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EXACT TAIL ASYMPTOTICS FOR A THREE-DIMENSIONAL BROWNIAN-DRIVEN TANDEM QUEUE WITH INTERMEDIATE INPUTS

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Abstract. In this paper, we consider a three-dimensional Brownian-driven tandem queue with intermediate inputs, which corresponds to a three-dimensional semimartingale reflecting Brownian motion whose reflection matrix is triangular. For this three-node tandem queue, no closed form formula is known, not only for its stationary distribution but also for the corresponding transform. We are interested in exact tail asymptotics for stationary distributions. By generalizing the kernel method, and using extreme value theory and copula, we obtain exact tail asymptotics for the marginal stationary distribution of the buffer content in the third buffer and for the joint stationary distribution.

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1. Introduction

Traffic engineering greatly benefits from models that are capable of accurately describing and predicting the performance of the system. The network nodes are usually modeled as queues, and queueing theory can be used to analyze the performance of these nodes. However, most studies address performance issues for single-node models. The single-node models can offer valuable insights, but are an oversimplification of reality, since traffic streams usually traverse concatenations of nodes (rather than just a single node). In this paper, we consider a tandem queueing model. Tandem queues consist of a very important type of queueing systems, which have numerous applications in many fields, including manufacturing, telecommunications, computer network management, supply network management, health care among others. For example, tandem queues are perfect models for manufacturing (product) assembly lines (say a car or aircraft product/assembly line), where intermediate inputs represent various parts or components arrived to different stages of the assembly line; in telecommunications, information (with the form of emails, documents, live conversations, videos, internet requests, and many others) is often partitioned into packets, which are transmitted, through routers, over telecommunication networks according to QoS criteria. The whole path is a model of a tandem queue, where intermediate inputs are traffic of other sources arrived to routers. More applications either in the above mentioned or other areas can

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also be easily found (for example, see Bhaskar and Lallement [4], Giambene [12], Govil and Fu [13], Robertazzi [29]).

Typical queues possess a "discrete" nature largely due to the discrete nature of the number of customers and the number of servers. Therefore, a direct modeling usually gives us a stochastic (often Markovian) process with a discrete state space. However, processes with continuous states have become a standard tool in queueing modeling and analysis since, for example: (1) empirical evidences from measurement studies suggest that in many cases, network traffic can be approximated by Gaussian processes, *e.g.*, Brownian motions. In fact, under some very general conditions, input processes of a broad class of (short-range dependent) network traffic converge to Brownian motions (for example, see Whitt [36]); and (2) Semimartingale reflecting Brownian motions (SRBM) are often used to approximate the behavior of open networks under heavy traffic conditions. See, for example, Harrison and Williams [17] and Williams [35].

All above considerations underscore the importance of analyzing tandem queues with Brownian inputs. Closely related to our work in this paper, Mandjes [24] studied a two-node Brownian-driven tandem queue without intermediate inputs and got the joint distribution function of the first queue and the total queue length, whereas Debicki *et al.* [8] focused on the stationary distribution of the second queue. Miyazawa and Rolski [27] obtained exact tail asymptotics for marginal distributions of a two-dimensional Brownian-driven tandem queues with intermediate Brownian inputs, and also tried to discuss higher-dimensional cases. However only the stationary equation was obtained in terms of moment generating functions, and tail asymptotic properties for the marginal stationary distributions were left for future work.

One can expect that as the model becomes more general, exact solution becomes less possible. In this case, tail asymptotic properties and approximations become more important. Exact tail asymptotics of stationary distributions of two-dimensional SRBM have been studied intensively. For example, Dai *et al.* [5] studied exact tail asymptotics for boundary measures of a two-dimensional SRBM. Dai and Miyazawa [7] obtained exact tail asymptotics for marginal stationary distributions of a two-dimensional SRBM. Franceschi and Kurkova [11] obtained the asymptotic expansion of the stationary distribution density along all paths. We note that all aforementioned results are only for two-dimensional SRBM, and we could not find results on higher ($\geq$3)-dimensional cases, except for some very special cases. In fact, dealing with asymptotics of the stationary distribution in 3-dimension is always very challenging. Motivated by these, we consider a three-dimensional Brownian-driven tandem queue with intermediate inputs and discuss the exact tail behavior, not only for marginal stationary distributions, but also for the joint stationary distribution. This queueing system is a special SRBM since its reflection matrix is triangular. In this work, we successfully derive exact tail aysmptotics for the marginal stationary distribution of the third buffer content, since exact tail asymptotic results for the first two buffer contents can be obtained directly from results for two-dimensional SRBM. Furthermore, we present exact asymptotic properties for the joint stationary distribution. For exact tail asymptotic properties of the marginal stationary distributions, we extend the kernel method (which is available for two-dimensional cases) and show how to reduce a three-dimensional problem to a two-dimensional one. Roughly speaking, the kernel function corresponding to our model is a ternary function (see Eqs. (2.13) and (3.1)). In view of the kernel method, a binary alternative is constructed (see Eq. (3.9)). Then the kernel method is applied to study the binary alternative. Significant efforts will be made in this direction. Based on the exact tail asymptotics for the marginal distributions, we further obtain exact tail asymptotic properties of the joint distribution. However, for this purpose, the kernel method is not a proper method. Instead, we apply extreme value theory to study upper tail dependence for the joint stationary distribution (see Lem. 5.6), and then apply copula to get exact tail asymptotic properties of the joint stationary distribution (see Thm. 5.8).

