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MARTINGALE SOLUTIONS OF THE STOCHASTIC 2D PRIMITIVE EQUATIONS WITH ANISOTROPIC VISCOSITY*

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Abstract. The stochastic 2D primitive equations with anisotropic viscosity are studied in this paper. The existence of the martingale solutions and pathwise uniqueness of the solutions are obtained. The proof is based on anisotropic estimates, the compactness method, tightness criteria and the Jakubowski version of the Skorokhod theorem for nonmetric spaces.

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1. Introduction

The primitive equations (PE) for oceanic and atmospheric dynamics are fundamental models in meteorology, which derived from the Navier-Stokes equations assuming a hydrostatic balance for the pressure term in the vertical direction. For more information on physical background and geophysical applications of the primitive equations, we refer the reader to [9, 25, 46], for example.

Known results: For the deterministic primitive equations with full viscosity, *i.e.*, the diffusion term $-\Delta$ is defined by $-\partial_x^2 - \partial_y^2 - \partial_z^2$, the mathematical analysis of the initial value problem has been started by Lions, Temam and Wang [40–42], where the notions of weak and strong solutions were defined and the global existence of weak solutions was proved, however, the uniqueness of weak solutions is still unclear. By decomposing the velocity into barotropic and baroclinic components, a breakthrough result has been proven by Cao-Titi [8], where the global well-posedness of strong solutions in H^1 to the three dimensional primitive equations has been obtained, see also Kobelkov [33] and Kukavica-Ziane [36]. One can see some other literatures (for instance [21, 26, 30]-[34, 48, 49]) for the well-posed results with different space dimensions, initial data and boundary conditions. In particular, about uniqueness of weak solutions, by introducing the notion of z-weak solution (see [2, 47]), *i.e.*, weak solutions with additional regularity in the vertical direction, researchers have found some results [31, 34, 37, 44].

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In the models of geophysical flows, due to uncertainties in the physics derivations and intrinsic heat fluctuations, white noise driven random term was introduced to the geophysical equations (see for instance [45]). Research on stochastic primitive equations with full viscosity is a classical topic and has been studied extensively in a number of literatures. For the well-posedness, regularity, random attractor and existence and regularity of invariant measures, we refer the readers to [10]-[18, 22, 51]. For the deviation principles and small time asymptotics of the primitive equations, see [12, 13].

Due to the presence of strong turbulent mixing in the horizontal direction in the large scale atmosphere, the viscosity in the horizontal direction is much stronger than that in the vertical direction. As a result, it is necessary to investigate the primitive equations with anisotropic viscosity, *i.e.*, the diffusion term $-\Delta$ is replaced by $-\nu_1\partial_x^2 - \nu_2\partial_y^2 - \nu_3\partial_z^2$, where $\nu_1, \nu_2, \nu_3 \geq 0$, and, in particular, the system that with only horizontal viscosities, *i.e.*, the diffusion term $-\Delta$ is replaced by $-\Delta_H$, where $\Delta_H = \nu_1\partial_x^2 + \nu_2\partial_y^2$, $\nu_1, \nu_2 > 0$ (see [6, 7, 38]). Mathematically, for the primitive equations with partial viscosity, we would like to emphasis that the system is not purely parabolic anymore. Cao *et al.* [6, 7] studied the 3D deterministic primitive equations with only horizontal viscosity analytically. They tackled this problem in a periodical setting by considering a vanishing vertical viscosity limit and obtained global strong well-posedness results for initial data with regularity near H^1, and local well-posedness for initial data in H^1. Instead of considering a vanishing vertical viscosity limit, by a direct approach which in particular avoids unnecessary boundary conditions on top and bottom, Hussein *et al.* [28] studied the initial value and the time-periodic problem for the 3D deterministic primitive equations with horizontal viscosity and obtained existence and uniqueness of local z-weak solutions for initial data in $H_z^1 L_{xy}^2$ and local strong solutions for initial data in H^1. Furthermore, if $\partial_z v_0 \in L^q$ for $q > 2$, the local z-weak solutions extended to a global strong solution. For the case of full hyper-viscosity or only horizontal hyper-viscosity, *i.e.*, the diffusion term $-\Delta$ is replaced by $-\Delta + \varepsilon(-\Delta)^l$ or by $-\Delta + \varepsilon(-\Delta_H)^l$, respectively, where $\varepsilon > 0, l > 1$, strong convergence for $\varepsilon \to 0$ of hyper-viscous solutions to a weak solution of the 3D deterministic Navier-Stokes and primitive equations, respectively, was obtained by Hussein in [27].

For the 3D deterministic primitive equations without viscosity, *i.e.*, $\nu_1 = \nu_2 = \nu_3 = 0$, blow-up result was obtained by Cao *et al.* in [5], see also Wong [52]. Ill-posedness result was obtained in Sobolev spaces by Han-Kwan and Nguyen in [24], where the solution map was not (Hölder) continuous with respect to initial data. Local well-posedness has been proven only for analytical data by Kukavica *et al.* in [35].

Stochastic 2D primitive equations with only horizontal viscosity: The primary goal of this paper is to study the well-posedness of 2D stochastic primitive equations with horizontal viscosity driven by multiplicative white noise. For simplicity, a period setting is considered here. We consider the domain $M := (0, 1) \times (-1, 0)$ and denote by $x \in (0, 1)$ the horizontal coordinate and by $z \in (-1, 0)$ the vertical one. Let (v, w) be the velocity with horizontal component v and vertical component w, and p be the pressure. Then the 2D stochastic primitive equations are given by

$$\begin{cases} dv + (v \cdot \partial_x v + w\partial_z v - \partial_x^2 v + \partial_x p)\mathrm{d}t = f dt + \sigma(t, v)\mathrm{d}W, & \text{in } M \times (0, T), \\ \partial_z p = 0, & \text{in } M \times (0, T), \\ \partial_x v + \partial_z w = 0, & \text{in } M \times (0, T), \\ p \ \text{ periodic in } \ x, z, & \\ v, w \ \text{ periodic in } \ x, z, \ \text{ even and odd,} & \text{in } \ z, \text{respectively}, \\ v(0) = v_0, & \text{in } M, \end{cases} \tag{1.1}$$

where σ is the random external forces, f is an external force term and W is a cylindrical Wiener process, the definitions of which will be introduced in Section 2. As in [27], note that the vertical periodicity and parity conditions in (1.1) correspond in to an equivalent set of equations on $(0, 1) \times (-1, 0)$ with lateral periodicity and

$$\partial_z v|_{z=0} = \partial_z v|_{z=-1} = 0, \ \ w|_{z=0} = w|_{z=-1} = 0. \tag{1.2}$$

With these boundary conditions, we suppose with no loss of generality that (see also in [19])

$$\int_{-1}^{0} f \mathrm{d}z = \int_{-1}^{0} \sigma \mathrm{d}z = \int_{-1}^{0} v \mathrm{d}z = 0 \tag{1.3}$$

and

$$w(x,z,t) = -\int_{-1}^{z} \partial_x v(x,\xi,t) \mathrm{d}\xi. \tag{1.4}$$

In the rest of paper, in order to focus our attention on the difficulties arising from the nonlinear term, we ignore the external force term f. The whole process of proof shows that the existence of f does not affect our conclusion.

Studies on stochastic geophysical fluid equations with anisotropic viscosity have attracted more and more attention in recent years. Especially, for the stochastic Navier-Stokes equations, by adding a term of Brinkman-Forchheimer type, Bessaih and Millet [1] established the existence and uniqueness of global weak solutions (in the PDE sense) in the whole space $\mathbb{R}^3$. Liang, Zhang and Zhu [39] investigated the existence of the martingale solutions and pathwise uniqueness of the solutions in a given anisotropic Sobolev space on $\mathbb{R}^2$ or on the two dimensional torus $\mathbb{T}^2$. Comparing with the Navier-Stokes equations, it is worth to point out that the system of primitive equations is generally harder to deal with, the nonlinear term $w\partial_z v$ is a more difficult version in contrast to the nonlinearity of the Navier-Stokes equations since $w = w(v)$ given by (1.4) involves a first order derivative. Therefore, no matter which type of viscosity is considered, how to deal with the estimates of nonlinear term is a challenge.

Returning to the study of this paper, for the stochastic primitive equations with partial viscosity, fewer works have been done. To the best of our knowledge, only the literature [50] was involved on this topic. In [50], the existence and uniqueness of pathwise solutions in H^1 to the stochastic 3D primitive equations with only horizontal viscosity and diffusivity driven by transport noise were established. Firstly, the global existence of martingale solutions was established for a modified equations with a cut-off acting on $L_z^\infty L_{xy}^4$-norm of the solution. After a standard argument using the theorems by Prokhorov and Skorokhod, the existence of maximal solutions up to a strictly positive stopping time was established. Then, to establish uniqueness, more regular initial data was needed in the vertical direction. Finally, the global existence was established using the logarithmic Sobolev embedding and an iterated stopping time argument.

Generally, what works in 3D does not necessarily work in 2D, in this paper, we study the stochastic 2D primitive equations with horizontal viscosity. By new anisotropic estimations and compactness method differently from the above mentioned methods in [50], global existence of martingale solutions has been obtained. In the proof of the tightness argument of Galerkin schemes, an Aldous's condition (first introduced in [3]) is applied to deal with the H_{xz}^{-1} norm. On the other hand, compared to [50], where the solutions were strong in the PDE sense, the solutions considered here are z-weak meanings (in the PDE sense), moreover, to obtain the uniqueness of z-weak solutions, we don't need more regular initial value.

