ESAIM: PS 28 (2024) 1–21
https://doi.org/10.1051/ps/2023019

ESAIM: Probability and Statistics
www.esaim-ps.org

WELLPOSEDNESS OF SECOND ORDER REFLECTED BSDES: A NEW FORMULATION

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Abstract. The aim of this article is to provide a complete theory of second order reflected backward stochastic differential equation (2RBSDE). We reformulate the notion of 2RBSDE by introducing a new type of minimality condition. Unlike the previous works, our minimality condition is more reasonable in the sense that it is suitable for dynamics with any kind of generators and it allows us to consider the 2RBSDE as a natural extension of 2BSDE. We prove the existence and uniqueness of solution to the 2RBSDE under Lipschitz-type assumptions on the generator. Moreover, we apply the 2RBSDE to obtain the super-hedging price of American options in the uncertain, incomplete, nonlinear financial market.

Mathematics Subject Classification. 60H10, 60H30.

Received April 19, 2021. Accepted November 6, 2023.

1. Introduction

Since their first introduction by Bismut [1] in the linear case and the nonlinear extension by Pardoux and Peng [2], backward stochastic differential equations (BSDEs) have been developed rapidly with various types of generalizations in the last decades. In 1997, El Karoui *et al.* [3] introduced a notion of reflected backward stochastic differential equation (RBSDE), which is a kind of BSDE with constraints. More precisely, the solution Y of the BSDE is constrained to stay above a given obstacle process L. In order to achieve this, a non-decreasing process K is added to the solution

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s + K_T - K_t,$$

$$Y_t \geq L_t, \int_0^T (Y_t - L_t)\mathrm{d}K_t = 0,$$

where the second condition is called the Skorohod-type minimality condition. It means that the process K only increases when Y reaches the obstacle L. Afterwards, Hamadéne [4] and Lepeltier, Matoussi and Xu [5] studied

Keywords and phrases: Second order backward stochastic differential equation, reflected backward stochastic differential equation.

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the RBSDEs in the case where the obstacle process is no longer continuous and is assumed only càdlàg. In that case, the minimality condition is replaced by

$$Y_t \geq L_t, \ \int_0^T (Y_{t-} - L_{t-}) \mathrm{d}K_t = 0. \tag{1.1}$$

With the Skorohod-type minimality condition, the investigation of RBSDEs is rich enough.

In the very recent article [6], the authors of the present paper introduced the following non-Skorohod-type minimality condition:

$$K_t = \operatorname*{ess\,inf}_{\tau \in \mathbb{T}^{t,T}} \mathbb{E}[Y_\tau - \widetilde{L}_\tau + K_\tau | \mathcal{F}_t], \tag{1.2}$$

where $\widetilde{L}_\tau := L_\tau \mathbf{1}_{\tau < T} + \xi \mathbf{1}_{\tau = T}$ and $\mathbb{T}^{t,T}$ is the set of all stopping times taking values in $[t, T]$. Then, we proved the existence and uniqueness of solutions to RBSDEs by using purely stochastic control method. It is also proved in [6] that two minimality conditions (1.1) and (1.2) are in fact equivalent in the framework of classical linear expectations.

On the other hand, the quasi-sure stochastic analysis of Denis and Martini [7] pushed the BSDE theory as well as the RBSDE theory to their very limits–fully nonlinear extensions. Hence, Soner, Touzi and Zhang [8] proposed a rigorous formulation of second order BSDE (2BSDE), which is defined under a family of possibly non-dominated probability measures. Then, Possamaï, Tan and Zhou [9] developed the 2BSDE theory without any regularity assumptions on the generator and terminal condition. As a fully nonlinear extension of RBSDE, Matoussi, Possamaï and Zhou [10, 11] introduced a notion of second order RBSDE (2RBSDE). Then, following [9], the PhD thesis of Noubiagain [12] was contributed to the development of the 2RBSDE theory in an irregular setting.

We also would like to mention that the G-expectation theory introduced by Peng [13] is also closely connected to the fully nonlinear PDE. Hence, Hu, Ji, Peng and Song [14] introduced a notion of G-BSDE, which is defined under G-framework. Afterwards, Li, Peng and Soumana Hima [15] introduced a notion of the G-RBSDE (see also the PhD thesis of Soumana Hima [16] with slightly different definitions). The G-expectation theory is heavily based on PDE arguments, which is unlikely to be compatible with a theory without any regularity, since the PDEs need at the very least to have a continuous solution. On the other hand, a general construction of conditional G-expectation of a Borel-measurable random variable, suggested by Nutz and van Handel [17] involves a more hope of the probabilistic approach of the 2BSDEs and 2RBSDEs. However, their arguments correspond to the case of null-generators, and the extension to the nonlinear generators is still in development.

In their definition of 2BSDE in [8], the authors imposed the minimality condition for the non-decreasing component in order to guarantee the uniqueness of solution. Hence, the minimality condition plays a crucial role to define various types of dynamics under uncertainty. In [11], the authors proposed two different types of minimality conditions for the 2RBSDEs. The first one is obtained by modifying the original minimality condition of 2BSDE taking into account the non-decreasing component of solution to the standard RBSDE. Another minimality condition was motivated by definition of G-RBSDEs in [15, 16], where the authors introduced the Skorohod-type minimality condition under G-expectation. Hence, it is a generalization of the classical Skorohod-type minimality condition to the uncertainty setting.

The main disadvantage of the first minimality condition is that it is heavily based on the linearization arguement and thus is only suitable for the 2RBSDEs with Lipschitz or monotonic generators. The main drawback of the second minimality condition is that it is somewhat strong in the sense that it does not allows us to consider the 2RBSDE as a natural extension of the 2BSDE, whereas the RBSDE is a natural extension of the BSDE.

The aim of this paper is to provide a complete theory of 2RBSDE. We introduce a new type of minimality condition for the 2RBSDE, by extending the non-Skorohod-type minimality condition (1.2) for the standard RBSDE to the uncertainty setting. Our minimality condition is nice in the sense that it is suitable for dynamics with arbitrary generators and it allows us to consider the 2RBSDE as a natural extension of the 2BSDE (see Sect. 3 for details). With a new type of minimality condition, we recover the existence and uniqueness result for 2RBSDE under the Lipschitz-type assumptions on generator. In particular, we work without any regularity assumptions on generator and terminal condition like in [9]. This is the main purpose of Section 4. Finally, in Section 5, we apply the 2RBSDE to obtain a duality result for the robust pricing of American contingent claims in the uncertain, nonlinear and incomplete financial markets. Some concluding remarks are also given at the end of the paper.

2. Preliminaries

2.1. Canonical space

Let us fix $T>0$ and $d\in\mathbb{N}^*$. Let $\Omega:=\{\omega\in C([0,T],\mathbb{R}^d):\omega_0=0\}$ be the canonical space equipped with the uniform convergence norm $\|\omega\|_\infty:=\sup_{0\le t\le T}\|\omega\|_t$, X the canonical process, *i.e.* $X_t(\omega)=\omega_t$ for all $\omega\in\Omega$ and $\mathbb{P}_0$ the Wiener measure, $\mathbb{F}:=(\mathcal{F}_t)_{0\le t\le T}$ the natural filtration generated by X, and $\mathbb{F}_+:=(\mathcal{F}_{t+})_{0\le t\le T}$ the right limit of $\mathbb{F}$. Let $\mathbb{M}_1$ be the collection of all probability measures on $(\Omega,\mathcal{F}_T)$. Notice that $\mathbb{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathcal{B}$ its Borel σ-field. For any $\mathbb{P}\in\mathbb{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}_t$ under $\mathbb{P}$, $\mathbb{F}^{\mathbb{P}}=(\mathcal{F}_t^{\mathbb{P}})_{0\le t\le T}$ the completed filtration and $\mathbb{F}_+^{\mathbb{P}}$ its right limit. The filtration $\mathbb{F}_+^{\mathbb{P}}$ is the coarsest filtration satsfying the usual conditions. For $\mathcal{P}\subset\mathbb{M}_1$, we define

$$\mathbb{F}^U:=(\mathcal{F}_t^U)_{0\le t\le T},\ \mathbb{F}^{\mathcal{P}}:=(\mathcal{F}_t^{\mathcal{P}})_{0\le t\le T}\text{ and }\mathbb{F}_+^{\mathcal{P}}:=(\mathcal{F}_{t+}^{\mathcal{P}})_{0\le t\le T}\text{ such that}$$

$$\mathcal{F}_t^U:=\bigcap_{\mathbb{P}\in\mathbb{M}_1}\mathcal{F}_t^{\mathbb{P}},\ \mathcal{F}_t^{\mathcal{P}}:=\bigcap_{\mathbb{P}\in\mathcal{P}}\mathcal{F}_t^{\mathbb{P}},\ \text{and }\mathcal{F}_{t+}^{\mathcal{P}}:=\bigcap_{s>t}\mathcal{F}_t^{\mathcal{P}},\ t\in[0,T),\ \text{and }\mathcal{F}_{T+}^{\mathcal{P}}:=\mathcal{F}_T^{\mathcal{P}}.$$

2.2. The semi-martingale measures

We call a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$ a semi-martingale measure if X is a semimartingale under $\mathbb{P}$. As in Karandikar [18], we can define a pathwise version of $d\times d$-matrix valued process $\langle X\rangle$, which coincides with the quadratic variation of X under each semi-martingale measure $\mathbb{P}$. We then introduce the density process

$$\widehat{a}_t:=\limsup_{\varepsilon\downarrow0}\frac{1}{\varepsilon}(\langle X\rangle_t-\langle X\rangle_{t-\varepsilon}).$$

For every $t\in[0,T]$, let $\mathcal{P}_t^W$ denote the collection of all probability measures $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$ such that

- $(X_s)_{s\in[t,T]}$ is a $(\mathbb{P},\mathbb{F})$-semi-martingale admitting the canonical decomposition

$$X_s=X_t+\int_t^s b_r^{\mathbb{P}}+X_s^{c,\mathbb{P}},\ s\in[t,T],\ \mathbb{P}-a.s.,$$

  where $b^{\mathbb{P}}$ is a $\mathbb{F}^{\mathbb{P}}$-predictable $\mathbb{R}^d$-valued process, and $X^{c,\mathbb{P}}$ is the canonical local martingale part of X under $\mathbb{P}$.
- $(\langle X\rangle)_{s\in[t,T]}$ is absolutely continuous in s with respect to Lebesgue measure, and $\widehat{a}$ takes vlaues in $\mathbb{S}_d^{\ge0}$, $\mathbb{P}-a.s.$ Here $\mathbb{S}_d^{\ge0}$ is the set of all symmetric positive semi-definite $d\times d$-matrices.

