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ASYMPTOTIC BEHAVIOR FOR A TIME-INHOMOGENEOUS KOLMOGOROV TYPE DIFFUSION

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Abstract. We study a kinetic stochastic model with a non-linear time-inhomogeneous friction force and a Brownian-type random force. More precisely, a Kolmogorov type diffusion (V, X) is considered: here, X is the position of the particle, and V is its velocity. The process V is solution to a stochastic differential equation driven by a one-dimensional Brownian motion, with a drift of the form $t^{-\beta}F(v)$. The function F satisfies some homogeneity condition, and β is a real number. The behavior in large time of the process (V, X) is proved by using stochastic analysis tools.

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1. Introduction

In several domains as fluids dynamics, statistical mechanics, or biology, a number of models are based on the Fokker-Planck and Langevin equations driven by Brownian motion, could be non-linear or driven by other random processes. For example, in [5] the persistent turning walker model was introduced, inspired by the modelling of fish motion. An associated two-component Kolmogorov type diffusion solves a kinetic system based on an Ornstein-Uhlenbeck Gaussian process, and the authors study the large-time behavior of this model by using appropriate tools from stochastic analysis. One of the natural questions is to understand the behavior in large-time of the solution to the corresponding stochastic differential equation (SDE). Although the tools of partial differential equations allow us to ask this kind of question, and since these models are probabilistic, tools based on stochastic processes could be more natural to use.

In the last decade, the asymptotic study of solutions to non-linear Langevin's type was the subject of an important number of papers, see [6], [8] and [9]. For instance, in [9] the following system is studied

$$V_t = v_0 + B_t - \frac{\rho}{2}\int_0^t F(V_s)\,\mathrm{d}s \quad \text{and} \quad X_t = x_0 + \int_0^t V_s\,\mathrm{d}s.$$

In other words, one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like $(1+|v|^2)^{-\rho/2}$, as $|v| \to +\infty$. The authors prove that for large-time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of ρ. Results have

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been extended to additive functional of V in [4]. It should be noticed that these cited papers use the standard tools associated with time-homogeneous equations, as invariant measure, scale function, and speed measure. Several of them will not be available when the drag force depends explicitly on time. In [11], a non-linear SDE driven by a Brownian motion but having time-inhomogeneous drift coefficient was studied, and its large-time behavior was described. Moreover, sharp rates of convergence are proved for the 1-dimensional marginal of the solution. In the present paper, we consider the velocity process as satisfying the same kind of SDE.

Let us describe our framework: we consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by $(X_t)_{t>0}$ the process describing the position of a particle at time t and having the velocity V_t. The velocity process $(V_t)_{t>0}$ is supposed to solve a Brownian-driven SDE with a drag force, varying in time:

$$\mathrm{d}V_t = \mathrm{d}B_t - b(t, V_t)\,\mathrm{d}t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s.$$

This system can be viewed as a perturbation of the classical two-component Kolmogorov diffusion

$$\mathrm{d}V_t = \mathrm{d}B_t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s.$$

In the present paper the drift is supposed to grow slowly to infinity, and it will be supposed to be of the form $t^{-\beta}F(v)$, with $\beta \in \mathbb{R}$ and F satisfying some homogeneity condition. It describes a one-dimensional particle subject to a friction force and undergoing many small random shocks. A natural question is to understand the behavior of the process (V, X) in large time. More precisely we look for the limit in distribution of $(r_{\varepsilon,V}V_{t/\varepsilon}, r_{\varepsilon,X}X_{t/\varepsilon})_t$, as $\varepsilon \to 0$, for some rates of convergence. Our results are proved on the product of path spaces and consequently contain those of [11].

If $F = 0$, it is not difficult to see that the rescaled position process $(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}}X_{t/\varepsilon})_t$ converges in distribution towards the Kolmogorov diffusion $(B_t, \int_0^t B_s\,\mathrm{d}s)_t$. We prove that this kinetic behavior still holds for sufficiently "small at infinity" drag force. The strategy to tackle this problem is based on estimates of moments of the velocity process. The main result can then be extended to the case of a drift being equally weighted in some sense as the random noise. It either offsets the random noise (critical regime) or swings with it (sub-critical regime).

As suggested at the beginning of the introduction, other random noises can be considered. In [10], the case of a Lévy random noise is analyzed. The case of a stochastic system evolving in a quadratic potential is the purpose of another work (see [7]).

The organization of our paper is as follows: in the next section, we introduce notations, and we state our main results. Results about existence and non-explosion of solutions are stated in Section 3. Estimates of the moments of the velocity process are given in Section 4 while the proofs of our main results are presented in Section 5.

2. Notations and main results

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, β a real number and F a continuous function which is supposed to satisfy either

$$\text{for some } \gamma \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\gamma F(v), \tag{H_1^γ}$$

or

$$|F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_1^\gamma). \tag{H_2^γ}$$

Each assumption implies that there exist a positive constant K such that, for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^\gamma$. Obviously (H_2^γ) is a generalization of (H_1^γ). In the following, sgn is the sign function with convention $\mathrm{sgn}(0) = 0$. As an example of function satisfying (H_1^γ) one can keep in mind $F : v \mapsto \mathrm{sgn}(v)\,|v|^\gamma$ (see also [11]), and as an example of function satisfying (H_2^γ) (with $\gamma = 0$) $F : v \mapsto v/(1+v^2)$ (see also [9]).

Remark 2.1. If a function π satisfies (H_1^γ), then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\mathrm{sgn}(x))\,|x|^\gamma$.

We consider the following one-dimensional stochastic kinetic model, for $t \geq t_0 > 0$,

$$\mathrm{d}V_t = \mathrm{d}B_t - t^{-\beta} F(V_t)\,\mathrm{d}t,\ V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t,\ X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

Most of the convergences take place in the space of continuous functions $\mathcal{C}((0,+\infty),\mathbb{R})$ endowed by the uniform topology

$$\mathrm{d}_u : (f,g) \in \mathcal{C}((0,+\infty),\mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\Big(1, \sup_{[\frac{1}{n},n]} |f-g|\Big).$$

For a family $((Z_t^{(\varepsilon)})_{t>0})_{\varepsilon>0}$ of continuous processes, we write

$$(Z_t^{(\varepsilon)})_{t>0} \quad \underset{\varepsilon\to 0}{\Longrightarrow} \quad (Z_t)_{t>0},$$

if $(Z_t^{(\varepsilon)})_{t>0}$ converges in distribution to $(Z_t)_{t>0}$ in $\mathcal{C}((0,+\infty),\mathbb{R})$, as $\varepsilon \to 0$.

We write

$$(Z_t^{(\varepsilon)})_{t>0} \quad \overset{\text{f.d.d.}}{\underset{\varepsilon\to 0}{\Longrightarrow}} \quad (Z_t)_{t>0},$$

if for all finite subsets $S \subset (0,+\infty)$, the vector $(Z_t^{(\varepsilon)})_{t\in S}$ converges in distribution to $(Z_t)_{t\in S}$ in $\mathbb{R}^S$, as $\varepsilon \to 0$.

Let us state our main results. Set $q := \dfrac{\beta}{\gamma+1}$.