This paper is organized as follows: In Section 2, some basic definitions and notations are given for a periodic setting such as function spaces, assumptions and the definition of martingale solutions. In Section 3, the main results with proofs are formulated.

2. Preliminaries

For $s \in [0, +\infty)$, one defines the Bessel potential spaces

$$H^s_{per}(M) = \overline{C^\infty_{per}(\overline{M})}^{\|\cdot\|_{H^s}} \quad \textit{and} \quad H^{-s}_{per}(M) = \left(H^s_{per}(M)\right)',$$

where $C^{\infty}_{per}(\overline{M})$ denotes the spaces of smooth functions which are periodic of any order (cf. [27]) in all directions on ∂M and $(\cdot)^{'}$ denotes the dual of the corresponding space. Moreover, $H^s_{per}(M)$ can be characterized by means of Fourier series as

$$\|v\|^2_{H^s_{per}(M)} = \sum_{k\in\mathbb{Z}^2}(1+|k|^2)^s|\widehat{v}(k)|^2,$$

where

$$\widehat{v}(k) = \frac{1}{2}\int_M v(x,z)e^{\pi i k_1 x}e^{2\pi i k_2 z}\mathrm{d}x\mathrm{d}z, \qquad k=(k_1,k_2)\in\mathbb{Z}^2.$$

Furthermore, we recall the anisotropic Sobolev norms,

$$\begin{aligned}\|v\|^2_{H^{s,s'}_{per}(M)} &= \sum_{k\in\mathbb{Z}^2}(1+|k_1|^2)^s(1+|k_2|^2)^{s^{'}}|\widehat{v}(k)|^2,\\ \|v\|^2_{\dot{H}^{s,s'}_{per}(M)} &= \sum_{k\in\mathbb{Z}^2}|k_1|^{2s}(1+|k_2|^2)^{s^{'}}|\widehat{v}(k)|^2.\end{aligned}$$

For the sake of simplicity, we abbreviate $\|\cdot\|^2_{H^{s,s'}_{per}}$, $\|\cdot\|^2_{\dot{H}^{s,s'}_{per}}$ to $\|\cdot\|^2_{H^{s,s'}}$, $\|\cdot\|^2_{\dot{H}^{s,s'}}$, respectively. We will be working on these spaces,

$$\begin{aligned}&H := \{v\in L^2(M)\mid \int_{-1}^{0} v\mathrm{d}z = 0\},\\ &V := \{v\in H^1_{per}(M)|\int_{-1}^{0} v\mathrm{d}z = 0\},\\ &\widetilde{H}^s := \{v\in H^s_{per}(M)\mid \int_{-1}^{0} v\mathrm{d}z = 0\},\\ &\tilde{H}^{s,s'} := \{v\in H^{s,s'}_{per}(M)|\int_{-1}^{0} v\mathrm{d}z = 0\}.\end{aligned}$$

Note that $V \equiv \widetilde{H}^1$, moreover, the scalar product $(\cdot,\cdot)$ is denoted by

$$(u,v) = (u,v)_{L^2(M)} = \int_M u(x,z)v(x,z)\mathrm{d}x\mathrm{d}z.$$

We use $(\cdot,\cdot)_{H^{0,1}}$ or $(\cdot,\cdot)_{0,1}$ to denote the inner product

$$(u,v)_{H^{0,1}(M)} = \int_M (u(x,z)v(x,z) + \partial_z u(x,z)\partial_z v(x,z))\mathrm{d}x\mathrm{d}z.$$

The Leray operator P_H is the orthogonal projection of $L^2(M)$ on to H. The action of this operator is given by

$$P_H v = v - \int_{-1}^{0} v\mathrm{d}z.$$

Let $e_k, k \geq 1$ be an orthonormal basis of H whose elements belong to H^2 and orthogonal in $\widetilde{H}^{0,1}$ and $\widetilde{H}^{1,0}$ (hence also $\widetilde{H}^{1,1}$). For integers $k, l \geq 1$ with $k \neq l$, we deduce that for $i = x, z$,

$$(\partial_i^2 e_k, e_l) = -(\partial_i e_k, \partial_i e_l) = 0.$$

Therefore, $\partial_i^2 e_k$ is a constant multiple of e_k. For example, for $k = (k_1, k_2)$, the eigenfunctions and associated eigenvalues can be identified,

$$e_k(x,z) = \left\{\sqrt{2}\sin(k_1\pi x)\cos(k_2\pi z)\right\}_{k_1,k_2\geq 1}, \qquad \left\{\pi^2(k_1^2 + k_2^2)\right\}_{k_1,k_2\geq 1}.$$

In accordance with equality (1.4), we take

$$\mathbb{W}(v) := -\int_{-1}^{z} \partial_x v(x, \widetilde{z})\mathrm{d}\tilde{z}.$$

And let

$$B(u,v) := u\partial_x v + \mathbb{W}(u)\partial_z v,$$

where $u, v \in V$ and denote $B(u,u) = B(u)$.

Define the bilinear operator $B(u,v) : V \times V \to V'$ according to

$$\langle B(u,v), w\rangle = b(u,v,w),$$

where

$$b(u,v,w) = \int_M (u\partial_x v w + \mathbb{W}(u)\partial_z v w) dM.$$

Due to boundary conditions (1.2), we have the following lemma.

Lemma 2.1 (Anisotropic estimates)**.** *The trilinear forms b and B have the following properties. There exists a constant $C > 0$ such that*

$$\begin{aligned} |b(u,v,w)| \leq C\Big(&\|u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}}\|\partial_x v\|_{L^2}\|w\|_{L^2}^{\frac{1}{2}}\|\partial_x w\|_{L^2}^{\frac{1}{2}} \\ &+\|\partial_x u\|_{L^2}\|\partial_z v\|_{L^2}\|w\|_{L^2}^{\frac{1}{2}}\|\partial_x w\|_{L^2}^{\frac{1}{2}}\Big) \quad (2.1)\\ &\textit{for any } u,v,w \in V, \end{aligned}$$

$$b(u,v,v) = 0, \ \textit{for any } u,v,w \in H, \quad (2.2)$$

$$\langle B(u,u), \partial_{zz}u\rangle = 0, \ \textit{for any } u \in \widetilde{H}^{0,2}. \quad (2.3)$$

Proof. The properties of (2.2) and (2.3) have been obtained in [19], here, by Hölder inequality, we give the estimate of (2.1) anisotropically.

$$\begin{aligned} |b(u,v,w)| \leq &\int_0^1 \Big(\sup_{z\in[-1,0]}|u|\int_{-1}^0 (\partial_x v\cdot w)\mathrm{d}z\Big)\mathrm{d}x \\ &+\int_0^1 \Big(\sup_{z\in[-1,0]}\big(\int_{-1}^z \partial_x u \mathrm{d}\widetilde{z}\big)\int_{-1}^0 (\partial_z v\cdot w)\mathrm{d}z\Big)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^1\Big(\int_{-1}^0|u|^2\mathrm{d}z\Big)^{\frac14}\Big(\int_{-1}^0|\partial_z u|^2\mathrm{d}z\Big)^{\frac14}\Big(\int_{-1}^0|\partial_x v|^2\mathrm{d}z\Big)^{\frac12}\Big(\int_{-1}^0|w|^2\mathrm{d}z\Big)^{\frac12}\mathrm{d}x\\
&\quad +C\int_0^1\Big(\int_{-1}^0|\partial_x u|^2\mathrm{d}z\cdot\int_{-1}^0|\partial_z v|^2\mathrm{d}z\cdot\int_{-1}^0|w|^2\mathrm{d}z\Big)^{\frac12}\mathrm{d}x\\
&\le C\sup_{x\in[0,1]}\Big(\int_{-1}^0|w|^2\mathrm{d}z\Big)^{\frac12}\|\partial_x v\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_z u\|_{L^2}^{\frac12}\\
&\quad +C\sup_{x\in[0,1]}\Big(\int_{-1}^0|w|^2\mathrm{d}z\Big)^{\frac12}\|\partial_x u\|_{L^2}\|\partial_z v\|_{L^2}\\
&\le C\Big(\|u\|_{L^2}^{\frac12}\|\partial_z u\|_{L^2}^{\frac12}\|\partial_x v\|_{L^2}\|w\|_{L^2}^{\frac12}\|\partial_x w\|_{L^2}^{\frac12}\\
&\qquad +\|\partial_x u\|_{L^2}\|\partial_z v\|_{L^2}\|w\|_{L^2}^{\frac12}\|\partial_x w\|_{L^2}^{\frac12}\Big),
\end{aligned}
$$

where we make use of the boundary conditions and get the following estimates,

$$
\begin{aligned}
\sup_{z\in[-1,0]}|u|^2 &= \sup_{z\in[-1,0]}\int_{-1}^z\partial_z(u^2)\mathrm{d}z = 2\sup_{z\in[-1,0]}\int_{-1}^z u\cdot\partial_z u\mathrm{d}z\\
&\le C\Big(\int_{-1}^0|u|^2\mathrm{d}z\Big)^{\frac12}\Big(\int_{-1}^0|\partial_z u|^2\mathrm{d}z\Big)^{\frac12},
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{x\in[0,1]}\Big(\int_{-1}^0|w|^2\mathrm{d}z\Big) &= \sup_{x\in[0,1]}\Big(\int_0^x\partial_x\int_{-1}^0|w|^2\mathrm{d}z\mathrm{d}x\Big)\\
&\le 2\|w\|_{L^2}\|\partial_x w\|_{L^2}.
\end{aligned}
$$

