\mathbf{R}^d)$ is the space of functions f such that for all $R > 0$, $f \in L^p(B_R)$.

- $W^{m,k}(\mathcal{O})$ is the Sobolev space of functions $u \in L^k(\mathcal{O})$ admitting distributional derivatives, in the Sobolev sense, of order between 1 and m in $L^k(\mathcal{O})$, where $\mathcal{O}$ is open in $\mathbf{R}^d$. It is equipped with the norm

$$\|u\|_{W^{m,k}(\mathcal{O})} = \sum_{\alpha \in \mathbf{N}^d,\, |\alpha| \le m} \|\partial^\alpha u\|_{L^k(\mathcal{O})}.$$

– $W^{m,k}_{\text{loc}}(\mathbf{R}^d)$ is the space of functions u such that for all $R > 0$, u belongs to $W^{m,k}(B_R)$.
– $(\rho_n)_n$ is a mollifying sequence on $\mathbf{R}^d$, that is a sequence of non-negative $\mathcal{C}^\infty$ functions, such that for all n, $\int_{\mathbf{R}^d} \rho_n(x)\,\mathrm{d}x = 1$ and ρ_n is equal to 0 outside $B_{1/n}$. We assume that $\rho_n(x) = \rho_n(-x)$ for all x.
– $*$ denotes the convolution product between two functions, when it is well-defined, or two probability measures.
– $\mathcal{B}(E)$ is the Borel σ-algebra of a given metric space E.
– A^* denotes the transpose of the matrix $A \in \mathbf{R}^{d\times d}$.
– $A \cdot B$ denotes the usual scalar product of two matrices $A, B \in \mathbf{R}^{d\times d}$ given by $A \cdot B := \mathrm{Tr}(A^* B)$.

2.2. Spaces of measures and linear derivative

The set $\mathcal{P}(\mathbf{R}^d)$ is the space of probability measures on $\mathbf{R}^d$ equipped with the topology of weak convergence. The Wasserstein space $\mathcal{P}_2(\mathbf{R}^d)$ denotes the set of measures $\mu \in \mathcal{P}(\mathbf{R}^d)$ such that $\int_{\mathbf{R}^d} |x|^2\,\mathrm{d}\mu(x) < +\infty$, equipped with the 2-Wasserstein distance W_2 defined for $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^d)$ by

$$W_2(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{\mathbf{R}^d \times \mathbf{R}^d} |x-y|^2\,\mathrm{d}\pi(x,y) \right)^{1/2},$$

where $\Pi(\mu,\nu)$ is the subset of $\mathcal{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$ with marginal distributions μ and ν. As mentioned at the beginning of the article, we will work with the standard notion of linear derivative for functions of measures. To this aim, let us start by specifying a general essential notion of linear derivative we will consider from here on.

Definition 2.1 (Linear derivative)**.** A function $u : \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ is said to have a linear derivative if there exists a continuous function $(\mu, v) \in \mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d \mapsto \frac{\delta u}{\delta m}(\mu)(v) \in \mathbf{R}$, satisfying the following properties.

1. For all compact $\mathcal{K} \subset \mathcal{P}_2(\mathbf{R}^d)$ $\sup_{v \in \mathbf{R}^d} \sup_{\mu \in \mathcal{K}} \left\{ (1+|v|^2)^{-1} \left| \frac{\delta u}{\delta m}(\mu)(v) \right| \right\} < +\infty.$
2. For all $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^d)$, $u(\mu) - u(\nu) = \displaystyle\int_0^1 \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(t\mu + (1-t)\nu)(v)\, d(\mu-\nu)(v)\,\mathrm{d}t.$

Remark 2.2. The linear derivative $\frac{\delta u}{\delta m}(\mu)$ is only defined up to an additive constant. We do not precise the choice of the additive constant, as in [3], since it does not play any role. Let us just mention that a usual choice of the constant can be $\int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(\mu)(v)\,\mathrm{d}\mu(v)$ to guarantee that $\int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(\mu)(v)\,\mathrm{d}\mu(v) = 0$.

Remark 2.3. Let us point out that the property (2) is equivalent to assume that for all $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^d)$, $t \in [0,1] \mapsto u(t\mu + (1-t)\nu)$ is of class $\mathcal{C}^1$ with

$$\forall t \in [0,1],\ \frac{\mathrm{d}}{\mathrm{d}t} u(t\mu + (1-t)\nu) = \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(t\mu + (1-t)\nu)(v)\, d(\mu-\nu)(v).$$

One can find more details in Chapter 5 of [3], in particular the connection with the L-derivative.

Definition 2.4. Let us define $\Pi(\mathbf{R}^d)$ as the space of measures $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ which admit a density $\frac{\mathrm{d}\mu}{\mathrm{d}x}$ with respect to the Lebesgue measure belonging to $L^{(d+1)'}(\mathbf{R}^d)$. We endow $\Pi(\mathbf{R}^d)$ with a general distance $d_{\Pi(\mathbf{R}^d)}$ satisfying the following properties.

(H1) For any $n \geq 1$, $\mu \in (\mathcal{P}_2(\mathbf{R}^d), W_2) \mapsto \mu * \rho_n \in (\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)})$ is continuous, where $(\rho_n)_n$ is a mollifying sequence introduced in the notations.
(H2) For any $\mu \in \Pi(\mathbf{R}^d)$, $\mu * \rho_n \underset{n\to+\infty}{\longrightarrow} \mu$ for $d_{\Pi(\mathbf{R}^d)}$.

Note that for all $n \geq 1$ and for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $\mu * \rho_n \in \Pi(\mathbf{R}^d)$. Indeed, its density is given by $x \mapsto \rho_n * \mu(x) = \int_{\mathbf{R}^d} \rho_n(x-y)\,\mathrm{d}\mu(y)$. Jensen's inequality ensures that it belongs to $L^{(d+1)'}(\mathbf{R}^d)$. Considering the space $(\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)})$ comes in a natural way considering Assumptions **(A)** and **(B)** on the Itô process X. In view of Krylov's estimate (see Thm. 4.1 below), it implies the existence of a density $p \in L^1([0,T]\times\mathbf{R}^d;\mathbf{R}^+) \cap L^{(d+1)'}([0,T]\times\mathbf{R}^d;\mathbf{R}^+)$ such that for almost all $t \in [0,T]$, the law of X_t is equal to $p(t,\cdot)\,\mathrm{d}x$ and belongs to $\Pi(\mathbf{R}^d)$ (see Prop. 4.3). Let us give two examples for the distance $d_{\Pi(\mathbf{R}^d)}$.

Example 2.5. The Wasserstein distance W_2 clearly satisfies Assumptions **(H1)** and **(H2)** in Definition 2.4. Another family of examples is given by the distance $d_{k'}$ defined, for $k \in [d+1,+\infty[, \mu,\nu \in \Pi(\mathbf{R}^d)$, by

$$d_{k'}(\mu,\nu) = \left\| \frac{\mathrm{d}\mu}{\mathrm{d}x} - \frac{\mathrm{d}\nu}{\mathrm{d}x} \right\|_{L^{k'}(\mathbf{R}^d)}.$$

Note that $d_{k'}$ is well-defined since for any $\mu \in \Pi(\mathbf{R}^d)$, $\frac{\mathrm{d}\mu}{\mathrm{d}x} \in L^1(\mathbf{R}^d) \cap L^{(d+1)'}(\mathbf{R}^d)$ which is included in $L^{k'}(\mathbf{R}^d)$ by interpolation. The proof is postponed to the Appendix (Sect. A.1).

3. Main results and examples

3.1. Itô-Krylov's formula for functions defined on $\mathcal{P}_2(\mathbf{R}^d)$

Let us introduce now the Sobolev-type space of functions on $\mathcal{P}_2(\mathbf{R}^d)$ for which we will prove Itô's formula for a flow of measures.

Definition 3.1. Let $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ be the space of continuous functions $u : \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ having a linear derivative $\frac{\delta u}{\delta m}$ such that for all $\mu \in \Pi(\mathbf{R}^d)$, the function $\frac{\delta u}{\delta m}(\mu)(\cdot)$ admits distributional derivatives of order 1 and 2 in $L^k(\mathbf{R}^d)$, for a certain $k \geq d+1$, and satisfies the following properties.

1. The map $\mu \in (\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)}) \mapsto \partial_v \frac{\delta u}{\delta m}(\mu)(\cdot) \in \left(W^{1,k}(\mathbf{R}^d)\right)^d$ is continuous for a certain distance $d_{\Pi(\mathbf{R}^d)}$ satisfying **(H1)** and **(H2)**.
2. There exists $\alpha \in \mathbf{N}$ such that $k \geq (1+\alpha)d$ and for all compact $\mathcal{K} \subset \mathcal{P}_2(\mathbf{R}^d)$ and for any $\mu \in \mathcal{K} \cap \Pi(\mathbf{R}^d)$

$$\left\| \partial_v \frac{\delta u}{\delta m}(\mu)(\cdot) \right\|_{L^k(\mathbf{R}^d)} + \left\| \partial_v^2 \frac{\delta u}{\delta m}(\mu)(\cdot) \right\|_{L^k(\mathbf{R}^d)} \leq C_{\mathcal{K}} \left(1 + \left\| \frac{\mathrm{d}\mu}{\mathrm{d}x} \right\|^{\alpha}_{L^{k'}(\mathbf{R}^d)} \right).$$

Remark 3.2. –The space $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ contains the functions which satisfy Assumption (1) in Definition 3.1 with $(\mathcal{P}_2(\mathbf{R}^d), W_2)$ instead of $(\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)})$. Indeed, the second point is clearly satisfied with $\alpha = 0$ since $\mathcal{K}$ is compact.

–Assumption (2) in Definition 3.1 allows to control the growth of $\left\|\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)\right\|_{W^{1,k}(\mathbf{R}^d)}$ with respect to the measure μ. It allows us to take advantage of the continuity of the flow in $\mathcal{P}_2(\mathbf{R}^d)$ (because the control is assumed on compact subsets of $\mathcal{P}_2(\mathbf{R}^d)$), but also of its integrability properties proved in Lemma 4.4. The form of the inequality suggests the integration of functions in $L^k(\mathbf{R}^d)$ with respect to μ, at least when the function u is linear in μ.

–The Sobolev embedding theorem (see Cor. 9.14 in [24]) ensures that for all $\mu \in \Pi(\mathbf{R}^d)$, $\frac{\delta u}{\delta m}(\mu)(\cdot)$ belongs to $\mathcal{C}^1(\mathbf{R}^d;\mathbf{R})$ and that $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$ is bounded and γ-Hölder, where $\gamma := 1 - \frac{d}{k}$. Note that in view of (1.2), the absence of any assumption on the L^k-integrability of $\frac{\delta u}{\delta m}(\mu)$ is natural.

Having this definition at hand, we can now state Itô-Krylov's formula for functions in $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$.

Theorem 3.3 (Itô-Krylov's formula). *Let u be a function in $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ in Definition 3.1. Then we have for all $t \in [0,T]$*

$$u(\mu_t) = u(\mu_0) + \int_0^t \mathbf{E}\left(\partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s)\cdot b_s\right) \mathrm{d}s + \frac{1}{2}\int_0^t \mathbf{E}\left(\partial_v^2 \frac{\delta u}{\delta m}(\mu_s)(X_s)\cdot a_s\right) \mathrm{d}s, \tag{3.1}$$

where $\partial_v^2 \frac{\delta u}{\delta m}(\mu_s)(X_s)\cdot a_s := Tr\Big(\partial_v^2 \frac{\delta u}{\delta m}(\mu_s)(X_s) a_s\Big)$ is the usual scalar product on $\mathbf{R}^{d\times d}$.

Let us briefly describe the strategy of the proof of Theorem 3.3. The idea is to regularize u by setting $u^n := u(\cdot * \rho_n)$, which converges towards u as $n \to +\infty$. It regularizes the linear of u by convolution since for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ and $v \in \mathbf{R}^d$

$$\frac{\delta}{\delta m} u^n(\mu)(v) = \frac{\delta u}{\delta m}(\mu * \rho_n) * \rho_n(v).$$

In particular, since $\frac{\delta u}{\delta m}(\mu)$ has only distributional derivatives in the Sobolev sense, the convolution by ρ_n ensures that $\frac{\delta}{\delta m} u^n(\mu)$ is of class $\mathcal{C}^2$. We can thus write Itô's formula for u^n and finally take the limit $n \to +\infty$ using Krylov's estimate (see Cor. 4.2) to conclude the proof of Theorem 3.3.

Remark 3.4. Notice that a function $u \in \mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ is assumed to have a linear derivative on the whole space $\mathcal{P}_2(\mathbf{R}^d)$. This seems a bit strong at first sight in comparison with the assumptions on its spatial derivatives that are only made for measures $\mu \in \Pi(\mathbf{R}^d)$ (it is possible essentially because in Itô's formula (3.1), these derivatives only appear under integrals along the flow $(\mu_s)_{s\in[0,T]}$, which belongs to $\Pi(\mathbf{R}^d)$ for almost all $s \in [0,T]$). However, in order to establish Itô-Krylov's formula by regularization, the function u needs to be continuous on the whole space $\mathcal{P}_2(\mathbf{R}^d)$ and not only on $\Pi(\mathbf{R}^d)$. Indeed, the flow $s \in [0,T] \mapsto \mu_s \in \mathcal{P}_2(\mathbf{R}^d)$ is continuous but μ_s does not necessarily belong to $\Pi(\mathbf{R}^d)$ for all $s \in [0,T]$. That is why we have chosen to suppose the existence of a linear derivative on the whole space $\mathcal{P}_2(\mathbf{R}^d)$.

However, we could have supposed the existence of a linear derivative only on the space of densities, as done for example in [11]. More precisely, we could assume that u is continuous on $\mathcal{P}_2(\mathbf{R}^d)$ and that there exists a function $\frac{\delta u}{\delta m} : \Pi(\mathbf{R}^d) \times \mathbf{R}^d \to \mathbf{R}$ satisfying the following properties.

(1) There exists $k \geq d+1$ such that the map $\mu \in (\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)}) \mapsto \frac{\delta u}{\delta m}(\mu)(\cdot) \in L^k(\mathbf{R}^d)$ is continuous, for some distance $d_{\Pi(\mathbf{R}^d)}$ satisfying Assumptions **(H1)** and **(H2)**.

(2) For all $\mu, \nu \in \Pi(\mathbf{R}^d)$, $u(\mu) - u(\nu) = \displaystyle\int_0^1 \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(t\mu + (1-t)\nu)(v)\,\mathrm{d}(\mu-\nu)(v)\,\mathrm{d}t.$

The only difference induced by this different assumption deals with Itô's formula for the regularized function $u^n := u(\cdot * \rho_n)$, which can be proved using a sequence of subdivisions having a mesh converging to 0. We have not chosen to work under this assumption because we have not found an example of a continuous function defined on $\mathcal{P}_2(\mathbf{R}^d)$ which has only a linear derivative on the space $\Pi(\mathbf{R}^d)$, as defined above, and not on the whole space $\mathcal{P}_2(\mathbf{R}^d)$. Moreover, this definition of linear derivative imposes the integrability of the linear derivative of u while in some examples, this is not necessarily the case (see Ex. 3.9).

Let us also mention that in the case where X_0 has not a finite moment of order 2, we cannot work in $\mathcal{P}_2(\mathbf{R}^d)$. In a completely similar way, we can work with continuous functions defined on $\mathcal{P}(\mathbf{R}^d)$ and having a linear derivative on $\mathcal{P}(\mathbf{R}^d)$. In fact, the structure of $\mathcal{P}_2(\mathbf{R}^d)$ does not play any particular role. We have chosen to work with continuous functions on $\mathcal{P}_2(\mathbf{R}^d)$ rather than with continuous functions on $\mathcal{P}(\mathbf{R}^d)$ because it allows to consider a larger class of functions.

3.2. Itô-Krylov's formula for functions defined on $[0,T] \times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d)$

We now deal with the extension of Itô's formula for functions depending also on the time and space variables. First, we define the space of functions generalizing the space $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$.