Theorem 2.2. *Consider $\gamma \geq 0$, and $q > \frac{1}{2}$. Assume that either (H_1^γ) or (H_2^γ) is satisfied. Let $(V_t, X_t)_{t\geq t_0}$ be the solution to (SKE) and $(\mathcal{B}_t)_{t\geq 0}$ be a standard Brownian motion. Furthermore, if $\gamma \geq 1$, we suppose that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.*

Then,

$$\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon}\right)_{t\geq \varepsilon t_0} \quad \underset{\varepsilon\to 0}{\Longrightarrow} \quad \left(\mathcal{B}_t, \int_0^t \mathcal{B}_s\,\mathrm{d}s\right)_{t\geq 0}.$$

Theorem 2.3. *Consider $\gamma \geq 0$ and $q = \frac{1}{2}$. Assume that (H_1^γ) is satisfied. Let $(V_t, X_t)_{t\geq t_0}$ be the solution to (SKE). If $\gamma \geq 1$, we suppose furthermore that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.*

Call $\widetilde{H}$ the eternal ergodic process, solution to the homogeneous SDE

$$\mathrm{d}H_s = \mathrm{d}W_s - \frac{H_s}{2}\,\mathrm{d}s - F\big(H_s\big)\,\mathrm{d}s,$$

such that the distribution of $\widetilde{H}_{-\infty}$ is the invariant measure, where $(W_t)_{t\geq 0}$ is again a standard Brownian motion. Setting $\Lambda_{F,t_1,\cdots,t_d}$ for the finite dimensional distributions (f.d.d.) of $\widetilde{H}$, we call $(\mathcal{V}_t)_{t\geq 0}$ the process whose f.d.d. are $T * \Lambda_{F,\log(t_1),\cdots,\log(t_d)}$, the pushforward measure of $\Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ by the linear map $T(u_1,\cdots,u_d) := (\sqrt{t_1}u_1,\cdots,\sqrt{t_d}u_d)$. Indeed, we have $(\mathcal{V}_t)_{t\geq 0} = (\sqrt{t}\widetilde{H}_{\log(t)})_{t\geq 0}$.

Then,

$$\left(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon},\varepsilon^{\frac{3}{2}}X_{t/\varepsilon}\right)_{t\geq\varepsilon t_0} \underset{\varepsilon\to 0}{\Longrightarrow} \left(\mathcal{V}_t,\int_0^t \mathcal{V}_s\,\mathrm{d}s\right)_{t\geq 0}.$$

Remark 2.4. The one-dimensional distribution of $(\mathcal{V}_t)_{t\geq 0}$ has already been explicitly computed (see Thm. 4.1 in [11]).

Theorem 2.5. *Consider $\gamma\geq 1$ and $q<\frac{1}{2}$. Assume that $F: v\mapsto \rho\,\mathrm{sgn}(v)\,|v|^{\gamma}$ with $\rho>0$. Let $(V_t,X_t)_{t\geq t_0}$ be the solution to (SKE). Call $\widehat{H}$ the ergodic process, solution to the homogeneous SDE*

$$\mathrm{d}H_s=\mathrm{d}W_s-F\left(H_s\right)\mathrm{d}s,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. Call Π_F its invariant measure. We call $(\mathscr{V}_t)_{t\geq 0}$ the process whose f.d.d. are $T*\left(\Pi_F^{\otimes d}\right)$, the pushforward measure of $\Pi_F^{\otimes d}$ by the linear map $T(u_1,\cdots,u_d):=({t_1}^q u_1,\cdots,{t_d}^q u_d)$.

Then,

$$\left(\varepsilon^q V_{t/\varepsilon}\right)_{t\geq\varepsilon t_0} \underset{\varepsilon\to 0}{\overset{f.d.d.}{\Longrightarrow}} (\mathscr{V}_t)_{t\geq 0}.$$

Moreover, in the linear case (i.e. $\gamma=1$) and if $\beta>-\frac{1}{2}$, we define $(\mathscr{X}_t)_{t\geq 0}$ the centered Gaussian process with covariance function $K(s,t):=(\rho^2(1+2\beta))^{-1}(s\wedge t)^{1+2\beta}$.

Then, as $\varepsilon\to 0$,

$$\left(\varepsilon^{\beta+\frac{1}{2}}X_{t/\varepsilon}\right)_{t\geq\varepsilon t_0} \underset{\varepsilon\to 0}{\overset{f.d.d.}{\Longrightarrow}} (\mathscr{X}_t)_{t\geq 0}. \tag{2.1}$$

Remark 2.6. If $\beta=0$, one can prove using the martingale method, that $(\sqrt{\varepsilon}X_{t/\varepsilon})_{t\geq 0}$ converges towards a Brownian motion. Assume, by way of contradiction, that the process $(\varepsilon^q V_{t/\varepsilon})_{t\geq\varepsilon t_0}$ would converge (*i.e.* were tight), then by the continuous mapping theorem, the process $(\varepsilon X_{t/\varepsilon})_{t\geq 0}$ should converge. This is a contradiction with (2.1). Here is why we deal only with the finite-dimensional convergence of the velocity process.

3. Changed-of-time processes

In the following, we suppose that $\gamma>-1$ and set $\Omega=\overline{\mathcal{C}}([t_0,+\infty))$ the set of continuous functions, that equal $+\infty$ after their (possibly infinite) explosion time. Following the idea used in [11], we first perform a change of time in (SKE) in order to produce at least one time-homogeneous coefficient in the transformed equation. For every $\mathcal{C}^2$-diffeomorphism $\varphi:[0,t_1)\to[t_0,+\infty)$, let introduce the scaling transformation Φ_φ defined, for $\omega\in\Omega$, by

$$\Phi_\varphi(\omega)(s):=\frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}},\quad \text{with } s\in[0,t_1).$$

The result containing the change of time transformation can be found in Proposition 2.1 p. 187 in [11].

Let V be solution to the equation (SKE). Thanks to Lévy's characterization theorem of the Brownian motion, $(W_t)_{t\geq 0}:=\left(\int_0^t\frac{\mathrm{d}B_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right)_{t\geq 0}$ is a standard Brownian motion. Then, by a change of variable $t=\varphi(s)$, one gets

$$V_{\varphi(t)}-V_{\varphi(0)}=\int_0^t\sqrt{\varphi'(s)}\,\mathrm{d}W_s-\int_0^t\frac{F(V_{\varphi(s)})}{\varphi(s)^{\beta}}\varphi'(s)\,\mathrm{d}s.$$

The integration by parts formula yields

$$\mathrm{d}\left(\frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right) = \mathrm{d}W_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta} F(V_{\varphi(s)})\,\mathrm{d}s - \frac{\varphi''(s)}{2\varphi'(s)}\frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}}\,\mathrm{d}s.$$

As a consequence, we can state the following result in our context.

Proposition 3.1. *If V is a solution to the equation (SKE), then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$\mathrm{d}V_s^{(\varphi)} = \mathrm{d}W_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta} F(\sqrt{\varphi'(s)}V_s^{(\varphi)})\,\mathrm{d}s - \frac{\varphi''(s)}{\varphi'(s)}\frac{V_s^{(\varphi)}}{2}\,\mathrm{d}s, \quad V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\sqrt{\varphi'(0)}}, \tag{3.1}$$

where $W_t = \int_0^t \frac{\mathrm{d}B_{\varphi(s)}}{\sqrt{\varphi'(s)}}$.
If $V^{(\varphi)}$ is a solution to (3.1), then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation (SKE), where $B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\varphi'\circ\varphi^{-1})(s)}\,\mathrm{d}W_{\varphi^{-1}(s)}$.
Furthermore, uniqueness in law, pathwise uniqueness or strong existence hold for the equation (SKE) if and only if they hold for the equation (3.1).

In the following, we will use two particular changes of time, depending on which term of (3.1) should become time-homogeneous.