The rest of this paper is organized as follows: In Section 2, a three-dimensional Brownian-driven tandem queue with intermediate inputs is introduced. To apply the kernel method for asymptotic properties for the marginal stationary distribution of the buffer content in the third buffer, we study the kernel equation and the analytic continuation of moment generating functions in Section 3. Asymptotic results for the unknown functions and the tail asymptotic results for the marginal distributions are presented in Section 4. In Section 5, we study exact asymptotic properties of the joint stationary distribution using extreme value theory and copula. The final section contributes to concluding remarks and discussions.

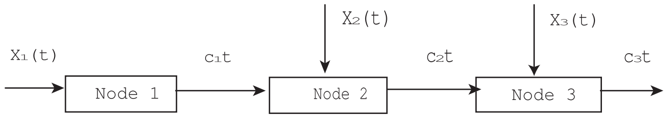

FIGURE 1. A tandem queue with 3 nodes.

Before we conclude the introduction, it is worthwhile to point out that on one hand, we anticipate that the tools developed in this paper could be useful in analyzing a general d-dimensional SRBM (for example, for the rough decay rate), but on the other hand, one has to overcome more technical challenges for a more general case (with non-triangular reflection matrix, or with dimension $d > 3$). More detailed discussions are provided in the last section.

2. Model and preliminaries

In this section, we introduce a three-dimensional Brownian-driven tandem queue with intermediate inputs and establish a stationary equation satisfied by stationary probabilities. This tandem queue has three nodes, numbered as 1, 2, 3, each of which has an exogenous input process and a constant processing rate. Outflow from the node 1 goes to node 2, and the outflow from node 2 goes to node 3. Finally, outflow from node 3 leaves the system (see Fig. 1). We assume that the exogenous inputs are independent Brownian processes of the form:

$$X_i(t) = \lambda_i t + B_i(t),\ i = 1, 2, 3, \tag{2.1}$$

where $\lambda_i > 0$ is a positive constant, and $B_i(t)$ is a Brownian motion with variance σ_i^2 and no drift. Denote the processing rate at node i by $c_i > 0$. Let $L_i(t)$ be the buffer content at node i at time $t \geq 0$ for $i = 1, 2, 3$, which are formally defined as

$$L_1(t) = L_1(0) + X_1(t) - c_1 t + Y_1(t), \tag{2.2}$$
$$L_i(t) = L_i(0) + X_i(t) + c_{i-1}t - c_i t - Y_{i-1}(t) + Y_i(t),\ i \geq 2, \tag{2.3}$$

where the local time $Y_i(t)$ is a regulator at node i, that is, a minimal nondecreasing process for $L_i(t)$ to be nonnegative.

In this paper, all vectors are supposed to be column vectors. For a vector v, we write v' to denote the transpose of it. To simplify the notation, let

$$L(t) = \big(L_1(t), L_2(t), L_3(t)\big)',\ Y(t) = \big(Y_1(t), Y_2(t), Y_3(t)\big)',$$

and

$$\tilde{X}(t) = B(t) + \Lambda t, \tag{2.4}$$

where $\Lambda = \big(\lambda_1 - c_1, \lambda_2 + c_1 - c_2, \lambda_3 + c_2 - c_3\big)'$ and $B(t) = \big(B_1(t), B_2(t), B_3(t)\big)'$. It is obvious that $\{\tilde{X}(t)\}$ is a $\mathbb{R}^3$-valued Brownian motion with mean vector Λ. Then, (2.2) and (2.3) can be rewritten as

$$L(t) = \tilde{X}(t) + RY(t) + L(0), \tag{2.5}$$

where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \tag{2.6}$$

Namely, $L(t)$ is a generated by a reflection mapping from net flow processes $\tilde{X}(t)$ with reflection matrix R.

Remark 2.1. Here we point out that the reflection matrix R is triangular. In this sense, our model is degenerate. This triangular structure of the reflection matrix R is crucial for us to carry out analytic continuation and find dominant singularities in Section 3.

Without any difficulty, we can obtain that the tandem queue $\{L(t)\}$ has a stationary distribution if and only if

$$\sum_{i=1}^{j} \lambda_i < c_j,\ 1 \le j \le 3. \tag{2.7}$$

Moreover, by Harrison and Williams [17], we can get that the stationary distribution of $\{L(t)\}$ is unique. Throughout this paper, we denote this stationary distribution by π. In order to simplify the discussion, we refine the stability condition (2.7) to assume that

$$\lambda_1 < c_1,\ \text{and}\ \lambda_i + c_{i-1} < c_i,\ i = 2, 3. \tag{2.8}$$

Remark 2.2. From the proofs of the main results of this paper, it is clear that under the more general stability condition (2.7), we can use the same argument to discuss tail asymptotics. The only difference is that we need to discuss possible relationships between the parameters λ_i and c_i, $i = 1, 2, 3$, before we use the arguments in the proofs in this paper. For each of the possible relationships, we repeat the method applied in this paper to study tail asymptotics.

We are interested in the asymptotic tail behavior of the stationary distribution. Recall that a positive function $g(x)$ is said to have exact tail asymptotic $h(x)$, if

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1.$$

Our main aim is to find exact tail asymptotics for various stationary distributions. Moment generating functions will play an important role in determining these exact tail asymptotics. We first introduce moment generating functions for stationary distributions. Let $L = (L_1, L_2, L_3)'$ be the stationary random vector with stationary distribution π. The moment generating function $\phi(\cdot)$ for L is given by:

$$\phi(x, y, z) = \mathbb{E}\big[e^{xL_1 + yL_2 + zL_3}\big],\ \text{for any}\ (x, y, z)' \in \mathbb{R}^3. \tag{2.9}$$