Remark 2.1. If $\widehat{a}_s$ is non-degenerate for all $s\in[0,T]$, then we can construct a Brownian motion $W^{\mathbb{P}}$ on Ω by

$$W_t^{\mathbb{P}}:=\int_0^t\widehat{a}_s^{-1/2}\mathrm{d}X_s^{c,\mathbb{P}},\ t\in[0,T],\ \mathbb{P}-a.s. \tag{2.1}$$

Otherwise we define the enlarged canonical space $\overline{\Omega} := \Omega \times \Omega'$, where Ω' is identical to Ω and set (X, B) its canonical process, *i.e.*, $X_t(\overline{\omega}) := \omega_t$, $B_t(\overline{\omega}) := \omega'_t$ for all $\overline{\omega} := (\omega, \omega') \in \overline{\Omega}$. The extension from Ω to $\overline{\Omega}$ of a random variable or a process λ is defined by

$$\lambda(\overline{\omega}) := \lambda(\omega),\ \forall \overline{\omega} = (\omega, \omega') \in \overline{\Omega}. \tag{2.2}$$

In particular, $\widehat{a}$ can be extended on $\overline{\Omega}$. For $\mathbb{P} \in \mathcal{P}_t^W$, a probability measure on the enlarged space $\overline{\Omega}$ is denoted by $\overline{\mathbb{P}}$ with $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$. We also consider like in [9], the canonical filtration $\overline{\mathbb{F}}$ generated by (X, B), the filtration $\overline{\mathbb{F}}^X$ generated by X, the right-continuous filtrations $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtrations $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$ given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$. With this construction, it follows that X in $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}}, \overline{\mathbb{F}})$ is a semi-martingale with the same triplet of characteristics as X in $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{F})$, B is a $\overline{\mathbb{F}}$-Brownian motion and X is independent of B. Then by Theorem 4.5.2 of Stroock and Varadhan [19], for every $\mathbb{P} \in \mathcal{P}_t^W$, there is some $\mathbb{R}^d$-valued $\overline{\mathbb{F}}$-Brownian motion $W^{\mathbb{P}}$ such that

$$X_s = \int_t^s b_r^{\mathbb{P}} + \int_t^s \widehat{a}_r^{1/2} \mathrm{d}W_r^{\mathbb{P}},\ s \in [t, T],\ \overline{\mathbb{P}} - a.s., \tag{2.3}$$

where the definition of $b^{\mathbb{P}}$ and $\widehat{a}$ are extended on $\overline{\Omega}$. Then, using the equivalence results in [12] (see Sect. A.2 therein), one can deal with the degenerate case with obvious changes. For this reason, we will always assume that $\widehat{a}$ is non-degenerate. This is only for the ease of presentation.

2.3. Regular conditional probability and concatenation of probability measures

We recall that for every probability measure $\mathbb{P}$ on Ω and $\mathbb{F}$-stopping time τ taking values in $[0, T]$, there exists a regular conditional probability distribution (*r.c.p.d.* for short) $(\mathbb{P}_\omega^\tau)_{\omega \in \Omega}$, (see, *e.g.*, Stroock and Varadhan [19]), satisfying

- For every $\omega \in \Omega$, $\mathbb{P}_\omega^\tau$ is a probability measure on $(\Omega, \mathcal{F})$.
- For every $E \in \mathcal{F}$, the mapping $\omega \mapsto \mathbb{P}_\omega^\tau(E)$ is $\mathcal{F}_\tau$-measurable.
- The family $(\mathbb{P}_\omega^\tau)_{\omega \in \Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, that is, for every integrable $\mathcal{F}_T$-measurable random variable ξ, we have $\mathbb{E}^{\mathbb{P}}[\xi | \mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}_\omega^\tau}[\xi]$, $\mathbb{P} - a.s.$
- For every $\omega \in \Omega$, $\mathbb{P}_\omega^\tau(\Omega_\tau^\omega) = 1$, where $\Omega_\tau^\omega := \{\bar{\omega} \in \Omega : \bar{\omega}(s) = \omega(s),\ o \leq s \leq \tau(\omega)\}$.

Given some $\mathbb{P}$ and a family $(\mathbb{Q}_\omega)_{\omega \in \Omega}$ such that $\omega \mapsto \mathbb{Q}_\omega$ is $\mathcal{F}_\tau$-measurable and $\mathbb{Q}_\omega(\Omega_\tau^\omega) = 1$ for all $\omega \in \Omega$, we define a concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}$ by

$$\mathbb{P} \otimes_\tau \mathbb{Q}[A] := \int_\Omega \mathbb{Q}_\omega[A] \mathbb{P}(\mathrm{d}\omega),\ A \in \mathcal{F}.$$

2.4. Spaces and norms

We are given a fixed family $(\mathcal{P}(t, \omega))_{(t,\omega) \in [0,T] \times \Omega}$ of sets of probability measures on $(\Omega, \mathcal{F}_T)$, where $\mathcal{P}(t, \omega) \subset \mathcal{P}_t^W$. Fix some $(t, \omega) \in [0, T] \times \Omega$. In what follows, $\mathbb{X} := (\mathcal{X}_s)_{t \leq s \leq T}$ will denote an arbitrary filtration on Ω, $\mathcal{X}$ an arbitrary σ-algebra on Ω, and $\mathbb{P}$ an arbitrary element in $\mathcal{P}(t, \omega)$. We denote by $\mathbb{X}^{\mathbb{P}}$ the $\mathbb{P}$-augmented filtration associated with $\mathbb{X}$. We also fix a constant $p \geq 1$.

- $\mathbb{L}_{t,\omega}^p(\mathcal{X})$ (resp. $\mathbb{L}_{t,\omega}^p(\mathcal{X}, \mathbb{P})$) denotes the space of all $\mathcal{X}$-measurable scalar random variable ξ with

$$\|\xi\|_{\mathbb{L}_{t,\omega}^p}^p := \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < \infty,\ \left(\text{resp. } \|\xi\|_{\mathbb{L}_{t,\omega}^p(\mathbb{P})}^p := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < \infty\right).$$

- $\mathbb{H}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{H}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-predictable $\mathbb{R}^d$-valued processes Z such that

$$\|Z\|^p_{\mathbb{H}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\left(\widehat{a}_s^{1/2}\right)^{\top} Z_s\right\|^2 \mathrm{d}s\right)^{p/2}\right] < +\infty,$$

$$\left(\text{resp. } \|Z\|^p_{\mathbb{H}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\left(\widehat{a}_s^{1/2}\right)^{\top} Z_s\right\|^2 \mathrm{d}s\right)^{p/2}\right] < +\infty\right).$$

- $\mathbb{M}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ denotes the space of all $(\mathbb{X},\mathbb{P})$-optional martingales M with $\mathbb{P}-a.s.$ càdlàg paths on $[t,T]$, with $M_t=0,\ \mathbb{P}-a.s.$ and

$$\|M\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[[M]_T^{p/2}\right] < +\infty.$$

- We say that a family $(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{M}^p_{t,\omega}(\{\mathbb{X}^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}(t,\omega)})$, if for any $\mathbb{P}\in\mathcal{P}(t,\omega)$, $M^{\mathbb{P}}\in\mathbb{M}^p_{t,\omega}(\mathbb{X}^{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|M^{\mathbb{P}}\right\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

- $\mathbb{I}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ (resp. $\mathbb{I}^{o,p}_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-predictable (resp. $\mathbb{X}$-optional) processes K with $\mathbb{P}-a.s.$ càdlàg and non-decreasing paths on $[t,T]$ with $K_t=0,\ \mathbb{P}-a.s.$, and

$$\|K\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K_T^{\mathbb{P}}\right] < +\infty \text{ (resp. } \|K\|^p_{\mathbb{I}^{o,p}_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K_T^{\mathbb{P}}\right] < +\infty).$$

- We say that a family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{I}^p_{t,\omega}((\mathbb{X}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ (resp. $\mathbb{I}^{o,p}_{t,\omega}((\mathbb{X}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$) if for any $\mathbb{P}\in\mathcal{P}(t,\omega)$, $K^{\mathbb{P}}\in\mathbb{I}^p_{t,\omega}(\mathbb{X}^{\mathbb{P}},\mathbb{P})$ (resp. $K^{\mathbb{P}}\in\mathbb{I}^{o,p}_{t,\omega}(\mathbb{X}^{\mathbb{P}},\mathbb{P})$) and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} < +\infty, \ \left(\text{resp. } \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|^p_{\mathbb{I}^{o,p}_{t,\omega}(\mathbb{P})} < +\infty\right).$$

- $\mathbb{D}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{D}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-progressively measurable real-valued processes Y with $\mathcal{P}(t,\omega)-q.s.$ (resp. $\mathbb{P}-a.s.$) càdlàg paths on $[t,T]$, with

$$\|Y\|^p_{\mathbb{D}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq s\leq T}|Y_s|^p\right] < +\infty,$$

$$\left(\text{resp. } \|Y\|^p_{\mathbb{D}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq s\leq T}|Y_s|^p\right] < +\infty\right).$$

- When $t=0$, we simplify the notations by suppressing the ω-dependence and write $\mathbb{L}^p_0(\mathbb{X}), \mathbb{L}^p_0(\mathbb{X},\mathbb{P}), \mathbb{H}^p_0(\mathbb{X})$, $\mathbb{H}^p_0(\mathbb{X},\mathbb{P})$ and so on.
- $\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{X})$ is the subset of probability measures defined by

$$\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{X}) := \left\{\mathbb{P}'\in\mathcal{P}(t,\omega),\ \mathbb{P}'=\mathbb{P} \text{ on } \mathcal{X}_s\right\}.$$

When $t = 0$, we simply denote $\mathcal{P}_{0,\omega}(s, \mathbb{P}, \mathbb{X})$ by $\mathcal{P}_0(s, \mathbb{P}, \mathbb{X})$.