- *The exponential change of time*: setting $\varphi_e : t \mapsto t_0 e^t$, the exponential scaling transformation is defined by $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega_{t_0 e^s}}{\sqrt{t_0} e^{\frac{s}{2}}}$, for $\omega \in \Omega$. Thanks to Proposition 3.1, the process $V^{(e)} := \Phi_e(V)$ satisfies the equation

$$\mathrm{d}V_s^{(e)} = \mathrm{d}W_s - \frac{V_s^{(e)}}{2}\,\mathrm{d}s - t_0^{\frac{1}{2}-\beta} e^{(\frac{1}{2}-\beta)s} F\big(\sqrt{t_0} e^{\frac{s}{2}} V_s^{(e)}\big)\,\mathrm{d}s,$$

  where $(W_t)_{t\geq 0}$ is a standard Brownian motion.
- *The power change of time*: for $q = \frac{\beta}{\gamma+1}$, consider $\varphi_q \in \mathcal{C}^2([0,t_1))$ the solution to the Cauchy problem

$$\varphi_q' = \varphi_q^{2q}, \quad \varphi_q(0) = t_0.$$

  Clearly, $\varphi_q(t) = \big(t_0^{1-2q} + (1-2q)t\big)^{1/(1-2q)}$, when $2q \neq 1$, and $\varphi_q = \varphi_e$, when $2q = 1$.
  The time t_1 satisfies $t_1 = +\infty$, when $2q \leq 1$, and $t_1 = t_0^{1-2q}(2q-1)^{-1}$, when $2q > 1$. The power scaling transformation is defined by $\Phi_q(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega(\varphi_q(s))}{\varphi_q(s)^q}$. The process $V^{(q)} := V^{(\varphi_q)}$ satisfies the equation

$$\mathrm{d}V_s^{(q)} = \mathrm{d}W_s - \varphi_q^{-\gamma q}(s) F\left(\sqrt{\varphi_q'(s)} V_s^{(q)}\right)\mathrm{d}s - q\varphi_q^{2q-1}(s) V_s^{(q)}\,\mathrm{d}s, \tag{3.2}$$

  where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

Adapting the proof of Propositions 3.2, 3.6 and 3.7 p. 188 in [11], one can prove the following proposition.

Proposition 3.2. *For $\gamma \geq 0$, there exists a pathwise unique strong solution to (SKE), defined up to the explosion time τ_∞ of V.*

- *When $\gamma \leq 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$, then the explosion time of V is a.s. infinite.*

- *If $2q > 1$, then $\mathbb{P}(\tau_\infty = +\infty) > 0$.*
- *Under (H_1^γ), if $\gamma > 1$ and $(F(-1), F(1)) \in ((0, +\infty)) \times [0, +\infty)) \cup (\mathbb{R} \times (-\infty, 0))$, then we have $\mathbb{P}(\tau_\infty = +\infty) < 1$, where τ_∞ denotes the explosion time of V.*

Remark 3.3. Assume that (H_1^γ) is satisfied. In the linear case ($\gamma = 1$), the drift and the diffusion terms are Lipschitz and satisfy locally linear growth condition. The existence and non-explosion of V follow from Thm. 2.9 p. 289 in [13].

For more details, we refer to [14].

4. Moment estimates of the velocity process

In this section, we give estimates for the moment of the velocity process. It will be useful to control some stochastic terms appearing later.

Proposition 4.1. *Assume that $\gamma \geq 0$ and $\beta \in \mathbb{R}$. The inequality*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{2}}$$

holds for

- *$\kappa \in [0, 1]$, when $\gamma < 1$ and $\beta \geq \frac{\gamma+1}{2}$,*
- *$\kappa \geq 0$, when for all $v \in \mathbb{R}$, $vF(v) \geq 0$.*

If $\kappa \in [0, 1]$, $\gamma < 1$ and $\beta < \frac{\gamma+1}{2}$, then

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\kappa\frac{1-\beta}{1-\gamma}}.$$

Remark 4.2. When $-1 < \gamma < 0$, it can be proved that for all $t \geq t_0$, $\mathbb{E}\left[|V_t|\right] \leq C_{\gamma,\beta,t_0}\sqrt{t}$, without hypothesis of the positivity of the function $v \mapsto vF(v)$.

Proof. Step 1. Assume that $\gamma \geq 1$ and that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.
Define, for all $n \geq 0$, the stopping time $T_n := \inf\{t \geq t_0,\ |V_t| \geq n\}$. By Itô's formula, for all $t \geq t_0$, we have

$$\begin{aligned} V_{t\wedge T_n}^2 &= v_0^2 + \int_{t_0}^{t\wedge T_n} 2V_s \, \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} 2s^{-\beta} V_s F(V_s) \, \mathrm{d}s + (t \wedge T_n - t_0) \\ &= v_0^2 + \int_{t_0}^{t} \mathbb{1}_{\{s \leq T_n\}} 2V_s \, \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} 2s^{-\beta} V_s F(V_s) \, \mathrm{d}s + (t \wedge T_n - t_0) \\ &\leq v_0^2 + \int_{t_0}^{t} \mathbb{1}_{\{s \leq T_n\}} 2V_s \, \mathrm{d}B_s + (t - t_0). \end{aligned}$$

Since $\int_{t_0}^t 4\mathbb{1}_{\{s \leq T_n\}} V_s^2 \, \mathrm{d}s \leq 4n^2(t - t_0) < +\infty$, taking expectation yields

$$\mathbb{E}\left[V_{t\wedge T_n}^2\right] \leq v_0^2 + (t - t_0) \leq C_{t_0} t.$$

Set $\kappa \in [0, 2]$, we obtain by Jensen's inequality that

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|^2\right]^{\frac{\kappa}{2}} \leq \left(\liminf_{n\to+\infty} \mathbb{E}\left[V_{t\wedge T_n}^2\right]\right)^{\frac{\kappa}{2}} \leq C_{\kappa,t_0} t^{\frac{\kappa}{2}}. \tag{4.1}$$

When $\kappa > 2$, the function $v \mapsto |v|^\kappa$ is $\mathcal{C}^2$, so by Itô's formula, we can write for all $t \geq t_0$,

$$|V_{t\wedge T_n}|^\kappa = |v_0|^\kappa + \int_{t_0}^{t\wedge T_n} \kappa \operatorname{sgn}(V_s)\, |V_s|^{\kappa-1}\, \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} \kappa s^{-\beta}\, |V_s|^{\kappa-1} \operatorname{sgn}(V_s) F(V_s)\, \mathrm{d}s + \int_{t_0}^{t\wedge T_n} \frac{\kappa(\kappa-1)}{2}\, |V_s|^{\kappa-2}\, \mathrm{d}s.$$

In addition, using the hypothesis on the sign of F, we have

$$|V_{t\wedge T_n}|^\kappa \leq |v_0|^\kappa + \int_{t_0}^{t} \mathbb{1}_{\{s\leq T_n\}} \kappa \operatorname{sgn}(V_s)\, |V_s|^{\kappa-1}\, \mathrm{d}B_s + \int_{t_0}^{t\wedge T_n} \frac{\kappa(\kappa-1)}{2}\, |V_s|^{\kappa-2}\, \mathrm{d}s. \tag{4.2}$$

We observe that $\int_{t_0}^t \kappa^2 V_s^{2\kappa-2} \mathbb{1}_{\{s\leq T_n\}}\, \mathrm{d}s \leq \kappa^2 n^{2\kappa-2}(t-t_0) < +\infty$. Taking expectation in (4.2), we obtain

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq \liminf_{n\to+\infty} \mathbb{E}\left[|V_{t\wedge T_n}|^\kappa\right] \leq |v_0|^\kappa + \int_{t_0}^{t} \frac{\kappa(\kappa-1)}{2} \mathbb{E}\left[|V_s|^{\kappa-2}\right] \mathrm{d}s.$$

When $0 \leq \kappa - 2 \leq 2$, we can upper bound $\mathbb{E}\left[|V_s|^{\kappa-2}\right]$ by injecting (4.1) and get

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq |v_0|^\kappa + \int_{t_0}^{t} \frac{\kappa(\kappa-1)}{2} C_{\kappa,t_0} s^{\frac{\kappa-2}{2}}\, \mathrm{d}s \leq C_{\kappa,t_0} s^{\frac{\kappa}{2}}.$$

The same method is then applied inductively to prove the inequality for all $\kappa > 2$.
Step 2. Assume now that $\gamma \in [0,1)$.
Fix $\kappa \in [0,1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|\right]^\kappa$, hence it suffices to verify the inequality only for $\kappa = 1$.
Define, for all $n \geq 0$, the stopping time $T_n := \inf\{t \geq t_0,\ |V_t| \geq n\}$ and let us recall that under both hypotheses (H_1^γ) or (H_2^γ), there exists a positive constant K, such that $|F(v)| \leq K|v|^\gamma$. We can write, for $t \geq t_0$ and $n \geq 0$,

$$\begin{aligned}
|V_{t\wedge T_n}| &\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} s^{-\beta}\, |F(V_{s\wedge T_n})|\, \mathrm{d}s \\
&\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} K s^{-\beta}\, |V_{s\wedge T_n}|^\gamma\, \mathrm{d}s.
\end{aligned}$$