We apply the kernel method to study tail asymptotics for marginal stationary distributions. In order to apply the kernel method, we need to establish a relationship between the moment generating function $\phi(\cdot)$ for the stationary distribution π and the moment generating functions for the boundary measures defined below. For any Borel set $A \subset \mathscr{B}(\mathbb{R}^3)$, we define the boundary measures $V_i(\cdot)$, $i = 1, 2, 3$, by

$$V_i(A) = \mathbb{E}_\pi\Big[\int_0^1 I_{\{L(u)\in A\}} \mathrm{d}Y_i(u)\Big]. \tag{2.10}$$

Moreover, due to Harrison and Williams [17], we obtain that the density functions for V_i, $i = 1, 2, 3$, exist. Then, their moment generating functions are defined by

$$\phi_i(x,y,z) = \int_{\mathbb{R}^3_+} e^{\langle w,\theta\rangle} V_i(\mathrm{d}\theta) = \mathbb{E}\Big[\int_0^1 e^{\langle w,\, L(u)\rangle}\mathrm{d}Y_i(u)\Big], \quad i = 1,2,3, \tag{2.11}$$

where $w = (x, y, z)' \in \mathbb{R}^3$ and $\langle w, \theta\rangle$ denotes the inner product of vectors w and θ.

The following lemma is a particular case of Theorem 4 in Konstantopoulous, Last and Lin [18].

Lemma 2.3. *For each* $(x, y, z)' \in \mathbb{R}^3$ *with* $\phi(x, y, z) < \infty$ *and* $\phi_i(x, y, z) < \infty$, $i = 1, 2, 3$, *we have*

$$H(x,y,z)\phi(x,y,z) = H_1(x,y)\phi_1(x,y,z) + H_2(y,z)\phi_2(x,y,z) + H_3(z)\phi_3(x,y,z), \tag{2.12}$$

where

$$\begin{aligned} H(x,y,z) &= -\frac{1}{2}\big(\sigma_1^2x^2 + \sigma_2^2y^2 + \sigma_3^2z^2\big) \\ &\quad +(c_1-\lambda_1)x + (c_2-\lambda_2-c_1)y + (c_3-\lambda_3-c_2)z, & (2.13)\\ H_1(x,y) &= x-y, & (2.14)\\ H_2(y,z) &= y-z, & (2.15)\\ H_3(z) &= z. & (2.16) \end{aligned}$$

Proof of Lemma 2.3. To make the paper self-contained, here we apply the Itô's formula to prove this lemma directly. Let $C^2(\mathbb{R}^3)$ be the set of all functions from $\mathbb{R}^3$ to $\mathbb{R}$ with continuous second-order partial derivatives. Let $f(\theta) \in C^2(\mathbb{R}^3)$, then by Itô's formula

$$\begin{aligned} f(L(t)) - f(L(0)) &= \sum_{i=1}^3 \Lambda_i \int_0^t \frac{\partial f}{\partial\theta_i}(L(u))\mathrm{d}u + \sum_{i,j=1}^3 \int_0^t r_{ji}\frac{\partial f}{\partial\theta_j}\big(L(u)\big)\mathrm{d}Y_i(u) \\ &\quad + \sum_{i=1}^3 \int_0^t \frac{\partial f}{\partial\theta_i}\big(L(u)\big)\mathrm{d}B_i(u) + \frac{1}{2}\sum_{i=1}^3 \sigma_i^2 \int_0^t \frac{\partial^2 f}{\partial\theta_i\partial\theta_i}\big(L(u)\big)\mathrm{d}u, \end{aligned} \tag{2.17}$$

where r_{ji} is the (j,i)-th entry of the reflection matrix $R = (r_{ji})_{3\times3}$. Take the expectation at the both sides of (2.17) for $t = 1$, given that $L(0)$ follows the stationary distribution, and denote this expectation by $\mathbb{E}_\pi$. Then, as long as all expectations are finite, we have

$$\sum_{i=1}^3 \Lambda_i\mathbb{E}_\pi\left[\int_0^1 \frac{\partial f}{\partial\theta_i}(L(u))\mathrm{d}u\right] + \sum_{i,j=1}^3 \mathbb{E}_\pi\left[\int_0^1 r_{ji}\frac{\partial f}{\partial\theta_j}\big(L(u)\big)\mathrm{d}Y_i(u)\right] + \frac{1}{2}\sum_{i=1}^3 \sigma_i^2\mathbb{E}_\pi\left[\int_0^1 \frac{\partial^2 f}{\partial\theta_i\partial\theta_i}\big(L(u)\big)\mathrm{d}u\right] = 0. \tag{2.18}$$

Therefore, taking $f(\theta) = \exp\{\theta_1 x + \theta_2 y + \theta_3 z\}$ with $\phi(x, y, z) < \infty$ and $\phi_i(x, y, z) < \infty$, $i = 1, 2, 3$, in equation (2.18) completes the the proof of this lemma. □

From Lemma 2.3, we can prove the following lemma.

Lemma 2.4. *For* $1 \le j \le 3$, *we have*

$$\phi_j(0,0,0) = \mathbb{E}_\pi[Y_j(1)] = c_j - \sum_{i=1}^j \lambda_i > 0. \tag{2.19}$$

Before we prove this lemma, we first introduce the following notation for convenience:

$$c_0 = 0 \text{ and } \phi_0 = 0.$$

Proof of Lemma 2.4. From (2.11), we get that (2.12) makes sense for $w \in \{(x,y,z)' : x \leq 0, y \leq 0 \text{ and } z \leq 0\}$. Let $w = (0, x_j, 0)'$ with $x_j < 0$. From (2.12), we get

$$\Big((c_j - \lambda_j - c_{j-1}) - \frac{1}{2}\sigma_j^2 x_j\Big)\phi(0, x_j, 0) = -\phi_{j-1}(0, x_j, 0) + \phi_j(0, x_j, 0). \tag{2.20}$$