- For $p > \kappa > 1$, $(t, \omega) \in [0, T] \times \Omega$,

$$\mathbb{L}^{p,\kappa}_{t,\omega}(\mathcal{F}_T) := \{\xi \in \mathbb{L}^p_{t,\omega}(\mathcal{F}_T) : \|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,\omega}} < \infty\},$$

where

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\underset{t\le s\le T}{\overset{\mathbb{P}}{\operatorname{ess\,sup}}} \left(\underset{\mathbb{P}'\in\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,sup}}} \mathbb{E}^{\mathbb{P}'}\left[|\xi|^\kappa|\mathcal{F}_{s+}\right]\right)^{p/\kappa}\right].$$

2.5. Basic assumptions

We consider a random variable $\xi : \Omega \to \mathbb{R}$ and a generator function

$$f : (t, \omega, y, z, a, b) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^{\geq 0}_d \times \mathbb{R}^d \to \mathbb{R}.$$

For simplicity of notation, we denote

$$\widehat{f}^{\mathbb{P}}_s(y, z) := f(s, X_{\cdot\wedge s}, y, z, \widehat{a}_s, b^{\mathbb{P}}_s) \text{ and } \widehat{f}^{\mathbb{P},0}_s := f(s, X_{\cdot\wedge s}, 0, 0, \widehat{a}_s, b^{\mathbb{P}}_s).$$

Recall that we are given a family $(\mathcal{P}(t, \omega))_{(t,\omega)\in[0,T]\times\Omega}$. Following [9], we make the following assumption.

Assumption 2.2. (1) The random variable ξ is $\mathcal{F}_T$-measurable, the generator function f is jointly Borel measurable and for every fixed (y, z, a, b), the map $(t, \omega) \mapsto f(t, \omega, y, z, a, b)$ is $\mathbb{F}$-progressively measurable. Moreover, f is Lipschitz in y and z, that is, there exists a constant $L_f > 0$ such that for every $(t, \omega, y, y', z, z', a, b) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{\geq 0}_d \times \mathbb{R}^d$,

$$|f(t, \omega, y, z, a, b) - f(t, \omega, y', z', a, b)| \leq L_f(|y - y'| + |z - z'|).$$

(2) For the fixed constant $p > 1$, one has for every $(t, \omega) \in [0, T] \times \Omega$,

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p + \int_t^T |\widehat{f}^{\mathbb{P},0}_s|^p \mathrm{d}s + \sup_{t\le s\le T} |L_s|^p\right] < +\infty.$$

(3) For every $(t, \omega) \in [0, T] \times \Omega$, one has $\mathcal{P}(t, \omega) = \mathcal{P}(t, \omega_{\cdot\wedge t})$ and $\mathbb{P}(\Omega^\omega_t) = 1$ whenever $\mathbb{P} \in \mathcal{P}(t, \omega)$, where $\Omega^\omega_t := \omega' \in \Omega : \omega'(s) = \omega(s),\ 0 \leq s \leq t$. The graph $[[\mathcal{P}]]$ of $\mathcal{P}$ defined by $[[\mathcal{P}]] := \{(t, \omega, \mathbb{P}) : \mathbb{P} \in \mathcal{P}(t, \omega)\}$, is analytic in $[0, T] \times \Omega \times \mathbb{M}_1$.

(4) $\mathcal{P}$ is stable under conditioning, *i.e.* for every $(t, \omega) \in [0, T] \times \Omega$ and every $\mathbb{P} \in \mathcal{P}(t, \omega)$ together with an $\mathbb{F}$-stopping time τ taking values in $[t, T]$, there is a family of r.c.p.d. $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ such that $\mathbb{P}_{\mathsf{w}} \in \mathcal{P}(\tau(\mathsf{w}), \mathsf{w})$ for $\mathbb{P} - a.e.$ $\mathsf{w} \in \Omega$.

(5) $\mathcal{P}$ is stable under concatenation, *i.e.* for every $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$ together with and $\mathbb{F}$-stopping time τ taking values in $[t, T]$, let $(\mathbb{Q}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ be a family of probability measures such that $\mathbb{Q}_{\mathsf{w}} \in \mathcal{P}(\tau(\mathsf{w}), \mathsf{w})$ for all $\mathsf{w} \in \Omega$ and $\mathsf{w} \mapsto \mathbb{Q}_{\mathsf{w}}$ is $\mathcal{F}_\tau$-measurable, then the concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{P}(t, \omega)$.

Remark 2.3. Conditions (3)–(5) in Assumption 2.2 are essentially used to prove the dynamic programming principle for the stochastic control problem over a family of non-dominated probability measures. We also notice that for $t = 0$, we have $\mathcal{P}_0 := \mathcal{P}(0, \omega)$ for any $\omega \in \Omega$.

In particular, let us consider the case where the sets $\mathcal{P}(t, \omega)$ are induced by controlled diffusion processes. Let U be some (non-empty) Polish space, $\mathcal{U}$ denote the collection of all U-valued and $\mathbb{F}$-progressively measurable processes, $(\mu,\sigma):[0,T]\times\Omega\times U\to\mathbb{R}^d\times\mathbb{S}^d$ be the drift and volatility coefficient function which are assumed to be such that for some constant $C>0$, $\|(\mu,\sigma)(t,0,u)\|\leq C$ and

$$\|(\mu,\sigma)(t,\omega,u)-(\mu,\sigma)(t',\omega',u)\|\leq C\left(\sqrt{t-t'}+\|\omega_{t\wedge\cdot}-\omega'_{t'\wedge\cdot}\|\right).$$

We denote by $S^{t,\omega,\nu}$ the unique (strong) solution of the SDE

$$S_s^{t,\omega,\nu}=\omega_t+\int_t^s\mu(r,S^{t,\omega,\nu},\nu_r)\mathrm{d}r+\int_t^s\sigma(r,S^{t,\omega,\nu},\nu_r)\mathrm{d}X_r,\ \ s\in[t,T],\ \mathbb{P}_0-a.s.,$$

with initial condition $S_s^{t,\omega,\nu}=\omega_s$ for all $s\in[0,t]$ and $\nu\in\mathcal{U}$. Then the collection $\mathcal{P}^{\mathcal{U}}(t,\omega)$ of sets of measures defined by

$$\mathcal{P}^{\mathcal{U}}(t,\omega):=\{\mathbb{P}_0\circ(S^{t,\omega,\nu})^{-1},\ \nu\in\mathcal{U}\}$$

satisfies Assumption 2.2 (3)–(5) (see [20]).

Remark 2.4. Using the Lipschitz property of f in the first assumption, we can define bounded functions $\lambda:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d\to\mathbb{R}$ and $\eta:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d\to\mathbb{R}^d$ such that for any (t,ω,y,y',z,z',a,b),

$$f(t,\omega,y,z,a,b)-f(t,\omega,y',z',a,b)=\lambda_t(\omega,y,y',z,z',a,b)(y-y')+\eta_t(\omega,y,y',z,z',a,b)\cdot(z-z').$$

For simplicity of notation, we denote

$$\lambda_t^{\mathbb{P}}(y,y',z,z'):=\lambda_t(X_{\cdot\wedge t},y,y',z,z',\widehat{a}_t,b_t^{\mathbb{P}})\text{ and }\eta_t^{\mathbb{P}}(y,y',z,z'):=\eta_t(X_{\cdot\wedge t},y,y',z,z',\widehat{a}_t,b_t^{\mathbb{P}}).\tag{2.4}$$

3. The second order reflected BSDEs

3.1. A new formulation of 2RBSDEs

Our lower obstacle is represented by the process L. We will always assume that L is càdlàg, $L_T\leq\xi$ and $L^+\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$. We consider the following 2RBSDE with lower obstacle L:

$$Y_t=\xi+\int_t^T\widehat{f}_s^{\mathbb{P}}(Y_s,(\widehat{a}_s^{1/2})^{\intercal}Z_s)\mathrm{d}s-\int_t^TZ_s\cdot\mathrm{d}X_s^{c,\mathbb{P}}-\int_t^TdM_s^{\mathbb{P}}+K_T^{\mathbb{P}}-K_t^{\mathbb{P}},\quad 0\leq t\leq T,\ \mathcal{P}_0-q.s.\tag{3.1}$$

We recall that $\widetilde{L}_\tau:=L_\tau\mathbf{1}_{\tau<T}+\xi\mathbf{1}_{\tau=T}$ and $\mathbb{T}^{t,T}$ is the set of all stopping times taking values in $[t,T]$.

Definition 3.1. For $\xi\in\mathbb{L}_0^{p,\kappa}$, we say that $(Y,Z,(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0},(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\times\mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ is a solution to the 2RBSDE (3.1) if:

(i) $Y_T=\xi$, $\mathcal{P}_0-q.s.$
(ii) $\forall\mathbb{P}\in\mathcal{P}_0$, the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P}-a.s.$

$$K_t^{\mathbb{P}}:=Y_0-Y_t-\int_0^t\widehat{f}_s^{\mathbb{P}}(Y_s,(\widehat{a}_s^{1/2})^{\intercal}Z_s)\mathrm{d}s+\int_0^tZ_s\cdot\mathrm{d}X_s^{c,\mathbb{P}}+\int_0^t\mathrm{d}M_s^{\mathbb{P}},\ t\in[0,T].$$

(iii) We have the following minimality condition: $\forall\mathbb{P}\in\mathcal{P}_0$,

$$K_t^{\mathbb{P}}=\underset{\tau\in\mathbb{T}^{t,T}}{\operatorname*{ess\,inf}^{\mathbb{P}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname*{ess\,inf}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_\tau-\widetilde{L}_\tau+K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right],\ 0\leq t\leq T,\ \mathbb{P}-a.s.\tag{3.2}$$

Remark 3.2. By taking $\tau = t$ in (3.2), we obtain for any $\mathbb{P} \in \mathcal{P}_0$,

$$Y_t \geq L_t, \quad \forall t \in [0,T],\ \mathbb{P} - a.s.$$

Remark 3.3. If we assume that $b^{\mathbb{P}} = 0$, $\mathbb{P} - a.s.$ for any $\mathbb{P} \in \mathcal{P}_0$, then we have that $X^{c,\mathbb{P}} = X$, $\mathbb{P} - a.s.$ for any $\mathbb{P} \in \mathcal{P}_0$. Then using the general result given by Nutz [21], the family of semimartingales $(K^{\mathbb{P}} - M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ can always be aggregated into a universal semimartingale $K - M$ under the acceptance of Zermelo–Fraenkel set theory with axiom of choice together with the continuum hypothesis. Then by the uniqueness of decomposition of semimartingales, the processes $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ and $(\mathbb{M}^p)_{\mathbb{P}\in\mathcal{P}_0}$ can be aggregated into processes K and M.

3.2. Two alternative formulations of 2RBSDEs

We recall the previous definitions of 2RBSDE (see [11, 12]).