By noting that $\gamma \in [0,1[$ and that $(B_t^2 - t)_{t\geq 0}$ is a martingale, taking expectation we get

$$\begin{aligned}
\mathbb{E}\left[|V_{t\wedge T_n}|\right] &\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \mathbb{E}\left[|B_{t\wedge T_n}|\right] + \int_{t_0}^{t} K s^{-\beta} \mathbb{E}\left[|V_{s\wedge T_n}|^\gamma\right] \mathrm{d}s \\
&\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \sqrt{\mathbb{E}\left[B_{t\wedge T_n}^2\right]} + \int_{t_0}^{t} K s^{-\beta} \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\gamma \mathrm{d}s \\
&\leq \mathbb{E}\left[|v_0 - B_{t_0}|\right] + \sqrt{\mathbb{E}\left[t\wedge T_n\right]} + \int_{t_0}^{t} K s^{-\beta} \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\gamma \mathrm{d}s \\
&\leq C_{t_0}\sqrt{t} + \int_{t_0}^{t} K s^{-\beta} \mathbb{E}\left[|V_{s\wedge T_n}|\right]^\gamma \mathrm{d}s.
\end{aligned}$$

The function $g_n : t \mapsto \mathbb{E}\left[|V_{t\wedge T_n}|\right]$ is bounded by n. Applying the Grönwall-type lemma stated below (Lem. 4.3) and Fatou's lemma, for $\beta \neq 1$ and for all $t \geq t_0$, we end up with

$$\mathbb{E}\left[|V_t|\right] \leq \liminf_{n\to+\infty} \mathbb{E}\left[|V_{t\wedge T_n}|\right] \leq C_\gamma \left[C_{t_0}\sqrt{t} + \left(\frac{1-\gamma}{1-\beta}K(t^{1-\beta} - t_0^{1-\beta})\right)^{\frac{1}{1-\gamma}}\right]$$
$$\leq C_{\gamma,\beta,t_0} \begin{cases} \sqrt{t} & \text{if } \beta \geq \frac{\gamma+1}{2}, \\ t^{\frac{1-\beta}{1-\gamma}} & \text{else.} \end{cases}$$

The case $\beta = 1$ can be treated similarly.

□

Lemma 4.3 (Grönwall-type lemma)**.** *Fix $r \in [0,1)$ and $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is continuous. If*

$$\forall t \geq t_0,\ g(t) \leq a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s, \tag{4.3}$$

then,

$$\forall t \geq t_0,\ g(t) \leq 2^{\frac{1}{1-r}}\left[a(t) + \left((1-r)\int_{t_0}^t b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}}\right].$$

Proof. For $t \geq t_0$, since $r \geq 0$,

$$g(t)^r \leq \left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r,$$

then, multiplying by $b(t) > 0$,

$$b(t)g(t)^r \leq b(t)\left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r.$$

Now, let us make appear the derivative of H

$$a'(t) + b(t)g(t)^r \leq a'(t) + b(t)\left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r,$$

that is

$$\frac{a'(t) + b(t)g(t)^r}{\left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r} \leq b(t) + \frac{a'(t)}{\left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^r} \leq b(t) + \frac{a'(t)}{a(t)^r}.$$

Integrating, since $r \neq 1$, we obtain

$$(1-r)^{-1}\left[\left(a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s\right)^{1-r} - a(t_0)^{1-r}\right] \leq (1-r)^{-1}\left[a(t)^{1-r} - a(t_0)^{1-r}\right] + \int_{t_0}^t b(s)\,\mathrm{d}s$$

or equivalently, setting H for the right-hand side of (4.3) and using that $r < 1$, we get

$$H(t)^{1-r} \leq a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s.$$

Since $\frac{1}{1-r} > 0$ and using (4.3)

$$g(t) \leq \left(a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}} \leq C_r\left[a(t) + \left((1-r)\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{\frac{1}{1-r}}\right].$$

This concludes the proof of the lemma. □

Remark 4.4. Call H the right-hand side of (4.3). If g is not continuous, note that the function H is continuous and satisfies (4.3) (since b is positive and $g \leq H$). Therefore, one can apply the lemma to H and then use the inequality $g \leq H$.

5. Proof of the asymptotic behavior of the solution

This section is devoted to the proof of our main results.

5.1. Asymptotic behavior in the super-critical regime under both assumptions

In this section, we assume that $\gamma \geq 0$ and $q > \frac{1}{2}$.

Proof of Theorem 2.2. We split the proof into three steps.

Step 1. We note that it is enough to prove that the process

$$(V_t^{(\varepsilon)})_{t\geq 0} := (\sqrt{\varepsilon}V_{t/\varepsilon})_{t\geq 0}$$

converges in distribution to a Brownian motion in the space of continuous functions $\mathcal{C}([0,+\infty))$ endowed by the uniform topology. In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{C}([0,+\infty))$, let us state for all $s \in [0, \varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \sqrt{\varepsilon}v_0$.
For every $\varepsilon \in (0,1]$ and $t \geq \varepsilon t_0$, we can write

$$\varepsilon^{\frac{3}{2}}X_{t/\varepsilon} = \varepsilon^{\frac{3}{2}}x_0 + \int_{\varepsilon t_0}^{t} V_s^{(\varepsilon)}\,\mathrm{d}s.$$

Clearly, the theorem will be proved once we show that $g_\varepsilon(V_\bullet^{(\varepsilon)}) := (V_\bullet^{(\varepsilon)}, \int_{\varepsilon t_0}^{\bullet} V_s^{(\varepsilon)}\,\mathrm{d}s)$ converges weakly in $\mathcal{C}([0,+\infty))$ endowed by the uniform topology. Here the mapping $g_\varepsilon : v \mapsto \left(v_t, \int_{\varepsilon t_0}^{t} v_s\,\mathrm{d}s\right)_{t\geq 0}$ is defined and valued on $\mathcal{C}((0,+\infty))$. This mapping is converging, as $\varepsilon \to 0$, to the continuous mapping $g : v \mapsto \left(v_t, \int_0^t v_s\,\mathrm{d}s\right)_{t\geq 0}$.
We have, for every $\varepsilon \in (0,1]$ and $t \geq \varepsilon t_0$,

$$\begin{aligned} V_t^{(\varepsilon)} = \sqrt{\varepsilon}V_{t/\varepsilon} &= \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon}B_{t/\varepsilon} - \sqrt{\varepsilon}\int_{t_0}^{t/\varepsilon} F(V_s)s^{-\beta}\,\mathrm{d}s \\ &= \sqrt{\varepsilon}(v_0 - B_{t_0}) + B_t^{(\varepsilon)} - \varepsilon^{\beta-\frac{1}{2}}\int_{\varepsilon t_0}^{t} F(V_{u/\varepsilon})u^{-\beta}\,\mathrm{d}u. \end{aligned}$$

By self-similarity, $B^{(\varepsilon)} := (\sqrt{\varepsilon} B_{t/\varepsilon})_{t\geq 0}$ has the same distribution as a standard Brownian motion.
Assume that the convergence of the rescaled velocity process is proved in the strong way, that is

$$\forall T>0, \sup_{\varepsilon t_0 \leq t\leq T} \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| \quad \underset{\varepsilon\to 0}{\overset{\mathbb{P}}{\longrightarrow}} \quad 0. \tag{5.1}$$

Then it suffices to prove that $g_\varepsilon(B^{(\varepsilon)}) \Longrightarrow g(\mathcal{B})$ and $\mathrm{d}_u\left(g_\varepsilon(V^{(\varepsilon)}), g_\varepsilon(B^{(\varepsilon)})\right) \overset{\mathbb{P}}{\longrightarrow} 0$, as $\varepsilon \to 0$ (see Thm. 3.1 p. 27 in [2]).
On the one hand, the process $B^{(\varepsilon)}$ being a Brownian motion and $\|\cdot\|$ denoting a norm on $\mathbb{R}^2$, the first convergence follows from

$$\forall T>0, \sup_{\varepsilon t_0 \leq t\leq T} \| g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t) \| \quad \underset{\varepsilon\to 0}{\overset{\mathbb{P}}{\longrightarrow}} \quad 0. \tag{5.2}$$