Definition 3.5. Let $\mathcal{W}^{1,2,2}([0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d))$ be the set of continuous functions $u : [0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ satisfying the following properties for a certain distance $d_{\Pi(\mathbf{R}^d)}$ satisfying **(H1)** and **(H2)**.

1. For all $(x,\mu) \in \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d)$, $u(\cdot,x,\mu) \in \mathcal{C}^1$ and $\partial_t u$ is continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$.
2. There exists $k_1 \geq d+1$ such that for all $(t,\mu) \in [0,T]\times\Pi(\mathbf{R}^d)$, $u(t,\cdot,\mu) \in W^{2,k_1}_{\mathrm{loc}}(\mathbf{R}^d)$ and for all $t\in[0,T]$ and $R>0$
$$\mu \in (\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)}) \mapsto \partial_x u(t,\cdot,\mu) \in \left(W^{1,k_1}(B_R)\right)^d,$$
is continuous and $\partial_x u$ and $\partial_x^2 u$ are measurable with respect to $(t,x,\mu)\in[0,T]\times\mathbf{R}^d\times\Pi(\mathbf{R}^d)$.
3. For all $(t,x)\in[0,T]\times\mathbf{R}^d$, $u(t,x,\cdot)$ admits a linear derivative $\frac{\delta u}{\delta m}(t,x,\cdot)(\cdot)$ which is continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)\times\mathbf{R}^d$, and such that for all $\mathcal{K}\subset\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$ compact and $t\in[0,T]$, there exists $C>0$ such that for all $v\in\mathbf{R}^d$
$$\sup_{(x,\mu)\in\mathcal{K}} \left|\frac{\delta u}{\delta m}(t,x,\mu)(v)\right| \leq C(1+|v|^2).$$
4. There exists $k_2 \geq 2d$ such that for all $(t,x,\mu)\in[0,T]\times\mathbf{R}^d\times\Pi(\mathbf{R}^d)$, $\frac{\delta u}{\delta m}(t,x,\mu)(\cdot)$ admits distributional derivatives of order 1 and 2 such that $(x,v)\in\mathbf{R}^d\times\mathbf{R}^d \mapsto (\partial_v\frac{\delta u}{\delta m}(t,x,\mu)(v), \partial_v^2\frac{\delta u}{\delta m}(t,x,\mu)(v)) \in (L^{k_2}(B_R\times\mathbf{R}^d))^2$ for any $R>0$. Moreover, for all t and $R>0$
$$\mu\in(\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)}) \mapsto \left(\partial_v\frac{\delta u}{\delta m}(t,\cdot,\mu)(\cdot)\,,\,\partial_v^2\frac{\delta u}{\delta m}(t,\cdot,\mu)(\cdot)\right) \in (L^{k_2}(B_R\times\mathbf{R}^d))^d\times(L^{k_2}(B_R\times\mathbf{R}^d))^{d\times d},$$
is continuous and $\partial_v\frac{\delta u}{\delta m}$ and $\partial_v^2\frac{\delta u}{\delta m}$ are Borel measurable with respect to $(t,x,\mu,v)\in[0,T]\times\mathbf{R}^d\times\Pi(\mathbf{R}^d)\times\mathbf{R}^d$.
5. There exists $\alpha_1,\alpha_2\in\mathbf{N}$ with $k_1\geq(2\alpha_1+1)d$, $k_2\geq(\alpha_2+2)d$ such that for all $\mathcal{K}\subset\mathcal{P}_2(\mathbf{R}^d)$ compact and $R>0$, there exists $C_{\mathcal{K},R}>0$ such that for all $\mu\in\mathcal{K}\cap\Pi(\mathbf{R}^d)$
$$\begin{cases} \sup_{t\leq T}\left\{\|\partial_x u(t,\cdot,\mu)\|_{L^{k_1}(B_R)} + \left\|\partial_x^2 u(t,\cdot,\mu)\right\|_{L^{k_1}(B_R)}\right\} \leq C_{\mathcal{K},R}\left(1+\left\|\frac{\mathrm{d}\mu}{\mathrm{d}x}\right\|^{\alpha_1}_{L^{k_1'}(\mathbf{R}^d)}\right) \\ \sup_{t\leq T}\left\{\left\|\partial_v\frac{\delta u}{\delta m}(t,\cdot,\mu)(\cdot)\right\|_{L^{k_2}(B_R\times\mathbf{R}^d)} + \left\|\partial_v^2\frac{\delta u}{\delta m}(t,\cdot,\mu)(\cdot)\right\|_{L^{k_2}(B_R\times\mathbf{R}^d)}\right\} \leq C_{\mathcal{K},R}\left(1+\left\|\frac{\mathrm{d}\mu}{\mathrm{d}x}\right\|^{\alpha_2}_{L^{k_2'}(\mathbf{R}^d)}\right). \end{cases}$$

Remark 3.6. – The space $\mathcal{W}^{1,2,2}([0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d))$ contains the functions satisfying the four first assumptions of Definition 3.5 with $(\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)})$ replaced by $(\mathcal{P}_2(\mathbf{R}^d), W_2)$ and also assuming that the functions in Assumptions (2) and (4) are continuous with respect to $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbf{R}^d)$. Indeed, Assumption (5) is automatically satisfied with $\alpha_1=\alpha_2=0$ because $\mathcal{K}$ is compact.
– The bound in Assumption (3) is quite natural. If the supremum in this bound was taken only over a compact set of $\mathcal{P}_2(\mathbf{R}^d)$, it would be the definition of the linear derivative. But we also need to control $\frac{\delta u}{\delta m}$ locally uniformly in the space variable $x\in\mathbf{R}^d$ because of our regularization procedure through a convolution both in the space

and measure variables. Assumptions (2), (4) and (5) are generalizations of those in Definition 3.1 adapted to the presence of the space and time variables. In Assumption (5), the condition on k_2 and α_2 changes a bit compared to the analogous assumption in Definition 3.1, essentially because it deals with functions on $\mathbf{R}^{2d}$ instead of $\mathbf{R}^d$. Let us mention that Assumption (5) in Definition 3.5 can be replaced by the integrability properties (6.1) established in Step 1 of the proof of the next theorem (see Sect. 6).

The next theorem is the natural extension of the formula for functions in $\mathcal{W}^{1,2,2}([0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d))$. Let $(\eta_s)_{s\in[0,T]}$ and $(\gamma_s)_{s\in[0,T]}$ be two progressively measurable processes, taking values respectively in $\mathbf{R}^d$ and $\mathbf{R}^{d\times d_1}$ and satisfying Assumptions **(A)** and **(B)**. Let us also consider β a Brownian motion of dimension d_1. We set, for all $t\le T$

$$\xi_t=\xi_0+\int_0^t\eta_s\,\mathrm{d}s+\int_0^t\gamma_s\,\mathrm{d}\beta_s,$$

where ξ_0 is a $\mathcal{F}_0$-measurable random variable with values in $\mathbf{R}^d$.

Theorem 3.7 (Extension of Itô-Krylov's formula)**.** *Let u be a function in $\mathcal{W}^{1,2,2}([0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d))$, which was defined in Definition 3.5. We have almost surely, for all $t\in[0,T]$*

$$\begin{aligned}u(t,\xi_t,\mu_t)=u(0,\xi_0,\mu_0)&+\int_0^t(\partial_t u(s,\xi_s,\mu_s)+\partial_x u(s,\xi_s,\mu_s)\cdot\eta_s)\,\mathrm{d}s+\frac{1}{2}\int_0^t\partial_x^2 u(s,\xi_s,\mu_s)\cdot\gamma_s\gamma_s^*\,\mathrm{d}s\\&+\int_0^t\tilde{\mathbf{E}}\left(\partial_v\frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b}_s\right)\mathrm{d}s+\frac{1}{2}\int_0^t\tilde{\mathbf{E}}\left(\partial_v^2\frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{a}_s\right)\mathrm{d}s\\&+\int_0^t\partial_x u(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s),\end{aligned}\tag{3.2}$$

where $(\tilde{X},\tilde{b},\tilde{\sigma})$ is an independent copy of (X,b,σ) defined on another filtered space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}})$.

The idea of the proof is the following. We mollify the function u by setting, for all $n\ge1$, $t\in[0,T]$, $x\in\mathbf{R}^d$ and $\mu\in\mathcal{P}_2(\mathbf{R}^d)$

$$u^n(t,x,\mu):=u(t,\cdot,\mu*\rho_n)*\rho_n(x).$$

It regularizes the spatial derivatives as well as the linear derivative with respect to its space variable. Then, it remains to take the limit $n\to+\infty$ in Itô's formula (3.2) for u^n using Krylov's estimate.

Let us now discuss the regularity assumption with respect to time.

Remark 3.8. In the abstract, we said that our Itô-Krylov's formula for a flow of measures was the almost analogue of the standard Itô-Krylov formula. We used the word "almost" because Assumption (1) in Definition 3.5 is not completely satisfactory. Indeed, we do not assume Sobolev regularity with respect to time, as it is the case in Itô-Krylov's formula for functions defined on $[0,T]\times\mathbf{R}^d$. Weakening the regularity assumption with respect to time is more difficult in the present framework. Let us give some general assumptions which guarantee that Itô-Krylov's formula (3.2) holds true for functions u that are not of class $\mathcal{C}^1$ with respect to time.

Let us consider a continuous function u defined on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$ such that for any $\mu\in\mathcal{P}_2(\mathbf{R}^d)$, $x\in\mathbf{R}^d$, $u(\cdot,x,\mu)\in W^{1,k}([0,T])$ for some $k\ge d+1$. We assume moreover that for any $R>0$ and any compact $\mathcal{K}\subset\mathcal{P}_2(\mathbf{R}^d)$

$$\int_0^T\sup_{\mu\in\mathcal{K}}\|\partial_s u(t,\cdot,\mu)\|_{L^k(B_R)}^k\,\mathrm{d}t<+\infty,\tag{3.3}$$

and that the following Lebesgue continuity condition holds true

$$\int_0^T \sup_{\mu\in\mathcal{K}} \|\partial_s u(s-r,\cdot,\mu)-\partial_s u(s,\cdot,\mu)\|^k_{L^k(B_R)}\,\mathrm{d}s \underset{r\to 0}{\to} 0. \tag{3.4}$$

Let us define, for $n\geq 1$,

$$u^n(t,x,\mu):=u(\cdot,x,\mu)*\rho^1_n(t),$$

where $(\rho^1_n)_n$ is a mollifying sequence on $\mathbf{R}$. Note that this makes sense since one can extend $u(\cdot,x,\mu)$ to $W^{1,k}(\mathbf{R})$ (see Thm. 8.6 in [24]). The function u^n is of class $\mathcal{C}^1$ with respect to time and converges towards u when $n\to+\infty$. We assume that Itô's formula (3.2) holds true for u^n. This is the case for example if Theorem 3.7 can be applied. Let us now take the limit $n\to+\infty$ in the term involving the time-derivative of u^n in (3.2). We can assume that the Itô process $(\xi_t)_{t\in[0,T]}$ is bounded by R thanks to a standard localization argument. Noting that $\mathcal{K}:=\{\mu_s,\, s\in[0,T]\}$ is compact in $\mathcal{P}_2(\mathbf{R}^d)$, Krylov's estimate (Cor. 4.2) and Minkowski's inequality yield

$$\begin{aligned}\mathbf{E}\left|\int_0^t \partial_s u^n(s,\xi_s,\mu_s)-\partial_s u(s,\xi_s,\mu_s)\,\mathrm{d}s\right| &\leq \left(\int_0^t\int_{B_R}\left|\partial_s u(\cdot,x,\mu_s)*\rho^1_n(s)-u(s,x,\mu_s)\right|^k\,\mathrm{d}s\,\mathrm{d}x\right)^{\frac{1}{k}}\\ &\leq \int_{\mathbf{R}}\left(\int_0^t\int_{B_R}\left|\partial_s u(s-r,x,\mu_s)-u(s,x,\mu_s)\right|^k\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{k}}\rho^1_n(r)\,dr\\ &\leq \sup_{|r|<\frac{1}{n}}\left(\int_0^t\sup_{\mu\in\mathcal{K}}\|\partial_s u(s-r,\cdot,\mu)-\partial_s u(s,\cdot,\mu)\|^k_{L^k(B_R)}\,\mathrm{d}s\right)^{\frac{1}{k}}.\end{aligned}$$

The upper-bound converges to 0 by (3.4). We can take the limit $n\to+\infty$ in the other terms of Itô's formula for u^n under assumptions, that we do not detail, completely similar to (3.3) and (3.4) for the derivatives of u with respect to space and measure variables.

However, Assumptions (3.3) and (3.4) are not optimal. Let us consider the case of a function u defined on $[0,T]\times\mathcal{P}_2(\mathbf{R}^d)$ to simplify. Assume that u is of the form $u(t,\mu)=\int_{\mathbf{R}^d} g(t,x)\,\mathrm{d}\mu(x)$ with $g\in\mathcal{C}^0([0,T]\times\mathbf{R}^d;\mathbf{R})$ at most of quadratic growth in x uniformly in t, and such that the distributional derivatives $\partial_t g$, $\partial_x g$ and $\partial^2_x g$ are in $L^k([0,T]\times\mathbf{R}^d)$ for some $k\geq d+1$. Then, Itô-Krylov's formula holds true for u even if Assumptions (5) and (6) are not satisfied.

Let us give the idea of the proof. We regularize u by setting $u^n(t,\mu):=\int_{\mathbf{R}^d} g*\rho_n(t,x)\,\mathrm{d}\mu(x)$, where $(\rho_n)_n$ is a mollifying sequence on $\mathbf{R}\times\mathbf{R}^d$. The function u^n clearly satisfies the assumptions of the standard Itô formula for a flow of measures (see Prop. 5.102 in [3]). It ensures that for all $t\in[0,T]$

$$\begin{aligned}u^n(t,\mu_t)=u^n(0,\mu_0)&+\int_0^t\mathbf{E}(\partial_s g*\rho_n(s,X_s))\,\mathrm{d}s+\int_0^t\mathbf{E}\left(\partial_x g*\rho_n(s,X_s)\cdot b_s\right)\,\mathrm{d}s\\ &+\frac{1}{2}\int_0^t\mathbf{E}\left(\partial^2_x g*\rho_n(s,X_s)\cdot a_s\right)\,\mathrm{d}s.\end{aligned}\tag{3.5}$$

As g is continuous, $(g*\rho_n)_n$ converges to g uniformly on compact sets. It follows from the growth assumption on g that u^n converges point-wise to u. Using that $(\partial_s g*\rho_n)_n$ converges in $L^k([0,T]\times\mathbf{R}^d)$ to $\partial_s g$ as $n\to+\infty$, we deduce with Krylov's estimate in Corollary 4.2 that for all $t\in[0,T]$

$$\int_0^t\mathbf{E}(\partial_s g*\rho_n(s,X_s))\,\mathrm{d}s\to\int_0^t\mathbf{E}(\partial_s g(s,X_s))\,\mathrm{d}s.$$

The same holds with the two other integrals in (3.5). Taking the limit $n \to +\infty$ in (3.5) yields for all $t \in [0,T]$

$$u(t,\mu_t) = u(0,\mu_0) + \int_0^t \mathbf{E}(\partial_s g(s,X_s))\,\mathrm{d}s + \int_0^t \mathbf{E}\left(\partial_x g(s,X_s)\cdot b_s\right)\,\mathrm{d}s + \frac{1}{2}\int_0^t \mathbf{E}\left(\partial_x^2 g(s,X_s)\cdot a_s\right)\,\mathrm{d}s.$$

3.3. Examples and comments on the results

Let us first give examples of functions belonging to $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$. Let us start with the linear case.

Example 3.9 (Linear functional)**.** Fix $g \in \mathcal{C}^0(\mathbf{R}^d;\mathbf{R})$ at most of quadratic growth and admitting a distributional derivative such that $\nabla g \in (W^{1,k}(\mathbf{R}^d))^d$ for some $k \geq d+1$. Then, the function

$$u : \begin{cases} \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ \mu & \mapsto \displaystyle\int_{\mathbf{R}^d} g(x)\,\mathrm{d}\mu(x), \end{cases}$$

belongs to the space $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = W_2$.