This completes the proof. □

Let $(W(t), t\ge 0)$ be a Y-cylindrical Wiener process on a stochastic basis $(\Omega,\mathcal{F},P)$, where Y is a separable Hilbert space. Let $\mathcal{L}_2(Y,\mathbb{U})$ denote the Hilbert-Schmit norms from Y to $\mathbb{U}$. For a Polish space $\mathbb{V}$, let $\mathcal{B}(\mathbb{V})$ denote its Borel σ-algebra and $\mathcal{P}(\mathbb{V})$ denote all the probability measures on $(\mathbb{V},\mathcal{B}(\mathbb{V}))$. Let σ be a measurable mapping from $([0,T]\times\tilde{H}^{1,1},\mathcal{B}([0,T]\times\tilde{H}^{1,1}))$ to $(\mathcal{L}_2(Y,\tilde{H}^{1,1}),\mathcal{B}(\mathcal{L}_2(Y,\tilde{H}^{1,1})))$. Then we introduce martingale solutions. Set

$$F(v):=-B(v)+\partial_x^2 v.$$

Definition 2.2 (Martingale solution.)**.** We say that a probability measure

$$P\in\mathcal{P}(C([0,T];H^{-1}))$$

is called a martingale solution of (1.1) with initial value v_0 if

(M1) $P(v(0)=v_0, v\in L^\infty(0,T;\tilde{H}^{0,1})\cap L^2(0,T;\tilde{H}^{1,1}))=1$, and

$$P\{v\in C([0,T],H^{-1}):\int_0^T\|F(v(s))\|_{H^{-1}}\mathrm{d}s+\int_0^T\|\sigma(s,v(s))\|^2_{\mathcal{L}_2(Y,H)}\mathrm{d}s<+\infty\}=1.$$

(M2) For every $\psi \in C^{\infty}_{per}(\overline{M})$ with $\int_{-1}^{0} \psi \mathrm{d}z = 0$, the process

$$M_{\psi}(t, v) = \langle v(t), \psi\rangle - \int_{0}^{t} \langle F(v(s)), \psi\rangle \mathrm{d}s$$

is a continuous square integrable $\mathcal{F}_t-$martingale with respect to P, whose quadratic variation process is $\int_0^t \|\sigma^*(s, v(s))(\psi)\|_Y^2 \mathrm{d}s$, where the asterisk denotes the adjoint operator of $\sigma(s, \tilde{v}(s))$.

(M3) We have

$$E^P(\sup_{t\in[0,T]} \|v(t)\|_{L^2}^2 + \int_0^T \|v(t)\|_{H^{1,0}}^2 \mathrm{d}t) \le C_T(1 + \|v_0\|_{L^2}^2).$$

Hypothesis 2.3 (Conditions). We assume that the diffusion coefficient σ is a measurable mapping from $([0,T]\times \tilde{H}^{1,1}, \mathcal{B}([0,T]\times \tilde{H}^{1,1}))$ to $(\mathcal{L}_2(Y, \tilde{H}^{1,1}), \mathcal{B}(\mathcal{L}_2(Y, \tilde{H}^{1,1})))$ and satisfies the following conditions.

(i) Growth condition

There exist nonnegative constants K'_i, K_i and $\tilde{K}_i (i = 0, 1$ *or* $2)$ such that for every $t \in [0, T]$ and $v \in \tilde{H}^{1,1}$,

$$\|\sigma(t, v)\|^2_{\mathcal{L}_2(Y,H^{-1})} \le K'_0 + K'_1\|v\|^2_{L^2}, \tag{2.4}$$

$$\|\sigma(t, v)\|^2_{\mathcal{L}_2(Y,H)} \le K_0 + K_1\|v\|^2_{L^2} + K_2\|\partial_x v\|^2_{L^2}, \tag{2.5}$$

$$\|\sigma(t, v)\|^2_{\mathcal{L}_2(Y,H^{0,1})} \le \tilde{K}_0 + \tilde{K}_1\|v\|^2_{H^{1,1}}. \tag{2.6}$$

(ii) Lipschitz condition

There exists a constant L_1 such that for $t \in [0, T]$ and $u, v \in \tilde{H}^{1,0}$,

$$\|\sigma(t, u) - \sigma(t, v)\|^2_{\mathcal{L}_2(Y,H)} \le L_1\|\partial_x(u - v)\|^2_{L^2}. \tag{2.7}$$

3. Main results

In this section, we state two theorems about the well-posedness of equation (1.1), which will be proved in the following subsections.

Theorem 3.1. *Assume that v_0 is a random variable in $\tilde{H}^{0,1}$ and suppose that σ satisfies Hypothesis 2.3 with $K_2 < \frac{2}{11}$ and $\tilde{K}_1 < \frac{2}{7}$. Then (1.1) has a global martingale solution.*

Theorem 3.2. *(Pathwise uniqueness). Assume that v_0 is a random variable in $\tilde{H}^{0,1}$. Suppose that σ satisfies Hypothesis 2.3 with $K_2 < \frac{2}{11}, \tilde{K}_1 < \frac{2}{7}$ and $L_1 < 1$. If v_1, v_2 are two weak solutions on the same stochastic basis $(\Omega, \mathcal{F}, P)$. Then we have $v_1 = v_2$ $P - a.s.$.*

3.1. Galerkin approximation and a priori estimates

Let $\mathcal{H}_n = span(e_1, \ldots, e_n)$ and $P_n(resp.\ \tilde{P}_n, \overline{P}_n)$ denote the orthogonal projection from H $(resp. \tilde{H}^{0,1}, \tilde{H}^1)$ to $\mathcal{H}_n$. As in [39], we have

$$P_n v = \tilde{P}_n v, \quad \text{for} \quad v \in \tilde{H}^{0,1}.$$

Furthermore, for $u \in \mathcal{H}_n$, we have $\partial_z^2 u \in \mathcal{H}_n$ and for any $v \in \tilde{H}^{0,1}$,

$$(P_n v, u) = (v, u) \quad \text{and} \quad (\partial_z P_n v, \partial_z u) = -(P_n v, \partial_z^2 u) = -(v, \partial_z^2 u) = (\partial_z v, \partial_z u).$$

Hence, given $v \in \tilde{H}^{0,1}$, we have

$$(P_n v, u)_{0,1} = (v, u)_{0,1}, \quad for \quad any \quad u \in \mathcal{H}_n.$$

This proves that P_n and $\tilde{P}_n$ coincide on $\tilde{H}^{0,1}$. Similarly, we can prove P_n, $\widetilde{P}_n$, and $\overline{P}_n$ coincide on $\widetilde{H}^1$.

We consider the following stochastic ordinary differential equations on $\mathcal{H}_n$,

$$v_n(0) = P_n v_0$$

and for $t \in [0,T], u \in \mathcal{H}_n$

$$d(v_n(t), u) = \langle P_n F(v_n(t)), u\rangle \mathrm{d}t + \langle P_n \sigma(t, v_n(t)) dW_n(t), u\rangle. \tag{3.1}$$

Then for $k = 1, \ldots, n$, $t \in [0,T]$, we have

$$d(v_n(t), e_k) = \langle P_n F(v_n(t)), e_k\rangle \mathrm{d}t + \langle P_n \sigma(t, v_n(t)) \mathrm{d}W_n(t), e_k\rangle.$$

Note that since it is in finite dimensions, there exists a constant $C(n)$ such that $\|v\|_{H^2} \leq C(n)\|v\|_{L^2}$ for $v \in \mathcal{H}_n$. Hence by Theorem 3.1.1 in [43], there exists a unique global solution $v_n(t)$ to (3.1). Moreover, $v \in C(\mathbb{R}^+, \mathcal{H}_n)$, $\mathbb{P} - a.s.$.

Lemma 3.3. *We have the following energy estimates under the hypothesis of Theorem 3.1,*

$$E(\sup_{t\in[0,T]} \|v_n(t)\|_{L^2}^2) + E\int_0^T \|v_n(t)\|_{H^{1,0}}^2 \mathrm{d}t \leq C(K_0, K_1, K_2, T)(1 + E\|v_0\|_{L^2}^2).$$

Proof. By Itô formula, we have

$$\begin{aligned} \|v_n(t)\|_{L^2}^2 = \|P_n v_0\|_{L^2}^2 &+ 2\int_0^t \langle \sigma(s, v_n(s)) \mathrm{d}W_n(s), v_n(s)\rangle \\ &- 2\int_0^t \|\partial_x v_n(s)\|_{L^2}^2 \mathrm{d}s + \int_0^t \|P_n \sigma(s, v_n(s))\|_{\mathcal{L}_2(Y,H)}^2 \mathrm{d}s. \end{aligned} \tag{3.2}$$

The growth condition implies that

$$\int_0^t \|P_n \sigma(s, v_n(s))\|_{\mathcal{L}_2(Y,H)}^2 \mathrm{d}s \leq \int_0^t [K_0 + K_1\|v_n(t)\|_{L^2}^2 + K_2\|\partial_x v_n(t)\|_{L^2}^2] \mathrm{d}s. \tag{3.3}$$

The Burkholder-Davis-Gundy inequality and the Young inequality as well as the growth condition imply that,

$$\begin{aligned} &E(\sup_{s\leq t} |2\int_0^s (P_n \sigma(r, v_n(r)) \mathrm{d}W_n(r), v_n(r))|) \\ &\leq 4E\{\int_0^t \|P_n \sigma(r, v_n)(r)\|_{\mathcal{L}_2(Y,H)}^2 \|v_n(r)\|_{L^2}^2 \mathrm{d}r\}^{\frac{1}{2}} \\ &\leq \beta E(\sup_{s\leq t} \|v_n(s)\|_{L^2}^2) + \frac{4}{\beta} E\int_0^t [K_0 + K_1\|v_n(s)\|_{L^2}^2 + K_2\|\partial_x v_n(t)\|_{L^2}^2] \mathrm{d}s, \end{aligned} \tag{3.4}$$

since $K_2 < \frac{2}{11}$, we can choose $0 < \beta < 1$ such that $(\frac{4}{\beta} + 1)K_2 - 2 < 0$.