Definition 3.4. For $\xi \in \mathbb{L}_0^{p,\kappa}$, we say that $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ is a solution to the 2RBSDE (3.1) if:

(i) $Y_T = \xi$ and $Y_t \geq L_t$, $t \in [0,T], \mathcal{P}_0 - q.s.$
(ii) $\forall \mathbb{P} \in \mathcal{P}_0$, the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P} - a.s.$

$$K_t^{\mathbb{P}} := Y_0 - Y_t - \int_0^t \widehat{f}_s^{\mathbb{P}}(Y_s, (\widehat{a}_s^{1/2})^{\mathsf{T}} Z_s)\mathrm{d}s + \int_0^t Z_s \cdot \mathrm{d}X_s^{c,\mathbb{P}} + \int_0^t \mathrm{d}M_s^{\mathbb{P}}, \quad 0 \leq t \leq T,\ \mathbb{P} - a.s.$$

(iii) We have the following minimality condition: $\forall \mathbb{P} \in \mathcal{P}_0$

$$\underset{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T G_s^{t,\mathbb{P}'} d(K_s^{\mathbb{P}'} - \mathsf{K}_s^{\mathbb{P}'}) \big| \mathcal{F}_{t+}^{\mathbb{P}'}\right] = 0,\ 0 \leq t \leq T,\ \mathbb{P} - a.s., \tag{3.3}$$

where for any $t \in [0,T]$ and for any $\mathbb{P} \in \mathcal{P}_0$, the process $G^{t,\mathbb{P}}$ is defined by

$$G_s^{t,\mathbb{P}} := \exp\left(\int_t^s (\lambda_u^{\mathbb{P}} - \frac{1}{2}\|\eta_u^{\mathbb{P}}\|^2)(Y_u, \mathsf{Y}_u^{\mathbb{P}}, Z_u, \mathsf{Z}_u^{\mathbb{P}})du + \int_t^s \eta_u^{\mathbb{P}}(Y_u, \mathsf{Y}_u^{\mathbb{P}}, Z_u, \mathsf{Z}_u^{\mathbb{P}}) \cdot \mathrm{d}W_u^{\mathbb{P}}\right),$$

where the processes $\lambda^{\mathbb{P}}$ and $\eta^{\mathbb{P}}$ are introduced in (2.4) and $(\mathsf{Y}^{\mathbb{P}}, \mathsf{Z}^{\mathbb{P}}, \mathsf{M}^{\mathbb{P}}, \mathsf{K}^{\mathbb{P}})$ denotes the solution to the following standard RBSDE with lower obstacle L (existence and uniqueness have been proved under Lipschitz assumptions by Bouchard *et al.* in [22]):

$$\begin{cases} \mathsf{Y}_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(\mathsf{Y}_s^{\mathbb{P}}, (\widehat{a}_s^{1/2})^{\mathsf{T}}\mathsf{Z}_s^{\mathbb{P}})\mathrm{d}s - \int_t^T \mathsf{Z}_s^{\mathbb{P}} dX_s^{c,\mathbb{P}} - \int_t^T \mathrm{d}\mathsf{M}_s^{\mathbb{P}} + \mathsf{K}_T^{\mathbb{P}} - \mathsf{K}_t^{\mathbb{P}},\ 0 \leq t \leq T,\ \mathbb{P} - a.s., \\ \mathsf{Y}_t^{\mathbb{P}} \geq L_t,\ \int_0^T (\mathsf{Y}_{s-}^{\mathbb{P}} - L_{s-})\mathrm{d}\mathsf{K}_s^{\mathbb{P}} = 0, \quad \mathbb{P} - a.s. \end{cases} \tag{3.4}$$

Definition 3.5. For $\xi \in \mathbb{L}_0^{p,\kappa}$, we say that $(Y, Z, (M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ is a solution to the 2RBSDE (3.1) if:

(i) The conditions (i) and (ii) in Definition 3.4 still hold true.
(ii) We have the following minimality condition:

$$\underset{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T (Y_{s-} - L_{s-})\mathrm{d}K_s^{\mathbb{P}'} \bigg| \mathcal{F}_{t+}^{\mathbb{P}'}\right] = 0, \quad 0 \leq t \leq T,\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_0. \tag{3.5}$$

Definition 3.5 was motivated by the definition of the G-RBSDE studied in [15, 16], where the authors introduced the Skorohod-type minimality condition with respect to nonlinear expectation. The main disadvantages of the above definitions are as follows:

- The minimality condition (3.3) is only suitable for 2RBSDEs with Lipschitz or monotonic generators, since it is mainly based on the linearization technique. The Lipschitz or monotonicity conditions are not satisfied for RBSDEs studied in [23] or 2BSDEs studied in [24]. Indeed, in those papers, the authors considered extended monotonicity condition, which is far away from the linearization procedure. Therefore, Definition 3.4 should be modified for further development of 2RBSDEs.
- The drawback of Definition 3.5 is that it is somewhat strong in the sense that it does not allow us to consider the 2RBSDE as a natural extension of the 2BSDE, whereas standard RBSDE is a natural extension of the BSDE. Indeed, if we set $L := -\infty$ then the minimality condition (3.5) implies that $K^{\mathbb{P}} = 0$, so that the 2RBSDE (3.1) does not become the 2BSDE.

The new minimality condition (3.2) does not involve the disadvantages of Definitions 3.4 and 3.5. Indeed, it can be reasonable for any kind of generators which may be non-Lipschitz or non-monotonic. Moreover, if we set $L = -\infty$ then the 2RBSDE (3.1) becomes the 2BSDE (see Sect. 3.4 below). The following proposition shows that our minimality condition is weaker than the Skorohod-type minimality condition.

Proposition 3.6. *The Skorohod-type minimality condition* (3.5) *implies the minimality condition* (3.2)*.*

Proof. Suppose Skorohod condition holds true. Fix $\mathbb{P} \in \mathcal{P}_0$ and $t \in [0,T]$. Since $Y_t \geq L_t$ and $K^{\mathbb{P}}$ is non-decreasing, it follows that

$$\underset{\tau\in\mathbb{T}^{t,T}}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] \geq \underset{\tau\in\mathbb{T}^{t,T}}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] \geq K_t^{\mathbb{P}}, \quad \mathbb{P}-a.s.$$

For the reverse inequality, fix $\varepsilon > 0$ and define the following stopping time:

$$\tau_\varepsilon := \inf\{t \leq s \leq T:\ Y_{s-} - Ls- \leq \varepsilon\} \wedge T, \quad \mathbb{P}-a.s.$$

We then obtain

$$0 = \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[\int_t^T (Y_{s-} - L_{s-})\mathrm{d}K_s^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] \geq \varepsilon\cdot \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_{\tau_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right],$$

which implies

$$\underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_{\tau_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] = 0, \quad \mathbb{P}-a.s.$$

It follows that

$$\begin{aligned}
0 &\leq \underset{\tau\in\mathbb{T}^{t,T}}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] \\
&\leq \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_{\tau_\varepsilon} - \widetilde{L}_{\tau_\varepsilon} + K_{\tau_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] \\
&\leq \varepsilon + \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_{\tau_\varepsilon}^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] = \varepsilon.
\end{aligned}$$

By the arbitrariness of ε, we deduce that

$$K_t^{\mathbb{P}} = \underset{\tau\in\mathbb{T}^{t,T}}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right], \quad \mathbb{P}-a.s.$$

□

Remark 3.7. In general, the reverse argument of the above proposition does not hold true. As it was mentioned in [11], if one wants to recover the Skorohod condition then we need additional assumption on the obstacle, which basically asks that the variations of obstacle are not too 'extreme' (see Asm. 2.1 and Sect. 2.5.2 (i) in that paper).

3.3. Connection with the standard RBSDEs

Let the family $\mathcal{P}_0$ be singleton, *i.e.* $\mathcal{P}_0 = \{\mathbb{P}_0\}$. In this case, the minimality condition (3.2) implies

$$K_t = \underset{\tau\in\mathbb{T}^{t,T}}{\operatorname{ess\,inf}}\ \mathbb{E}^{\mathbb{P}_0}\left[Y_\tau - \widetilde{L}_\tau + K_\tau\middle|\mathcal{F}_{t+}\right],\ 0\le t\le T,\ \mathbb{P}_0-a.s.$$

Hence, the 2RBSDE (3.1) is equivalent to th following standard RBSDE:

$$Y_t = \xi + \int_t^T f_s(Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s\cdot \mathrm{d}X_s - \int_t^T \mathrm{d}M_s + K_T - K_t, \quad 0\le t\le T,\ \mathbb{P}_0-a.s. \tag{3.6}$$

3.4. Connection with the 2BSDEs

If we set $L := -\infty$, then we obtain

$$K_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[Y_T - \widetilde{L}_T + K_T^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] = \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\overset{\mathbb{P}}{\operatorname{ess\,inf}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right],\ \forall\mathbb{P}\in\mathcal{P}_0,\ \mathbb{P}-a.s.$$

which is the minimality condition for the 2BSDEs. Hence the 2RBSDE (3.1) becomes the 2BSDE.

3.5. Connection with the standard BSDEs

If $\mathcal{P}_0 = \{\mathbb{P}_0\}$ and $L = -\infty$, then the minimality condition (3.2) implies

$$0 = K_0 = \mathbb{E}^{\mathbb{P}_0}[K_T|\mathcal{F}_{t+}] \text{ and thus } K = 0,\ \mathbb{P}_0-a.s.$$

Hence, the 2RBSDE (3.1) is equivalent to the following standard BSDE:

$$Y_t = \xi + \int_t^T f_s(Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s\cdot \mathrm{d}X_s - \int_t^T \mathrm{d}M_s, \quad 0\le t\le T,\ \mathbb{P}_0-a.s. \tag{3.7}$$

3.6. Integrability assumption on obstacle process

Recall the integrability assumption: $L^+ \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$.