Remark 3.10. Let us mention that with the particular choice of the additive constant for the linear derivative explained in Remark 2.2, one has $\frac{\delta u}{\delta m}(\mu) = g - \int_{\mathbf{R}^d} g\,\mathrm{d}\mu$.

Indeed, the Sobolev embedding theorem (see Cor. 9.14 in [24]) implies that $\nabla g \in L^\infty(\mathbf{R}^d)$ since $k \geq d+1$. Thus g is at most of linear growth so that for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $\frac{\delta u}{\delta m}(\mu) = g$, which clearly satisfies Assumptions (1) and (2) (with $\alpha = 0$) in Definition 3.1.

Let us now focus on the multi-linear case.

Example 3.11 (Polynomials on the Wasserstein space)**.** Fix $N \geq 2$ and $g \in \mathcal{C}^0((\mathbf{R}^d)^N;\mathbf{R})$ such that

- there exists $C > 0$ such that for all $\boldsymbol{x} = (x_1,\ldots,x_N) \in (\mathbf{R}^d)^N$, $|g(\boldsymbol{x})| \leq C(1+|x_1|^2+\cdots+|x_N|^2)$,
- the distributional derivative ∇g belongs to $(W^{1,k}((\mathbf{R}^d)^N))^{Nd}$ for a certain $k \in [Nd,+\infty[$.

Then, the function

$$u : \begin{cases} \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ \mu & \mapsto \displaystyle\int_{(\mathbf{R}^d)^N} g(x_1,\ldots,x_N)\,\mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_N), \end{cases}$$

belongs to the space $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = d_{k'}$.

The proof is postponed to the Appendix (Sect. A.2).

Remark 3.12. In Definition 3.1, the distributional derivatives of the linear derivative $\frac{\delta u}{\delta m}(\mu)$ are not necessarily integrable functions for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Of course, in Example 3.9, it is the case for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ as the linear derivative does not depend on the measure μ (up to an additive constant which disappears for the spatial derivatives of the linear derivative). However, in Example 3.11 for $N = 2$, the linear derivative is given by

$$\frac{\delta u}{\delta m}(\mu)(v) = \int_{\mathbf{R}^d} g(v,y)\,\mathrm{d}\mu(y) + \int_{\mathbf{R}^d} g(y,v)\,\mathrm{d}\mu(y). \tag{3.6}$$

Formally, the derivative with respect to v of the first integral in (3.6) is

$$\int_{\mathbf{R}^d} \partial_v g(v,y)\,\mathrm{d}\mu(y).$$

This term is not well-defined for general measures $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ because we have only assumed that $\nabla g \in (W^{1,k}(\mathbf{R}^{2d}))^{2d}$ with $k \geq 2d$. Indeed, for $k = 2d$, we just know by the Sobolev embedding theorem that ∇g belongs to $(L^r(\mathbf{R}^{2d}))^{2d}$ with $r \in [2d, +\infty[$ (see Cor. 9.11 in [24]). As we will see in the proof (Sect. A.2 of the Appendix), it is well-defined as an integrable function of v if we restrict to measures $\mu \in \Pi(\mathbf{R}^d)$. This also justifies why we have chosen to work with the linear derivative instead of the L-derivative. Indeed, the L-derivative of u would be equal to the gradient of the linear derivative $\partial_v \frac{\delta u}{\delta m}(\mu)(\cdot)$, which is not well-defined for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Thus, the function u does not need to be L-differentiable in the usual sense in our setting.

The next example focuses on the particular case of convolution which has to be treated differently than in Example 3.11 with $N = 2$ because of the structure of the convolution which mixes the two variables.

Example 3.13. Let $f \in \mathcal{C}^0(\mathbf{R}^d; \mathbf{R})$ be a function such that the distributional derivative ∇f belongs to $(W^{1,k+1}(\mathbf{R}^d))^d$, for a certain $k \geq d$. Then, the function

$$u : \begin{cases} \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ \mu & \mapsto \displaystyle\int_{\mathbf{R}^d} f * \mu \,\mathrm{d}\mu, \end{cases}$$

belongs to $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = W_2$.

Here, the particular structure of convolution enables us to work on the whole space $\mathcal{P}_2(\mathbf{R}^d)$ instead of $\Pi(\mathbf{R}^d)$, as explained in the first point of Remark 3.2. The proof is postponed to the Appendix (Sect. A.3).

Finally, we give a non-linear example of functions belonging to $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$.

Example 3.14. Let $F \in \mathcal{C}^1(\mathbf{R}; \mathbf{R})$ and $g \in \mathcal{C}^0(\mathbf{R}^d; \mathbf{R})$ be such that the distributional derivative ∇g belongs to $(W^{1,k}(\mathbf{R}^d))^d$ for some $k \geq d + 1$. Then

$$u : \begin{cases} \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ \mu & \mapsto F\left(\int_{\mathbf{R}^d} g \,\mathrm{d}\mu\right) \end{cases}$$

belongs to $\mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = W_2$.

The proof is again postponed in the Appendix (Sect. A.4).

Let us now give examples of functions belonging to the space $\mathcal{W}^{1,2,2}([0,T] \times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d))$.

Example 3.15. Let $g \in \mathcal{C}^0(\mathbf{R}^{2d}; \mathbf{R})$ be a function such that its distributional derivative ∇g belongs to $(W^{1,k}(\mathbf{R}^{2d}))^{2d}$ for some $k \geq 5d$. Then, the function

$$u : \begin{cases} \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ (x,\mu) & \mapsto \displaystyle\int_{\mathbf{R}^d} g(x,y)\,\mathrm{d}\mu(y) \end{cases}$$

belongs to $\mathcal{W}^{1,2,2}([0,T] \times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = d_{k'}$.

The proof is postponed to the Appendix (Sect. A.5).

Example 3.16. Let $F \in \mathcal{C}^1(\mathbf{R}^d \times \mathbf{R}; \mathbf{R})$ be a function such that for all $R > 0$

$$y \in \mathbf{R} \mapsto \nabla F(\cdot, y) \in (W^{1,k_1}(B_R))^{d+1},$$

is well-defined and continuous for some $k_1 \geq d+1$. Let $g \in \mathcal{C}^0(\mathbf{R}^d; \mathbf{R})$ be such that the distributional derivative ∇g belongs to $(W^{1,k_2}(\mathbf{R}^d))^d$ for some $k_2 \geq 2d$. Then

$$u : \begin{cases} \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d) & \to \mathbf{R} \\ (x, \mu) & \mapsto F\left(x, \int_{\mathbf{R}^d} g \, \mathrm{d}\mu\right) \end{cases}$$

belongs to $\mathcal{W}^{1,2,2}([0,T] \times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d))$ for $d_{\Pi(\mathbf{R}^d)} = W_2$.

The proof is again postponed to the Appendix (Sect. A.6).

4. Preliminaries

4.1. Krylov's estimate and densities

The key element to prove the theorem is Krylov's estimate. We recall it in the next theorem taken from [4] (see Thm. 4 in Sect. 2.3).

Theorem 4.1 (Krylov's estimate)**.** *Let $b : \mathbf{R}^+ \times \Omega \to \mathbf{R}^d$ and $\sigma : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d \times d_1}$ be two progressively measurable functions satisfying Assumptions (**A**) and **(B)** on $\mathbf{R}^+$ instead of the time interval $[0,T]$. We assume that $p, d_1 \geq d$.*

For X_0 a $\mathbf{R}^d$-valued $\mathcal{F}_0$-measurable random variable, we define the Itô process $X = (X_t)_t$, for all $t \in \mathbf{R}^+$, by

$$X_t = X_0 + \int_0^t b_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s.$$

*Let $\lambda > 0$ be a positive constant. Then, there exists a constant $N = N(d, p, \lambda, \delta, K)$, δ and K being the constants in Assumptions **(A)** and **(B)**, such that for all measurable function $f : \mathbf{R}^+ \times \mathbf{R}^d \to \mathbf{R}$*

$$\mathbf{E} \int_0^\infty e^{-\lambda t} |f(t, X_t)| \, \mathrm{d}t \leq N \|f\|_{L^{p+1}(\mathbf{R}^+ \times \mathbf{R}^d)}.$$

We will use the following immediate corollary for a finite horizon of time.

Corollary 4.2. *If b and σ satisfy Assumptions **(A)** and **(B)**, there exists $N_1 = N_1(d, p, \delta, K, T)$ such that for all measurable function $f : [0,T] \times \mathbf{R}^d \to \mathbf{R}$, we have*

$$\mathbf{E} \int_0^T |f(s, X_s)| \, \mathrm{d}s \leq N_1 \|f\|_{L^{p+1}([0,T] \times \mathbf{R}^d)}.$$

Krylov's estimate also provides the existence of a density with respect to the Lebesgue measure for μ_s, the law of X_s, for almost all $s \in [0,T]$.

Proposition 4.3. *Under Assumptions **(A)** and **(B)** on the coefficients b and σ, there exists a function $p \in L^1([0,T] \times \mathbf{R}^d; \mathbf{R}^+) \cap L^{(d+1)'}([0,T] \times \mathbf{R}^d; \mathbf{R}^+)$ such that for all $f : [0,T] \times \mathbf{R}^d \to \mathbf{R}^+$ measurable*

$$\int_0^T \mathbf{E} f(s, X_s) \, \mathrm{d}s = \int_{[0,T] \times \mathbf{R}^d} f(s,x) p(s,x) \, \mathrm{d}x \, \mathrm{d}s. \tag{4.1}$$

If τ is a stopping time such that $(X_t)_{t\in[0,T]}$ belongs to B_R almost surely on the set $\{\tau > 0\}$, then

$$\mathbf{E}\int_0^{\tau\wedge T} f(s,X_s)\,\mathrm{d}s \le \int_{[0,T]\times B_R} f(s,x)p(s,x)\,\mathrm{d}x\,\mathrm{d}s. \tag{4.2}$$

Moreover, for almost all $s\in[0,T]$, $\mu_s = \mathcal{L}(X_s)$ is equal to $p(s,\cdot)\,\mathrm{d}x$. Thus $\mu_s \in \Pi(\mathbf{R}^d)$ for almost all $s\in[0,T]$.

We do not prove this proposition since it can be easily deduced from Krylov's estimate and the fact that the dual space of $L^{d+1}([0,T]\times\mathbf{R}^d)$ is $L^{(d+1)'}([0,T]\times\mathbf{R}^d)$.

We now prove the following lemma dealing with the integrability of the density p.

Lemma 4.4. *Let p and q be two densities of two Itô processes of the form (1.1) and satisfying **(A)** and **(B)** given by Proposition 4.3. We have*

- *for $k\ge d+1$*

$$s\in[0,T]\mapsto \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)} \in L^{k/d}([0,T]),$$

- *for $k,\alpha\in\mathbf{N}$ such that $k\ge\max\{d+1, d(\alpha+1)\}$*

$$\int_0^T \|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)}\|q(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}\,\mathrm{d}s < +\infty.$$

Proof. For the first point, we use Jensen's inequality since $\frac{k}{k'} = k-1\ge d$

$$\begin{aligned}\int_0^T\left(\int_{\mathbf{R}^d} p(s,x)^{k'}\,\mathrm{d}x\right)^{\frac{k}{dk'}}\mathrm{d}s &= \int_0^T\left(\int_{\mathbf{R}^d} p(s,x)^{k'-1}p(s,x)\,\mathrm{d}x\right)^{\frac{k}{dk'}}\mathrm{d}s\\ &\le \int_0^T\int_{\mathbf{R}^d} p(s,x)^{\frac{k}{dk'}(k'-1)+1}\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^T\int_{\mathbf{R}^d} p(s,x)^{\frac{1}{d}+1}\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

which is finite since $(d+1)' = \frac{1}{d}+1$ and $p\in L^{(d+1)'}([0,T]\times\mathbf{R}^d)$.

For the second point, we have proved that the function $s\mapsto\|q(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}$ belongs to $L^1([0,T])\cap L^{k/d}([0,T])$. Using Hölder's inequality, the proof is complete once we prove that $s\mapsto\|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)}$ belongs to $L^r([0,T])$ for some $r\ge\left(\frac{k}{d}\right)'$. However the function $s\mapsto\|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)}\in L^{\frac{k}{\alpha d}}([0,T])$. Thus, we have to prove that $\left(\frac{k}{d}\right)'\le\frac{k}{\alpha d}$. This is equivalent to our assumption $k\ge d(\alpha+1)$. □

4.2. Classical results on convolution and regularization

Fix $p\in[1+\infty[$. We will need the two following basic lemmas, which we recall for the sake of clarity.

Lemma 4.5 (Convolution).
- *For all $f\in L^p(\mathbf{R}^d)$ and for all $g\in L^1(\mathbf{R}^d)$, the convolution $f*g$ is well-defined and belongs to $L^p(\mathbf{R}^d)$. Moreover, we have $\|f*g\|_{L^p}\le\|f\|_{L^p}\|g\|_{L^1}$.*
- *For all $f\in L^p(\mathbf{R}^d)$ and for all $g\in L^{p'}(\mathbf{R}^d)$, the convolution $f*g$ is well-defined and belongs to $L^\infty(\mathbf{R}^d)$. Moreover, we have $\|f*g\|_{L^\infty}\le\|f\|_{L^p}\|g\|_{L^{p'}}$.*

Lemma 4.6 (Regularization). *Recall that $(\rho_n)_n$ is a mollifying sequence.*

– *Let* $f \in L^1_{loc}(\mathbf{R}^d)$ *and* $\rho \in \mathcal{C}^\infty_c(\mathbf{R}^d)$*. Then* $f * \rho \in \mathcal{C}^\infty(\mathbf{R}^d)$ *and* $\forall \alpha \in \mathbf{N}^d$, $\partial^\alpha (f * \rho) = f * \partial^\alpha \rho$.
– *If* $f \in L^p(\mathbf{R}^d)$*, then* $f * \rho_n \xrightarrow{L^p} f$*, and if* $f \in \mathcal{C}^0(\mathbf{R}^d)$, $f * \rho_n \to f$ *uniformly on compact sets.*
– *If* $f \in L^p_{loc}(\mathbf{R}^d)$*, then for all* $R > 0$, $f * \rho_n \to f$ *in* $L^p(B_R)$.

The following proposition will also be useful.

Proposition 4.7. *Let* $f \in \mathcal{C}^0(\mathbf{R}^d)$ *be a function admitting distributional derivatives of order* 1 *et* 2 *in* $L^1_{loc}(\mathbf{R}^d)$*. Then* $f * \rho_n \in \mathcal{C}^\infty(\mathbf{R}^d)$ *and for all* $i, j \in \{1, \dots d\}$

$$\begin{cases} \partial_{x_i}(f * \rho_n) & = \partial_{x_i} f * \rho_n \\ \partial_{x_i\, x_j}(f * \rho_n) & = \partial_{x_i\, x_j} f * \rho_n. \end{cases}$$

The next lemma deals with the convolution of a function $f \in L^p$ with $\mu \in \mathcal{P}(\mathbf{R}^d)$.