Letting x_j go to 0 in (2.20), we get that the left-hand side of equation (2.20) equals to

$$c_j - \lambda_j - c_{j-1}, \tag{2.21}$$

since $\phi(0,0,0) = 1$. Hence,

$$(c_j - \lambda_j - c_{j-1}) + \phi_{j-1}(0,0,0) = \phi_j(0,0,0). \tag{2.22}$$

Let $j = 1$. Then, one can easily get that

$$\phi_1(0,0,0) = c_1 - \lambda_1. \tag{2.23}$$

By (2.22) and (2.23), the proof is completed. □

In general, it is difficult or impossible to obtain the explicit expression for the stationary distribution π, or its moment generating function. Hence, our focus is on its tail asymptotics. There are a few available methods for studying tail asymptotics, for example, in terms of large deviations and boundary value problems. In this paper, we study tail asymptotics of the marginal distribution $\mathbb{P}\big(L_3 < z\big)$ *via* the kernel method introduced by Li and Zhao [20] and asymptotic properties of the joint stationary distribution by extreme value theory and copula.

Remark 2.5. The kernel method introduced by Li and Zhao [20] is an analytic method for studying stationary tail behaviour of two-dimensional queueing systems. This method is a combination of analytic continuation and asymptotic analysis of complex functions. For more information about this method, refer the readers to the survey paper [37]. Here we also note that the term "kernel method" has also been used by others, for example, Banderier *et al.* [2], Bousquet-Mélou [3] among others. The kernel method in [2, 3] continued the work of Fayolle *et al.* [10] and is used to solve one-dimensional unknown probability sequences (or functions) first through the kernel equation and then joint probability. Their methods focus on a complete determination of the unknown function and therefore involve much more work. The kernel method used here only requires the location of the dominant singularity of the unknown function and the asymptotic property at the dominant singularity.

At the end of this section, we recall the Tauberian-like Theorem for moment generating functions introduced in Dai, *et al.* [5]. Let $\widetilde{g}$ be the $\mathbb{L}$-transformation of a nonnegative, continuous and integrable function $\widetilde{f}$ on $[0, \infty)$, *i.e.*,

$$\widetilde{g}(s) = \int_0^\infty e^{st}\widetilde{f}(t)\mathrm{d}t.$$

Then, $\widetilde{g}(s)$ is analytic on the left half-plane. Let $\mathbb{C}$ denote the complex plane. Moreover, for a point $z_0 \in \mathbb{C}$ and $\delta \geq 0$, define

$$\mathbb{G}_\delta(z_0) = \big\{z \in \mathbb{C} : z \neq z_0, |\arg(z - z_0)| > \delta\big\}, \tag{2.24}$$

where $\arg(z) \in (-\pi,\ \pi)$ is the principal part of the argument of a complex number z. The following lemma comes from Dai *et al.* [5].

Lemma 2.6. *(Tauberian-like Theorem) Assume that $\widetilde{g}(z)$ satisfies the following conditions:*

(1) The left-most singularity of $\widetilde{g}(z)$ is α_0 with $\alpha_0 > 0$. Furthermore, we assume that as $z \to \alpha_0$,

$$\widetilde{g}(z) \sim (\alpha_0 - z)^{-\lambda}$$

for a real number $\lambda \notin \{0, -1, -2, \dots\}$;

(2) $\widetilde{g}(z)$ is analytic on $\mathbb{G}_{\tilde{\delta}_0}(\alpha_0)$ for some $\tilde{\delta}_0 \in (0,\ \frac{\pi}{2}]$;

(3) $\widetilde{g}(z)$ is bounded on $\mathbb{G}_{\tilde{\delta}_1}(\alpha_0)$ for some $\tilde{\delta}_1 > 0$.

Then, as $t \to \infty$,

$$\widetilde{f}(t) \sim e^{-\alpha_0 t} \frac{t^{\lambda-1}}{\Gamma(\lambda)}, \tag{2.25}$$

where $\Gamma(\cdot)$ is the Gamma function.

3. Kernel equation, dominant singularities and analytic continuation

In this paper, we apply the kernel method to study tail asymptotics for the marginal stationary measure $\mathbb{P}(L_3 < z)$. The original kernel method applies easily to one-dimensional problems, and for two-dimensional problems (or random walks in the quarter plane), we refer the readers to Li and Zhao [21]. The problem of interest in this paper is a three-dimensional problem. Significant efforts are required in order to apply the kernel method to our problem, which will be addressed in this and the next sections before we can use the Tauberian-like Theorem (see Lem. 2.6) to connect the asymptotic properties of the unknown function and the corresponding tail asymptotic properties of $\mathbb{P}(L_3 < z)$. Specifically, since $\phi(0,0,z)$ is the transformation function for the marginal stationary measure for L_3, we need to study analytic properties of the moment generating function $\phi(0,0,z)$. In this section, we address analytic continuation and defer the singularity analysis in the next section.

3.1. Kernel equation and branch points

To study analytic properties of the moment generating functions, we first focus on the kernel equation and the corresponding branch points. For this purpose, we consider the kernel equation:

$$H(x,y,z) = 0, \tag{3.1}$$

which is critical in our analysis.

Since tail asymptotics for $\mathbb{P}(L_3 < z)$ is our focus, we first treat z in $(x,y,z)' \in \mathbb{R}^3$ as a variable. Inspired by the procedure of applying the kernel method, for example, see Li and Zhao [20, 21], we first construct the relationship between z and x, y. The kernel equation in (3.1) defines an implicit function z in variables x and y when we only consider non-negative values for z.