Let us prove (5.2). For every $\varepsilon t_0 \leq t \leq T$, we get

$$\| g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t) \| = \left| \int_0^{\varepsilon t_0} \mathcal{B}_s \,\mathrm{d}s \right| \quad \leq \int_0^{\varepsilon t_0} |\mathcal{B}_s| \,\mathrm{d}s.$$

Hence,

$$\mathbb{E}\left[\sup_{\varepsilon t_0 \leq t\leq T} \| g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t) \| \right] \leq \int_0^{\varepsilon t_0} \mathbb{E}\,|\mathcal{B}_s| \,\mathrm{d}s \leq C \int_0^{\varepsilon t_0} \sqrt{s}\,\mathrm{d}s \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad 0.$$

On the other hand, we prove that

$$\forall T>0, \sup_{\varepsilon t_0 \leq t\leq T} \left\| g_\varepsilon(V_t^{(\varepsilon)}) - g_\varepsilon(B_t^{(\varepsilon)}) \right\| \quad \underset{\varepsilon\to 0}{\overset{\mathbb{P}}{\longrightarrow}} \quad 0. \tag{5.3}$$

For every $\varepsilon t_0 \leq t \leq T$, using (5.1)

$$\begin{aligned} \left\| g_\varepsilon(V_t^{(\varepsilon)}) - g_\varepsilon(B_t^{(\varepsilon)}) \right\| &= \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| + \left| \int_{\varepsilon t_0}^t V_s^{(\varepsilon)} - B_s^{(\varepsilon)} \,\mathrm{d}s \right| \\ &\leq (1+T-\varepsilon t_0) \sup_{\varepsilon t_0 \leq t\leq T} \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| \quad \underset{\varepsilon\to 0}{\overset{\mathbb{P}}{\longrightarrow}} \quad 0. \end{aligned}$$

Step 2. Let us prove now (5.1). Recall that under both hypothesis (H_1^γ) and (H_2^γ), there exists a positive constant K, such that $(\sqrt{\varepsilon})^\gamma \left| F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) \right| \leq K \left| V_u^{(\varepsilon)} \right|^\gamma$. Modifying the factor in front of the integral part, we get

$$V_t^{(\varepsilon)} = \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon} B_{t/\varepsilon} - \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^t (\sqrt{\varepsilon})^\gamma F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta}\,\mathrm{d}u.$$

It follows that, for all $T > 0$,

$$\sup_{\varepsilon t_0 \le t \le T} \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| \le \sqrt{\varepsilon}\, |v_0 - B_{t_0}| + \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \sup_{\varepsilon t_0 \le t \le T} \left| \int_{\varepsilon t_0}^{t} (\sqrt{\varepsilon})^{\gamma} F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}} \right) u^{-\beta}\, \mathrm{d}u \right|$$
$$\le \sqrt{\varepsilon}\, |v_0 - B_{t_0}| + \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^{T} K \left| V_u^{(\varepsilon)} \right|^{\gamma} u^{-\beta}\, \mathrm{d}u.$$

Taking the expectation and using moment estimates (Prop. 4.1), we obtain, when $\beta \neq \frac{\gamma}{2} + 1$ and since $\beta > \frac{\gamma+1}{2}$,

$$\varepsilon^{\beta - \frac{(\gamma+1)}{2}} \mathbb{E}\left[\int_{\varepsilon t_0}^{T} K \left| V_u^{(\varepsilon)} \right|^{\gamma} u^{-\beta}\, \mathrm{d}u \right] = \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^{T} K \mathbb{E}\left[\left| V_u^{(\varepsilon)} \right|^{\gamma} \right] u^{-\beta}\, \mathrm{d}u$$
$$\le \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^{T} C_{\gamma,\beta,t_0} u^{\frac{\gamma}{2} - \beta}\, \mathrm{d}u$$
$$\le C \left(\varepsilon^{\beta - \frac{(\gamma+1)}{2}} T^{\frac{\gamma}{2} - \beta + 1} - t_0^{\frac{\gamma}{2} - \beta + 1} \sqrt{\varepsilon} \right) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Hence, setting $r = \min(\frac{1}{2}, \beta - \frac{(\gamma+1)}{2}) > 0$,

$$\mathbb{E}\left[\sup_{\varepsilon t_0 \le t \le T} \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| \right] = \underset{\varepsilon \to 0}{O}(\varepsilon^r).$$

The case $\beta = \frac{\gamma}{2} + 1$ can be treated similarly to get

$$\mathbb{E}\left[\sup_{\varepsilon t_0 \le t \le T} \left| V_t^{(\varepsilon)} - B_t^{(\varepsilon)} \right| \right] = \underset{\varepsilon \to 0}{O}(\sqrt{\varepsilon} \ln(\varepsilon)).$$

This concludes the proof.

□

Remark 5.1. One can observe that the only moment in this proof when we need the condition "$\gamma < 1$ or for all $v \in \mathbb{R},\ vF(v) \ge 0$" is when we are proving the moment estimates.

5.2. Asymptotic behavior in the critical regime under (H_1^γ)

Assume in this section that $\beta = \frac{\gamma+1}{2}$ and (H_1^γ) is satisfied.

Proof of Theorem 2.3. Step 1. As in the first step of the previous section, it suffices to prove the convergence of the rescaled velocity process $(\sqrt{\varepsilon} V_{t/\varepsilon})_t$.
Keeping same notations, we prove that $g_\varepsilon(V^{(\varepsilon)})$ converges in distribution in $\mathcal{C}([0,+\infty))$ to $g(\mathcal{V})$. In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{C}([0,+\infty))$, let us set for all $s \in [0, \varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \sqrt{\varepsilon} v_0$. Call P_ε, P the distribution of $V^{(\varepsilon)}$ and $\mathcal{V}$ respectively. Then, using Portmanteau theorem (see Thm. 2.1 p. 16 in [2]), it suffices to prove that for all bounded and uniformly continuous function $h : \mathcal{C}([0,+\infty)) \times \mathcal{C}([0,+\infty)) \to \mathbb{R}$,

$$\int_{\mathcal{C}([0,+\infty))^2} h(g_\varepsilon(\omega))\, \mathrm{d}P_\varepsilon(\mathrm{d}\omega) \underset{\varepsilon \to 0}{\longrightarrow} \int_{\mathcal{C}([0,+\infty))^2} h(g(\omega))\, \mathrm{d}P(\mathrm{d}\omega).$$

Take a bounded and uniformly continuous function h. By assumption, one knows that $P_\varepsilon \underset{\varepsilon \to 0}{\Longrightarrow} P$, hence, by Problem 4.12 p. 64 in [13], it suffices to prove that the uniformly bounded sequence $(h \circ g_\varepsilon)$ of continuous

functions on $\mathcal{C}([0,+\infty))$ converges uniformly on compact subsets of $\mathcal{C}([0,+\infty))$ to the continuous function $h\circ g$. Let K be a compact set of $\mathcal{C}([0,+\infty))$. Then, for all $\omega\in K$, $\max_{[0,\varepsilon t_0]}|\omega|$ is uniformly bounded by a constant, called M.
Fix $\eta>0$. By the uniform continuity of h, there exists $\delta>0$ such that for all $\omega\in K$,

$$\mathrm{d}_u(g_\varepsilon(\omega),g(\omega))\le\delta \quad\Longrightarrow\quad |h\circ g_\varepsilon(\omega),h\circ g(\omega)|\le\eta.$$

However, there exists $\varepsilon_1>0$ small enough, such that for all $\varepsilon\le\varepsilon_1$ and for all $\omega\in K$,

$$\mathrm{d}_u(g_\varepsilon(\omega),g(\omega))\le C\left|\int_0^{\varepsilon t_0}\omega(s)\,\mathrm{d}s\right|\le C\varepsilon t_0 M\le\delta.$$