Let $(\bar{Y}, \bar{Z}, (\bar{M}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}, (\bar{K}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ be the solution of the standard 2BSDE corresponding to 2RBSDE (3.1). Fix $\mathbb{P}\in\mathcal{P}_0$ and $t\in[0,T]$. For each $\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$, denote $\delta Y = \bar{Y} - Y$, $\delta Z = \bar{Z} - Z$, $\delta M^{\mathbb{P}'} = \bar{M}^{\mathbb{P}'} - M^{\mathbb{P}'}$. By Lipschitz assumption, there exist two bounded processes $\lambda^{\mathbb{P}'}$

and $\eta^{\mathbb{P}'}$ such that $\mathbb{P}' - a.s.$

$$\delta Y_t = \int_t^T \left(\lambda_s^{\mathbb{P}'} \delta Y_s + \eta_s^{\mathbb{P}'} \cdot (\widehat{a}_s^{1/2})^{\intercal} \delta Z_s\right) \mathrm{d}s - \int_t^T \delta Z_s \cdot (\widehat{a}_s^{1/2})^{\intercal} \mathrm{d}W_s^{\mathbb{P}'} - \int_t^T d(\delta M_s^{\mathbb{P}'}) + \int_t^T d(\bar{K}^{\mathbb{P}'} - K^{\mathbb{P}'})_s.$$

Define for $t \leq s \leq T$ the following preocess

$$\Delta_s^{\mathbb{P}'} := \exp\left(\int_t^s \left(\lambda_s^{\mathbb{P}'} - \frac{1}{2}\|\eta_s^{\mathbb{P}'}\|^2\right) \mathrm{d}s - \int_t^s \eta_s^{\mathbb{P}'} \cdot \mathrm{d}W_s^{\mathbb{P}'}\right), \quad \mathbb{P}' - a.s.$$

Since $\lambda^{\mathbb{P}'}$ and $\eta^{\mathbb{P}'}$ are bounded, we have for all $p \geq 1$, for some $C_p > 0$.

$$\mathbb{E}^{\mathbb{P}'}\left[\sup_{t\leq s\leq T} (\Delta_s^{\mathbb{P}'})^p + \sup_{t\leq sleT} (\Delta_s^{\mathbb{P}'})^{-p} \middle| \mathcal{F}_{t+}\right] \leq C_p.$$

By Itô's formula, we obtain

$$\delta Y_t = \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \Delta_s^{\mathbb{P}'} d(\bar{K}^{\mathbb{P}'} - K^{\mathbb{P}'})_s \middle| \mathcal{F}_{t+}\right] \leq \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \Delta_s^{\mathbb{P}'} \mathrm{d}\bar{K}_s^{\mathbb{P}'} \middle| \mathcal{F}_{t+}\right].$$

We then deduce that

$$\begin{aligned} \delta Y_t &\leq \left(\mathbb{E}^{\mathbb{P}'}\left[\sup_{t\leq s\leq T} (\Delta_s^{\mathbb{P}'})^{\frac{p+1}{p-1}} \middle| \mathcal{F}_{t+}\right]\right)^{\frac{p-1}{p+1}} \left(\mathbb{E}^{\mathbb{P}'}\left[(\bar{K}_T^{\mathbb{P}'} - \bar{K}_t^{\mathbb{P}'})^{\frac{p+1}{2}} \middle| \mathcal{F}_{t+}\right]\right)^{\frac{2}{p+1}} \\ &\leq C(C_t^{\mathbb{P}'})^{\frac{1}{p+1}} \left(\mathbb{E}^{\mathbb{P}'}\left[\bar{K}_T^{\mathbb{P}'} - \bar{K}_t^{\mathbb{P}'} \middle| \mathcal{F}_{t+}\right]\right)^{\frac{1}{p+1}}, \end{aligned}$$

where $C_t^{\mathbb{P}}$ is defined as follows (see the proof of Thm. 4.2 (ii) in [9]):

$$C_t^{\mathbb{P}} = \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\left(\bar{K}_T^{\mathbb{P}'} - \bar{K}_t^{\mathbb{P}'}\right)^p \middle| \mathcal{F}_{t+}\right] < +\infty, \quad \mathbb{P} - a.s.$$

By the arbitrariness of $\mathbb{P}'$ and the minimality condition for $\bar{K}$, we deduce that

$$\delta Y_t \leq 0, \quad \mathbb{P} - a.s.$$

Hence $Y \geq \bar{Y}$. We also note that this fact can be proved without linearization technique by applying Itø-Tanaka's formula for the process $(\delta Y_t^+)^{\mathsf{p}}$ with $1 < \mathsf{p} < p$ (see [24]). So, we can replace the obstacle process L_t by $L_t \vee \bar{Y}_t$. Consequently, we may assume that without loss of generality that $L \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$.

4. Wellposedness of 2RBSDEs

Following [9, 10, 12], in addition to Assumption 2.2, we introduce the following assumption.[1]

[1] This assumption is mainly due to the fact that it is still an open problem to obtain an efficient form of Doob's inequality under nonlinear expectation. It is essentially contributed to obtain *a priori* estimate of solutions to 2RBSDEs.

Assumption 4.1. For the given $p > \kappa > 1$, one has

$$\phi_f^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\le t\le T}\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\int_0^T |\widehat{f}_s^{\mathbb{P},0}|^\kappa \mathrm{d}s \Big| \mathcal{F}_{t+}\right]\right)^{p/\kappa}\right] < +\infty, \tag{4.1}$$

$$\psi_L^{p,\kappa} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\le t\le T}\left(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\sup_{0\le s\le T} |L_s^+|^\kappa \Big| \mathcal{F}_{t+}\right]\right)^{p/\kappa}\right] < +\infty. \tag{4.2}$$

For every $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$, a stopping time τ and $\mathcal{F}_\tau^U$-measurable random variable $\zeta \in \mathbb{L}^p_{t,\omega}(\mathcal{F}^U_\tau,\mathbb{P})$, let $(\mathcal{Y}^{\mathbb{P}},\mathcal{Z}^{\mathbb{P}},\mathcal{M}^{\mathbb{P}}) := (\mathcal{Y}^{\mathbb{P}}(\tau,\zeta),\mathcal{Z}^{\mathbb{P}}(\tau,\zeta),\mathcal{M}^{\mathbb{P}}(\tau,\zeta))$ denotes the solution to the following standard BSDE:

$$\mathcal{Y}_s^{\mathbb{P}} = \zeta + \int_s^\tau \widehat{f}_r^{\mathbb{P}}(\mathcal{Y}_r^{\mathbb{P}},(\widehat{a}_r^{1/2})^\intercal \mathcal{Z}_r^{\mathbb{P}})\mathrm{d}r - \int_s^\tau \mathcal{Z}_r^{\mathbb{P}}\cdot \mathrm{d}X_r^{c,\mathbb{P}} - \int_s^\tau \mathrm{d}\mathcal{M}_r^{\mathbb{P}},\ t\le s\le\tau,\ \mathbb{P}-a.s. \tag{4.3}$$

4.1. Representation and uniqueness of the solution

Theorem 4.2. *Let Assumptions 2.2 and 4.1 hold. Let $\xi \in \mathbb{L}_0^{p,\kappa}$ and $(Y,Z,(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0},(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ be a solution to the 2RBSDE (3.1). For any $\mathbb{P}\in\mathcal{P}_0$, let $(\mathcal{Y}^{\mathbb{P}},\mathcal{Z}^{\mathbb{P}},\mathcal{M}^{\mathbb{P}}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ be the solutions to the corresponding BSDE (4.3). Then for any $\mathbb{P}\in\mathcal{P}_0$ and $0\le t\le T$,*

$$Y_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\tau\in\mathbb{T}^{t,T}} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau),\ \mathbb{P}-a.s. \tag{4.4}$$

Consequently, the 2RBSDE (3.1) has at most one solution in $\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}_0})\times\mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\times\mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$.

Proof. We first prove the last statement about uniqueness. So suppose that (4.4) holds. Then the uniqueness of Y is immediate by the representation formula (4.4). Then, since

$$\langle Y,Z\rangle_t = \int_0^t \widehat{a}_s Z_s \mathrm{d}s,\ \mathbb{P}-a.s.,$$

Z is also uniquely defined $\widehat{a}_t\mathrm{d}t\otimes\mathcal{P}_0 - q.s.$ We therefore deduce that the processes $M^{\mathbb{P}}-K^{\mathbb{P}}$ are also uniquely defined for any $\mathbb{P}\in\mathcal{P}_0$. Since $K^{\mathbb{P}}$ is a non-decreasing process and $M^{\mathbb{P}}$ is a martingale, we observe that for any $\mathbb{P}\in\mathcal{P}_0$, $M^{\mathbb{P}}-K^{\mathbb{P}}$ is a $(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$-supermartingale. Furthermore, $(K_t^{\mathbb{P}},M_t^{\mathbb{P}})\in\mathbb{L}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{L}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ for any $t\in[0,T]$, and $K^{\mathbb{P}}$ is $\mathbb{F}_+^{\mathbb{P}}$-predictable. Then the uniqueness of $M^{\mathbb{P}}$ and $K^{\mathbb{P}}$ is a simple consequence of the uniqueness in the Doob-Meyer decomposition of supermartingales.

It remains to prove (4.4).

(i) We first prove the forward inequality " $\le$ " in (4.4). Fix $t\in[0,T]$, $\tau\in\mathbb{T}^{t,T}$ and $\mathbb{P}\in\mathcal{P}_0$. For any $\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$, note that

$$Y_t = Y_\tau + \int_t^\tau \widehat{f}_s^{\mathbb{P}'}(Y_s,(\widehat{a}_s^{1/2})^\intercal Z_s)\mathrm{d}s - \int_t^\tau Z_s\cdot\widehat{a}_s^{1/2}\mathrm{d}W_s^{\mathbb{P}'} - \int_t^\tau \mathrm{d}M_s^{\mathbb{P}'} + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ \mathbb{P}'-a.s., \tag{4.5}$$

and that $K^{\mathbb{P}'}$ is non-decreasing $\mathbb{P}'-a.s.$ Since $Y_\tau \ge L_\tau$, $\mathbb{P}'-a.s.$, by the general comparison principle for BSDEs (see Lem. A.3 of [9]), we deduce immediately that

$$Y_t \ge \mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau),\ \mathbb{P}'-a.s.$$

Then, since $\mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau)$ is $\mathcal{F}_{t+}$-measurable and since Y_t is $\mathcal{F}_{t+}^{\mathcal{P}_0}$-measurable, we deduce that the aboe inequality also holds $\mathbb{P}-a.s.$, by definition of $\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$ and the fact that measures extend uniquely to the completed σ-algebras. By the arbitrariness of τ and $\mathbb{P}'$, we deduce that

$$Y_t = \underset{\tau\in\mathbb{T}^{t,T}}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau),\ \mathbb{P}-a.s.$$

(ii) We now prove the reverse inequality. Define for any $\tau\in\mathbb{T}^{t,T}$ and $\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$,

$$\Delta Y := Y - \mathcal{Y}^{\mathbb{P}'}(\tau,\widetilde{L}_\tau),\ \Delta Z := Z - \mathcal{Z}^{\mathbb{P}'}(\tau,\widetilde{L}_\tau),\ \text{and}\ \Delta M := M^{\mathbb{P}'} - \mathcal{M}^{\mathbb{P}'}(\tau,\widetilde{L}_\tau).$$