Lemma 4.8. *Let* $f \in L^p(\mathbf{R}^d)$*. Then* $\mu \in \mathcal{P}(\mathbf{R}^d) \mapsto f * \mu \in L^p(\mathbf{R}^d)$ *is continuous.*

Proof. Note that the convolution $f * \mu$ is well-defined as an element of $L^p(\mathbf{R}^d)$ thanks to Jensen's inequality which shows that

$$\forall f \in L^p(\mathbf{R}^d),\ \forall \mu \in \mathcal{P}(\mathbf{R}^d),\ \|f * \mu\|_{L^p} \leq \|f\|_{L^p}.$$

Let $(\mu_n)_n$ be a sequence of $\mathcal{P}(\mathbf{R}^d)$ weakly convergent to $\mu \in \mathcal{P}(\mathbf{R}^d)$. Using Skorokhod's representation theorem (see Thm. 6.7 in [25]), there exists a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, a sequence of random variables $(X_n)_n$ converging $\mathbf{P}'$-almost surely to a random variable X such that, the law of X_n is μ_n for all n and the law of X if μ. For any $a \in \mathbf{R}^d$, let us denote by $\tau_a f$ the translation of f defined, for all $x \in \mathbf{R}^d$, by $\tau_a f(x) := f(x - a)$. Jensen's inequality and Fubini-Tonelli's theorem yield

$$\begin{aligned} \|f * \mu_n - f * \mu\|^p_{L^p} &= \int_{\mathbf{R}^d} |\mathbf{E}'(f(x - X_n) - f(x - X))|^p \, \mathrm{d}x \\ &\leq \int_{\mathbf{R}^d} \mathbf{E}'(|f(x - X_n) - f(x - X)|^p) \, \mathrm{d}x \\ &= \mathbf{E}'(\|\tau_{X_n - X} f - f\|^p_{L^p}). \end{aligned}$$

It follows from the almost sure convergence of $(X_n)_n$ to X and the continuity of the translation operator in L^p that $\|\tau_{X_n - X} f - f\|^p_{L^p} \xrightarrow{a.s.} 0$. Moreover, the inequality

$$\begin{aligned} \|\tau_{X_n - X} f - f\|^p_{L^p} &\leq 2^{p-1}(\|\tau_{X_n - X} f\|^p_{L^p} + \|f\|^p_{L^p}) \\ &= 2^p \|f\|^p_{L^p}, \end{aligned}$$

enables us to conclude with the dominated convergence theorem. □

4.3. Convolution of probability measures

Lemma 4.9 (Contraction inequality)**.** *Fix* $\mu, \nu, m \in \mathcal{P}_2(\mathbf{R}^d)$*. Then, we have*

$$W_2(\mu * m, \nu * m) \leq W_2(\mu, \nu).$$

Proof. Let $\pi \in \mathcal{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$ be an optimal coupling between μ and ν. We consider a couple of random variables (X, Y) with law π, and a random variable Z independent of (X, Y) with law m. The law of $X + Z$ being $\mu * m$

and the law of $Y+Z$ being $\nu * m$, one has

$$W_2(\mu * m, \nu * m) \leq \|(X+Z)-(Y+Z)\|_{L^2} = W_2(\mu,\nu).$$

□

The next corollary follows from the fact that $\rho_n \xrightarrow{W_2} \delta_0$.

Corollary 4.10. *For all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $\mu * \rho_n \xrightarrow{W_2} \mu$.*

4.4. Measurability

We will need the following lemma to guarantee that, for $u \in \mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$, we can find versions of $\partial_v \frac{\delta u}{\delta m}$ and $\partial_v^2 \frac{\delta u}{\delta m}$ that are measurable with respect to $(\mu, v) \in \Pi(\mathbf{R}^d) \times \mathbf{R}^d$.

Lemma 4.11. *Let $u : E \to L^k(\mathbf{R}^d)$ be a continuous function, where E is a metric space and $k > 1$. Then, for all $x \in E$, we can find a version of $u(x)$ such that $(x,v) \in E \times \mathbf{R}^d \mapsto u(x)(v)$ is measurable with respect to $\mathcal{B}(E) \otimes \mathcal{B}(\mathbf{R}^d)$.*

Proof. For $(x,v) \in E \times \mathbf{R}^d$, we define

$$\tilde{u}(x,v) = \varliminf_{n \to +\infty} \frac{1}{\lambda(B(v,1/n))} \int_{B(v,1/n)} u(x)(y)\,dy = \varliminf_{n\to+\infty} u^n(x,v),$$

where λ denotes the Lebesgue measure on $\mathbf{R}^d$. From Lebesgue differentiation theorem (see Thm. 7.7 in [26]), we deduce that for all $x \in E$, $\tilde{u}(x,\cdot) = u(x)$ λ-almost everywhere. We prove that for all $n \geq 1$, u^n is continuous. Note that $\frac{1}{\lambda(B(v,1/n))}$ does not depend on v. The continuity of u^n follows from the continuity of $x \in E \mapsto u(x) \in L^k(\mathbf{R}^d)$, $v \in \mathbf{R}^d \mapsto \mathbf{1}_{B(v,1/n)} \in L^{k'}(\mathbf{R}^d)$ (coming from the dominated convergence theorem), and of $(f,g) \in L^k(\mathbf{R}^d) \times L^{k'}(\mathbf{R}^d) \mapsto \int_{\mathbf{R}^d} fg\,\mathrm{d}x$. □

5. Proof of Theorem 3.3

The proof will be divided into three parts. Step 1 is dedicated to prove that all the terms in Itô-Krylov's formula (3.1) are well-defined. In Step 2, we regularize u by convolution of the measure argument with a mollifying sequence $(\rho_n)_n$. The effect of replacing $u(\mu)$ by $u(\mu * \rho_n)$ is that the linear derivative is regularized by convolution, in its space variable. Then, we apply the standard Itô's formula for a flow of measure. We finally take the limit $n \to +\infty$ in Step 3 with the help of Krylov's estimate.

Step 1: All the terms in (3.1) are well-defined.

Let us show that the two integrals in (3.1) are well-defined.

Measurability. Thanks to Lemma 4.11, we can find a version of $\partial_v \frac{\delta u}{\delta m}$ which is measurable with respect to $(\mu, v) \in \Pi(\mathbf{R}^d) \times \mathbf{R}^d$. To conclude, we prove that $s \mapsto \mu_s \in \Pi(\mathbf{R}^d)$ is measurable. Indeed if it is the case, the function $(s,\omega) \in [0,T] \times \Omega \mapsto \partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s(\omega)).b_s(\omega)$ will be measurable by composition. First, note that $\mu_s \in \Pi(\mathbf{R}^d)$ for almost all $s \in [0,T]$ (see Prop. 4.3) so we can change μ_s on a negligible set of times s to ensure that $\mu_s \in \Pi(\mathbf{R}^d)$ for all $s \in [0,T]$. But $\mu_s = \lim_{n\to+\infty} \mu_s * \rho_n$ for $d_{\Pi(\mathbf{R}^d)}$ by Assumption **(H2)** in Definition 2.4. It remains to show that $s \mapsto \mu_s * \rho_n \in \Pi(\mathbf{R}^d)$ is continuous and thus mesurable for all n. This follows from the continuity of $s \mapsto \mu_s \in \mathcal{P}_2(\mathbf{R}^d)$ and also from Assumption **(H1)** in Definition 2.4.

Integrability. We can omit the coefficients b and a to prove the integrability properties because they are uniformly bounded. Taking advantage from the existence of a density coming from Proposition 4.3, we have by Hölder's inequality

$$\begin{aligned}\int_0^T \mathbf{E}\left|\partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s)\right| \mathrm{d}s &= \int_0^T \int_{\mathbf{R}^d} \left|\partial_v \frac{\delta u}{\delta m}(\mu_s)(x)\right| p(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\leq \int_0^T \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s)(\cdot)\right\|_{L^k(\mathbf{R}^d)} \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}\,\mathrm{d}s \\ &\leq \int_0^T C\left(1+\|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)}\right)\|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}\,\mathrm{d}s,\end{aligned}$$

for some constant C coming from Assumption (2) in Definition 3.1 because the flow $(\mu_s)_{s\leq T}$ is compact in $\mathcal{P}_2(\mathbf{R}^d)$ and belongs to $\Pi(\mathbf{R}^d)$ for almost all s. The last bound is finite thanks to Lemma 4.4 since $k \geq \max\{d(\alpha+1), d+1\}$. The same properties hold for the term involving $\partial_v^2 \frac{\delta u}{\delta m}$.

Step 2: Itô's formula for the mollification of $\boldsymbol{u}$.

For $n \geq 1$, we set $u^n : \mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto u(\mu * \rho_n)$. By standard arguments, for each $n \geq 1$, u^n has a linear derivative given by

$$\frac{\delta u^n}{\delta m}(\mu)(v) = \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(\mu * \rho_n)(x)\rho_n(v-x)\,\mathrm{d}x = \frac{\delta u}{\delta m}(\mu * \rho_n) * \rho_n(v).$$

Now, we aim at applying the standard Itô formula for a flow of probability measures (see for example Thm. 5.99 in Chap. 5 of [3] with the L-derivative) to u^n for a fixed $n \geq 1$.

(i) Regularity of $\boldsymbol{\frac{\delta u^n}{\delta m}(\mu)}$ for a fixed $\boldsymbol{\mu \in \mathcal{P}_2(\mathbf{R}^d)}$. Since for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $\mu * \rho_n \in \Pi(\mathbf{R}^d)$, Proposition 4.7 implies that $\frac{\delta u^n}{\delta m}(\mu)(.) \in \mathcal{C}^\infty(\mathbf{R}^d)$ and for all $i,j \in \{1,\dots,\mathrm{d}\}$

$$\partial_{v_i}\frac{\delta u^n}{\delta m}(\mu) = \partial_{v_i}\frac{\delta u}{\delta m}(\mu * \rho_n) * \rho_n \quad \text{and} \quad \partial_{v_i\,v_j}\frac{\delta u^n}{\delta m}(\mu) = \partial_{v_i\,v_j}\frac{\delta u}{\delta m}(\mu * \rho_n) * \rho_n. \tag{5.1}$$

(ii) Continuity of $\boldsymbol{\partial_v \frac{\delta u^n}{\delta m}}$ and $\boldsymbol{\partial_v^2 \frac{\delta u^n}{\delta m}}$ with respect to $\boldsymbol{(\mu, v)}$. Let $i \in \{1,\dots,d\}$, $(\mu_m)_m \in \mathcal{P}_2(\mathbf{R}^d)^{\mathbf{N}}$ and $(v_m)_m \in (\mathbf{R}^d)^{\mathbf{N}}$ be sequences converging respectively to μ and v. We have

$$\begin{aligned}&\left|\partial_{v_i}\frac{\delta u^n}{\delta m}(\mu_m)(v_m) - \partial_{v_i}\frac{\delta u^n}{\delta m}(\mu)(v)\right| \\ &\leq \left|\partial_{v_i}\frac{\delta u^n}{\delta m}(\mu_m)(v_m) - \partial_{v_i}\frac{\delta u^n}{\delta m}(\mu)(v_m)\right| + \left|\partial_{v_i}\frac{\delta u^n}{\delta m}(\mu)(v_m) - \partial_{v_i}\frac{\delta u^n}{\delta m}(\mu)(v)\right| \\ &=: D_1 + D_2\end{aligned}$$

D_2 converges to 0 when $m \to +\infty$ by (i). For D_1, Hölder's inequality gives that

$$\begin{aligned}D_1 &= \left|\partial_{v_i}\frac{\delta u}{\delta_m}(\mu_m * \rho_n) * \rho_n(v_m) - \partial_{v_i}\frac{\delta u}{\delta_m}(\mu * \rho_n) * \rho_n(v_m)\right| \\ &\leq \left\|\partial_{v_i}\frac{\delta u}{\delta_m}(\mu_m * \rho_n) - \partial_{v_i}\frac{\delta u}{\delta_m}(\mu * \rho_n)\right\|_{L^k} \|\rho_n\|_{L^{k'}}.\end{aligned}$$

Assumption **(H1)** in Definition 2.4 provides that $\mu_m * \rho_n \overset{d_{\Pi(\mathbf{R}^d)}}{\longrightarrow} \mu * \rho_n$ when $m \to +\infty$. Finally, using the first assumption in Definition 3.1, we conclude that D_1 converges to 0 when $m \to +\infty$. This shows the continuity of $\partial_v \frac{\delta u^n}{\delta m}$ on $\mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d$. The same reasoning proves the joint continuity of $\partial_v^2 \frac{\delta u^n}{\delta m}$.

(iii) Boundedness of $\partial_v \frac{\delta u^n}{\delta m}$ and $\partial_v^2 \frac{\delta u^n}{\delta m}$. Let $\mathcal{K} \subset \mathcal{P}_2(\mathbf{R}^d)$ be a compact set. For $\mu \in \mathcal{K}$ and $v \in \mathbf{R}^d$, one has

$$\left|\partial_{v_i} \frac{\delta u^n}{\delta m}(\mu)(v)\right| \leq \left\|\partial_{v_i} \frac{\delta u}{\delta_m}(\mu * \rho_n)\right\|_{L^k} \|\rho_n\|_{L^{k'}}.$$

The set $\{\mu * \rho_n,\, \mu \in \mathcal{K}\}$ is compact in $(\Pi(\mathbf{R}^d), d_{\Pi(\mathbf{R}^d)})$ as the image of the compact $\mathcal{K}$ by the application $\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \mu * \rho_n \in \Pi(\mathbf{R}^d)$ which is continuous by Assumption (**H1**) in Definition 2.4. The first assumption in Definition 3.1 guarantees that $\sup_{\mu\in\mathcal{K}} \left\|\partial_{v_i} \frac{\delta u}{\delta_m}(\mu * \rho_n)\right\|_{L^k(\mathbf{R}^d)} < +\infty$ and thus

$$\sup_{v\in\mathbf{R}^d} \sup_{\mu\in\mathcal{K}} \left|\partial_v \frac{\delta u^n}{\delta m}(\mu)(v)\right| < \infty.$$

The same property holds for $\partial_v^2 \frac{\delta u^n}{\delta m}$.

We can thus apply Itô's formula of [3] to obtain that for all $n \geq 1$ and for all $t \in [0, T]$

$$u^n(\mu_t) = u^n(\mu_0) + \int_0^t \mathbf{E}\left(\partial_v \frac{\delta u^n}{\delta m}(\mu_s)(X_s) \cdot b_s\right) \mathrm{d}s + \frac{1}{2}\int_0^t \mathbf{E}\left(\partial_v^2 \frac{\delta u^n}{\delta m}(\mu_s)(X_s) \cdot a_s\right) \mathrm{d}s. \tag{5.2}$$

Step 3: Letting $n \to +\infty$.