Step 2. We first prove the f.d.d. convergence.
The exponential scaling process $V^{(e)}$ satisfies the time-homogeneous equation

$$\mathrm{d}V_s^{(e)}=\mathrm{d}W_s-\frac{V_s^{(e)}}{2}\,\mathrm{d}s-F\big(V_s^{(e)}\big)\,\mathrm{d}s, \tag{5.4}$$

where $(W_t)_{t\ge0}$ is a standard Brownian motion.
Using the bijection induced by the exponential change of time (Prop. 3.1), we get

$$\left(\frac{V_{t_0e^t}}{\sqrt{t_0}e^{t/2}}\right)_{t\ge0}=(H_t)_{t\ge0},$$

as solutions to the same SDE, starting at the same point. This can also be written as

$$\left(\frac{V_t}{\sqrt{t}}\right)_{t\ge t_0}=(H_{\log(t/t_0)})_{t\ge t_0}.$$

So, we have, for all $\varepsilon>0$, and $(t_1,\cdots,t_d)\in[\varepsilon t_0,+\infty)^d$,

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{\sqrt{\varepsilon^{-1}t_1}},\cdots,\frac{V_{\varepsilon^{-1}t_d}}{\sqrt{\varepsilon^{-1}t_d}}\right)=\big(H_{\log(t_1)+\log((\varepsilon t_0)^{-1})},\cdots,H_{\log(t_d)+\log((\varepsilon t_0)^{-1})}\big). \tag{5.5}$$

As in [11], the scale function and the speed measure of H are respectively

$$\mathtt{p}(x):=\int_0^x\exp\left(\frac{y^2}{2}+\frac{2}{\gamma+1}\operatorname{sgn}(y)F(\operatorname{sgn}(y))\,|y|^{\gamma+1}\right)\mathrm{d}y$$

and

$$\nu_F(\mathrm{d}x):=\exp\left(-\frac{x^2}{2}-\frac{2}{\gamma+1}\operatorname{sgn}(x)F(\operatorname{sgn}(x))\,|x|^{\gamma+1}\right)\mathrm{d}x.$$

By the ergodic theorem (Thm. 23.15 p. 465 in [12]), H is Λ_F-ergodic, where Λ_F is the probability measure associated to ν_F. Call $\widetilde{H}$ the solution to the time homogeneous equation (5.4) such that the initial condition $\widetilde{H}_{-\infty}$ has the distribution Λ_F.
For $(t_1,\cdots,t_d)\in[\varepsilon t_0,+\infty)^d$, let $\Lambda_{F,t_1,\cdots,t_d}:=\mathcal{L}(\widetilde{H}_{t_1},\cdots,\widetilde{H}_{t_d})$ be the distribution of the vector $(\widetilde{H}_{t_1},\cdots,\widetilde{H}_{t_d})$. Then, for all $s\in\mathbb{R}$, $\Lambda_{F,t_1,\cdots,t_d}=\Lambda_{F,t_1+s,\cdots,t_d+s}$. Indeed, thanks to the invariance property of Λ_F, $(\widetilde{H}_t)_{t\in\mathbb{R}}$ and

$(\widetilde{H}_{t+s})_{t\in\mathbb{R}}$ satisfy the same SDE, starting at the same distribution. As a consequence, for all $\varepsilon > 0$,

$$\mathcal{L}\left(\widetilde{H}_{\log(t_1)+\log((\varepsilon t_0)^{-1})}, \cdots, \widetilde{H}_{\log(t_d)+\log((\varepsilon t_0)^{-1})}\right) = \Lambda_{F,\log(t_1),\cdots,\log(t_d)}. \tag{5.6}$$

Moreover, by exponential ergodicity, we can prove that for every continuous and bounded function $\psi : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\left[\psi\left(H_{\log(t_1/(t_0\varepsilon))}, \cdots, H_{\log(t_d/(t_0\varepsilon))}\right)\right] - \mathbb{E}\left[\psi\left(\widetilde{H}_{\log(t_1/(t_0\varepsilon))}, \cdots, \widetilde{H}_{\log(t_d/(t_0\varepsilon))}\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.7}$$

We postpone the proof of this convergence in Step 3.
To conclude this step, gather (5.5), (5.6) and (5.7) to get

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{\sqrt{\varepsilon^{-1}t_1}}, \cdots, \frac{V_{\varepsilon^{-1}t_d}}{\sqrt{\varepsilon^{-1}t_d}}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \Lambda_{F,\log(t_1),\cdots,\log(t_d)}.$$

This can be written as

$$\left(\sqrt{\varepsilon}V_{t_1/\varepsilon}, \cdots, \sqrt{\varepsilon}V_{t_d/\varepsilon}\right) \underset{\varepsilon\to 0}{\Longrightarrow} T * \Lambda_{F,\log(t_1),\cdots,\log(t_d)},$$

where $T * \Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ is the pushforward of the measure $\Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ by the linear map $T(u_1, \cdots, u_d) := (\sqrt{t_1}u_1, \cdots, \sqrt{t_d}u_d)$.
Step 3. Let us now prove (5.7).
Pick $\varepsilon t_0 \le s \le t$. Set $h_0 = v_0 t_0^{-\frac{1}{2}}$. Actually, we prove a more general result, which will also be useful in the last regime. The convergence (5.7) will be a direct consequence of this lemma.

Lemma 5.2. *Let H be an exponential ergodic process with invariant measure ν, solution to a SDE driven by a Brownian motion. Pick a continuous function $\phi : [t_0, +\infty) \to \mathbb{R}$ satisfying $\lim_{s\to+\infty} \phi(s) = +\infty$.*
Then, for all integer $d \ge 1$, every continuous and bounded function $\psi : \mathbb{R}^d \to \mathbb{R}$, all $h_0 \in \mathbb{R}$ and all $(t_1, \cdots, t_d) \in [\varepsilon t_0, +\infty)^d$,

$$\mathbb{E}\left[\psi\left(H_{\phi(\varepsilon^{-1}t_1)}, \cdots, H_{\phi(\varepsilon^{-1}t_d)}\right)\Big|H_0 = h_0\right] - \mathbb{E}\left[\psi\left(H_{\phi(\varepsilon^{-1}t_1)}, \cdots, H_{\phi(\varepsilon^{-1}t_d)}\right)\Big|H_0 \sim \nu\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof. For the sake of clarity, let us give a proof for $d = 2$. The general case $d \ge 2$ is similar.
Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous and bounded function.
We set $\mu_\varepsilon := \mathcal{L}\left(H_{\phi(\varepsilon^{-1}s)}\Big|H_0 = h_0\right)$ and use the generalized Markov property of solution to SDEs driven by a Brownian motion (see Cor. 16.9 p. 313 in [12]. This leads to

$$\mathbb{E}\left[\psi\left(H_{\phi(\varepsilon^{-1}s)}, H_{\phi(\varepsilon^{-1}t)}\right)\Big|H_0 = h_0\right] = \mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0 \sim \mu_\varepsilon\right]$$

and, since Λ_F is invariant,

$$\mathbb{E}\left[\psi\left(H_{\phi(\varepsilon^{-1}s)}, H_{\phi(\varepsilon^{-1}t)}\right)\Big|H_0 \sim \nu\right] = \mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0 \sim \nu\right].$$

Then, we are reduced to prove

$$\mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0 \sim \mu_\varepsilon\right] - \mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0 \sim \nu\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Hence, setting $p(t,x,\mathrm{d}y) := \mathbb{P}_x(H_t \in \mathrm{d}y)$ and $\|.\|_{TV}$ for the total variation norm, we get

$$\begin{aligned}
&\left|\mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0\sim\mu_\varepsilon\right] - \mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0\sim\nu\right]\right| \\
&\qquad \leq \left|\int_{\mathbb{R}} \mathbb{E}\left[\psi\left(H_0, H_{\phi(\varepsilon^{-1}t)-\phi(\varepsilon^{-1}s)}\right)\Big|H_0=y\right](\mu_\varepsilon(\mathrm{d}y)-\nu(\mathrm{d}y))\right| \\
&\qquad \leq \|\psi\|_\infty \int_{\mathbb{R}} \left|p\left(\phi(\varepsilon^{-1}s), h_0, \mathrm{d}y\right) - \nu(\mathrm{d}y)\right| \\
&\qquad \leq \|\psi\|_\infty \left\|p\left(\phi(\varepsilon^{-1}s), h_0, \cdot\right) - \nu\right\|_{TV}.
\end{aligned}$$