In view of the kernel method for the bivariate case, we locate the maximum $z^{\max}$ of z on $H(x,y,z) = 0$. In order to do it, taking the derivative with respect to x at the both sides of (3.1) yields

$$\big(-z\sigma_3^2 - \lambda_3 - c_2 + c_3\big)\frac{\partial z}{\partial x} + \big(-x\sigma_1^2 - \lambda_1 - c_0 + c_1\big) = 0. \tag{3.2}$$

Let

$$\frac{\partial z}{\partial x} = 0, \tag{3.3}$$

and solve the system of equations (3.2) and (3.3), and then we have

$$x_{z^{\max}} = \frac{c_1 - \lambda_1 - c_0}{\sigma_1^2}. \tag{3.4}$$

Similarly, take the derivative with respect to y,

$$\frac{\partial z}{\partial y} = 0, \tag{3.5}$$

to obtain

$$y_{z^{\max}} = \frac{c_2 - \lambda_2 - c_1}{\sigma_2^2}. \tag{3.6}$$

It is easy to check that at the point $\left(x_{z^{\max}}, y_{z^{\max}}\right)$, z attains the maximum value $z^{\max}$. From (3.4) and (3.6), we can get that on the point $(x_{z^{\max}}, y_{z^{\max}}, z^{\max})$, the coordinates x and y satisfy

$$x_{z^{\max}} = k_1 y_{z^{\max}}, \tag{3.7}$$

where

$$k_1 = \frac{(c_1 - \lambda_1 - c_0)\sigma_2^2}{(c_2 - \lambda_2 - c_1)\sigma_1^2}. \tag{3.8}$$

Remark 3.1. Without loss of generality, we assume that $k_1 \neq 1$ in the rest of this paper. For the special case $k_1 = 1$, the discussion can be carried out by using the same idea, which is much simpler than the general case due to the fact that when $k_1 = 1$, the terms including $k_1 - 1$ in most equations will disappear.

From the above arguments, we obtain the maximum $z^{\max}$ on the plane $H(k_1 y, y, z) = 0$. Now, we consider the new equation:

$$H(k_1 y, y, z) = 0. \tag{3.9}$$

From (2.8) and (3.1), we can easily know that (3.9) defines an ellipse. Thus, for fixed z, there are two solutions to (3.9) for y, which are given:

$$Y_{\max,0}(z) = \frac{(c_1 - \lambda_1)k_1 + (c_2 - \lambda_2 - c_1) - \sqrt{\Delta(z)}}{(\sigma_1^2 k_1^2 + \sigma_2^2)} \tag{3.10}$$

and

$$Y_{\max,1}(z) = \frac{(c_1 - \lambda_1)k_1 + (c_2 - \lambda_2 - c_1) + \sqrt{\Delta(z)}}{(\sigma_1^2 k_1^2 + \sigma_2^2)}, \tag{3.11}$$

where

$$\Delta(z) = \Big((c_1 - \lambda_1)k_1 + c_2 - \lambda_2 - c_1\Big)^2 + 2\big(\sigma_1^2 k_1^2 + \sigma_2^2\big)\big(-\frac{1}{2}\sigma_3^2 z^2 + (c_3 - \lambda_3 - c_2)z\big). \tag{3.12}$$

Moreover, these two solutions are distinct except if $\Delta(z) = 0$. We call a point z a branch point if $\Delta(z) = 0$. For branch points, we have the following properties.

Lemma 3.2.

(i) $\Delta(z)$ has two real zeros, one of which is $z^{\max}$, and the other is denoted by $z^{\min}$. Moreover, they satisfy

$$z^{\min} < 0 < z^{\max}. \tag{3.13}$$

(ii) $\Delta(z) > 0$ in $(z^{\min},\ z^{\max})$ and $\Delta(z) < 0$ in $(-\infty,\ z^{\min}) \cup (z^{\max},\ \infty)$.

Proof. From (3.12), we obtain

$$\Delta(0) = \Big(\sum_{i=1}^{2}(c_i - \lambda_i - c_{i-1})k_i\Big)^2 > 0, \tag{3.14}$$

where $k_2 = 1$ and $c_0 = 0$. On the other hand,

$$\sum_{i=1}^{2} \sigma_i^2 k_i^2 > 0. \tag{3.15}$$

From (3.14) and (3.15), we get (3.13).

By properties of quadratic functions, we can get that (ii) holds. The proof of the lemma is completed now. □

Remark 3.3. From Lemma 3.2, we can evaluate $z^{\max}$ and $z^{\min}$. Specifically, we have

$$\begin{aligned} z^{\max} = {} & \frac{c_3 - \lambda_3 - c_2}{\sigma_3^2} \\ & + \frac{\sqrt{\big(\sigma_1^2 + \sigma_2^2\big)^2 (c_3 - \lambda_3 - c_2)^2 + \sigma_3^2(\sigma_1 k^2 + \sigma_2^2)\big((c_1 - \lambda_1)k_1 + c_2 - \lambda_2 - c_1\big)^2}}{\sigma_3^2(\sigma_1^2 k^2 + \sigma_2^2)}, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} z^{\min} = {} & \frac{c_3 - \lambda_3 - c_2}{\sigma_3^2} \\ & - \frac{\sqrt{\big(\sigma_1^2 + \sigma_2^2\big)^2 (c_3 - \lambda_3 - c_2)^2 + \sigma_3^2(\sigma_1 k^2 + \sigma_2^2)\big((c_1 - \lambda_1)k_1 + c_2 - \lambda_2 - c_1\big)^2}}{\sigma_3^2(\sigma_1^2 k^2 + \sigma_2^2)}. \end{aligned} \tag{3.17}$$