Our aim is now to take the limit $n \to +\infty$ in (5.2). As for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $\mu * \rho_n \overset{W_2}{\longrightarrow} \mu$ and u is continuous on $\mathcal{P}_2(\mathbf{R}^d)$, we deduce that $(u^n)_n$ converges pointwise to u. It remains to take the limit in the two integrals of (5.2). We show that

$$\int_0^t \mathbf{E}\left(\partial_v \frac{\delta u^n}{\delta m}(\mu_s)(X_s) \cdot b_s\right) \mathrm{d}s \to \int_0^t \mathbf{E}\left(\partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s) \cdot b_s\right) \mathrm{d}s. \tag{5.3}$$

Using (5.1) and since b is uniformly bounded, it is enough to prove that

$$\mathbf{E}\int_0^T \left|\partial_v \frac{\delta u}{\delta m}(\mu * \rho_n) * \rho_n(X_s) - \partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s)\right| \mathrm{d}s \to 0.$$

By Proposition 4.3, Hölder's inequality and then the convolution embedding $L^1 * L^k \hookrightarrow L^k$, one has

$$\begin{aligned}
&\mathbf{E}\int_0^T \left|\partial_v \frac{\delta u}{\delta m}(\mu_s * \rho_n) * \rho_n(X_s) - \partial_v \frac{\delta u}{\delta m}(\mu_s)(X_s)\right| \mathrm{d}s \\
&\leq \int_0^T \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s * \rho_n) - \partial_v \frac{\delta u}{\delta m}(\mu_s)\right\|_{L^k(\mathbf{R}^d)} \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)} \mathrm{d}s \\
&\quad + \int_0^T \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s) * \rho_n - \partial_v \frac{\delta u}{\delta m}(\mu_s)\right\|_{L^k(\mathbf{R}^d)} \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)} \mathrm{d}s \\
&=: I_1 + I_2.
\end{aligned}$$

The integrand in I_1 converges to 0 for almost all s using Assumption (1) in Theorem 3.3 and the fact that $\mu_s * \rho_n \overset{d_{\Pi(\mathbf{R}^d)}}{\longrightarrow} \mu_s$ for almost all s thanks to Assumption **(H2)** in Definition 2.4. Let us now prove that the dominated convergence theorem applies. The integrand is bounded by

$$\left[\sup_{n\geq 1}\left\|\partial_v \frac{\delta u}{\delta m}(\mu_s * \rho_n)\right\|_{L^k(\mathbf{R}^d)} + \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s)\right\|_{L^k(\mathbf{R}^d)}\right] \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}.$$

Note that the set $\{\mu_s * \rho_n,\ s\in[0,T],\ n\geq 1\}\cup\{\mu_s,\ s\in[0,T]\}$ is compact in $\mathcal{P}_2(\mathbf{R}^d)$. Indeed, if $(s_k)_k\in[0,T]^{\mathbf{N}}$ and $(n_k)_k\in\mathbf{N}^{\mathbf{N}}$ are two sequences, we have to find a convergent subsequence from $(\mu_{s_k} * \rho_{n_k})_k$. Up to an extraction, we can assume that $(s_k)_k$ converges to some $s\in[0,T]$. There are two cases. If there exists l such that $n_k = l$ infinitely often, then $\mu_{s_k} * \rho_l \overset{W_2}{\longrightarrow} \mu_s * \rho_l$ by the contraction inequality (see Lem. 4.9). Otherwise, we can assume that $(n_k)_k$ converges to $+\infty$. We use the triangle inequality to get

$$W_2(\mu_{s_k} * \rho_{n_k}, \mu_s) \leq W_2(\mu_{s_k} * \rho_{n_k}, \mu_s * \rho_{n_k}) + W_2(\mu_s * \rho_{n_k}, \mu_s).$$

The last term converges to 0 owing to Lemma 4.10, and the first is bounded by $W_2(\mu_{s_k},\mu_s)$ by the contraction inequality (see Lem. 4.9), which converges to 0. Thus Assumption (2) in Definition 3.1 ensures that there exists $C>0$ such that for almost all $s\in[0,T]$ and for all n

$$\left\|\partial_v \frac{\delta u}{\delta m}(\mu_s * \rho_n)\right\|_{L^k(\mathbf{R}^d)} + \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s)\right\|_{L^k(\mathbf{R}^d)} \leq C(1+\|p(s,\cdot) * \rho_n\|^{\alpha}_{L^{k'}(\mathbf{R}^d)} + \|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)}).$$

It follows from the the convolution embedding $L^1 * L^{k'} \hookrightarrow L^{k'}$ that for almost all s

$$\begin{aligned}&\left[\sup_{n\geq 1}\left\|\partial_v \frac{\delta u}{\delta m}(\mu_s * \rho_n)\right\|_{L^k(\mathbf{R}^d)} + \left\|\partial_v \frac{\delta u}{\delta m}(\mu_s)\right\|_{L^k(\mathbf{R}^d)}\right] \|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)}\\ &\leq 2C(1+\|p(s,\cdot)\|^{\alpha}_{L^{k'}(\mathbf{R}^d)})\|p(s,\cdot)\|_{L^{k'}(\mathbf{R}^d)},\end{aligned}$$

which is integrable on $[0,T]$ thanks to Lemma 4.4 since $k\geq\max\{d(\alpha+1),d+1\}$. We conclude by the dominated convergence theorem that I_1 converges to 0. The term I_2 also converges to 0 following the same method. Indeed, for almost all s, $\partial_v \frac{\delta u}{\delta m}(\mu_s)(\cdot)\in L^k(\mathbf{R}^d)$ thus the integrand converges to 0 by Lemma 4.6 and we conclude with the dominated convergence theorem. Therefore (5.3) is proved. Following the same lines, we take the limit $n\to+\infty$ in the last integral of (5.2) to obtain that for all $t\in[0,T]$

$$\int_0^t \mathbf{E}\left(\partial_v^2 \frac{\delta u^n}{\delta m}(\mu_s)(X_s)\cdot a_s\right)\mathrm{d}s \to \int_0^t \mathbf{E}\left(\partial_v^2 \frac{\delta u}{\delta m}(\mu_s)(X_s)\cdot a_s\right)\mathrm{d}s.$$

This concludes the proof of Theorem 3.3. □

6. Proof of Theorem 3.7

The strategy of the proof is the following. In Step 1, we prove some integrability results coming from Assumption (5) in Definition 3.5. Step 2 is devoted to prove that all the terms in Itô-Krylov's formula (3.2) are well-defined using a localization argument, Krylov's estimate, and Step 1. Moreover, we see that it is enough to prove the formula up to random times localizing the process ξ. Step 3 is dedicated to regularize u using convolutions both in space and measure variables. In Step 4 and 5, we follow the strategy of the proof of Theorem 5.102 in [3] to prove Itô-Krylov's formula for u^n, the mollified version of u. Finally, Step 6 aims at taking the limit $n\to+\infty$ thanks to Krylov's estimate.

Note that there are three kind of integrals in Itô's formula (3.2): the terms involving standard time and space derivatives in the first line, those involving the linear derivative in the second line and the martingale term in the third line. We will treat them separately.

Step 1: Useful integrability results.
Let us denote by $q(s,\cdot)$ the density of ξ_s, which exists for almost all $s\in[0,T]$ by Proposition 4.3. It follows from Assumption (5) in Definition 3.5 and Lemma 4.4 that for any $M>0$ the following quantities are finite:

$$\begin{aligned}
J_1(M) &:= \int_0^T \left[\sup_{n\geq 1}\|\partial_x u(s,\cdot,\mu_s*\rho_n)\|_{L^{k_1}(B_M)} + \sup_{n\geq 1}\left\|\partial_x^2 u(s,\cdot,\mu_s*\rho_n)\right\|_{L^{k_1}(B_M)}\right]\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s, \qquad (6.1)\\
J_2(M) &:= \int_0^T \sup_{n\geq 1}\|\partial_x u(s,\cdot,\mu_s*\rho_n)\|^2_{L^{2k_1}(B_M)}\,\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s,\\
J_3(M) &:= \int_0^T \sup_{n\geq 1}\left\|\partial_v\frac{\delta u}{\delta m}(s,\cdot,\mu_s*\rho_n)(\cdot)\right\|_{L^{k_2}(B_M\times\mathbf{R}^d)}\|q(s,\cdot)\|_{L^{k_2'}(B_M)}\|p(s,\cdot)\|_{L^{k_2'}(\mathbf{R}^d)}\,\mathrm{d}s,\\
J_4(M) &:= \int_0^T \sup_{n\geq 1}\left\|\partial_v^2\frac{\delta u}{\delta m}(s,\cdot,\mu_s*\rho_n)(\cdot)\right\|_{L^{k_2}(B_M\times\mathbf{R}^d)}\|q(s,\cdot)\|_{L^{k_2'}(B_M)}\|p(s,\cdot)\|_{L^{k_2'}(\mathbf{R}^d)}\,\mathrm{d}s.
\end{aligned}$$

To prove this, we follow the method employed in Step 3 of the preceding proof to justify the dominated convergence theorem. We just give details for $J_2(M)$ since it requires a bit more attention. Owing to Assumption (2) in Definition 3.5, we know that for all $(t,\mu)\in[0,T]\times\Pi(\mathbf{R}^d)$, $\partial_x u(t,\cdot,\mu)\in W^{1,k_1}(B)$. The Sobolev embedding theorem (see Cor. 9.14 in [24]) ensures that the embedding $W^{1,k_1}(B_M)\hookrightarrow L^{2k_1}(B_M)$ is continuous since $k_1\geq d+1$. Thus there exists $C>0$ such that

$$\forall t\in[0,T],\ \forall\mu\in\Pi(\mathbf{R}^d),\ \|\partial_x u(t,\cdot,\mu)\|_{L^{2k_1}(B_M)}\leq C\left(\|\partial_x u(t,\cdot,\mu)\|_{L^{k_1}(B_M)}+\|\partial_x^2 u(t,\cdot,\mu)\|_{L^{k_1}(B_M)}\right).$$

Thanks to Assumption (5) in Definition 3.5, there exists a constant $C_M>0$ such that for almost all s and for all $n\geq 1$

$$\sup_{n\geq 1}\|\partial_x u(s,\cdot,\mu_s*\rho_n)\|^2_{L^{2k_1}(B_M)}\leq C_M\left(1+\|p(s,\cdot)\|^{2\alpha_1}_{L^{k_1'}(\mathbf{R}^d)}\right),$$

where we used the fact that $\{\mu_s*\rho_n,\ s\in[0,T],\ n\geq 1\}$ is relatively compact in $\mathcal{P}_2(\mathbf{R}^d)$ and the convolution embedding $L^{k_1'}*L^1\hookrightarrow L^{k_1'}$. We conclude with Lemma 4.4 since $k_1\geq\max\{d(2\alpha_1+1),d+1\}$. Note that these integrability properties remain true if we replace $\mu_s*\rho_n$ by μ_s and remove the supremum. We justify it only for the second point. It follows from the continuity assumption (2) in Definition 3.5 that for almost all $s\in[0,T]$

$$\partial_x u(s,\cdot,\mu_s*\rho_n)\overset{W^{1,k_1}(B_M)}{\longrightarrow}\partial_x u(s,\cdot,\mu_s),$$

because $\mu_s*\rho_n\overset{d_{\Pi(\mathbf{R}^d)}}{\longrightarrow}\mu_s$ for almost all s. The Sobolev embedding theorem guarantees that

$$\|\partial_x u(t,\cdot,\mu*\rho_n)\|_{L^{2k_1}(B_M)}\to\|\partial_x u(t,\cdot,\mu)\|_{L^{2k_1}(B_M)}.$$

Thus we obtain

$$\int_0^T\|\partial_x u(s,\cdot,\mu_s)\|^2_{L^{2k_1}(B_M)}\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s\leq J_2(M)<+\infty.$$

Step 2: Meaning of the terms in (3.2) and localization.

Let $(T_M)_M$ be the sequence of stopping times converging almost surely to T defined by

$$T_M = \inf\{t \in [0,T],\, |\xi_t| \geq M\} \wedge T.$$

Let $\xi_t^M = \xi_{t\wedge T_M}$, which is bounded by M on the set $\{T_M > 0\}$.

(i) Terms involving standard derivatives in (3.2). We prove that almost surely

$$\int_0^T |\partial_x u(s,\xi_s,\mu_s)\cdot \eta_s|\,\mathrm{d}s < +\infty.$$

By Proposition 4.3 and Hölder's inequality, one has

$$\begin{aligned}\mathbf{E}\int_0^{T\wedge T_M} |\partial_x u(s,\xi_s,\mu_s)|\,\mathrm{d}s &\leq \int_0^T \int_{B_M} |\partial_x u(s,x,\mu_s)| q(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\leq \int_0^T \|\partial_x u(s,\cdot,\mu_s)\|_{L^{k_1}(B_M)} \|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s \\ &\leq J_1(M),\end{aligned}$$

which is finite (see (6.1) in Step 1). We deduce that almost surely, for all $M \geq 1$

$$\int_0^{T\wedge T_M} |\partial_x u(s,\xi_s,\mu_s)|\,\mathrm{d}s < \infty.$$

But it is clear that for almost all $\omega \in \Omega$ and for M bigger than some random constant $M(\omega) \geq 1$, $T_M(\omega) = T$. Thus, since η is uniformly bounded, $\int_0^T |\partial_x u(s,\xi_s,\mu_s).\eta_s|\,\mathrm{d}s$ is finite almost surely. The other terms in the first line of Itô's formula (3.2) are treated with the same method.

(ii) Martingale term in (3.2). We need to prove that $\int_0^T |\partial_x u(s,\xi_s,\mu_s)|^2\,\mathrm{d}s$ is almost surely finite. Reasoning as before, it is a consequence of the fact that J_2 is finite since we have

$$\int_0^T \|\partial_x u(s,\cdot,\mu_s)\|^2_{L^{2k_1}(B_M)} \|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s \leq J_2(M).$$

Therefore the martingale term in (3.2) is well-defined.

(iii) Terms involving the linear derivative in (3.2). We remark that $\tilde{X}$ and ξ can be seen as independent processes on the product space $\Omega\times\tilde{\Omega}$ with $\mathcal{L}(\tilde{X}_s) = p(s,\cdot)\,\mathrm{d}x$ and $\mathcal{L}(\xi_s) = q(s,\cdot)\,\mathrm{d}x$ for almost all s. Hölder's inequality gives that

$$\begin{aligned}&\mathbf{E}\int_0^{T\wedge T_M} \tilde{\mathbf{E}}\left|\partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\right|\,\mathrm{d}s \\ &\leq \int_0^T \int_{B_M\times\mathbf{R}^d} \left|\partial_v \frac{\delta u}{\delta m}(s,x,\mu_s)(v)\right| q(s,x)p(s,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ &\leq \int_0^T \left\|\partial_v \frac{\delta u}{\delta m}(s,\cdot,\mu_s)(\cdot)\right\|_{L^{k_2}(B_M\times\mathbf{R}^d)} \|q(s,\cdot)\|_{L^{k_2'}(B_M)} \|p(s,\cdot)\|_{L^{k_2'}(\mathbf{R}^d)}\,\mathrm{d}s \\ &= J_3(M),\end{aligned}$$

which was defined in (6.1) and is finite. We deduce as previously that $\int_0^T \tilde{\mathbf{E}} \left| \partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s).\tilde{b_s}\right| \mathrm{d}s$ is almost surely finite. The term involving $\partial_v^2 \frac{\delta u}{\delta m}$ is dealt similarly.

Since all the terms in (3.2) are well-defined, it is enough to prove Itô-Krylov's formula for $u(t \wedge T_M, \xi_{t\wedge T_M}, \mu_{t\wedge T_M})$ almost surely for all $t \in [0,T]$, and then take the limit $M \to +\infty$ using the continuity of the integrals in Itô-Krylov's formula with respect to t. So we fix $\tau := T_M$ for $M \geq 1$ and we want to prove the formula up to time τ.

Step 3: Mollification of u.