By (3.2)–(3.4) and dropping some negative terms, we deduce

$$(1-\beta)E\sup_{s\in[0,t]}\|v_n(s)\|_{L^2}^2\le E\|v(0)\|_{L^2}^2+CK_0T+CE\int_0^t K_1\|v_n(s)\|_{L^2}^2\mathrm{d}s.$$

Gronwall's lemma implies that

$$E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^2)\le C, \tag{3.5}$$

where C is a constant depending on K_0, K_1, K_2, T but not n.

Inserting (3.5) back to (3.2)–(3.4) yields

$$E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^2)+E\int_0^t\|v_n(t)\|_{H^{1,0}}^2\mathrm{d}s\le C(K_0,K_1,K_2,T)(1+E\|v_0\|_{L^2}^2).$$

This completes the proof. □

However, we also need the $L^4(\Omega)$ uniform estimates of v_n.

Lemma 3.4. *We have the following uniform estimates under the hypothesis of Theorem 3.1,*

$$E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^4)+E\int_0^T\|v_n(t)\|_{L^2}^2\|v_n(t)\|_{H^{1,0}}^2\mathrm{d}t\le C(K_0,K_1,K_2,T)(1+E\|v_0\|_{L^2}^4).$$

Proof. Applying once more the Itô formula to the square of $\|\cdot\|_{L^2}^2$, we obtain

$$\begin{aligned}\|v_n(t)\|_{L^2}^4=&\|P_nv_0\|_{L^2}^4-4\int_0^t\|\partial_xv_n(s)\|_{L^2}^2\|v_n(s)\|_{L^2}^2\mathrm{d}s\\&+I_1+I_2+I_3,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}I_1&=4\int_0^t\langle\sigma(s,v_n(s))dW_n(s),v_n(s)\rangle\|v_n(s)\|_{L^2}^2,\\I_2&=2\int_0^t\|P_n\sigma(s,v_n(s))\|_{\mathcal{L}_2(Y,H)}^2\|v_n(s)\|_{L^2}^2\mathrm{d}s,\\I_3&=4\int_0^t\|(P_n\sigma(s,v_n(s)))^*(v_n)\|_Y^2\mathrm{d}s.\end{aligned}$$

The growth condition implies that

$$I_2(t)+I_3(t)\le6\int_0^t(K_0+K_1\|v_n(s)\|_{L^2}^2+K_2\|\partial_xv_n(t)\|_{L^2}^2)\|v_n(s)\|_{L^2}^2\mathrm{d}s. \tag{3.7}$$

The Burkholder-Davis-Gundy inequality, the growth condition and the Young inequality imply that

$$E(\sup_{s\le t}I_1(s))\le8E\{\int_0^t\|\sigma(r,v_n(r))\|_{\mathcal{L}_2(Y,H)}^2\|v_n(r)\|_{L^2}^6\mathrm{d}r\}^{\frac{1}{2}}$$

$$\begin{aligned}&\leq \gamma E(\sup_{s\leq t}\|v_n(s)\|_{L^2}^4)\\&+\frac{16}{\gamma}E\int_0^t(K_0+K_1\|v_n(s)\|_{L^2}^2+K_2\|\partial_x v_n(t)\|_{L^2}^2)\|v_n(s)\|_{L^2}^2\mathrm{d}s,\end{aligned}\tag{3.8}$$

since $K_2<\frac{2}{11}$, we can choose $0<\gamma<1$, such that $6K_2+\frac{16}{\gamma}K_2-4<0$.

Thus, combining (3.7)–(3.8) and dropping some negative terms on the right of the inequality, we have:

$$(\frac{1}{2}-\gamma)E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^4)\leq E\|v(0)\|_{L^2}^4+E\int_0^t C_1\|v_n(s)\|_{L^2}^4+C_2\|v_n(s)\|_{L^2}^2\mathrm{d}s,$$

since we have obtained $E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^2)\leq C$, the Gronwall's inequality yields

$$E(\sup_{t\in[0,T]}\|v_n(t)\|_{L^2}^4)<\infty.$$

We complete the proof. □

Lemma 3.5. *We have the following uniform estimates under the hypothesis of Theorem 3.1,*

$$E\sup_{t\in[0,T]}\|v_n(t)\|_{H^{0,1}}^2+E\int_0^T\|v_n(t)\|_{H^{1,1}}^2\mathrm{d}t\leq C(\tilde{K}_0,\tilde{K}_1,T)(1+E\|v_0\|_{H^{0,1}}^2).\tag{3.9}$$

Proof. Using again the Itô formula to $\|v_n(t)\|_{H^{0,1}}^2$, we obtain

$$\|v_n(t)\|_{H^{0,1}}^2+2\int_0^t\|v_n(t)\|_{H^{1,1}}^2\mathrm{d}s=\|P_nv(0)\|_{H^{0,1}}^2+\sum_{j=1}^2J_j(t),\tag{3.10}$$

where

$$\begin{aligned}J_1(t)&=2\int_0^t(\sigma(s,v_n(s))\mathrm{d}W_n(s),v_n(s))_{H^{0,1}},\\J_2(t)&=\int_0^t\|P_n\sigma(s,v_n(s))\|_{\mathcal{L}_2(Y,H^{0,1})}^2\mathrm{d}s.\end{aligned}$$

The growth condition implies that

$$J_2(t)\leq\int_0^t\tilde{K}_0+\tilde{K}_1\|v_n(s)\|_{H^{1,1}}^2\mathrm{d}s,\tag{3.11}$$

and

$$\begin{aligned}&E(\sup_{s\leq t}|2\int_0^s(\sigma(r,v_n(r))\mathrm{d}W_n(r),v_n(r))_{H^{0,1}}|)\\&\leq 6E\{\int_0^s\|P_n\sigma(r,v_n)(r)\|_{\mathcal{L}_2(Y,H^{0,1})}^2\|v_n(r)\|_{H^{0,1}}^2\mathrm{d}r\}^{\frac{1}{2}}\\&\leq\tilde{\beta}E(\sup_{s\leq t}(\|v_n(s)\|_{H^{0,1}}^2)+\frac{6}{\tilde{\beta}}E\int_0^t[\tilde{K}_0+\tilde{K}_1\|v_n(s)\|_{H^{1,1}}^2]\mathrm{d}s.\end{aligned}\tag{3.12}$$

Since $\tilde{K}_1 < \frac{2}{7}$, we can choose $0 < \tilde{\beta} < 1$ such that $(\frac{6}{\tilde{\beta}} + 1)\tilde{K}_1 - 2 < 0$. By (3.10)–(3.12) and dropping some negative terms, we deduce

$$(1-\tilde{\beta})E\sup_{s\in[0,t]}\|v_n(s)\|^2_{H^{0,1}} \le E\|v(0)\|^2_{H^{0,1}} + CK_0T + \frac{1}{2}E\int_0^t \|v_n\|^2_{H^{0,1}}\mathrm{d}s.$$

Gronwall's lemma implies that

$$E(\sup_{t\in[0,T]}\|v_n(t)\|^2_{H^{0,1}}) \le C, \tag{3.13}$$

which complete the proof.