In order to use the Tauberian-like Theorem, we consider the analytic continuation of the moment generating functions in the complex plane $\mathbb{C}$. The function $\sqrt{\Delta(z)}$ plays an important role in the process of the analytic continuation. Hence, we first study its analytic continuation. By Lemma 3.2, $\sqrt{\Delta(z)}$ is well defined for $z \in [z^{\min},\ z^{\max}]$. Moreover, it is a multi-valued function in the complex plane. For convenience, in the sequel, $\sqrt{\Delta(z)}$ denotes the principal branch, that is $\Delta(z) = \Delta(Re(z))$ for $z \in (z^{\min},\ z^{\max})$. In the following, we continue $\sqrt{\Delta(z)}$ to the cut plane $\mathbb{C} \setminus \big\{(-\infty,\ z^{\min}] \cup [z^{\max}, \infty)\big\}$. In fact, we have

Lemma 3.4. *$\sqrt{\Delta(z)}$ is analytic in the cut plane $\mathbb{C}\setminus\big\{(-\infty,\ z^{\min}]\cup[z^{\max},\infty)\big\}$.*

The proof of Lemma 3.4 is standard. For example, see Dai and Miyazawa [7], and Dai, *et al.* [5]. Here we omit the proof.

Corollary 3.5. *Both $Y_{\max,0}(z)$ and $Y_{\max,1}(z)$ are analytic in the cut plane $\mathbb{C}\setminus\big\{(-\infty,\ z^{\min}]\cup[z^{\max},\infty)\big\}$.*

Symmetrically, we can treat the kernel equation in (3.9) as a quadratic function in z, and obtain the parallel results to those in Lemmas 3.2 and 3.4, and Corollary 3.5, respectively. We list them below. Before stating them, we first introduce the following notation. Define

$$\bar{\Delta}(y)=(c_3-\lambda_3-\lambda_2)^2+2\sigma_3^2\Big(y\sum_{i=1}^{2}(c_i-\lambda_i-c_{i-1})k_i-\frac{1}{2}y^2\sum_{i=1}^{2}\sigma_i^2k_i^2\Big). \tag{3.18}$$

For fixed y, there are two solutions to (3.9), which are given by

$$Z_{\max,1}(y)=\frac{(c_3-\lambda_3-c_2)+\sqrt{\bar{\Delta}(y)}}{\sigma_3^2} \tag{3.19}$$

and

$$Z_{\max,0}(y)=\frac{(c_3-\lambda_3-c_2)-\sqrt{\bar{\Delta}(y)}}{\sigma_3^2}. \tag{3.20}$$

Similarly to Lemmas 3.2 and 3.4, and Corollary 3.5, we have:

Lemma 3.6.

(i) *$\bar{\Delta}(y)$ has two real zeros, denoted by $y^{\min}$ and $y^{\max}$, respectively, satisfying*

$$y^{\min}<0<y^{\max}. \tag{3.21}$$

(ii) *$\bar{\Delta}(y)>0$ in $(y^{\min},\ y^{\max})$ and $\bar{\Delta}(y)<0$ in $(-\infty,\ y^{\min})\cup(y^{\max},\ \infty)$.*
(iii) *$Z_{\max,0}(y)$ are analytic in the cut plane $\mathbb{C}\setminus\{(-\infty,y^{\min}]\cup[y^{\max},\infty)\}$.*

In order to get the analytic continuation of the moment generating functions, we need some technical lemmas. For the function $Y_{\max,0}(z)$, we have the following properties.

Lemma 3.7. *For $Y_{\max,0}(z)$, we have*

(i) *$Re\big(Y_{\max,0}(z)\big)\le Y_{\max,0}\big(Re(z)\big)$ for $Re(z)\in(z^{\min},\ z^{\max})$.*
(ii) *$Re\big(Y_{\max,0}(z)\big)<y^{\max}$ for $z\in\mathbb{G}_{\delta_0}(z^{\max})\cap\big\{z\in\mathbb{C}:\ z^{\min}<Re(z)\big\}$ with some $\delta_0\in[0,\ \frac{\pi}{2})$.*

Proof. Since $z^{\min}$ and $z^{\max}$ are two zeros of $\Delta(z)=0$, we have

$$\Delta(z)=\Big(\sum_{i=1}^{2}\sigma_i^2k_i^2\Big)\sigma_3^2(z-z^{\min})(z^{\max}-z). \tag{3.22}$$

It follows from (3.10) and (3.22) that

$$Re\ (Y_{\max,0}(z))-Y_{\max,0}\big(Re(z)\big)=\sqrt{\sigma_3^2\sum_{i=1}^{2}\sigma_i^2k_i^2}$$

$$\times\left[\sqrt{(Re(z)-z^{\min})(z^{\max}-Re(z))}-Re\Big(\sqrt{(z-z^{\min})(z^{\max}-z)}\Big)\right]. \tag{3.23}$$

By (3.23), in order to prove case (i), we only need to show

$$\sqrt{\Big(Re(z)-z^{\min}\Big)\Big(z^{\max}-Re(z)\Big)}-Re\Big(\sqrt{(z-z^{\min})(z^{\max}-z)}\Big)\le 0. \tag{3.24}$$

We also note that $(Re(z)-z^{\min})$ and $(z^{\max}-Re(z))$ are real parts of $(z-z^{\min})$ and $(z^{\max}-z)$, respectively, since $z^{\min}$ and $z^{\max}$ are real. Therefore,

$$\begin{aligned}\big(Re(z)-z^{\min}\big)\ &=|z-z^{\min}|\cos\big(\omega_{\min}(z)\big),\\ \big(z^{\max}-Re(z)\big)\ &=|z^{\max}-z|\cos\big(\omega_{\max}(z)\big).\end{aligned}$$