By (4.5) and Remark 2.4, there exist bounded processes $\lambda^{\mathbb{P}'}$ and $\eta^{\mathbb{P}'}$ such that

$$\begin{aligned}\Delta Y_t = Y_\tau - \widetilde{L}_\tau + \int_t^\tau (\lambda_s^{\mathbb{P}'}\Delta Y_s + \eta_s^{\mathbb{P}'}\cdot\widehat{a}_s^{1/2}\Delta Z_s)\mathrm{d}s - \int_t^\tau \Delta Z_s\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}'}\\ - \int_t^\tau d(\Delta M_s^{\mathbb{P}'}) + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'},\ \mathbb{P}'-a.s.\end{aligned}\tag{4.6}$$

Define the continuous process

$$\Gamma_s^{\mathbb{P}'} := \exp\left(\int_t^s\left(\lambda_r^{\mathbb{P}'} - \frac{1}{2}|\eta_r^{\mathbb{P}'}|^2\right)\mathrm{d}r + \int_t^s \eta_r^{\mathbb{P}'}\cdot\mathrm{d}W_r^{\mathbb{P}'}\right),\ t\leq s\leq\tau,\ \mathbb{P}'-a.s.\tag{4.7}$$

Note that since $\lambda^{\mathbb{P}'}$ and $\eta^{\mathbb{P}'}$ are bounded, we have for all $p\geq 1$, for some constant $C_p>0$ independent of $\mathbb{P}'$ and τ,

$$\mathbb{E}^{\mathbb{P}'}\left[\sup_{t\leq s\leq\tau}\left|\Gamma_s^{\mathbb{P}'}\right|^p + \sup_{t\leq s\leq\tau}\left|\Gamma_s^{\mathbb{P}'}\right|^{-p}\middle|\mathcal{F}_{t+}\right]\leq C_p,\ \mathbb{P}'-a.s.\tag{4.8}$$

Then, by Itô's formula, we get

$$\Delta Y_t = \mathbb{E}^{\mathbb{P}'}\left[\Gamma_\tau^{\mathbb{P}'}(Y_\tau - \widetilde{L}_\tau) + \int_t^\tau \Gamma_s^{\mathbb{P}'}\mathrm{d}K_s^{\mathbb{P}'}\middle|\mathcal{F}_{t+}\right]\tag{4.9}$$

because the martingale terms vanish when we take the conditional expectation. We therefore deduce that

$$\begin{aligned}\Delta Y_t &\leq \mathbb{E}^{\mathbb{P}'}\left[\left(\sup_{t\leq s\leq\tau}\Gamma_s^{\mathbb{P}'}\right)\cdot\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)\middle|\mathcal{F}_{t+}\right]\\ &\leq \left(\mathbb{E}^{\mathbb{P}'}\left[\sup_{t\leq s\leq\tau}\left|\Gamma_s^{\mathbb{P}'}\right|^{\frac{p-1}{p+1}}\middle|\mathcal{F}_{t+}\right]\right)^{\frac{p-1}{p+1}}\left(\mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^{\frac{p+1}{2}}\middle|\mathcal{F}_{t+}\right]\right)^{\frac{2}{p+1}}\\ &\leq C\left(\mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{p+1}}\left(\mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{p+1}}.\end{aligned}$$

We shall prove in step **(iii)** that

$$C_t^{\mathbb{P}} := \underset{\tau\in\mathbb{T}^{t,T}}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\middle|\mathcal{F}_{t+}\right] < +\infty,\ \mathbb{P}-a.s.\tag{4.10}$$

Then, it follows from the last inequality that

$$
\begin{aligned}
&Y_t - \underset{\tau\in\mathbb{T}^{t,T}}{\operatorname{ess\,sup}^{\mathbb{P}}} \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau) \\
&\leq C\left(C_t^{\mathbb{P}}\right)^{\frac{1}{p+1}} \underset{\tau\in\mathbb{T}^{t,T}}{\operatorname{ess\,inf}^{\mathbb{P}}} \underset{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,inf}^{\mathbb{P}}} \left(\mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{p+1}} = 0,\ \mathbb{P}-a.s.,
\end{aligned}
$$

by the minimality condition (3.2).

(iii) It remains to prove show that the estimate (4.10) holds. First of all, we have by definition

$$
\begin{aligned}
&\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p \leq C\left[\left(Y_\tau - \widetilde{L}_\tau\right)^p + \left(K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\right] \\
&\leq C\left[|Y_\tau|^p + \left|\widetilde{L}_\tau\right|^p + \left(K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\right] \\
&\leq C\Bigg(\sup_{t\leq s\leq T}|Y_s|^p + \sup_{t\leq s\leq T}|L_s|^p + \left(\int_t^T \left|\widehat{f}_s^{\mathbb{P}',0}\right|\mathrm{d}s\right)^p + \left(\int_t^T \left|(\widehat{a}_s^{1/2})^\top Z_s\right|\mathrm{d}s\right)^p \\
&\qquad + \left|\int_t^T Z_s\cdot\widehat{a}_s^{1/2}\mathrm{d}W_s^{\mathbb{P}'}\right|^p + \left|\int_t^T \mathrm{d}M_s^{\mathbb{P}'}\right|^p\Bigg).
\end{aligned}
$$

Therefore, we obtain by Burkholder-Davis-Gundy's inequality

$$
\mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\right] \leq C\left(\phi_f^{p,\kappa} + \|Y\|_{\mathbb{D}_0^p}^p + \|Z\|_{\mathbb{H}_0^p}^p + \|L\|_{\mathbb{D}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\left[[M^{\mathbb{P}}]_T^{p/2}\right]\right). \tag{4.11}
$$

Next, we claim that the family

$$
\left\{\mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K_\tau^{\mathbb{P}'} - K_t^{\mathbb{P}'}\right)^p\middle|\mathcal{F}_{t+}\right],\ (\tau,\mathbb{P}')\in\tau^{t,T}\times\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)\right\}
$$

is upward directed. Let us consider $(\mathbb{P}^1,\mathbb{P}^2)\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)\times\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$ and $(\tau^1,\tau^2)\in\mathbb{T}^{t,T}\times\mathbb{T}^{t,T}$. Let

$$
\begin{aligned}
A := \Bigg\{\omega\in\Omega : &\ \mathbb{E}^{\mathbb{P}^1}\left[\left(Y_{\tau^1} - \widetilde{L}_{\tau^1} + K_{\tau^1}^{\mathbb{P}^1} - K_t^{\mathbb{P}^1}\right)^p\middle|\mathcal{F}_{t+}\right](\omega) \\
&\geq \mathbb{E}^{\mathbb{P}^2}\left[\left(Y_{\tau^2} - \widetilde{L}_{\tau^2} + K_{\tau^2}^{\mathbb{P}^2} - K_t^{\mathbb{P}^2}\right)^p\middle|\mathcal{F}_{t+}\right](\omega)\Bigg\},
\end{aligned}
$$

Then, $A, A^c\in\mathcal{F}_{t+}^{\mathbb{P}}$ and we can define a stopping time $\tau^{1,2}\in\mathbb{T}^{t,T}$ and a probability measure $\mathbb{P}^{1,2}$ on $(\Omega,\mathcal{F}_T)$ by

$$
\tau^{1,2} := \tau^1\mathbf{1}_A + \tau^2\mathbf{1}_{A^c},
$$

and

$$
\mathbb{P}^{1,2}(B) := \mathbb{P}^1(A\cap B) + \mathbb{P}^2(A^c\cap B),\ \text{for any } B\in\mathcal{F}_T,
$$

respectively. By Assumption 2.2 (5), we know that $\mathbb{P}^{1,2} \in \mathcal{P}_0$, and by definition we further have $\mathbb{P}^{1,2} \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$. In particular, we have $\mathbb{P} - a.s.$,

$$\mathbb{E}^{\mathbb{P}^{1,2}}\left[\left(Y_{\tau^{1,2}} - \widetilde{L}_{\tau^{1,2}} + K^{\mathbb{P}^{1,2}}_{\tau^{1,2}} - K^{\mathbb{P}^{1,2}}_t\right)^p \middle| \mathcal{F}_{t+}\right]$$
$$= \mathbb{E}^{\mathbb{P}^1}\left[\left(Y_{\tau^1} - \widetilde{L}_{\tau^1} + K^{\mathbb{P}^1}_{\tau^1} - K^{\mathbb{P}^1}_t\right)^p \middle| \mathcal{F}_{t+}\right] \vee \mathbb{E}^{\mathbb{P}^2}\left[\left(Y_{\tau^2} - \widetilde{L}_{\tau^2} + K^{\mathbb{P}^2}_{\tau^2} - K^{\mathbb{P}^2}_t\right)^p \middle| \mathcal{F}_{t+}\right],$$

which proves the claim.

Therefore, by classical results for the essential supremum (see, *e.g.*, Neveu [25]), there exist two sequences $(\mathbb{P}_n) \subset \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)$ and $(\tau_m) \subset \mathbb{T}^{t,T}$ such that

$$\underset{\tau \in \mathbb{T}^{t,T}}{\operatorname{ess\,sup}^{\mathbb{P}}} \ \underset{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\left(Y_\tau - \widetilde{L}_\tau + K^{\mathbb{P}'}_\tau - K^{\mathbb{P}'}_t\right)^p \middle| \mathcal{F}_{t+}\right]$$
$$= \lim_{m,n \to \infty} \uparrow \mathbb{E}^{\mathbb{P}_n}\left[\left(Y_{\tau_m} - \widetilde{L}_{\tau_m} + K^{\mathbb{P}_n}_{\tau_m} - K^{\mathbb{P}_n}_t\right)^p \middle| \mathcal{F}_{t+}\right].$$

Then using (4.11) and the monotone convergence theorem under $\mathbb{P}$, we deduce that

$$\mathbb{E}^{\mathbb{P}}[C^{\mathbb{P}}_t] \leq \lim_{m,n \to \infty} \uparrow \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}_n}\left[\left(Y_{\tau_m} - \widetilde{L}_{\tau_m} + K^{\mathbb{P}_n}_{\tau_m} - K^{\mathbb{P}_n}_t\right)^p \middle| \mathcal{F}_{t+}\right]\right]$$
$$\leq C\left(\phi^{p,\kappa}_f + \|Y\|^p_{\mathbb{D}^p_0} + \|Z\|^p_{\mathbb{H}^p_0} + \|L\|^p_{\mathbb{D}^p_0} + \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[[M^{\mathbb{P}}]^{p/2}_T\right]\right) < +\infty,$$

which provides the desired result. □

The representation formula (4.4), combined with the comparison theorem for corresponding BSDEs (see, *e.g.*, Lem. A.3 in [9]), gives us the following comparison result for 2RBSDEs.