□

Lemma 3.6. *We have the following uniform estimates under the hypothesis of Theorem 3.1,*

$$E\sup_{t\in[0,T]}\|v_n(t)\|^4_{H^{0,1}} + E\int_0^T \|v_n(t)\|^2_{H^{1,1}}\|v_n(t)\|^2_{H^{0,1}}\mathrm{d}t \le C(\tilde{K}_0,\tilde{K}_1,T)(1+E\|v_0\|^4_{H^{0,1}}). \tag{3.14}$$

Proof. Using again the Itô formula to $\|v_n(t)\|^4_{H^{0,1}}$, we obtain

$$\|v_n(t)\|^4_{H^{0,1}} + 4\int_0^t \|v_n(t)\|^2_{H^{1,1}}\|v_n(t)\|^2_{H^{0,1}}\mathrm{d}s = \|P_nv(0)\|^4_{H^{0,1}} + \sum_{j=1}^3 T_j(t). \tag{3.15}$$

where

$$T_1(t) = 4\int_0^t (\sigma(s,v_n(s))\mathrm{d}W_n(s), v_n(s))_{H^{0,1}}\|v_n(t)\|^2_{H^{0,1}},$$
$$T_2(t) = 2\int_0^t \|P_n\sigma(s,v_n(s))\|^2_{\mathcal{L}_2(Y,H^{0,1})}\|v_n(t)\|^2_{H^{0,1}}\mathrm{d}s,$$
$$T_3(t) = 4\int_0^t \|(P_n\sigma(s,v_n(s)))^*(v_n)\|^2_Y\mathrm{d}s.$$

The growth condition implies that

$$\begin{aligned} T_2(t)+T_3(t) &\le 2\int_0^t [\tilde{K}_0 + \tilde{K}_1\|v_n(s)\|^2_{H^{1,1}}]\|v_n(t)\|^2_{H^{0,1}}\mathrm{d}s \\ &\le 2\tilde{K}_0\int_0^t \|v_n(s)\|^2_{H^{0,1}}\mathrm{d}s + 2\tilde{K}_1\int_0^t \|v_n(s)\|^2_{H^{1,1}}\|v_n(s)\|^2_{H^{0,1}}\mathrm{d}s. \end{aligned} \tag{3.16}$$

Similar, we have

$$\begin{aligned} &E(\sup_{s\le t}|2\int_0^s (\sigma(r,v_n(r))\mathrm{d}W_n(r), v_n(r))_{H^{0,1}}\|v_n(t)\|^2_{H^{0,1}}|) \\ &\le 12E\{\int_0^s \|P_n\sigma(r,v_n)(r)\|^6_{\mathcal{L}_2(Y,H^{0,1})}\|v_n(r)\|^2_{H^{0,1}}\mathrm{d}r\}^{\frac{1}{2}} \\ &\le \tilde{\beta}E(\sup_{s\le t}(\|v_n(s)\|^4_{H^{0,1}}) + \frac{12}{\tilde{\beta}}\tilde{K}_0E\int_0^t \|v_n\|^2_{H^{0,1}}\mathrm{d}s \end{aligned}$$

$$+\frac{12}{\tilde{\beta}}\tilde{K}_1 E\int_0^t \|v_n\|_{H^{1,1}}^2\|v_n\|_{H^{0,1}}^2\mathrm{d}s. \tag{3.17}$$

Since $\tilde{K}_1 < \frac{2}{7}$, we can choose $0<\tilde{\beta}<1$ such that $(\frac{12}{\tilde{\beta}}+2)\tilde{K}_1 - 4 < 0$. By (3.16)–(3.17) and dropping some negative terms, we deduce

$$(1-\tilde{\beta})E\sup_{s\in[0,t]}\|v_n(s)\|_{H^{0,1}}^4 \le E\|v(0)\|_{H^{0,1}}^4 + \frac{1}{2}E\int_0^t \|v_n\|_{H^{0,1}}^4\mathrm{d}s.$$

Gronwall's lemma implies that

$$E(\sup_{t\in[0,T]}\|v_n(t)\|_{H^{0,1}}^4) \le C. \tag{3.18}$$

□

3.2. Tightness of the family of laws for the Galerkin solutions

In this section, we will prove the tightness of the family of laws for the Galerkin solutions. Before we proceed further, we recall the following definition and theorem which we can refer to [3].

Let $(\mathbb{S},\varrho)$ be a separable and complete metric space.

Definition 3.7. A sequence $(X_n)_{n\ge1}$ satisfies the Aldous's condition $[A]$ iff for any $\varepsilon,\eta>0$, there exists $\delta>0$ such that for every sequence $(\tau_n)_{n\ge1}$ of $\mathcal{F}_t$ -stopping times with $\tau_n\le T$, we have

$$\sup_{n\in\mathbb{N}}\sup_{\theta\in[0,\delta]}\mathbb{P}(\varrho(X_n(\tau_n+\theta),X_n(\tau_n))\ge\eta)\le\varepsilon.$$

Theorem 3.8 (see [3]). *Assume that (X_n) satisfies Aldous's condition $[A]$. Let $\mathbb{P}_n$ be the law of X_n on $C([0,T],\mathbb{S})$, $n\in\mathbb{N}$. Then for every ε there exists a subset $A_\varepsilon\subset C([0,T],\mathbb{S})$ such that*

$$\sup_{n\in\mathbb{N}}\mathbb{P}_n(A_\varepsilon)\ge 1-\varepsilon$$

and

$$\lim_{\delta\to0}\sup_{u\in A_\varepsilon}\sup_{s,t\in[0,T],|t-s|\le\delta}\varrho(u(t),u(s))=0.$$

We now state and prove the following crucial result.

Lemma 3.9. *The family $(v_n)_{n\in\mathbb{N}}$ satisfies the Aldous's condition $[A]$ in $\mathbb{S}=H^{-1}$.*

Proof. For any sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_t$ -stopping times with $\tau_n\in[0,T]$, we have

$$\begin{aligned}\|v_n(\tau_n+\theta)&-v_n(\tau_n)\|_{H^{-1}}\\ &=\|\int_{\tau_n}^{\tau_n+\theta}(\partial_x^2 v_n(s)+P_nB(v_n,v_n))\mathrm{d}s+\int_{\tau_n}^{\tau_n+\theta}P_n\sigma(s,v_n(s))\mathrm{d}W\|_{H^{-1}}\\ &\le\|I_n^{(1)}\|_{H^{-1}}+\|I_n^{(2)}\|_{H^{-1}}+\|I_n^{(3)}\|_{H^{-1}}.\end{aligned} \tag{3.19}$$

Using Lemma 3.3, the definition of $\|\cdot\|_{H^{-1}}$ and the Hölder inequality, we infer that

$$\begin{aligned}E\|I_n^{(1)}\|_{H^{-1}} &= E\int_{\tau_n}^{\tau_n+\theta}\|\partial_x^2 v_n(s)\|_{H^{-1}}\mathrm{d}s\\&\le CE\int_{\tau_n}^{\tau_n+\theta}\sup_{\|\varphi\|_{H^1}<1}|\langle\partial_x^2 v_n(s),\varphi\rangle|\mathrm{d}s\\&\le \theta^{\frac12}(E\int_{\tau_n}^{\tau_n+\theta}\|v_n(s)\|^2_{H^{1,0}}\mathrm{d}t)^{\frac12}\\&\le C\theta^{\frac12}.\end{aligned}$$

By Lemma 3.4, Lemma 3.6 and (2.1), we have

$$\begin{aligned}E\int_0^T\|P_nB(v_n(t))\|^2_{H^{-1}}\mathrm{d}t &\le E\int_0^T\sup_{\|\psi\|_{H^1}<1}|b(v_n,\psi,v_n)|^2\mathrm{d}t\\&\le E\int_0^T\Big(\|v_n\|^2_{L^2}\|\partial_z v_n\|_{L^2}\|\partial_x v_n\|_{L^2}\\&\quad+\|v_n\|_{L^2}\|\partial_x v_n\|^3_{L^2}\Big)\mathrm{d}t\\&\le C\Big(E\int_0^T\|v_n\|^2_{L^2}\|\partial_z v_n\|^2_{L^2}\mathrm{d}t+E\int_0^T\|v_n\|^2_{L^2}\|\partial_x v_n\|^2_{L^2}\mathrm{d}t\\&\quad+E\int_0^T\|\partial_x v_n\|^2_{L^2}\mathrm{d}t+E\int_0^T\|v_n\|^4_{L^2}\mathrm{d}t\Big)\\&\le C\Big(E\sup_{t\in[0,T]}\|v_n\|^4_{L^2}+E\sup_{t\in[0,T]}\|\partial_z v_n\|^4_{L^2}\\&\quad+E\int_0^T\|\partial_x v_n\|^2_{L^2}\mathrm{d}t+E\int_0^T\|v_n\|^2_{L^2}\|\partial_x v_n\|^2_{L^2}\mathrm{d}t\Big)\\&<\infty,\end{aligned}$$

such that

$$\begin{aligned}E\|I_n^{(2)}\|_{H^{-1}} &= E\int_{\tau_n}^{\tau_n+\theta}\|B(v_n(s),v_n(s))\|_{H^{-1}}\mathrm{d}s\\&\le CE\int_{\tau_n}^{\tau_n+\theta}\sup_{\|\varphi\|_{H^1}<1}|\langle B(v_n(s)),\varphi\rangle|\mathrm{d}s\\&\le E(\int_{\tau_n}^{\tau_n+\theta}\|B(v_n(s))\|^2_{H^{-1}}\mathrm{d}s)^{\frac12}(\int_{\tau_n}^{\tau_n+\theta}\|\varphi\|^2_{H_1}\mathrm{d}s)^{\frac12}\\&\le \theta^{\frac12}(E\int_{\tau_n}^{\tau_n+\theta}\|B(v_n(s))\|^2_{H^{-1}}\mathrm{d}s)^{\frac12}\\&\le C\theta^{\frac12}.\end{aligned}$$

By Burkholder-Davis-Gundy inequality and (2.4), we have

$$E\|I_n^{(3)}\|_{H^{-1}} = E\Big\|\int_{\tau_n}^{\tau_n+\theta}\sigma(s,v_n(s))\mathrm{d}W\Big\|_{H^{-1}}$$

$$
\begin{aligned}
&\le \Big(E\Big\|\int_{\tau_n}^{\tau_n+\theta}\sigma(s,v_n(s))\mathrm{d}W\Big\|_{H^{-1}}^2\Big)^{\frac{1}{2}}\\
&\le C\Big(E\int_{\tau_n}^{\tau_n+\theta}\|\sigma(s,v_n(s))\|_{\mathcal{L}_2(Y,H^{-1})}^2\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\le C\theta^{\frac{1}{2}}.
\end{aligned}
$$