So,

$$\sqrt{(Re(z)-z^{\min})(z^{\max}-Re(z))}=\sqrt{|z-z^{\min}||z^{\max}-z|}\Big(\cos\big(\omega_{\min}(z)\big)\cos\big(\omega_{\max}(z)\big)\Big)^{\frac{1}{2}}. \tag{3.25}$$

Similarly, we have

$$(z-z^{\min})(z^{\max}-z)=|z-z^{\min}||z^{\max}-z|\exp\Big\{i(\omega_{\max}(z)+\omega_{\min}(z))\Big\}.$$

Thus,

$$\sqrt{(z-z^{\min})(z^{\max}-z)}=\sqrt{|z-z^{\min}||z^{\max}-z|}\exp\{i\frac{\omega_{\max}(z)+\omega_{\min}(z)}{2}\}.$$

Hence,

$$Re\Big(\sqrt{(z-z^{\min})(z^{\max}-z)}\Big)=\sqrt{|z-z^{\min}||z^{\max}-z|}\cos\big(\frac{\omega_{\max}(z)+\omega_{\min}(z)}{2}\big). \tag{3.26}$$

Since for $Re(z)\in(z^{\min},\ z^{\max})$,

$$\omega_{\max}(z)\in(-\frac{\pi}{2},\ 0), \tag{3.27}$$

$$\omega_{\min}(z)\in(0,\ \frac{\pi}{2}), \tag{3.28}$$

$$\frac{\omega_{\max}(z)+\omega_{\min}(z)}{2}\in(-\frac{\pi}{4},\ \frac{\pi}{4}). \tag{3.29}$$

From (3.25) to (3.29), in order to prove (3.24), we only need to prove

$$\Big(\cos\big(\omega_{\min}(z)\big)\cos\big(\omega_{\max}(z)\big)\Big)^{\frac{1}{2}}\le\cos\Big(\frac{\omega_{\max}(z)+\omega_{\min}(z)}{2}\Big), \tag{3.30}$$

which directly follows from the proof of the inequality (6.2) in Dai and Miyazawa [7].

Next, we prove case (ii). We first assume that $z^{\min} < Re(z) < z^{\max}$. From (3.10) and Lemma 3.2, we have

$$y^m := Y_{\max,0}(z^{\max}) = \frac{\sum_{i=1}^{2}(\lambda_i - c_i - c_{i-1})k_i}{\sum_{i=1}^{2}\sigma_i^2 k_i^2}, \tag{3.31}$$

since $\Delta(z^{\max}) = 0$. From (3.31) and case (i), in order to prove case (ii), we only need to show

$$\frac{\sum_{i=1}^{2}(c_i - \lambda_i - c_{i-1})k_i}{\sum_{i=1}^{2}\sigma_i^2 k_i^2} \le y^{\max}. \tag{3.32}$$

It follows from (3.18) and Lemma 3.6 that

$$2\frac{\sum_{i=1}^{2}(\lambda_i - c_i - c_{i-1})k_i}{\sum_{i=1}^{2}\sigma_i^2 k_i^2} \le y^{\max}. \tag{3.33}$$

Hence, (3.32) follows from (3.33).

Finally, we assume that $Re(z) \ge z^{\max}$. As $\delta \to \frac{\pi}{2}$, we have that

$$Re(z) \to z^{\max}. \tag{3.34}$$

It follows from Lemma 3.6, (3.33) and (3.34) that we can find $\delta_0 \in [0, \frac{\pi}{2})$ such that case (ii) holds. The proof of the lemma is completed. □

3.2. Dominant singularities and analytic continuation

The analytic continuation of the moment generating function $\phi_2(0,0,z)$ plays an important role in our analysis, which is the focus in this subsection. In order to carry out this, we need the following technical lemma.

Lemma 3.8. *For the moment generating functions $\phi_i(\cdot)$, $i = 1,2,3$, we have*

(i) $\phi_1(0,y,z)$ is finite on some region $\{(y,z)' : y < \epsilon, z < \epsilon\}$ with $\epsilon > 0$;
(ii) $\phi_2(0,0,z)$ is finite on some region $\{z : z < \epsilon\}$ with $\epsilon > 0$;
(iii) $\phi_2(x,0,z)$ is finite on some region $\{(x,z)' : z < \epsilon, x < \epsilon\}$ with $\epsilon > 0$;
(iv) $\phi_3(0,y,0)$ is finite on some region $\{y : y < \epsilon\}$ with $\epsilon > 0$.

Proof. We first prove case (i). In order to prove it, we first prove

$$\mathbb{E}\Big[\int_0^1 e^{yL_2(u)}\mathrm{d}Y_1(u)\Big] < \infty \tag{3.35}$$

for some $y > 0$, and

$$\mathbb{E}\Big[\int_0^1 e^{zL_3(u)}dY_1(u)\Big] < \infty \tag{3.36}$$

for some $z > 0$.

In fact,

$$\mathbb{E}\Big[\int_0^1 e^{yL_2(u)}\mathrm{d}Y_1(u)\Big] = \phi_1(0,y,0) \tag{3.37}$$

which suggests that we may restrict our analysis to the two-dimensional tandem queue $\{(L_1(t), L_2(t))'\}$ with the two nodes 1 and 2. We note that $(L_1(t), L_2(t))'$ is not affected by $L_3(t)$. Hence, (3.35) follows straightforwardly from Dai, *et al.* [5].