Corollary 4.3. *For $i = 1, 2$, let f^i, ξ^i and L^i be respectively a generator map, a terminal condition, and an obstacle process satisfying Assumptions 2.2 and 4.1. Let also Y^i be the first component of the solution to the 2RBSDE with generator f^i, terminal condition ξ^i and obstacle L^i. Suppose in addition that for any $\mathbb{P} \in \mathcal{P}_0$ we have*

(i) $\xi^1 \leq \xi^2$, $\mathbb{P} - a.s.$
(ii) $\widehat{f}^{1,\mathbb{P}}_s(y,z) \leq \widehat{f}^{2,\mathbb{P}}_s(y,z)$, $\forall (y,z)$, $\mathrm{d}s \times \mathrm{d}\mathbb{P} - a.e.$
(iii) $L^1_t \leq L^2_t$, $0 \leq t \leq T$, $\mathbb{P} - a.s.$

Then we have $Y^1_t \leq Y^2_t$, $t \in [0, T]$, $\mathcal{P}_0 - q.s.$

Remark 4.4. In view of the representation formula of solutions to standard RBSDEs (see, *e.g.*, [26, 27] or the recent work [6]), we have

$$\mathsf{Y}^{\mathbb{P}'}_t = \underset{\tau \in \mathbb{T}^{t,T}}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}^{\mathbb{P}'}_t(\tau, \widetilde{L}_\tau),$$

where $\mathsf{Y}^{\mathbb{P}'}$ is the first component of solution to the RBSDE (3.4). Hence, we can rewrite the representation formula (4.4) as

$$Y_t = \underset{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathsf{Y}^{\mathbb{P}'}_t, \ \mathbb{P} - a.s., \tag{4.12}$$

which coincides with the representation formula in [10–12]. Therefore, *a priori* estimate and stability results for 2RBSDEs established in [12] still hold true in our setting. On the other hand, the representation formula (4.4) gives us an alternative method to prove such results by using the corresponding ones for standard BSDEs, as it was done in [6] for the standard RBSDEs. Given the length of the paper, we leave such arguments to the interesting readers.

4.2. Existence

For every $(t, \omega) \in [0, T] \times \Omega$, we define the value process

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[\mathsf{Y}_t^{\mathbb{P}}].$$

We then define

$$V_t^+ := \limsup_{r \in \mathbb{Q} \cap [0,T], r \downarrow t} , \; V_T^+ := V_T.$$

It is proved in [12] that V^+ admits right- and left- limits outside a $\mathcal{P}_0$-polar set and

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap [0,T], r \downarrow t} V_t, \text{ outside a } \mathcal{P}_0 - polarset,$$

which implies that V^+ is right-continuous outside a $\mathcal{P}_0$-polar set (see Sect. 3.4.4 therein).

We further recall some important results proved in [12].

Lemma 4.5 ([12], Lemma. 3.4.8)**.** *Let Assumption 2.2 holds true. Then for any stopping times $0 \le \sigma \le \tau \le T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma^+ \ge \mathsf{Y}_\sigma^{\mathbb{P}}(\tau, V_\tau^+), \; \mathbb{P} - a.s.$$

In particular, V^+ is càdlàg, $\mathcal{P}_0 - q.s.$

Lemma 4.6 ([12], Lemma. 3.4.9)**.** *Let Assumption 2.2 holds true. Then for any $\mathbb{F}$-stopping times $0 \le \sigma \le \tau \le T$, for any $0 \le t \le T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})} \mathbb{E}^{\mathbb{P}'} \left[\mathsf{Y}_\sigma^{\mathbb{P}'}(\tau, \mathsf{Y}_\tau^{\mathbb{P}'}) \middle| \mathcal{F}_\sigma \right], \; \mathbb{P} - a.s. \text{ and } V_t^+ = \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)} \mathsf{Y}_t^{\mathbb{P}'}(T, \xi), \; \mathbb{P} - a.s.$$

In particular, if Assumption 4.1 holds, one has $V^+ \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0 +})$.

The above two lemmas, combining with the representation formula of solutions to standard RBSDEs, give us the following results.

Corollary 4.7. *Let Assumption 2.2 holds true. Then for any stopping times $0 \le \sigma \le \tau \le T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma^+ \ge \mathcal{Y}_\sigma^{\mathbb{P}}(\tau, V_\tau^+), \; \mathbb{P} - a.s.$$

Corollary 4.8. *Let Assumption 2.2 holds true. Then for any $0 \le t \le T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_t^+ = \operatorname*{ess\,sup^{\mathbb{P}}}_{\tau \in \mathbb{T}^{t,T}} \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}_0(t, \mathbb{P}, \mathbb{F}_+)} \mathcal{Y}_t^{\mathbb{P}'}(\tau, \widetilde{L}_\tau), \; \mathbb{P} - a.s.$$

Finally, we recall the result on the semimartingale decomposition of V^+ (see steps (i)–(ii) in the proof of Lem. 3.4.10 of [12]).

Lemma 4.9. *Let Assumption 2.2 holds true. Then for any $\mathbb{P}\in\mathcal{P}_0$, there exists $(Z^{\mathbb{P}},M^{\mathbb{P}},K^{\mathbb{P}})\in\mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ such that*

$$V_t^+=\xi+\int_t^T\widehat{f}_s^{\mathbb{P}}(V_s^+,\widehat{a}_s^{1/2}Z_s^{\mathbb{P}})\mathrm{d}s-\int_t^T Z_s^{\mathbb{P}}\cdot\mathrm{d}X_s^{c,\mathbb{P}}-\int_t^T\mathrm{d}M_s^{\mathbb{P}}+\int_t^T\mathrm{d}K_s^{\mathbb{P}},\ t\in[0,T],\ \mathbb{P}-a.s.$$

Moreover, there is some $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z which aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$.

We now in a position to state the main result of this paper.

Theorem 4.10. *Let Assumptions 2.2 and 4.1 hold. Then the 2RBSDE (3.1) has a unique solution $(Y,Z,(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0},(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$.*

Proof. The uniqueness was already proved in Theorem 4.2. To prove the existence, we shall show that V^+ is indeed a solution to the 2RBSDE. We already from Lemma 4.6 that $V^+\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$. By Lemma 4.9, there are some processes $(Z,(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0},(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\in\mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}_0})\times\mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\times\mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ satisfying (3.1). It only remains to check the minimality condition (3.2).

Fix $t\in[0,T]$, $\tau\in\mathbb{T}^{t,T}$ and $\mathbb{P}\in\mathcal{P}_0$. Let us denote for any $\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)$,

$$\Delta V_t^+:=V^+-\mathcal{Y}_t^{\mathbb{P}'}(\tau,\widetilde{L}_\tau).$$

By (4.9), we have

$$\Delta V_t^+=\mathbb{E}^{\mathbb{P}'}\left[\Gamma_\tau^{\mathbb{P}'}(V_\tau^+-\widetilde{L}_\tau)+\int_t^\tau\Gamma_s^{\mathbb{P}'}\mathrm{d}K_s^{\mathbb{P}'}\middle|\mathcal{F}_{t+}\right]\geq\mathbb{E}^{\mathbb{P}'}\left[\inf_{t\leq s\leq\tau}\Gamma_s^{\mathbb{P}'}\cdot\left(V_\tau^+-\widetilde{L}_\tau+K_\tau^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\middle|\mathcal{F}_{t+}\right],$$

where $\Gamma^{\mathbb{P}'}$ is defined in (4.7). Using Hölder's inequality, we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}'}\left[V_\tau^+-\widetilde{L}_\tau+K_\tau^{\mathbb{P}'}-K_t^{\mathbb{P}'}\middle|\mathcal{F}_{t+}\right]\\&\leq\left(\mathbb{E}^{\mathbb{P}'}\left[\inf_{t\leq s\leq\tau}\Gamma_s^{\mathbb{P}'}\cdot\left(V_\tau^+-\widetilde{L}_\tau+K_\tau^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{2}}\\&\qquad\times\left(\mathbb{E}^{\mathbb{P}'}\left[\left(V_\tau^+-\widetilde{L}_\tau+K_\tau^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^p\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{2p}}\left(\mathbb{E}^{\mathbb{P}'}\left[\left(\inf_{t\leq s\leq\tau}\Gamma_s^{\mathbb{P}'}\right)^{-q}\middle|\mathcal{F}_{t+}\right]\right)^{\frac{1}{2q}}\\&\leq C\left(C_t^{\mathbb{P}'}\right)^{\frac{1}{2p}}\left(\Delta V_t^+\right)^{\frac{1}{2}},\end{aligned}$$

with $q>1$ such that $1/p+1/q=1$. Then, the result follows immediately thanks to Corollary 4.8. □

5. Nonlinear optional decomposition and super–hedging duality

5.1. Saturated 2RBSDEs

Definition 5.1. The set $\mathcal{P}_0$ is said to be saturated[2] if, when $\mathbb{P}\in\mathcal{P}_0$, we have $\mathbb{Q}\in\mathcal{P}_0$ for every probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ which is equivalent to $\mathbb{P}$ and under which X is local martingale.

[2]For instance, the set $\overline{\mathcal{P}}_S$ of [8], whose measures only change the volatility of X is saturated.

We give now an alternative definition for 2RBSDEs of the form

$$Y_t = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(Y_s, (\widehat{a}_s^{1/2})^\intercal Z_s)\mathrm{d}s - \int_t^T Z_s \cdot dX_s^{c,\mathbb{P}} + K_T^{\mathbb{P}} - K_t^{\mathbb{P}},\ 0 \le t \le T. \tag{5.1}$$

Definition 5.2. We say $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}_0}) \times (\mathbb{I}_0^{o,p}(\mathbb{F}_+^{\mathbb{P}}))_{\mathbb{P}\in\mathcal{P}_0}$ is a saturated solution to 2RBSDE (5.1) if equation (5.1) holds $\mathcal{P}_0 - q.s.$ and if the family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_0\}$ satisfies the minimality condition (3.2).

We then have the following result.