We let $\varepsilon \to 0$, using the exponential ergodicity of H. □

STEP 4. Let us prove now the tightness of the family of distributions $V^{(\varepsilon)} = \left(\sqrt{\varepsilon}V_{t/\varepsilon}\right)_{t\geq\varepsilon t_0}$ on every compact interval $[m, M]$, $0 < m \leq M$. We check the Kolmogorov criterion stated in Problem 4.11 p. 64 in [13].
Take ε_0 small enough such that for all $\varepsilon \leq \varepsilon_0$, $\varepsilon t_0 \leq m$. Fix $m \leq s \leq t \leq M$ and $\alpha > 2$. Recalling that $B^{(\varepsilon)}$ is a Brownian motion, using Jensen's inequality, moment estimates (Prop. 4.1) and the relation $\beta = \frac{\gamma+1}{2}$, we can write

$$\begin{aligned}
\mathbb{E}\left[\left|V_t^{(\varepsilon)} - V_s^{(\varepsilon)}\right|^\alpha\right] &\leq C_\alpha \mathbb{E}\left[\left|B_t^{(\varepsilon)} - B_s^{\varepsilon}\right|^\alpha\right] + C_\alpha \mathbb{E}\left[\left|\sqrt{\varepsilon}\int_{s/\varepsilon}^{t/\varepsilon} F(V_u)u^{-\beta}\,\mathrm{d}u\right|^\alpha\right] \\
&\leq C_\alpha \mathbb{E}\left[|B_t - B_s|^\alpha\right] + C_\alpha \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1}\mathbb{E}\left[\int_{s/\varepsilon}^{t/\varepsilon} |F(V_u)|^\alpha u^{-\beta\alpha}\,\mathrm{d}u\right] \\
&\leq C_\alpha \mathbb{E}\left[|B_{t-s}|^\alpha\right] + C_\alpha \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1}\int_{s/\varepsilon}^{t/\varepsilon} u^{\frac{\gamma\alpha}{2}-\beta\alpha}\,\mathrm{d}u \\
&\leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_\alpha \varepsilon^{1-\frac{\alpha}{2}}(t-s)^{\alpha-1}\int_{s/\varepsilon}^{t/\varepsilon} u^{-\frac{\alpha}{2}}\,\mathrm{d}u \\
&\leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_\alpha (t-s)^{\alpha-1}(t^{1-\frac{\alpha}{2}} - s^{1-\frac{\alpha}{2}}) \\
&\leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_{\alpha,m,M}(t-s)^{\alpha-1} \\
&\leq C_{\alpha,m,M}(t-s)^{\frac{\alpha}{2}}.
\end{aligned}$$

Since $\alpha > 2$, then $\frac{\alpha}{2} > 1$ and the upper bound does not depend on ε. Furthermore, by moment estimates (Prop. 4.1),

$$\sup_{\varepsilon\leq\varepsilon_0} \mathbb{E}\left[\left|V_m^{(\varepsilon)}\right|\right] \leq \sqrt{m} < +\infty.$$

CONCLUSION. The previous steps yield weak convergence on every compact set (Thm. 13.1 p. 139 in [2]). The conclusion follows from Thm. 16.7 p. 174 in [2], since all processes considered are continuous.

□

Example 5.3. We will see that the limiting process $\mathcal{V}$ is more explicit in the linear case ($\gamma = 1$). Choose $F(1) = 1$, $F(-1) = -1$, the process $\widetilde{H}$ solution to (5.4) is in fact an Ornstein-Uhlenbeck process with invariant measure $\Lambda_F(dx) := e^{-\frac{3x^2}{2}}\,\mathrm{d}x$. It is a centered Gaussian process, hence for all $s_1, \cdots, s_d$, its f.d.d. $\Lambda_{F,s_1,\cdots,s_d}$ are Gaussian. As a consequence, knowing the covariance function K is enough to provide the distribution of the process. Since $\widetilde{H}$ is a stationary Ornstein-Uhlenbeck process, one has $K : (s,t) \mapsto \frac{1}{3}e^{-\frac{3}{2}|t-s|}$. Hence, the limiting process $\mathcal{V}$ having f.d.d $T * \Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ is a centered Gaussian process with covariance function $(s,t) \mapsto \frac{1}{3}\frac{(s\wedge t)^2}{s\vee t}$.

5.3. Asymptotic behavior in the sub-critical regime under (H_1^γ)

Assume in this section that $\beta < \frac{\gamma+1}{2}$ and $F : v \mapsto \rho\,\mathrm{sgn}(v)\,|v|^\gamma$ with $\gamma \geq 1$. For simplicity, we shall write φ instead of φ_q.

Proof of Theorem 2.5. STEP 1. We first prove the f.d.d. convergence of the velocity process $(V_t^{(\varepsilon)})_{t\geq\varepsilon t_0} := (\varepsilon^q V_{t/\varepsilon})_{t\geq\varepsilon t_0}$. Again we give a proof only for $d = 2$, since the general case $d \geq 2$ is similar.
The power scaling process $V^{(q)}$, solution to (3.2) satisfies

$$\mathrm{d}V_s^{(q)} = \mathrm{d}W_s - F\left(V_s^{(q)}\right)\mathrm{d}s - q\varphi^{2q-1}(s)V_s^{(q)}\,\mathrm{d}s.$$

We call H the ergodic process solution to the SDE

$$\mathrm{d}H_s = \mathrm{d}W_s - F\left(H_s\right)\mathrm{d}s, \quad \text{with } H_0 = h_0 := v_0 t_0^{-q}. \tag{5.8}$$

We denote by $\Pi_F(\mathrm{d}x) := e^{-\frac{2\rho}{\gamma+1}|x|^{\gamma+1}}\,\mathrm{d}x$ its invariant measure. Using the bijection induced by the power change of time (Prop. 3.1), as solutions to the same SDE starting at the same point, we have, for all $\varepsilon > 0$, and $(s,t) \in [\varepsilon t_0, +\infty)^2$,

$$\left(\varepsilon^q \frac{V_{\varepsilon^{-1}s}}{s^q}, \varepsilon^q \frac{V_{\varepsilon^{-1}t}}{t^q}\right) = \left(V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}s)}, V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}t)}\right).$$

Using Thm. 3.1 p. 27 in [2], it suffices to prove that for all $(s,t) \in [\varepsilon t_0, +\infty)^2$,
- $\left\|\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)}\right) - \left(V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}s)}, V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}t)}\right)\right\| \underset{\varepsilon\to 0}{\longrightarrow} 0.$
- $\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \Pi_F \otimes \Pi_F.$

STEP 2. We prove that $\mathbb{E}\left[\left(H_t - V_t^{(q)}\right)^2\right] \underset{t\to+\infty}{\longrightarrow} 0.$
We have

$$\mathrm{d}\left(H - V^{(q)}\right)_t = -\left(F(H_t) - F(V_t^{(q)})\right)\mathrm{d}t + q\varphi^{2q-1}(t)V_t^{(q)}\,\mathrm{d}t.$$

By straightforward differentiation, we can write

$$\mathrm{d}\left(H - V^{(q)}\right)_t^2 = -2\left(F(H_t) - F(V_t^{(q)})\right)\left(H_t - V_t^{(q)}\right)\mathrm{d}t + 2q\varphi^{2q-1}(t)V_t^{(q)}\left(H_t - V_t^{(q)}\right)\mathrm{d}t. \tag{5.9}$$

We set

$$g(t) := \mathbb{E}\left[\left(H_t - V_t^{(q)}\right)^2\right], \quad t \geq 0.$$