Collecting these estimates altogether and plugging them in (3.19), we obtain that there exists a constant $C>0$ such that for any $\delta\in(0,\infty)$,

$$
\sup_{n\in\mathbb{N}} E\sup_{\theta\in[0,\delta]}\|v_n(t+\theta)-v_n(t)\|_{H^{-1}}\le C\delta. \tag{3.20}
$$

By Chebyshev's inequality, for any $\varepsilon>0$ and $\vartheta>0$

$$
\sup_{n\in\mathbb{N}}\sup_{\theta\in[0,\delta]}\mathbb{P}(\{\|v_n(t+\theta)-v_n(t)\|_{H^{-1}}\ge\vartheta\})\le\frac{1}{\vartheta}E\|v_n(t+\theta)-v_n(t)\|_{H^{-1}}\le\frac{C}{\vartheta}\delta.
$$

Choosing δ in such a way that $\delta<C^{-1}\vartheta\varepsilon$, we infer that

$$
\sup_{n\in\mathbb{N}}\sup_{\theta\in[0,\delta]}\mathbb{P}(\|v_n(t+\theta)-v_n(t)\|_{H^{-1}}\ge\vartheta)<\varepsilon,
$$

from which we easily infer that $(v_n)_{n\in\mathbb{N}}$ satisfies the condition $[A]$ in H^{-1}. □

In the next lemma we will prove that the family of laws of $(v_n)_{n\in\mathbb{N}}$, denoted by $\hat{P}_n$ is tightness.

Lemma 3.10. *Under the hypothesis of Theorem 3.1, $\hat{P}_n$ is tight in the space*

$$
\mathcal{X}=C([0,T];H^{-1})\cap L^2([0,T];H)\cap L^2_w([0,T];H^{1,1})\cap L^\infty_{w^*}([0,T];H^{0,1}),
$$

where $L^2_w([0,T];H^{1,1})$ denotes $L^2([0,T];H^{1,1})$ with the weak topology and $L^\infty_{w^*}([0,T];H^{0,1})$ denotes $L^\infty([0,T];H^{0,1})$ with the weak star topology.

Proof. In order to get the tightness of $\hat{P}_n$, we should prove that there exists a compact subset K_ε of $\mathcal{X}$ such that

$$
\sup_{n\in\mathbb{N}}\hat{P}_n(K_\varepsilon)\ge 1-\varepsilon. \tag{3.21}
$$

By the Chebyshev inequality and Lemma 3.3, Lemma 3.5, we infer that for any $n\in\mathbb{N}$ and any $r_1,r_2>0$,

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{s\in[0,T]}\|v_n\|_{H^{0,1}}^2>r_1\Big)&\le\frac{E(\sup_{s\in[0,T]}\|v_n\|_{H^{0,1}}^2)}{r}\le\frac{C_1}{r_1},\\
\mathbb{P}\Big(\|v_n\|_{L^2([0,T];H^{1,1})}>r_2\Big)&\le\frac{E(\|v_n\|_{L^2([0,T];H^{1,1})}^2)}{r_2^2}\le\frac{C_2}{r_2^2}.
\end{aligned}
$$

Let R_1, R_2 be such that $\frac{C_1}{R_1}\le\frac{\varepsilon}{3}$ and $\frac{C_2}{R_2^2}\le\frac{\varepsilon}{3}$, we denote

$$
B_1:=\{v\in\mathcal{X}:\sup_{s\in[0,T]}\|v\|_{H^{0,1}}^2\le R_1\},
$$

$$B_2 := \{v \in \mathcal{X} : \|v\|_{L^2([0,T];H^{1,1})} \le R_2\}.$$

Then

$$\sup_{n\in\mathbb{N}} \mathbb{P}\Big(\sup_{s\in[0,T]} \|v_n\|^2_{H^{0,1}} > R_1\Big) \le \frac{\varepsilon}{3}, \tag{3.22}$$

$$\sup_{n\in\mathbb{N}} \mathbb{P}\Big(\|v_n\|_{L^2([0,T];H^{1,1})} > R_2\Big) \le \frac{\varepsilon}{3}. \tag{3.23}$$

By Theorem 3.8, there exists a subset $A_{\frac{\varepsilon}{3}} \subset C([0,T], H^{-1})$ such that

$$\mathbb{P}(A_{\frac{\varepsilon}{3}}) \ge 1 - \frac{\varepsilon}{3} \tag{3.24}$$

and

$$\lim_{\delta\to 0} \sup_{v_n\in A_{\frac{\varepsilon}{3}}} \sup_{s,t\in[0,T],|t-s|\le\delta} \|v_n(t) - v_n(s)\|_{H^{-1}} = 0. \tag{3.25}$$

It is sufficient to define K_ε as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\varepsilon}{3}}$ in $\mathcal{X}$, and by (3.22),(3.23) and (3.24), we can obtain (3.21).

By the definition, K_ε is uniformly bounded hence relatively compact in $L^2_w([0,T];H^{1,1})$ and $L^\infty_{w^*}([0,T];H^{0,1})$. Since L^2 could be embedded compactly in H^{-1}, as $n,m\to\infty$, we have

$$\begin{aligned}\int_0^t \|v_n - v_m\|^2_{L^2}\mathrm{d}t &\le \int_0^t \|v_n - v_m\|_{H^1}\|v_n - v_m\|_{H^{-1}}\mathrm{d}t\\ &\le C_{R_2,T} \sup_{t\in[0,T]} \|v_n - v_m\|^2_{H^{-1}}\\ &\to 0.\end{aligned} \tag{3.26}$$

By (3.25), argument analogously to the proof of the classical Arzelá-Ascoli Theorem, we can obtain that $v_n \in A_{\frac{\varepsilon}{3}}$ is compact in $C([0,T], H^{-1})$. To sum up, we conclude that K_ε is a compact subset of $\mathcal{X}$. The proof is thus completed. □

We will use the following Jakubowski's version of the Skorokhod theorem in the form given by Brzezniak and Ondrejat [4], see also [29].

Theorem 3.11. *Let $\mathcal{Y}$ be a topological space such that there exists a sequence f_m of continuous functions $f_m : \mathcal{Y} \to \mathbb{R}$ that separates points of $\mathcal{Y}$. Let us denote by $\mathcal{S}$ the σ -algebra generated by the maps f_m. Then*

(j1) every compact subset of $\mathcal{Y}$ is metrizable;

(j2) if (μ_m) is tight sequence of probability measures on $(\mathcal{Y},\mathcal{S})$, then there exists a subsequence (m_k), a probability space $(\Omega,\mathcal{F},P)$ with $\mathcal{Y}$ -valued Borel measurable variables ξ_k,ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a Radon measure.

As in [39], we can obtain that on each space appearing in the definition of $\mathcal{X}$ there exists a countable set of continuous real-valued functions separating points. Then all the conditions of the above Skorokhod theorem are satisfied. By Theorem 3.2, there exists another probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a subsequence $\hat P_{n_k}$ as well as random variables $\tilde v_{n_k}$ in the space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that

(i) $\tilde v_{n_k}$ has the law $\hat P_{n_k}$,

(ii) $\hat P_{n_k}$ converges weakly to some $\hat P$,

(iii) $\tilde v_{n_k} \to \tilde v$ in $\mathcal{X}$ $\tilde{\mathbb{P}} - a.s.$ and $\tilde v$ has the law $\hat P \in \mathcal{P}(C([0,T];H^{-1}))$.

Remark 3.12. since $\tilde{v}_{n_k}$ has the same law as v_{n_k}, we immediately have

$$E^{\tilde{\mathbb{P}}}(\sup_{t\in[0,T]}\|\tilde{v}(t)\|_{L^2}^2)+E^{\tilde{\mathbb{P}}}\int_0^T\|\tilde{v}(t)\|_{H^{1,0}}^2\mathrm{d}t\leq C(1+E^{\tilde{\mathbb{P}}}\|v_0\|_{L^2}^2),$$
$$E^{\hat{P}}(\sup_{t\in[0,T]}\|v(t)\|_{L^2}^2)+E^{\hat{P}}\int_0^T\|v(t)\|_{H^{1,0}}^2\mathrm{d}t\leq C(1+E^{\hat{P}}\|v_0\|_{L^2}^2). \tag{3.27}$$

Remark 3.13. Similarly, for $L^4(\Omega)$ estimates, we have

$$E^{\tilde{\mathbb{P}}}(\sup_{t\in[0,T]}\|\tilde{v}(t)\|_{L^2}^4)+E^{\tilde{\mathbb{P}}}\int_0^T\|\tilde{v}(t)\|_{L^2}^2\|\tilde{v}(t)\|_{H^{1,0}}^2\mathrm{d}t\leq C(1+E^{\tilde{\mathbb{P}}}\|v_0\|_{L^2}^4),$$
$$E^{\hat{P}}(\sup_{t\in[0,T]}\|v(t)\|_{L^2}^4)+E^{\hat{P}}\int_0^T\|v(t)\|_{L^2}^2\|v(t)\|_{H^{1,0}}^2\mathrm{d}t\leq C(1+E^{\hat{P}}\|v_0\|_{L^2}^4). \tag{3.28}$$

3.3. Pass to the limit and the proof of main theorems

Let us denote the subsequence $(\tilde{v}_{n_k})$ again by $(\tilde{v}_n)$ and pass the limit as $n\to\infty$.