Theorem 5.3. *Let Assumptions 2.2 and 4.1 hold. Assume in addition that the set $\mathcal{P}_0$ is saturated. Then there is a unique saturated solution of the 2RBSDE (5.1).*

Proof. This result can be proved by following exactly the same procedure as in the proof of Theorem 5.1 in [9]. □

5.2. A super-hedging duality for American options in ucertain, incomplete and nonlinear markets

We consider a standard market (possibly incomplete) consisting of one non-risky asset and n risky assets whose dynamics are uncertain but given as solutions of controlled SDEs $S^{t,\omega,\nu}$. The collection of the law of these dynamics, denoted by $\mathcal{P}_0^{\mathcal{U}}$, satisfies Assumptions 2.2 (3)–(5) (see Rem. 2.3). We further assume that $\mathcal{P}_0^{\mathcal{U}}$ is saturated. A portfolio strategy is then defined as a $\mathbb{R}^n$-valued and $\mathbb{F}^{\mathcal{P}_0^{\mathcal{U}}}$-predictable process $(Z_t)_{t\in[0,T]}$, such that Z_t^i describes the number of units of asset i in the portfolio of the investor at time t. It is well-known that under some constrained cases, the wealth $Y^{y_0,Z}$ associated to the strategy Z and initial capital $y_0 \in \mathbb{R}$ can be written, for every $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$, as

$$Y_t^{y_0,Z} := y_0 - \int_0^t \widehat{f}_s^{\mathbb{P}}(Y_s^{y_0,Z}, (\widehat{a}_s^{1/2})^\intercal Z_s)\mathrm{d}s + \int_0^t Z_s \cdot dX_s^{c,\mathbb{P}},\ t \in [0,T],\ \mathbb{P} - a.s.$$

For instance, the classical case corresponds to

$$\widehat{f}_s^{\mathbb{P}}(y,z) = r_s y + z \cdot \theta_s^{\mathbb{P}},$$

where r_s is the risk-free rate of the market and $\theta^{\mathbb{P}}$ is the risk premium vector under $\mathbb{P}$, defined by $\theta_s^{\mathbb{P}} := (\widehat{a}_s^{1/2})^{-1}(b_s^{\mathbb{P}} - r_s\mathbf{1}_n)$, where $(\widehat{a}_s^{1/2})^{-1}$ denotes the reverse of $\widehat{a}_s^{1/2}$. The simplest example of a nonlinear $\widehat{f}$ corresponds to the case where there are different lending and borrowing $\underline{r}_t \le \overline{r}_t$, in which (see Example 1.1 in [28])

$$\widehat{f}_s^{\mathbb{P}}(y,z) = \underline{r}_s y + z \cdot \theta_s^{\mathbb{P}} - (\overline{r}_s - \underline{r}_s)(y - z \cdot \mathbf{1}_n)^-.$$

We will always assume that the generator $\widehat{f}^{\mathbb{P}}$ satisfies our standing hypotheses in Assumptions 2.2 and 4.1. Let us now be given some stochastic process L belongs to $\mathbb{D}_0^{\mathbb{F}_+^{\mathcal{P}_0}}$. The problem of super-hedging L corresponds to finding its super-replication price defined as

$$P_{\sup}(L) := \inf\{y_0 \in \mathbb{R} : \exists Z \in \mathcal{H},\ Y_\tau^{y_0,Z} \ge L_\tau,\ \forall \tau \in \mathbb{T}^{0,T},\ \mathcal{P}_0^{\mathcal{U}} - q.s.\},$$

where the set of admissible trading strategies $\mathcal{H}$ is defined as the set of $\mathbb{F}^{\mathcal{P}_0^{\mathcal{U}}}$-predictable processes Z such that in addition, $\left(Y_t^{y_0,Z}\right)_{t\in[0,T]}$ is a nonlinear supermartingale under $\mathbb{P}$ for any $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$, in the sense that for any stopping times $0 \le \sigma \le \tau \le T$,

$$Y_\sigma^{y_0,Z} \ge \mathcal{Y}_\sigma^{\mathbb{P}}(\tau, Y_\tau^{y_0,Z}),\ \mathbb{P} - a.s.$$

When there is only volatility uncertainty in continuous-time setting, the authors in [10] provided an upper bound for the super-hedging price of an American options in the nonlinear market, as the initial value of a 2RBSDE. Afterwards, the authors in [11] mentioned that this can be actually the super-hedging price itself. Our aim is to obtain a super-hedging price of American options in the nonlinear, incomplete market with both drift and volatility uncertainty. We will follow Possamaï, Tan and Zhou [9] where the super-hedging price of European options has already been treated in that context.

Theorem 5.4. *Suppose that Assumptions 2.2 and 4.1 hold and the set $\mathcal{P}_0$ is saturated. Let (Y, Z) be the first two components of the saturated solution of the 2RBSDE with generator $\widehat{f}^{\mathbb{P}}$, terminal condition L_T and lower obstacle L. Then*

$$P_{\sup}(L) = \sup_{\mathbb{P}\in\mathcal{P}_0^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[Y_0],$$

and $Z \in \mathcal{H}$ is a super-hedging strategy for L.

Proof. First of all, assume that we have some $Z \in \mathcal{H}$ such that $Y_\tau^{y_0,Z} \geq L_\tau$ for all $\tau \in \mathbb{T}^{0,T}$, $\mathcal{P}_0^{\mathcal{U}} - q.s.$ Then, since $Y_t^{y_0,Z}$ is a non-linear super-martingale under $\mathbb{P}$ for any $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$, we have

$$y_0 \geq \mathcal{Y}_0^{\mathbb{P}}(\tau, Y_\tau^{y_0,Z}),\ \mathcal{P}_0^{\mathcal{U}} - q.s.$$

However, by comparison, we also have $\mathcal{Y}_0^{\mathbb{P}}(\tau, Y_\tau^{y_0,Z}) \geq \mathcal{Y}_0^{\mathbb{P}}(\tau, L_\tau)$, from which we deduce

$$y_0 \geq \mathcal{Y}_0^{\mathbb{P}}(\tau, L_\tau).$$

In particular, we deduce that

$$y_0 \geq \underset{\tau\in\mathbb{T}^{0,T}}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \underset{\mathbb{P}'\in\mathcal{P}_0^{\mathcal{U}}(0,\mathbb{P},\mathbb{F}_+)}{\operatorname{ess\,sup}^{\mathbb{P}}} \mathcal{Y}_0^{\mathbb{P}'}(\tau, L_\tau) = Y_0,\ \mathbb{P} - a.s.,$$

where we have used Corollary 4.8. It therefore directly implies, since y_0 is deterministic, that

$$y_0 \geq \sup_{\mathbb{P}\in\mathcal{P}_0^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[Y_0].$$

For the reverse inequality, let $(Y, Z, (K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0^{\mathcal{U}}}) \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0^{\mathcal{U}}}) \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}_0^{\mathcal{U}}}) \times \mathbb{I}_0^{o,p}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0^{\mathcal{U}}})$ be the unique saturated solution to the 2RBSDE with generator $\widehat{f}^{\mathbb{P}}$ and lower obstacle L. Then we have for any $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$ and $\tau \in \mathbb{T}^{0,T}$,

$$Y_0 - \int_0^\tau \widehat{f}_s^{\mathbb{P}}(Y_s, (\widehat{a}_s^{1/2})^\mathsf{T} Z_s)\mathrm{d}s + \int_0^\tau Z_s \cdot dX_s^{c,\mathbb{P}} = L_\tau + K_\tau^{\mathbb{P}} \geq L_\tau,\ \mathbb{P} - a.s.$$

However, since Y_0 is only $\mathcal{F}_{0+}^{\mathcal{P}_0^{\mathcal{U}}}$-measurable, it is not, in general, deterministic, so that we cannot conclude directly. Let us nonetheless consider, for any $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$, $y_0^{\mathbb{P}}$ the smallest constant which dominates Y_0, $\mathbb{P} - a.s.$. We therefore want to show that for any $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$,

$$y_0^{\mathbb{P}} \leq \sup_{\mathbb{P}\in\mathcal{P}_0^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[Y_0],$$

which can be done by following exactly the same arguments as in the proof of Theorem 3.2 in [29]. Finally, we do have $Z \in \mathcal{H}$, since by Corollary 4.7, Y is automatically a nonlinear supermartingale for every $\mathbb{P} \in \mathcal{P}_0^{\mathcal{U}}$. □

Concluding remarks

In this paper, we proposed a new type of minimality condition for the 2RBSDEs with lower obstacle and proved the existence and uniqueness of solution under Lipschitz-type assumptions on generator. We then applied this result to obtain the super-hedging price of American option in the uncertain financial market. We also note that the solution of 2RBSDE is characterized as the unique viscosity solution of obstacle problems for fully nonlinear path-dependent PDEs (PPDEs). This was actually done in Ekren [30], where the author proved that the value function, which is defined as the supremum of solutions of RBSDEs, becomes the unique viscosity solution of fully nonlinear variational inequality. In view of the representation formula (see Thm. 4.2 and Remark 4.4), we see that such value function is just the solution of 2RBSDE.

Finally, we further present some important remarks below.

- Following [6, 24], our result may be extended to the dynamics with extended monotonic generators. In particular, one can prove the representation formula without using the linearization argument, by just following exactly the same arguments as in [6, 24] (see the proof of Thm. 3.4 in [24] and Appendix part in [6]). We would like to mention that such extension is unlikey to be compatible with the original definition, which is heavily based on the linearization argument.
- As it was mentioned in [11, 12, 31], the 2RBSDE with upper obstacle has an essential difference with the 2RBSDE with lower obstacle, whereas they are symmetric for the standard RBSDEs (see also [32, 33] for G-RBSDEs with upper obstacles). In particular, the third component $K^{\mathbb{P}}$ of the solution, is not non-decreasing in general. In fact, this difference comes from the fact that there is a fundamental difference between nonlinear submartingales and supermartingales (see [34]). One may expect that the minimality condition for the 2RBSDE with upper obstacle U is

$$K_t^{\mathbb{P}} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\tau\in\mathbb{T}^{t,T}} \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{U}_\tau + K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right],\ 0\le t\le T,\ \mathbb{P}-a.s., \tag{5.2}$$

  where $\widetilde{U}_\tau := \mathbf{1}_{\tau<T}U_\tau + \mathbf{1}_{\tau=T}\xi$. However, since $K^{\mathbb{P}}$ is not non-decreasing, we cannot follow the standard approach in Section 4. Moreover, we need to clarify the relation

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\tau\in\mathbb{T}^{t,T}} \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{U}_\tau + K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right] = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} \operatorname*{ess\,sup}^{\mathbb{P}}_{\tau\in\mathbb{T}^{t,T}} \mathbb{E}^{\mathbb{P}'}\left[Y_\tau - \widetilde{U}_\tau + K_\tau^{\mathbb{P}'}\middle|\mathcal{F}_{t+}^{\mathbb{P}'}\right]. \tag{5.3}$$

  Under sufficient regularity assumptions, Nutz and Zhang [35] showed this result at $t = 0$ (see Thm. 3.4 therein). However, the generalization of their work to our irregular setting is still in development. We leave the study of 2RBSDEs with an upper obstacle to our future work.

Acknowledgements. We are sincerely grateful to the anonymous referee for careful reading and insightful comments which improved the first version of the paper.

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