Let u^n be the function defined by $u^n(t,x,\mu) := u(t,\cdot,\mu * \rho_n) * \rho_n(x)$. It is clearly continuous on $[0,T]\times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d)$, as u. Since $\partial_t u$ is jointly continuous, it follows from Leibniz's rule that u^n is $\mathcal{C}^1$ with respect to t and that we can differentiate under the integral *i.e.* for all $(t,x,\mu) \in [0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$

$$\partial_t u^n(t,x,\mu) = \partial_t u(t,\cdot,\mu*\rho_n)*\rho_n(x),$$

which is also jointly continuous. As a result of Lemma 4.6 and Proposition 4.7, u^n is $\mathcal{C}^2$ with respect to x and we have

$$\partial_x u^n(t,x,\mu) = \partial_x u(t,\cdot,\mu*\rho_n)*\rho_n(x) \quad \text{and} \quad \partial_x^2 u^n(t,x,\mu) = \partial_x^2 u(t,\cdot,\mu*\rho_n)*\rho_n(x).$$

These two functions are continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$ by the dominated convergence theorem and the fact that u is jointly continuous. We define $\tilde{\rho_n}$ by $\tilde{\rho_n}(x,v) := \rho_n(x)\rho_n(v)$ for all $x, v \in \mathbf{R}^d$. It is easy to see that $(\tilde{\rho_n})_n$ is a mollifying sequence on $\mathbf{R}^{2d}$. Next, we claim that for all $(t,x)\in[0,T]\times\mathbf{R}^d$, $u^n(t,x,\cdot)$ has a linear derivative given by

$$\frac{\delta u^n}{\delta m}(t,x,\mu)(v) := \frac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)*\tilde{\rho_n}(x,v). \tag{6.2}$$

This convolution is well-defined as $\frac{\delta u}{\delta m}$ is jointly continuous. To prove (6.2), note first that the bound of Assumption (3) in Definition 3.5 implies that for all $(t,x)\in[0,T]\times\mathbf{R}^d$, $\frac{\delta u^n}{\delta m}(t,x,\mu)(\cdot)$ is at most of quadratic growth, uniformly in μ on each compact set. Since for all $(t,x)\in[0,T]\times\mathbf{R}^d$, $\frac{\delta u}{\delta m}(t,x,\cdot)(\cdot)$ is continuous on $\mathcal{P}_2(\mathbf{R}^d)\times\mathbf{R}^d$, the dominated convergence theorem proves that $\frac{\delta u^n}{\delta m}(t,x,\cdot)(\cdot)$ is continuous. As explained in Remark 2.3, it is enough to compute, for $\mu,\nu\in\mathcal{P}_2(\mathbf{R}^d)$ and $\lambda\in[0,1]$, the derivative with respect to λ of $u^n(t,x,m_\lambda)$, where $m_\lambda = \lambda\mu + (1-\lambda)\nu$. As recalled in the proof of Theorem 3.3, when (t,x) are fixed

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} u(t,x,m_\lambda*\rho_n) = \int_{\mathbf{R}^d} \frac{\delta u}{\delta m}(t,x,m_\lambda*\rho_n)*\rho_n(v)\, d(\mu-\nu)(v).$$

Thanks to the bound Assumption (3) in Definition 3.5 for all compact $K\subset\mathbf{R}^d$, one has

$$\sup_{x\in K}\sup_{\lambda\in[0,1]} \left|\frac{\mathrm{d}}{\mathrm{d}\lambda} u(t,x,m_\lambda*\rho_n)\right| \leq C\left(1+\int_{\mathbf{R}^d}|v|^2\, d(\mu+\nu)(v)\right).$$

We can conclude with the help of Leibniz's rule and Fubini's theorem that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} u^n(t,x,m_\lambda) = \int_{\mathbf{R}^d}\frac{\delta u}{\delta m}(t,\cdot,m_\lambda*\rho_n)*\tilde{\rho}_n(x,v)\, d(\mu-\nu)(v).$$

It follows from the joint continuity of $\frac{\delta u}{\delta m}$ and Leibniz's rule that $\frac{\delta u^n}{\delta m}$ is $\mathcal{C}^2$ with respect to v and that

$$\begin{cases}\partial_v \dfrac{\delta u^n}{\delta m}(t,x,\mu)(v) = & \dfrac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)*\partial_v\tilde{\rho_n}(x,v) = \partial_v\dfrac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)*\tilde{\rho_n}(x,v)\\ \partial_v^2 \dfrac{\delta u^n}{\delta m}(t,x,\mu)(v) = & \dfrac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)*\partial_v^2\tilde{\rho_n}(x,v) = \partial_v^2\dfrac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)*\tilde{\rho_n}(x,v).\end{cases}$$

Note that $\partial_v\frac{\delta u^n}{\delta m}$ and $\partial_v^2\frac{\delta u^n}{\delta m}$ are continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)\times\mathbf{R}^d$ thanks to the dominated convergence theorem and the joint continuity of $\frac{\delta u}{\delta m}$. Moreover for all compact $\mathcal{K}\subset\mathcal{P}_2(\mathbf{R}^d)$ and for all $M>0$

$$\sup_{t\in[0,T]}\sup_{\mu\in\mathcal{K}}\sup_{|x|\le M}\sup_{v\in\mathbf{R}^d}\left|\partial_v\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|+\left|\partial_v^2\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|<+\infty. \tag{6.3}$$

Indeed, Hölder's inequality ensures that

$$\begin{aligned}&\sup_{|x|\le M}\sup_{v\in\mathbf{R}^d}\left|\partial_v\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|+\left|\partial_v^2\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|\\ &\le\left[\left\|\partial_v\frac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)\right\|_{L^{k_2}(B_{M+1}\times\mathbf{R}^d)}+\left\|\partial_v^2\frac{\delta u}{\delta m}(t,\cdot,\mu*\rho_n)(\cdot)\right\|_{L^{k_2}(B_{M+1}\times\mathbf{R}^d)}\right]\|\tilde{\rho_n}\|_{L^{k_2'}(\mathbf{R}^{2d})},\end{aligned}$$

the ball B_{M+1} coming from the fact that the support of $\tilde{\rho_n}$ is included in B_1. Since $\mathcal{K}*\rho_n$ is compact in $\mathcal{P}_2(\mathbf{R}^d)$ and included in $\Pi(\mathbf{R}^d)$, Assumption (5) in Definition 3.5 ensures that there exists $C>0$ such that for all $\mu\in\mathcal{K}$

$$\sup_{t\in[0,T]}\sup_{|x|\le M}\sup_{v\in\mathbf{R}^d}\left|\partial_v\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|+\left|\partial_v^2\frac{\delta u^n}{\delta m}(t,x,\mu)(v)\right|\le C\left(1+\left\|\frac{d\mu*\rho_n}{\mathrm{d}x}\right\|^{\alpha_2}_{L^{k_2'}(\mathbf{R}^d)}\right)\|\tilde{\rho_n}\|_{L^{k_2'}(\mathbf{R}^{2d})}.$$

But we know that $\frac{d\mu*\rho_n}{\mathrm{d}x}(x)=\int_{\mathbf{R}^d}\rho_n(x-y)\,\mathrm{d}\mu(y)$. We conclude with Jensen's inequality that

$$\left\|\frac{d\mu*\rho_n}{\mathrm{d}x}\right\|^{\alpha_2}_{L^{k_2'}(\mathbf{R}^d)}\le\|\rho_n\|^{\alpha_2}_{L^{k_2'}(\mathbf{R}^d)}.$$

This proves (6.3).

Step 4: Itô's formula (3.2) for u^n when the coefficients b and σ are continuous.

We assume that $b=b^m$ and $\sigma=\sigma^m$ are jointly continuous. We claim that $(t,x)\mapsto U^n(t,x):=u^n(t,x,\mu_t)\in\mathcal{C}^{1,2}([0,T]\times\mathbf{R}^d)$. The regularity with respect to x is clear with the preceding properties on u^n. Let us thus focus on the regularity with respect to the time variable. For $(t,x)\in[0,T]\times\mathbf{R}^d$ fixed, the regularity assumption on u with respect to t and the standard Itô formula for a flow of measures applied to $u^n(t,x,\cdot)$ (see Thm. 5.99 in Chap. 5 of [3]) ensure that we have for $h\in\mathbf{R}$ satisfying $t+h\ge0$

$$\begin{aligned}u^n(t+h,x,\mu_{t+h})-u^n(t,x,\mu_t)&=u^n(t+h,x,\mu_{t+h})-u^n(t,x,\mu_{t+h})+u^n(t,x,\mu_{t+h})-u^n(t,x,\mu_t)\\ &=\int_t^{t+h}\partial_t u^n(s,x,\mu_{t+h})\,\mathrm{d}s+\int_t^{t+h}\mathbf{E}\left(\partial_v\frac{\delta u^n}{\delta m}(t,x,\mu_s)(X_s)\cdot b_s^m\right)\mathrm{d}s\\ &\quad+\frac12\int_t^{t+h}\mathbf{E}\left(\partial_v^2\frac{\delta u^n}{\delta m}(t,x,\mu_s)(X_s)\cdot a_s^m\right)\mathrm{d}s.\end{aligned} \tag{6.4}$$

The function $(s,x,\mu) \in [0,T] \times \mathbf{R}^d \times \mathcal{P}_2(\mathbf{R}^d) \mapsto \partial_t u^n(s,x,\mu)$ is continuous so

$$\frac{1}{h}\int_t^{t+h} \partial_t u^n(s,x,\mu_{t+h})\,\mathrm{d}s \underset{h\to 0}{\longrightarrow} \partial_t u^n(t,x,\mu_t).$$

The two other terms in (6.4) can be dealt similarly. Indeed, the dominated convergence theorem justified by (6.3) ensures that the functions $(s,x) \in [0,T] \times \mathbf{R}^d \mapsto \mathbf{E}\left(\partial_v \frac{\delta u^n}{\delta m}(s,x,\mu_s)(X_s)\cdot b_s^m\right)$ and $(s,x) \in [0,T] \times \mathbf{R}^d \mapsto \mathbf{E}\left(\partial_v^2 \frac{\delta u^n}{\delta m}(s,x,\mu_s)(X_s)\cdot a_s^m\right)$ are continuous. Then, it follows that $U^n \in \mathcal{C}^{1,2}([0,T]\times\mathbf{R}^d)$ and that for all $(t,x) \in [0,T]\times\mathbf{R}^d$

$$\partial_t U^n(t,x) = \partial_t u^n(t,x,\mu_t) + \mathbf{E}\left(\partial_v \frac{\delta u^n}{\delta m}(t,x,\mu_t)(X_t)\cdot b_t^m\right) + \frac{1}{2}\mathbf{E}\left(\partial_v^2 \frac{\delta u^n}{\delta m}(t,x,\mu_t)(X_t)\cdot a_t^m\right).$$

We can now apply the classical Itô formula for U^n and ξ, up to the random time τ defined at the end of Step 2, to obtain that almost surely, for all $t \in [0,T]$

$$\begin{aligned} u^n(t\wedge\tau,\xi_{t\wedge\tau},\mu_{t\wedge\tau}) = u^n(0,\xi_0,\mu_0) &+ \int_0^{t\wedge\tau} \partial_t u^n(s,\xi_s,\mu_s) + \partial_x u^n(s,\xi_s,\mu_s)\cdot\eta_s + \frac{1}{2}\partial_x^2 u^n(s,\xi_s,\mu_s)\cdot\gamma_s\gamma_s^*\,\mathrm{d}s \\ &+ \int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v \frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot \tilde{b_s^m}\right)\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v^2 \frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot \tilde{a_s^m}\right)\mathrm{d}s \\ &+ \int_0^{t\wedge\tau} \partial_x u^n(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s). \end{aligned} \tag{6.5}$$

Note that (6.5) does not require Assumptions **(A)** and **(B)** on the Itô process X. These assumptions will only be used in Step 6.

Step 5: Removing the continuity hypothesis on the coefficients b and σ.

We consider $(b^m)_m$ and $(\sigma^m)_m$ two sequences of continuous and progressively measurable processes such that

$$\mathbf{E}\int_0^T |b_s^m - b_s|^2 + |\sigma_s^m - \sigma_s|^4\,\mathrm{d}s \to 0.$$

We set, for $t \le T$, $X_t^m := X_0 + \int_0^t b_s^m\,\mathrm{d}s + \int_0^t \sigma_s^m\,\mathrm{d}B_s$, and μ_t^m the law of X_t^m. Owing to Step 4, Itô's formula (6.5) holds true for X^m and ξ. Now, we aim at taking the limit $m \to +\infty$ in (6.5). Note that the set $\mathcal{K} := \{\mu_s^m,\ s \le T,\ m \ge 1\} \cup \{\mu_s,\ s \le T\}$ is compact in $\mathcal{P}_2(\mathbf{R}^d)$. Indeed, using Jensen's inequality and the Burkholder-Davis-Gundy (BDG) inequalities, it is clear that $\mathbf{E}\sup_{t\le T}|X_t^m - X_t|^2 \to 0$, thus $\sup_{t\le T} W_2(\mu_t^m,\mu_t) \to 0$. We deduce that almost surely, for all $t \in [0,T]$

$$u^n(t,\xi_t,\mu_t^m) \underset{m\to+\infty}{\longrightarrow} u^n(t,\xi_t,\mu_t).$$

Now, we take the limit $m \to +\infty$ in the integrals in Itô's formula (6.5).

(i) Martingale term in (6.5). Using BDG's inequality, there exists $C > 0$ such that

$$\begin{aligned}
&\mathbf{E}\sup_{t\leq T}\left|\int_0^{t\wedge\tau}(\partial_x u^n(s,\xi_s,\mu_s^m) - \partial_x u^n(s,\xi_s,\mu_s))\cdot(\gamma_s\,\mathrm{d}\beta_s)\right|^2 \\
&\leq C\mathbf{E}\int_0^{T\wedge\tau}|\partial_x u^n(s,\xi_s,\mu_s^m) - \partial_x u^n(s,\xi_s,\mu_s)|^2|\gamma_s|^2\,\mathrm{d}s \\
&\leq C\mathbf{E}\int_0^{T}|\partial_x u^n(s,\xi_s,\mu_s^m) - \partial_x u^n(s,\xi_s,\mu_s)|^2\mathbf{1}_{B_M}(\xi_s)|\gamma_s|^2\,\mathrm{d}s.
\end{aligned}$$

The dominated convergence theorem can be applied since γ is bounded and $\partial_x u^n$ is jointly continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$. It shows that, up to an extraction, almost surely

$$\forall t\leq T,\ \int_0^{t\wedge\tau}\partial_x u^n(s,\xi_s,\mu_s^m)\cdot(\gamma_s\,\mathrm{d}\beta_s)\underset{m\to+\infty}{\longrightarrow}\int_0^{t\wedge\tau}\partial_x u^n(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s).$$

(ii) Terms involving the linear derivative in (6.5). Let us write

$$\begin{aligned}
&\left|\int_0^{t\wedge\tau}\tilde{\mathbf{E}}\left(\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m})\cdot\tilde{b_s^m}\right)\mathrm{d}s - \int_0^{t\wedge\tau}\tilde{\mathbf{E}}\left(\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s}\right)\mathrm{d}s\right| \\
&\leq \int_0^{T\wedge\tau}\tilde{\mathbf{E}}\left|\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m})\right||\tilde{b_s^m}-\tilde{b_s}|\,\mathrm{d}s \\
&\quad+\int_0^{T\wedge\tau}\tilde{\mathbf{E}}\left|\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m}) - \partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\right||\tilde{b_s}|\,\mathrm{d}s \\
&=: I_1 + I_2
\end{aligned}$$

Cauchy-Schwarz's inequality ensures that

$$I_1\leq\left(\int_0^{T\wedge\tau}\tilde{\mathbf{E}}\left|\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m})\right|^2\mathrm{d}s\right)^{1/2}\left(\int_0^T\tilde{\mathbf{E}}|\tilde{b_s^m}-\tilde{b_s}|^2\,\mathrm{d}s\right)^{1/2}.$$

We conclude that I_1 converges to 0 thanks to the bound (6.3) proved in Step 3 and since ξ is bounded by M on the set $\{\tau > 0\}$. To show that $I_2\to 0$, we use the fact that b is bounded by K to get

$$I_2\leq K\int_0^{T\wedge\tau}\tilde{\mathbf{E}}\left|\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m}) - \partial_v\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\right|\mathrm{d}s.$$

The continuity of $\partial_v\frac{\delta u^n}{\delta m}$ and the convergence in L^2 of $(\tilde{X_s^m})_m$ to $\tilde{X}_s$ ensure that for all $\omega\in\Omega$, $\left|\partial_v\frac{\delta u^n}{\delta m}(s,\xi_s(\omega),\mu_s^m)(\tilde{X_s^m}) - \partial_v\frac{\delta u^n}{\delta m}(s,\xi_s(\omega),\mu_s)(\tilde{X_s})\right|$ converges in probability on $\tilde{\Omega}$ to 0 as m goes to infinity. Using a uniform integrability argument coming from (6.3), we deduce that I_2 converges to 0. Following the same strategy, one has for all $t\in[0,T]$

$$\int_0^{t\wedge\tau}\tilde{\mathbf{E}}\left(\partial_v^2\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s^m)(\tilde{X_s^m})\cdot\tilde{a_s^m}\right)\mathrm{d}s\underset{m\to+\infty}{\longrightarrow}\int_0^{t\wedge\tau}\tilde{\mathbf{E}}\left(\partial_v^2\frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{a_s}\right)\mathrm{d}s.$$

(iii) Terms involving standard derivatives in (6.5). It follows from the dominated convergence theorem that almost surely, for all $t \leq T$

$$\int_0^{t\wedge\tau} (\partial_t u^n(s,\xi_s,\mu_s^m) + \partial_x u^n(s,\xi_s,\mu_s^m)\cdot\eta_s)\,\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tau} \partial_x^2 u^n(s,\xi_s,\mu_s^m)\cdot\gamma_s\gamma_s^*\,\mathrm{d}s$$
$$\underset{m\to+\infty}{\longrightarrow} \int_0^{t\wedge\tau} (\partial_t u^n(s,\xi_s,\mu_s) + \partial_x u^n(s,\xi_s,\mu_s)\cdot\eta_s)\,\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tau} \partial_x^2 u^n(s,\xi_s,\mu_s)\cdot\gamma_s\gamma_s^*\,\mathrm{d}s.$$

Indeed the functions $\partial_t u^n$, $\partial_x u^n$ and $\partial_x^2 u^n$ are jointly continuous on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$ and thus uniformly bounded on $[0,T]\times B_M\times\{\mu_s^m,\ s\in[0,T],\ m\geq 1\}$. Moreover, η and γ are also uniformly bounded.