Proof of Theorem 3.1. Let us prove $\hat{P}$ satisfies $(M1),(M2)$ and $(M3)$.

For $(M1)$, noting that $v_n(0)\to v_0$ in H, we have

$$\hat{P}(v(0)=v_0)=\tilde{\mathbb{P}}(\tilde{v}(0)=v_0)=\lim_{n\to\infty}\tilde{\mathbb{P}}(\tilde{v}_n(0)=P_nv_0)=1,$$

$$\hat{P}\{v\in C([0,T],H^{-1}):\int_0^T\|F(v(s))\|_{H^{-1}}\mathrm{d}s+\int_0^T\|\sigma(s,v(s))\|_{\mathcal{L}_2(Y,H)}^2\mathrm{d}s<+\infty\}$$
$$=\tilde{\mathbb{P}}\{\tilde{v}\in C([0,T],H^{-1}):\int_0^T\|F(\tilde{v}(s))\|_{H^{-1}}\mathrm{d}s+\int_0^T\|\sigma(s,\tilde{v}(s))\|_{\mathcal{L}_2(Y,H)}^2\mathrm{d}s<+\infty\}.$$

Since

$$\tilde{v}_n\to\tilde{v}\ in\ \mathcal{X}\ \tilde{\mathbb{P}}-a.s.,$$
$$\tilde{v}\in L^2([0,T],H^{1,1})\cap L^\infty([0,T],H^{0,1})\ \tilde{\mathbb{P}}-a.s.,$$

thus by the growth condition of σ, we have

$$\int_0^T\|\sigma(s,\tilde{v}(s))\|_{\mathcal{L}_2(Y,H)}^2\mathrm{d}s\leq\int_0^T(K_0+K_1\|\tilde{v}\|_{L^2}^2+K_2\|\partial_x\tilde{v}\|_{L^2}^2)\mathrm{d}s<\infty,\ \tilde{\mathbb{P}}-a.s.$$

Similar as (3.19), we obtain $\int_0^T\|F(\tilde{v}(s))\|_{H^{-1}}\mathrm{d}s<+\infty,\tilde{\mathbb{P}}-a.s.$. Thus $(M1)$ is satisfied. Then we prove $(M2)$. The following key lemma should be given.

Lemma 3.14. *For all $s,t\in[0,T]$ such that $s\leq t$ and all $\psi\in C_{per}^\infty(\overline{M})$,*

$$(a)\ \lim_{n\to\infty}(\tilde{v}_n(t),P_n\psi)_{H^{0,1}}=(\tilde{v}(t),\psi)_{H^{0,1}},\quad\tilde{\mathbb{P}}-a.s.$$
$$(b)\ \lim_{n\to\infty}\int_s^t\langle\partial_{xx}\tilde{v}_n(\sigma),P_n\psi\rangle\mathrm{d}\sigma=\int_s^t\langle\partial_{xx}\tilde{v}(\sigma),\psi\rangle\mathrm{d}\sigma,\quad\tilde{\mathbb{P}}-a.s.$$

$$(c)\ \lim_{n\to\infty}\int_s^t \langle B(\tilde v_n(\sigma)), P_n\psi\rangle \mathrm{d}\sigma = \int_s^t \langle B(\tilde v(\sigma)),\psi\rangle \mathrm{d}\sigma, \quad \tilde{\mathbb{P}}-a.s.$$

Proof. Let us fix $s,t\in[0,T], s\le t$ and $\psi\in C^\infty_{per}(\overline{M})$. We know that

$$\tilde v_n \to \tilde v\ in\ \mathcal{X},\ \tilde{\mathbb{P}}-a.s.. \tag{3.29}$$

Thus $\tilde v_n\to\tilde v$ in $L_{w^*}([0,T];H^{0,1})$, $\tilde{\mathbb{P}}$-a.s., and $P_n\psi\to\psi$ in H^1, we infer that assertion (a) holds.

Let us move to (b). Since $\tilde v_n\to\tilde v$ in $L^2_w(0,T;H^{1,1})$, $\tilde{\mathbb{P}}$ -a.s., and $P_n\psi\to\psi$ in H^1, we infer that $\tilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}\int_s^t \langle \partial_{xx}\tilde v_n(\sigma), P_n\psi\rangle \mathrm{d}\sigma &= \int_s^t \langle \partial_x\tilde v_n(\sigma), P_n\partial_x\psi)\rangle \mathrm{d}\sigma \to \int_s^t \langle \partial_x\tilde v(\sigma),\partial_x\psi\rangle \mathrm{d}\sigma\\ &= \int_s^t \langle \partial_{xx}\tilde v(\sigma),\psi\rangle \mathrm{d}\sigma, \quad as \quad n\to\infty.\end{aligned}$$

We will prove now assertion (c). The proof is easily get if we are able to show that

$$\int_s^t \langle B(\tilde v_n)-B(\tilde v),\psi\rangle \mathrm{d}\sigma\to 0, \quad as \quad n\to\infty. \tag{3.30}$$

For this purpose, by (3.26), we obtain the strong convergence of $(\tilde v_n)_n$ in $L^2([0,T],L^2)$. This implies that in $L^1([0,T],L^1)$, we have the following weak convergence (see Rem. 3.15).

$$\tilde v_n\cdot\partial_x\tilde v_n \rightharpoonup \tilde v\cdot\partial_x\tilde v, \tag{3.31}$$

$$\mathbb{W}(\tilde v_n)\cdot\partial_z\tilde v_n \rightharpoonup \mathbb{W}(\tilde v)\cdot\partial_z\tilde v, \tag{3.32}$$

and by (3.31)–(3.32), it yields that

$$\begin{aligned}&\int_s^t \langle B(\tilde v_n)-B(\tilde v),\psi\rangle \mathrm{d}\sigma\\ &\quad= \int_s^t \langle \tilde v_n\cdot\partial_x\tilde v_n - \tilde v\cdot\partial_x\tilde v,\psi\rangle \mathrm{d}\sigma\\ &\qquad+ \int_s^t \langle \mathbb{W}(\tilde v_n)\cdot\partial_z\tilde v_n - \mathbb{W}(\tilde v)\cdot\partial_z\tilde v,\psi\rangle \mathrm{d}\sigma\\ &\quad\to 0, \quad as \quad n\to\infty.\end{aligned} \tag{3.33}$$

Then we have

$$\begin{aligned}&\int_s^t \langle B(\tilde v_n,\tilde v_n), P_n\psi\rangle \mathrm{d}\sigma\\ &= \int_s^t \langle B(\tilde v_n,\tilde v_n), P_n\psi-\psi\rangle \mathrm{d}\sigma + \int_s^t \langle B(\tilde v_n,\tilde v_n),\psi\rangle \mathrm{d}\sigma\\ &= S_1(n)+S_2(n).\end{aligned}$$

Since

$$|S_1(n)| \le \int_s^t \|B(\tilde v_n,\tilde v_n)\|_{H^{-1}}\mathrm{d}\sigma\cdot\|P_n\psi-\psi\|_{H^1}. \tag{3.34}$$

By (3.30) and (3.34), we infer that

$$\lim_{n\to\infty}\int_s^t \langle B(\tilde{v}_n(\sigma)), P_n\psi\rangle \mathrm{d}\sigma = \lim_{n\to\infty}(S_1(n)+S_2(n)) = \int_s^t \langle B(\tilde{v}(\sigma)), \psi\rangle \mathrm{d}\sigma, \quad \tilde{\mathbb{P}}-a.s..$$

This completes the proof of the lemma. □

By Lemma 3.14, we have

$$\int_0^T \langle F(\tilde{v}_n(s)), P_n\psi\rangle \mathrm{d}s \to \int_0^T F(\tilde{v}(s)), \psi\rangle \mathrm{d}s, \quad \tilde{\mathbb{P}}-a.s., \tag{3.35}$$

with (3.35) in hand, using a similar method as in [20, 39], $(M2)$ holds.

$(M3)$ is satisfied by (3.27).

Thus we complete the proof of Theorem 3.1. □

Remark 3.15. The proofs of (3.31)–(3.32) are similar. Let's take (3.32) as an example. By the boundary conditions and integrating by parts in x or z, we get