Taking expectation in (5.9), we get

$$g'(t) = -2\mathbb{E}\left[\left(F(H_t) - F(V_t^{(q)})\right)\left(H_t - V_t^{(q)}\right)\right] + 2q\varphi^{2q-1}(t)\mathbb{E}\left[V_t^{(q)}\left(H_t - V_t^{(q)}\right)\right].$$

Since $\gamma \geq 1$, the function F^{-1} is $\frac{1}{\gamma}$-Hölder, therefore there exists $C_\gamma > 0$ such that,

$$g'(t) \leq -C_\gamma \mathbb{E}\left[\left(H_t - V_t^{(q)}\right)^{1+\gamma}\right] + 2q\varphi^{2q-1}(t)\mathbb{E}\left[V_t^{(q)}\left(H_t - V_t^{(q)}\right)\right].$$

Then, by Jensen's inequality, since $\gamma \geq 1$,

$$g'(t) \leq -C_\gamma g(t)^{\frac{\gamma+1}{2}} + 2q\varphi^{2q-1}(t)\mathbb{E}\left[V_t^{(q)}\left(H_t - V_t^{(q)}\right)\right].$$

Using Cauchy-Schwarz inequality and moment estimates (Prop. 4.1), we have

$$g'(t) \leq -C_\gamma g(t)^{\frac{\gamma+1}{2}} + C\,|q|\,\varphi^{q-\frac{1}{2}}(t)\sqrt{g(t)}, \quad g(0) = 0.$$

Note that since $2q < 1$, then $\varphi^{q-\frac{1}{2}}(t) \underset{t\to+\infty}{\longrightarrow} 0$, therefore the conclusion follows from **??**. Besides, for all $t \geq \varepsilon t_0$, $\mathbb{E}\left[\left(H_{\varphi^{-1}(\varepsilon^{-1}t)} - V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}t)}\right)^2\right] = g\left(\varphi^{-1}(\varepsilon^{-1}t)\right) \underset{\varepsilon\to0}{\longrightarrow} 0$.

Step 3. Pick $(s,t) \in [\varepsilon t_0, +\infty)^2$. We prove that the solution H to (5.8) satisfies

$$\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)}\right) \underset{\varepsilon\to0}{\Longrightarrow} \Pi_F \otimes \Pi_F. \tag{5.10}$$

Observe that

$$\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s) = \frac{t^{1-2q} - s^{1-2q}}{\varepsilon^{1-2q}} \underset{\varepsilon\to0}{\longrightarrow} +\infty. \tag{5.11}$$

By Lemma 5.2, for every continuous and bounded function ψ, we can write

$$\mathbb{E}\left[\psi\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)}\right)\Big| H_0 = h_0\right] - \mathbb{E}\left[\psi\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)}\right)\Big| H_0 \sim \Pi_F\right] \underset{\varepsilon\to0}{\longrightarrow} 0.$$

Hence, it suffices to prove that for every bounded continuous functions $f, g : \mathbb{R} \to \mathbb{R}$, the following convergence holds

$$\lim_{\varepsilon\to0} \mathbb{E}\left[f\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}\right) g\left(H_{\varphi^{-1}(\varepsilon^{-1}t)}\right)\Big| H_0 \sim \Pi_F\right] = \Pi_F(f)\Pi_F(g).$$

The following reasoning is inspired from the proof of Lemma 3.2 p. 7-8 in [5]. Since H_0 is starting from the invariant measure, up to considering $f - \Pi_F(f)$ and $g - \Pi_F(g)$, we can assume that f and g have zero Π_F-mean. We call $(P_t)_{t\geq0}$ the semigroup of H, then we get, by invariance property of Π_F,

$$\begin{aligned}\mathbb{E}\left[f\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}\right) g\left(H_{\varphi^{-1}(\varepsilon^{-1}t)}\right)\Big| H_0 \sim \Pi_F\right] &= \int P_{\varphi^{-1}(\varepsilon^{-1}s)}\left(f P_{\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)} g\right) \mathrm{d}\Pi_F \\ &= \int f P_{\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)} g \,\mathrm{d}\Pi_F.\end{aligned}$$

Note that $U : v \mapsto \frac{|v|^{1+\gamma}}{1+\gamma}$ is a convex function, thus a λ-Poincaré inequality holds for the process H (see [3] p. 1904). This implies the exponential decay of the variance (see Thm. 4.2.5 p. 183 in [1]), *i.e.* there exists a constant $C > 0$ such that, since Π_F is a probability measure,

$$\begin{aligned}\left|\int f P_{\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)} g \,\mathrm{d}\Pi_F\right| &\leq \left\|f P_{\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)} g\right\|_2 \\ &\leq \|f\|_\infty \left\|P_{\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)} g\right\|_2 \\ &\leq C\|f\|_\infty \|g\|_\infty e^{-\lambda\left(\varphi^{-1}(\varepsilon^{-1}t)-\varphi^{-1}(\varepsilon^{-1}s)\right)}.\end{aligned}$$

We deduce (5.10) from (5.11).
Step 4. We prove the convergence of the f.d.d. of the position process.
We set $(X_t^{(\varepsilon)})_{t\geq\varepsilon t_0} := (\varepsilon^{\beta+\frac{1}{2}} X_{t/\varepsilon})_{t\geq\varepsilon t_0}$. Take $\gamma = 1$ and $\beta \in (-\frac{1}{2}, 1)$. Pick $t \geq \varepsilon t_0$. By Itô's formula applied to $t^\beta V_t$, we get

$$\rho X_t^{(\varepsilon)} = \varepsilon^{\beta+\frac{1}{2}}(t_0^\beta v_0 + x_0) - \varepsilon^{\frac{1-\beta}{2}} t^\beta V_t^{(\varepsilon)} + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} s^\beta \,\mathrm{d}B_s + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} V_s \,\mathrm{d}s.$$

Since $\beta > -\frac{1}{2}$, the first term converges to 0 in probability as $\varepsilon \to 0$. Moreover, by Itô's formula, for all $t \geq t_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[V_t^2\right] = -2\rho s^{-\beta}\mathbb{E}\left[V_s^2\right] + 1.$$

Hence, by comparison theorem for ordinary differential equation,

$$\mathbb{E}\left[V_t^2\right] \leq \exp(-2\rho\frac{t^{1-\beta}}{1-\beta})\left(v_0^2 + \int_{t_0}^t \exp(2\rho\frac{s^{1-\beta}}{1-\beta})\,\mathrm{d}s\right).$$

Using an integration by parts, we deduce that there exists a positive constant C such that, for all $t \geq t_0$,

$$\mathbb{E}\left[V_t^2\right] \leq Ct^\beta.$$

As a consequence, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|-\varepsilon^{\frac{1-\beta}{2}} t^\beta V_t^{(\varepsilon)} + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} V_s \,\mathrm{d}s\right|\right] &\leq \varepsilon^{\frac{1-\beta}{2}} t^\beta \mathbb{E}\left[\left|V_t^{(\varepsilon)}\right|\right] + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} \mathbb{E}\left[|V_s|\right] \mathrm{d}s \\ &\leq C\varepsilon^{\frac{1}{2}} t^{\frac{3\beta}{2}} + C\varepsilon^{\frac{1-\beta}{2}} t^{\frac{3\beta}{2}} - C\varepsilon^{\beta+\frac{1}{2}} t_0^{\frac{3\beta}{2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

It remains to study the centered Gaussian process $M_t^{(\varepsilon)} := \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} s^\beta \,\mathrm{d}B_s$. By Itô's isometry and since $\beta > -\frac{1}{2}$, for all $\varepsilon t_0 \leq s \leq t$, we can write

$$\mathrm{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) = \varepsilon^{2\beta+1} \int_{t_0}^{s/\varepsilon} u^{2\beta} \,\mathrm{d}s \underset{\varepsilon\to 0}{\sim} \frac{s^{1+2\beta}}{1+2\beta}.$$

Since the convergence of centered Gaussian processes is characterized by the convergence of their covariance function, the conclusion follows from Thm. 3.1 p. 27 in [2].

□

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