This concludes Step 5.

Step 6: Letting $n\to+\infty$.

From Step 5, we deduce that Itô's formula (6.5) in Step 4 holds for u^n up to time τ. To conclude the proof, we need to take the limit $n\to+\infty$ in each term of (6.5). Then it remains to remove the stopping time τ as explained at the end of Step 2 (*i.e.* letting $\tau\to T$). The continuity of u ensures that almost surely, for all $t\leq T$, $u^n(t,\xi_t,\mu_t)\to u(t,\xi_t,\mu_t)$. We now focus on the integrals in Itô's formula (6.5).

(i) Martingale term in (6.5). Thanks to BDG's inequality, Hölder's inequality, and the boundedness of γ, we have

$$\begin{aligned}
&\mathbf{E}\sup_{t\leq T}\left|\int_0^{t\wedge\tau}(\partial_x u^n(s,\xi_s,\mu_s)-\partial_x u(s,\xi_s,\mu_s))\cdot(\gamma_s\,\mathrm{d}\beta_s)\right|^2\\
&\leq C\mathbf{E}\int_0^T |\partial_x u(s,\cdot,\mu_s*\rho_n)*\rho_n(\xi_s)-\partial_x u(s,\xi_s,\mu_s)|^2\mathbf{1}_{B_M}(\xi_s)\,\mathrm{d}s\\
&= C\int_0^T\int_{B_M} |\partial_x u(s,\cdot,\mu_s*\rho_n)*\rho_n(x)-\partial_x u(s,x,\mu_s)|^2 q(s,x)\,\mathrm{d}x\,\mathrm{d}s\\
&\leq C\int_0^T \|\partial_x u(s,\cdot,\mu_s*\rho_n)*\rho_n-\partial_x u(s,\cdot,\mu_s)\|^2_{L^{2k_1}(B_M)}\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s\\
&\leq C\int_0^T \|\partial_x u(s,\cdot,\mu_s*\rho_n)*\rho_n-\partial_x u(s,\cdot,\mu_s)*\rho_n\|^2_{L^{2k_1}(B_M)}\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s\\
&\quad + C\int_0^T \|\partial_x u(s,\cdot,\mu_s)*\rho_n-\partial_x u(s,\cdot,\mu_s)\|^2_{L^{2k_1}(B_M)}\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s\\
&=: I_1+I_2.
\end{aligned}$$

We prove that I_1 and I_2 converge to 0. First note that, due to the convolution embedding $L^1*L^r\hookrightarrow L^r$, we have for $f\in L^r_{\mathrm{loc}}(\mathbf{R}^d)$ and for all $R>0$, $\|f*\rho_n\|_{L^r(B_R)}\leq\|f\|_{L^r(B_{R+1})}$. The control on B_{R+1} follows from the fact that the support of each ρ_n is included in B_1. Hence

$$I_1\leq C\int_0^T \|\partial_x u(s,\cdot,\mu_s*\rho_n)-\partial_x u(s,\cdot,\mu_s)\|^2_{L^{2k_1}(B_{M+1})}\|q(s,\cdot)\|_{L^{k_1'}(B_{M+1})}\,\mathrm{d}s =: \tilde{I}_1.$$

As a consequence of the Sobolev embedding theorem, for all t, the function

$$\mu\in(\Pi(\mathbf{R}^d),d_{\Pi(\mathbf{R}^d)})\mapsto\partial_x u(t,\cdot,\mu)\in L^\infty(B_{M+1})$$

is continuous. Since $\mu_s \in \Pi(\mathbf{R}^d)$ for almost all s and thanks to Assumption (2) in Definition 2.4, we deduce that the integrand in $\tilde{I}_1$ converges to 0 for almost all s. It follows from the dominated convergence theorem (see (6.1) in Step 1) that $\tilde{I}_1$ converges to 0, as well as I_1. We now focus on I_2. The integrand in I_2 converges to 0 for almost all s because $\partial_x u(s,\cdot,\mu_s) \in L^{2k_1}(B_M)$. We conclude with the dominated convergence theorem as previously. This shows that, up to an extraction, almost surely

$$\sup_{t\le T}\left|\int_0^{t\wedge\tau} \partial_x u^n(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s) - \int_0^{t\wedge\tau} \partial_x u(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s)\right| \to 0.$$

(ii) Terms involving the linear derivative in (6.5). Following the same strategy, we obtain using Hölder's inequality

$$\begin{aligned}
&\mathbf{E}\sup_{t\le T}\left|\int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v \frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s}\right)\mathrm{d}s - \int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s}\right)\mathrm{d}s\right|\\
&\le \mathbf{E}\tilde{\mathbf{E}}\int_0^{T\wedge\tau}\left|\partial_v \frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s).\tilde{b_s} - \partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s).\tilde{b_s}\right|\mathrm{d}s\\
&\le \mathbf{E}\tilde{\mathbf{E}}\int_0^{T}\left|\partial_v \frac{\delta u^n}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s} - \partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s}\right|\mathbf{1}_{B_M}(\xi_s)\,\mathrm{d}s\\
&\le C\int_0^T\int_{B_M\times\mathbf{R}^d}\left|\partial_v \frac{\delta u^n}{\delta m}(s,x,\mu_s)(v) - \partial_v \frac{\delta u}{\delta m}(s,x,\mu_s)(v)\right| q(s,x)p(s,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s\\
&\le C\int_0^T\left\|\partial_v \frac{\delta u}{\delta m}(s,\cdot,\mu_s*\rho_n)(\cdot)*\tilde{\rho_n} - \partial_v \frac{\delta u}{\delta m}(s,\cdot,\mu_s)(\cdot)\right\|_{L^{k_2}(B_M\times\mathbf{R}^d)}\|q(s,\cdot)\|_{L^{k_2'}(B_M)}\|p(s,\cdot)\|_{L^{k_2'}(\mathbf{R}^d)}\,\mathrm{d}s.
\end{aligned}$$

The dominated convergence theorem justified by Assumption (4) in Definition 3.5 and (6.1) in Step 1 ensures that this term converges to 0. The same argument holds true for the term involving $\partial_v^2\frac{\delta u}{\delta m}$.

(iii) Terms involving standard derivatives in (6.5). The convergence of the term involving $\partial_t u^n$ in (6.5) follows from the continuity of $\partial_t u$ on $[0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d)$ and the dominated convergence theorem since almost surely on the set $\{\tau>0\}$

$$\sup_{s\in[0,T]}\sup_{n\ge1}|\partial_t u^n(s,\xi_s,\mu_s)| \le \sup_{s\in[0,T]}\sup_{n\ge1}\sup_{|x|\le M+1}|\partial_t u(s,x,\mu_s*\rho_n)| < +\infty.$$

For the spatial derivatives, Hölder's inequality ensures that

$$\begin{aligned}
&\mathbf{E}\sup_{t\le T}\left|\int_0^{t\wedge\tau}\partial_x u^n(s,\xi_s,\mu_s)\cdot\eta_s\,\mathrm{d}s - \int_0^{t\wedge\tau}\partial_x u(s,\xi_s,\mu_s)\cdot\eta_s\,\mathrm{d}s\right|\\
&\le C\int_0^T\|\partial_x u(s,\cdot,\mu_s*\rho_n)*\rho_n - \partial_x u(s,\cdot,\mu_s)\|_{L^{k_1}(B_M)}\|q(s,\cdot)\|_{L^{k_1'}(B_M)}\,\mathrm{d}s.
\end{aligned}$$

The right-hand side term converges to 0 with same reasoning as before. This shows that, up to an extraction, one has almost surely

$$\sup_{t\le T}\left|\int_0^{t\wedge\tau}\partial_x u^n(s,\xi_s,\mu_s)\cdot\eta_s\,\mathrm{d}s - \int_0^{t\wedge\tau}\partial_x u(s,\xi_s,\mu_s)\cdot\eta_s\,\mathrm{d}s\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

The term involving $\partial_x^2 u$ in (6.5) is dealt similarly.

Taking the limit $n \to +\infty$ in (6.5), up to an extraction, we conclude that almost surely, for all $t \in [0,T]$

$$\begin{aligned} u(t\wedge\tau, \xi_{t\wedge\tau}, \mu_{t\wedge\tau}) &= u(0,\xi_0,\mu_0) \\ &\quad + \int_0^{t\wedge\tau} (\partial_t u(s,\xi_s,\mu_s) + \partial_x u(s,\xi_s,\mu_s)\cdot\eta_s)\,\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tau} \partial_x^2 u(s,\xi_s,\mu_s)\cdot\gamma_s\gamma_s^*\,\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{b_s}\right)\mathrm{d}s + \frac{1}{2}\int_0^{t\wedge\tau} \tilde{\mathbf{E}}\left(\partial_v^2 \frac{\delta u}{\delta m}(s,\xi_s,\mu_s)(\tilde{X}_s)\cdot\tilde{a_s}\right)\mathrm{d}s \\ &\quad + \int_0^{t\wedge\tau} \partial_x u(s,\xi_s,\mu_s)\cdot(\gamma_s\,\mathrm{d}\beta_s). \end{aligned}$$

This ends the proof as explained in Step 2. □

Appendix A

A.1 On Example 2.5

(1) It follows from the contraction inequality in Lemma 4.9 and Corollary 4.10.

(2) To prove (**H1**), we fix $n \geq 1$ and $\mu_j \xrightarrow{W_2} \mu \in \mathcal{P}_2(\mathbf{R}^d)$. For $\nu \in \mathcal{P}_2(\mathbf{R}^d)$, the density of $\nu * \rho_n$ is given by

$$x \in \mathbf{R}^d \mapsto \rho_n * \nu(x) = \int_{\mathbf{R}^d} \rho_n(x-y)\,\mathrm{d}\nu(y).$$

Hence,

$$d_{k'}(\mu_j * \rho_n, \mu * \rho_n) = \left\| \int_{\mathbf{R}^d} \rho_n(\cdot - y)\,\mathrm{d}\mu_j(y) - \int_{\mathbf{R}^d} \rho_n(\cdot - y)\,\mathrm{d}\mu(y) \right\|_{L^{k'}(\mathbf{R}^d)}.$$

Using Lemma 4.8, we conclude that $d_{k'}(\mu_j * \rho_n, \mu * \rho_n) \underset{j\to+\infty}{\longrightarrow} 0$. For **(H2)**, let $\mu \in \Pi(\mathbf{R}^d)$ and denote by $f \in L^{k'}(\mathbf{R}^d)$ the density of μ. For $n \geq 1$, we have

$$\frac{\mathrm{d}\mu * \rho_n}{\mathrm{d}x} = f * \rho_n \xrightarrow{L^{k'}} f,$$

owing to Lemma 4.6. □

A.2 On Example 3.11

Let us give the detailed proof in the bilinear case $N = 2$. It is standard (see Ex. 4 page 389 in Chap. 5 of [3]) that u has a linear derivative given, up to an additive constant depending on μ and chosen to be equal to 0 here, by

$$\frac{\delta u}{\delta m}(\mu)(v) = \int_{\mathbf{R}^d} g(v,y)\,\mathrm{d}\mu(y) + \int_{\mathbf{R}^d} g(y,v)\,\mathrm{d}\mu(y).$$

We will only treat the first term since the other one can be dealt similarly.

Computation of the distributional derivatives and continuity: Let $\mu \in \Pi(\mathbf{R}^d)$ and $f \in L^{(d+1)'}(\mathbf{R}^d)$ be its density. By interpolation, we know that $f \in L^{r'}(\mathbf{R}^d)$ for all $r \geq d+1$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbf{R}^d)$ and $i \in \{1,\ldots,d\}$.

Using Fubini's theorem, justified by the quadratic growth of g and the fact that $f\,\mathrm{d}x \in \mathcal{P}_2(\mathbf{R}^d)$, we have

$$\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} g(x,y)f(y)\,\mathrm{d}y\right)\partial_{v_i}\varphi(v)\,\mathrm{d}v = \int_{\mathbf{R}^d\times\mathbf{R}^d} g(v,y)f(y)\partial_{v_i}\varphi(v)\,\mathrm{d}y\,\mathrm{d}v.$$

Let us define $f_n(x) = \frac{1}{\mu(B_n)}(f\mathbf{1}_{B_n})*\rho_n(x)$, for n large enough to have $\mu(B_n)>0$. The function f_n is a probability density which is in $\mathcal{C}_c^\infty(\mathbf{R}^d)$. It easily follows from Lemma 4.5, Lemma 4.6 and the dominated convergence theorem that

$$f_n \xrightarrow{L^{k'}} f \quad \text{and} \quad f_n \xrightarrow{W_2} f. \tag{A.1}$$

For a fixed $n \geq 1$, we have by definition of the distributional derivative

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} g(v,y)f_n(y)\partial_{v_i}\varphi(v)\,\mathrm{d}y\,\mathrm{d}v = -\int_{\mathbf{R}^d\times\mathbf{R}^d}\partial_{v_i}g(v,y)f_n(y)\varphi(v)\,\mathrm{d}y\,\mathrm{d}v. \tag{A.2}$$

Our aim is to take the limit $n\to+\infty$ in both side of the previous equality. Using Fubini's theorem, the left-hand side term is equal to

$$\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} g(v,y)\partial_{v_i}\varphi(v)\,\mathrm{d}v\right)f_n(y)\,\mathrm{d}y.$$

Moreover, it converges to

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} g(v,y)\partial_{v_i}\varphi(v)f(y)\,\mathrm{d}y\,\mathrm{d}v.$$

Indeed, $f_n \xrightarrow{W_2} f$ and the function $y\mapsto \int_{\mathbf{R}^d} g(v,y)\partial_{v_i}\varphi(v)\,\mathrm{d}v$ is continuous and at most of quadratic growth. For the right-hand side term, we prove that

$$\int_{\mathbf{R}^d\times\mathbf{R}^d}\partial_{v_i}g(v,y)f_n(y)\varphi(v)\,\mathrm{d}y\,\mathrm{d}v \to \int_{\mathbf{R}^d\times\mathbf{R}^d}\partial_{v_i}g(v,y)f(y)\varphi(v)\,\mathrm{d}y\,\mathrm{d}v.$$

Note that the limit is well-defined using Hölder's inequality

$$\int_{\mathbf{R}^d\times\mathbf{R}^d}|\partial_{v_i}g(v,y)f(y)\varphi(v)|\,\mathrm{d}y\,\mathrm{d}v \leq \|f\|_{L^{k'}(\mathbf{R}^d)}\int_{\mathbf{R}^d}\varphi(v)\|\partial_{v_i}g(v,\cdot)\|_{L^k(\mathbf{R}^d)}\,\mathrm{d}v.$$

The right-hand side term is finite because $v\mapsto\|\partial_{v_i}g(v,\cdot)\|_{L^k(\mathbf{R}^d)}\in L^k(\mathbf{R}^d)$. The same inequality shows that

$$\left|\int_{\mathbf{R}^d\times\mathbf{R}^d}\partial_{v_i}g(v,y)(f_n(y)-f(y))\varphi(v)\,\mathrm{d}y\,\mathrm{d}v\right| \leq \|f_n-f\|_{L^{k'}(\mathbf{R}^d)}\int_{\mathbf{R}^d}\varphi(v)\|\partial_{v_i}g(v,\cdot)\|_{L^k(\mathbf{R}^d)}\,\mathrm{d}v \underset{n\to+\infty}{\longrightarrow} 0,$$

thanks to (A.1). Taking the limit $n\to+\infty$ in (A.2), we deduce that:

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} g(v,y)f(y)\partial_{v_i}\varphi(v)\,\mathrm{d}y\,\mathrm{d}v = -\int_{\mathbf{R}^d\times\mathbf{R}^d}\partial_{v_i}g(v,y)f(y)\varphi(v)\,\mathrm{d}y\,\mathrm{d}v.$$