$$\begin{aligned}
&|\int_0^t \langle \mathbb{W}(\tilde{v}_n)\cdot\partial_z\tilde{v}_n - \mathbb{W}(\tilde{v})\cdot\partial_z\tilde{v}, \psi\rangle \mathrm{d}s| \\
&= |\int_0^t \langle \mathbb{W}(\tilde{v}_n)\cdot\partial_z(\tilde{v}_n-\tilde{v}), \psi\rangle + \langle(\mathbb{W}(\tilde{v}_n)-\mathbb{W}(\tilde{v}))\cdot\partial_z\tilde{v}, \psi\rangle \mathrm{d}s| \\
&= \int_0^t |\langle \partial_z\mathbb{W}(\tilde{v}_n)\cdot(\tilde{v}_n-\tilde{v}), \psi\rangle|\mathrm{d}s + \int_0^t |\langle \mathbb{W}(\tilde{v}_n)\cdot(\tilde{v}_n-\tilde{v}), \partial_z\psi\rangle|\mathrm{d}s \\
&\quad + \int_0^t |\langle \int_{-1}^z (\tilde{v}_n-\tilde{v})\mathrm{d}z\cdot\partial_{xz}\tilde{v}, \psi\rangle|\mathrm{d}s + \int_0^t |\langle \int_{-1}^z (\tilde{v}_n-\tilde{v})\mathrm{d}z\cdot\partial_z\tilde{v}, \partial_x\psi\rangle|\mathrm{d}s \\
&\le C\Big(\|\tilde{v}_n\|_{L^2([0,T],H^{1,0})}\|\tilde{v}_n-\tilde{v}\|_{L^2([0,T],L^2)}\|\psi\|_{L^\infty} \\
&\quad + \|\mathbb{W}(\tilde{v}_n)\|_{L^2([0,T],L^2)}\|\tilde{v}_n-\tilde{v}\|_{L^2([0,T],L^2)}\|\partial_z\psi\|_{L^\infty} \\
&\quad + \Big\|\int_{-1}^z (\tilde{v}_n-\tilde{v})\mathrm{d}z\Big\|_{L^2([0,T],L^2)}\|\tilde{v}\|_{L^2([0,T],H^{1,1})}\|\psi\|_{L^\infty} \\
&\quad + \Big\|\int_{-1}^z (\tilde{v}_n-\tilde{v})\mathrm{d}z\Big\|_{L^2([0,T],L^2)}\|\tilde{v}\|_{L^2([0,T],H^{0,1})}\|\partial_x\psi\|_{L^\infty}\Big) \\
&\le C\|\tilde{v}_n-\tilde{v}\|_{L^2([0,T],L^2)}\Big(\|\tilde{v}_n\|_{L^2([0,T],H^{1,0})} + \|\tilde{v}\|_{L^2([0,T],H^{1,1})} + \|\tilde{v}\|_{L^2([0,T],H^{0,1})}\Big) \\
&\to 0, \quad as \quad n\to\infty,
\end{aligned}$$

where

$$\begin{aligned}
\|\mathbb{W}(\tilde{v}_n)\|_{L^2([0,T],L^2)} &= \int_0^1\int_{-1}^0 \Big|\int_{-1}^z \partial_x\tilde{v}_n\mathrm{d}z\Big|^2\mathrm{d}z\mathrm{d}x \\
&\le C\int_0^1\int_{-1}^0 \Big|\partial_x\tilde{v}_n\Big|^2\mathrm{d}z\mathrm{d}x \\
&\le C\|\tilde{v}_n\|_{L^2([0,T],H^{1,0})},
\end{aligned}$$

and

$$\begin{aligned}\Big\|\int_{-1}^{z}(\tilde{v}_n-\tilde{v})\mathrm{d}z\Big\|_{L^2([0,T],L^2)} &= \int_0^1\int_{-1}^0\Big|\int_{-1}^{z}(\tilde{v}_n-\tilde{v})\mathrm{d}z\Big|^2\mathrm{d}z\mathrm{d}x\\ &\le C\int_0^1\int_{-1}^0\Big|(\tilde{v}_n-\tilde{v})\Big|^2\mathrm{d}z\mathrm{d}x\\ &\le C\|\tilde{v}_n-\tilde{v}\|_{L^2([0,T],L^2)}.\end{aligned}$$

□

Finally let us turn to the proof of the pathwise uniqueness. To this end, we need the following lemma.

Lemma 3.16. *Assume that $u,v\in\tilde{H}^{1,1}$, we have*

$$\langle F(u)-F(v),u-v\rangle+\frac{1}{2}\|\partial_x(u-v)\|_{L^2}^2\le C\|u-v\|_{L^2}^2(1+\|\partial_x v\|_{L^2}^2+\|\partial_{xz}v\|_{L^2}^2+\|\partial_z v\|_{L^2}^4).$$

Proof. Set $\Phi=u-v$. We deduce

$$\begin{aligned}\langle F(u)-F(v),\Phi\rangle &= -\langle\partial_{xx}(u-v),\Phi\rangle-\langle B(u)-B(v),\Phi\rangle\\ &\equiv I_1+I_2.\end{aligned}$$

Integrating by parts, using a similar method as in Lemma 2.1, Hölder inequality and Young inequality imply

$$\begin{aligned}I_1 &= -\|\partial_x(u-v)\|_{L^2}^2,\\ I_2 &= b(u-v,v,u-v)\\ &\le\int_0^1\Big(\sup_{z\in[-1,0]}|\partial_x v|\int_{-1}^0\big[(u-v)\cdot(u-v)\big]\mathrm{d}z\Big)\mathrm{d}x\\ &\quad+\int_0^1\Big(\sup_{z\in[-1,0]}\big(\int_{-1}^{z}\partial_x(u-v)\mathrm{d}\tilde{z}\big)\int_{-1}^0\big(\partial_z v\cdot(u-v)\big)\mathrm{d}z\Big)\mathrm{d}x\\ &\le\|u-v\|_{L^2}\|\partial_x v\|_{L^2}^{\frac{1}{2}}\|\partial_{xz}v\|_{L^2}^{\frac{1}{2}}\|u-v\|_{L^2}^{\frac{1}{2}}\|\partial_x(u-v)\|_{L^2}^{\frac{1}{2}}\\ &\quad+\|\partial_x(u-v)\|_{L^2}\|\partial_z v\|_{L^2}\|u-v\|_{L^2}^{\frac{1}{2}}\|\partial_x(u-v)\|_{L^2}^{\frac{1}{2}}\\ &\le\frac{1}{2}\|\partial_x(u-v)\|_{L^2}^2+C\|u-v\|_{L^2}^2(\|\partial_x v\|_{L^2}^{\frac{2}{3}}\cdot\|\partial_{xz}v\|_{L^2}^{\frac{2}{3}}+\|\partial_z v\|_{L^2}^4)\\ &\le\frac{1}{2}\|\partial_x(u-v)\|_{L^2}^2+C\|u-v\|_{L^2}^2(1+\|\partial_x v\|_{L^2}^2+\|\partial_{xz}v\|_{L^2}^2+\|\partial_z v\|_{L^2}^4).\end{aligned}$$

Combining I_1 and I_2, we end the proof. □

Proof of Theorem 3.2. Set

$$\tilde{u}:=v_1-v_2,\quad\tilde{w}_u:=w_1-w_2.$$

Then $\tilde{u}$ satisfies the following equation

$$\mathrm{d}\tilde{u}=-\Big(\partial_{xx}\tilde{u}+\big(B(v_1(t))-B(v_2(t))\big)\Big)\mathrm{d}t+[\sigma(v_1(t))-\sigma(v_2(t))]\mathrm{d}W(t).\tag{3.36}$$

Let

$$r(t) = C\int_0^t (1+\|\partial_x v_2\|_{L^2}^2 + \|\partial_{xz} v_2\|_{L^2}^2 + \|\partial_z v_2\|_{L^2}^4)\mathrm{d}s, \quad t\in[0,T],$$

by the Itô formula for the term $e^{-r(t)}\|\widetilde{u}\|_L^2$, we get

$$\begin{aligned} & e^{-r(t)}\|\widetilde{u}(t)\|_{L^2}^2 \\ & \le \int_0^t e^{-r(s)}\Big(-r'(s)\|\widetilde{u}(s)\|_{L^2}^2 - 2\|\partial_x\widetilde{u}(s)\|_{L^2}^2 \\ & \qquad -2\langle B(v_1(s)) - B(v_2(s)), \widetilde{u}(s)\rangle\Big)\mathrm{d}s \\ & + \int_0^t e^{-r(s)}\|\sigma(v_1(s)) - \sigma(v_2(s))\|_{\mathcal{L}_2(Y,H)}^2\mathrm{d}s \\ & +2\int_0^t e^{-r(s)}\langle\sigma(v_1(s)) - \sigma(v_2(s)), \widetilde{u}(s)\mathrm{d}W(s)\rangle. \end{aligned} \tag{3.37}$$

Due to Lemma 3.16, we have

$$\begin{aligned} & 2|\langle B(v_1(s)) - B(v_2(s)), \widetilde{u}(s)\rangle| \le \|\partial_x\widetilde{u}(s)\|_{L^2}^2 \\ & \quad +C\|\widetilde{u}\|_{L^2}^2(1+\|\partial_x v_2\|_{L^2}^2 + \|\partial_{xz} v_2\|_{L^2}^2 + \|\partial_z v_2\|_{L^2}^4). \end{aligned}$$

By (2.7), hence we have

$$\begin{aligned} & e^{-r(t)}\|\widetilde{u}(t)\|_{L^2}^2 \le \|\widetilde{u}(0)\|_{L^2}^2 + (-1+L_1)\int_0^t e^{-r(s)}\|\partial_x\widetilde{u}(s)\|^2 ds \\ & +2\int_0^t e^{-r(s)}\langle\sigma(v_1(s)) - \sigma(v_2(s)), \widetilde{u}(s)\mathrm{d}W(s)\rangle. \end{aligned}$$

Taking expectation on both sides, by the martingales have zero averages, for $L_1 < 1$, we have

$$E\Big[e^{-r(t)}\|\widetilde{u}(t)\|_{L^2}^2 + (1-L_1)\int_0^t e^{-r(s)}\|\partial_x\widetilde{u}(s)\|^2 ds\Big] \le E\|\widetilde{u}(0)\|_{L^2}^2 = 0.$$

Thus we obtain the uniqueness of the solution. □

With martingale solutions and pathwise uniqueness of the solutions in hand, we obtain the existence of the pathwise solutions (meaning that the solutions are defined on the prescribed probability space) to this system by the infinite-dimensional-space extension of the Yamada-Watanabe theorem (see [23]).

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