Hence, the distributional derivative of $v \mapsto \int_{\mathbf{R}^d} g(v,y) f(y)\,\mathrm{d}y$ is given by the function

$$v \mapsto \int_{\mathbf{R}^d} \partial_v g(v,y) f(y)\,\mathrm{d}y.$$

Moreover, it belongs to $L^k(\mathbf{R}^d)$ because applying Hölder's inequality, one has

$$\begin{aligned}\int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \partial_v g(v,y) f(y)\,\mathrm{d}y \right|^k \mathrm{d}v &\leq \int_{\mathbf{R}^d} \|\partial_v g(v,\cdot)\|^k_{L^k(\mathbf{R}^d)} \|f\|^k_{L^{k'}(\mathbf{R}^d)}\,\mathrm{d}v \\ &= \|\partial_v g\|^k_{L^k(\mathbf{R}^d\times\mathbf{R}^d)} \|f\|^k_{L^{k'}(\mathbf{R}^d)}.\end{aligned}$$

Note that this inequality and the linearity in f justify that $\mu \in (\Pi(\mathbf{R}^d), d_{k'}) \mapsto \int_{\mathbf{R}^d} \partial_v g(\cdot,y)\,\mathrm{d}\mu(y) \in L^k(\mathbf{R}^d)$ is continuous with

$$\left\| \partial_v \frac{\delta u}{\delta m}(\mu) \right\|_{L^k(\mathbf{R}^d)} \leq \left\| \frac{\mathrm{d}\mu}{\mathrm{d}x} \right\|_{L^{k'}(\mathbf{R}^d)} \|\nabla g\|_{L^k(\mathbf{R}^d\times\mathbf{R}^d)}. \tag{A.3}$$

Following the same lines, we show that the distributional derivative of order 2 of $\frac{\delta u}{\delta m}(\mu)$, for $\mu \in \Pi(\mathbf{R}^d)$, is given by the $\mathbf{R}^{d\times d}$-valued function

$$v \mapsto \int_{\mathbf{R}^d} \partial_v^2 g(v,y)\,\mathrm{d}\mu(y) + \int_{\mathbf{R}^d} \partial_y^2 g(y,v)\,\mathrm{d}\mu(y).$$

It is also a continuous function from $(\Pi(\mathbf{R}^d), d_{k'})$ into $L^k(\mathbf{R}^d)$. Indeed, as previously, we obtain:

$$\left\| \partial_v^2 \frac{\delta u}{\delta m}(\mu) \right\|_{L^k(\mathbf{R}^d)} \leq \left\| \frac{\mathrm{d}\mu}{\mathrm{d}x} \right\|_{L^{k'}(\mathbf{R}^d)} \|\nabla^2 g\|_{L^k(\mathbf{R}^d\times\mathbf{R}^d)}. \tag{A.4}$$

Growth property: Using the inequalities (A.3) and (A.4) of the previous step, one has for all $\mu \in \Pi(\mathbf{R}^d)$

$$\left\| \partial_v \frac{\delta u}{\delta m}(\mu) \right\|_{L^k(\mathbf{R}^d)} + \left\| \partial_v^2 \frac{\delta u}{\delta m}(\mu) \right\|_{L^k(\mathbf{R}^d)} \leq \left\| \frac{\mathrm{d}\mu}{\mathrm{d}x} \right\|_{L^{k'}(\mathbf{R}^d)} \left[\|\nabla g\|_{L^k(\mathbf{R}^d\times\mathbf{R}^d)} + \|\nabla^2 g\|_{L^k(\mathbf{R}^d\times\mathbf{R}^d)} \right].$$

The second point in Definition 3.1 is thus satisfied with $\alpha = 1$ because we have supposed that $k \geq 2d$.

In the general case $N \geq 2$, one can show following the same lines that u admits a linear derivative and that for all $\mu \in \Pi(\mathbf{R}^d)$, its distributional derivative is given for all $v \in \mathbf{R}^d$ by

$$\partial_v \frac{\delta u}{\delta m}(\mu)(v) = \sum_{j=1}^N \int_{(\mathbf{R}^d)^{N-1}} \partial_{x_j} g(x_1,\ldots,x_{j-1},v,x_{j+1},\ldots,x_N)\,\mathrm{d}\mu(x_1)\ldots\,\mathrm{d}\mu(x_{j-1})\,\mathrm{d}\mu(x_{j+1})\ldots\,\mathrm{d}\mu(x_N).$$

Denoting by f the density of μ and using Hölder's inequality, we obtain as previously that for all $j \in \{1,\ldots,N\}$

$$\begin{aligned}&\int_{\mathbf{R}^d} \left| \int_{(\mathbf{R}^d)^{N-1}} \partial_{x_j} g(x_1,\ldots,x_{j-1},v,x_{j+1},\ldots,x_N)\,\mathrm{d}\mu(x_1)\ldots\,\mathrm{d}\mu(x_{j-1})\,\mathrm{d}\mu(x_{j+1})\ldots\,\mathrm{d}\mu(x_N) \right|^k \mathrm{d}v \\ &= \|\partial_{x_j} g\|^k_{L^k((\mathbf{R}^d)^N)} \|f\|^{(N-1)k}_{L^{k'}(\mathbf{R}^d)}.\end{aligned}$$

We easily show that $\mu \in (\Pi(\mathbf{R}^d), d_{k'}) \mapsto \partial_v \frac{\delta u}{\delta m}(\mu) \in L^k(\mathbf{R}^d)$ is continuous and the same properties hold for the distributional derivative of order two. We deduce that $\mu \in \Pi(\mathbf{R}^d)$

$$\left\|\partial_v \frac{\delta u}{\delta m}(\mu)\right\|_{L^k(\mathbf{R}^d)} + \left\|\partial_v^2 \frac{\delta u}{\delta m}(\mu)\right\|_{L^k(\mathbf{R}^d)} \leq \left\|\frac{\mathrm{d}\mu}{\mathrm{d}x}\right\|_{L^{k'}(\mathbf{R}^d)}^{N-1} \left[\|\nabla g\|_{L^k((\mathbf{R}^d)^N)} + \|\nabla^2 g\|_{L^k((\mathbf{R}^d)^N)}\right].$$

The second point in Definition 3.1 is thus satisfied with $\alpha = N-1$ because we have supposed that $k \geq Nd$.

□

A.3 On Example 3.13

Note that $f * \mu$ and $u(\mu)$ are well-defined for $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Indeed, it follows from the Sobolev embedding theorem (see Cor. 9.14 in [24]) that $f \in \mathcal{C}^1(\mathbf{R}^d, \mathbf{R})$ and $\partial_x f \in (L^\infty(\mathbf{R}^d))^d$. Thus f is at most of linear growth. Since f is continuous and at most of linear growth, it is easy to see that u has a linear derivative given by

$$\forall \mu \in \mathcal{P}_2(\mathbf{R}^d),\, \forall v \in \mathbf{R}^d,\, \frac{\delta u}{\delta m}(\mu)(v) = f * \mu(v) + \tilde{f} * \mu(v),$$

where $\tilde{f}(x) = f(-x)$ (see Ex. 2 page 386 in Chap. 5 of [3]). An easy computation based on Fubini's theorem shows that the distributional derivatives of order 1 and 2 of $\frac{\delta u}{\delta m}(\mu)$ are given by

$$\forall i, j \in \{1, \ldots, d\},\, \begin{cases} \partial_{v_i} \frac{\delta u}{\delta m}(\mu) & = \partial_{v_i} f * \mu + \partial_{v_i} \tilde{f} * \mu \\ \partial_{v_i\, v_j} \frac{\delta u}{\delta m}(\mu) & = \partial_{v_i\, v_j} f * \mu + \partial_{v_i\, v_j} \tilde{f} * \mu, \end{cases}$$

as elements of $L^{k+1}(\mathbf{R}^d)$. These functions are continuous with respect to $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ owing to Lemma 4.8. It remains to apply the first point in Remark 3.2 to conclude.

□

A.4 On Example 3.14

The function u is well-defined and continuous because $\nabla g \in L^\infty(\mathbf{R}^d)$ and is continuous thanks to the Sobolev embedding theorem. Thus g is at most of linear growth. It follows from the continuity of $\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \int_{\mathbf{R}^d} g \,\mathrm{d}\mu$ that the function u admits a linear derivative given by

$$\forall (\mu, v) \in \mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d,\, \frac{\delta u}{\delta m}(\mu)(v) = g(v) F'\left(\int_{\mathbf{R}^d} g \,\mathrm{d}\mu\right).$$

We thus have in the sense of distributions

$$\forall \mu \in \mathcal{P}_2(\mathbf{R}^d),\, \forall v \in \mathbf{R}^d,\, \partial_v \frac{\delta u}{\delta m}(\mu)(v) = \nabla g(v) F'\left(\int_{\mathbf{R}^d} g \,\mathrm{d}\mu\right).$$

Moreover, the function

$$\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \partial_v \frac{\delta u}{\delta m}(\mu)(\cdot) \in L^k(\mathbf{R}^d)$$

is continuous because $F \in \mathcal{C}^1(\mathbf{R}; \mathbf{R})$ and $\nabla g \in L^k(\mathbf{R}^d)$. The same reasoning proves that

$$\forall \mu \in \mathcal{P}_2(\mathbf{R}^d),\, \forall v \in \mathbf{R}^d,\, \partial_v^2 \frac{\delta u}{\delta m}(\mu)(v) = \nabla^2 g(v) F'\left(\int_{\mathbf{R}^d} g \,\mathrm{d}\mu\right),$$

and that the function

$$\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \partial_v^2 \frac{\delta u}{\delta m}(\mu)(\cdot) \in L^k(\mathbf{R}^d)$$

is continuous. We conclude that $u \in \mathcal{W}^2(\mathcal{P}_2(\mathbf{R}^d))$ with Remark 3.2.

□

A.5 On Example 3.15

The function u is well-defined and continuous. Indeed, the Sobolev embedding theorem implies that $\nabla g \in L^\infty(\mathbf{R}^{2d})$ and is continuous. Hence g is at most of linear growth. Following the same method as in the proof of Example 3.11, we obtain that

$$\forall \mu \in \Pi(\mathbf{R}^d),\ \partial_x u(\cdot,\mu) = \int_{\mathbf{R}^d} \partial_x g(\cdot,y)\,\mathrm{d}\mu(y).$$

Moreover

$$\forall \mu \in \Pi(\mathbf{R}^d),\ \|\partial_x u(\cdot,\mu)\|_{L^k(\mathbf{R}^d)} \le \|\nabla g\|_{L^k(\mathbf{R}^{2d})} \left\|\frac{\mathrm{d}\mu}{\mathrm{d}x}\right\|_{L^{k'}(\mathbf{R}^d)}.$$

This yields the continuity of the function

$$\mu \in (\Pi(\mathbf{R}^d), d_{k'}) \mapsto \partial_x u(\cdot,\mu) \in L^k(\mathbf{R}^d).$$

Moreover, keeping the notations of Definition 3.5, Assumption (2) is satisfied and setting $\alpha_1 = 1$, Assumption (5) is satisfied because we have supposed $k \ge 5d$. The same holds true for $\partial_x^2 u$. Since g is continuous and at most of linear growth, the linear derivative of u satisfies Assumption (3) in Definition 3.5 and is given, for all $x, v \in \mathbf{R}^d$ and for all $\mu \in \mathscr{P}_2(\mathbf{R}^d)$, by

$$\frac{\delta u}{\delta m}(x,\mu)(v) = g(x,v).$$

As $\nabla g \in (W^{1,k}(\mathbf{R}^{2d}))^d$, Assumption (4) in Definition 3.5 is satisfied, as well as the growth property in Assumption (5) with $\alpha_2 = 0$.

□

A.6 On Example 3.16

As in A.5, the function u is well-defined and continuous because $\nabla g \in L^\infty(\mathbf{R}^d)$ and is continuous. Thus g is at most of linear growth. It is clear with the assumption on ∇F that for all $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, $u(\cdot,\mu) \in W^{2,k_1}_{\mathrm{loc}}(\mathbf{R}^d)$. It follows from the continuity of $\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \int_{\mathbf{R}^d} g\,\mathrm{d}\mu$ that the function

$$\mu \in \mathcal{P}_2(\mathbf{R}^d) \mapsto \partial_x u(\cdot,\mu) = \partial_x F\left(\cdot, \int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right) \in (W^{1,k_1}(B_R))^d,$$

is also continuous for all $R > 0$. Moreover, it is easy to show with Remark 2.3 that for all $x \in \mathbf{R}^d$, $u(x,\cdot)$ admits a linear derivative given by

$$\forall (\mu,v) \in \mathcal{P}_2(\mathbf{R}^d) \times \mathbf{R}^d,\ \frac{\delta u}{\delta m}(x,\mu)(v) = g(v)\partial_y F\left(x, \int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right).$$

Assumption (3) in Definition 3.5 is clearly satisfied because $\partial_y F$ is continuous. Next, we compute the derivatives of $\frac{\delta u}{\delta m}(\cdot,\mu)(\cdot)$ with respect to v in the sense of distributions. For $\phi \in \mathcal{C}_c^\infty(\mathbf{R}^{2d})$ and $\mu \in \mathcal{P}_2(\mathbf{R}^d)$, Fubini's theorem ensures that

$$\begin{aligned}\int_{\mathbf{R}^{2d}} g(v)\partial_y F\left(x, \int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right)\partial_v\phi(x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} g(v)\partial_v\phi(x,v)\,\mathrm{d}v\right)\partial_y F\left(x,\int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right)\mathrm{d}x\\ &= -\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d}\nabla g(v)\phi(x,v)\,\mathrm{d}v\right)\partial_y F\left(x,\int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right)\mathrm{d}x\\ &= -\int_{\mathbf{R}^{2d}}\left(\nabla g(v)\partial_y F\left(x,\int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right)\right)\phi(x,v)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

This proves exactly that

$$\forall \mu\in\mathcal{P}_2(\mathbf{R}^d),\ \forall x,v\in\mathbf{R}^d,\ \partial_v\frac{\delta u}{\delta m}(x,\mu)(v) = \nabla g(v)\partial_y F\left(x,\int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right).$$

Since $\nabla g \in L^{k_2}(\mathbf{R}^d)$ and $\partial_y F\left(\cdot, \int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right) \in L^\infty(B_R)$, for all $R>0$ and $\mu\in\mathcal{P}_2(\mathbf{R}^d)$, the function

$$(x,v)\in B_R\times\mathbf{R}^d \mapsto \partial_v\frac{\delta u}{\delta m}(x,\mu)(v)$$

belongs to $L^{k_2}(B_R\times\mathbf{R}^d)$. Moreover, the function

$$\mu\in\mathcal{P}_2(\mathbf{R}^d)\mapsto \partial_v\frac{\delta u}{\delta m}(\cdot,\mu)(\cdot)\in L^{k_2}(B_R\times\mathbf{R}^d)$$

is continuous because $F\in\mathcal{C}^1(\mathbf{R}^d\times\mathbf{R};\mathbf{R})$ and thus $y\mapsto\partial_y F(\cdot,y)\in L^\infty(B_R)$ is continuous. The same reasoning proves that

$$\forall \mu\in\mathcal{P}_2(\mathbf{R}^d),\ \forall x,v\in\mathbf{R}^d,\ \partial_v^2\frac{\delta u}{\delta m}(x,\mu)(v) = \nabla^2 g(v)\partial_y F\left(x,\int_{\mathbf{R}^d} g\,\mathrm{d}\mu\right),$$

and that the function

$$\mu\in\mathcal{P}_2(\mathbf{R}^d)\mapsto \partial_v^2\frac{\delta u}{\delta m}(\cdot,\mu)(\cdot)\in L^{k_2}(B_R\times\mathbf{R}^d)$$

is continuous for all $R>0$. We conclude that $u\in\mathcal{W}^{1,2,2}([0,T]\times\mathbf{R}^d\times\mathcal{P}_2(\mathbf{R}^d))$ with Remark 3.6.

□

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