Next, we prove (3.36). Since Y_1 is a regulator,

$$\mathbb{E}\Big[\int_0^1 e^{zL_3(u)}dY_1(u)\Big] = \mathbb{E}\Big[\int_0^1 e^{xL_1(u)+0L_2(u)+zL_3(u)}dY_1(u)\Big] = \phi_1(x,0,z). \tag{3.38}$$

By (3.38) and (2.12), we get that the left-hand side of (3.36) satisfies

$$H(x,0,z)\phi(x,0,z) = x\phi_1(0,0,z) - z\phi_2(x,0,z) + z\phi_3(x,0,0). \tag{3.39}$$

Next, we study this system on the plane $y = 0$. We first consider the ellipse defined by

$$H(x,0,z) = 0. \tag{3.40}$$

For the point $(x,z)'$ on this ellipse, we have

$$x\phi_1(0,0,z) - z\phi_2(x,0,z) + z\phi_3(x,0,0) = 0. \tag{3.41}$$

For fixed x, we can find two solutions to (3.40) for z. Denote one of these two solutions by

$$Z_0(x) = \frac{(c_3-\lambda_3-c_2) - \sqrt{(c_3-\lambda_3-c_2)^2 + 2\sigma_3^2\big(-\frac{1}{2}\sigma_1^2x^2 + (c_1-\lambda_1)x\big)}}{\sigma_3^2}. \tag{3.42}$$

Using the same method as in the proof of Lemma 3.2, we can get that $Z_0(x)$ is well-defined in $[x^{\min},\ x^{\max}]$ with $x^{\min} < 0 < x^{\max}$ and

$$\Delta_1(x^{\min}) = \Delta_1(x^{\max}) = 0,$$

where

$$\Delta_1(x) = (c_3-\lambda_3-c_2)^2 + 2\sigma_3^2\big(-\frac{1}{2}\sigma_1^2x^2 + (c_1-\lambda_1)x\big).$$

Hence, from (3.41) and (3.42), we have

$$x\phi_1\big(0,0,Z_0(x)\big) - Z_0(x)\phi_2\big(x,0,Z_0(x)\big) + Z_0(x)\phi_3(x,0,0) = 0, \tag{3.43}$$

that is,

$$Z_0(x)\phi_3(x,0,0) = Z_0(x)\phi_2\big(x,0,Z_0(x)\big) - x\phi_1\big(0,0,Z_0(x)\big). \tag{3.44}$$

Hence, $Z_0(x)\phi_3(x,0,0)$ is finite if and only if the right-hand side of (3.44) is finite. On the other hand, from (3.42), we obtain that for $x \in [x^{\min},\ 0)$,

$$z = Z_0(x) > 0, \tag{3.45}$$

and

$$\phi_3(x,0,0) < \infty. \tag{3.46}$$

From (3.45) and (3.46), we obtain that

$$\phi_1(0,0,Z_0(x)) < \infty, \tag{3.47}$$

since $\infty > Z_0(x)\phi_2\big(x,0,Z_0(x)\big) \geq 0$ and $-x\phi_1\big(0,0,Z_0(x)\big) \geq 0$. Therefore (3.36) holds. On the other hand, noting that $e^{yL_2(u)} \geq 0$ and $e^{zL_3(u)} \geq 0$, we have

$$e^{yL_2(u)+zL_3(u)} = e^{yL_2(u)}e^{zL_3(u)} \leq \frac{1}{2}\Big(e^{2yL_2(u)} + e^{2zL_3(u)}\Big), \tag{3.48}$$

since $a^2+b^2 \geq 2ab$ for any $a,b>0$. Finally, we have

$$\begin{aligned}\phi_1(0,y,z) &= \mathbb{E}\Big[\int_0^1 e^{yL_2(u)+zL_3(u)}\mathrm{d}Y_1(u)\Big]\\ &\leq \frac{1}{2}\Big(\mathbb{E}\Big[\int_0^1 e^{2yL_2(u)}dY_1(u)\Big] + \mathbb{E}\Big[\int_0^1 e^{2zL_3(u)}dY_1(u)\Big]\Big).\end{aligned} \tag{3.49}$$

Combining (3.35), (3.36) and (3.49), we get that for some $y>0$ and $z>0$

$$\phi_1(0,y,z) < \infty.$$

Next, we prove case (ii). Since

$$\phi_2(0,0,z) = \mathbb{E}\Big[\int_0^1 e^{0L_1(u)+zL_3(u)}\mathrm{d}Y_2(u)\Big], \tag{3.50}$$

we can consider the problem on the plane $x=0$. It follows from (2.12) that

$$H(0,y,z)\phi(0,y,z) = -y\phi_1(0,y,z) + (y-z)\phi_2(0,0,z) + z\phi_3(0,y,0). \tag{3.51}$$

Then,

$$H(0,y,z) = 0 \tag{3.52}$$

defines an ellipse. For every fixed y, define

$$\bar{Z}_0(y) = \frac{(c_3-\lambda_3-c_2) - \sqrt{(c_3-\lambda_3-c_2)^2 + 2\sigma_3^2\big(-\frac{1}{2}\sigma_2^2y^2 + (c_2-\lambda_2-c_1)y\big)}}{\sigma_3^2}. \tag{3.53}$$

Then, (3.53) is a solution to equation (3.52). Similarly to Lemma 3.2, $\bar{Z}_0(y)$ is well-defined on some region $[a,\ b]$ with $a<0$ and $b>0$. It follows from (3.51) and (3.53) that

$$\big(y-\bar{Z}_0(y)\big)\phi_2(0,0,z) = y\phi_1\big(0,y,\bar{Z}_0(y)\big) - \bar{Z}_0(y)\phi_3(0,y,0). \tag{3.54}$$

Furthermore, from (3.53), we obtain that for $y \in \{y : a < y < 0\}$

$$\bar{Z}_0(y) > 0. \tag{3.55}$$

Hence, by case (i) and (3.55), we can choose $y < 0$ such that $z = \bar{Z}_0(y) > 0$ and

$$\phi_1\big(0, y, \bar{Z}_0(y)\big) < \infty. \tag{3.56}$$

It is also worthy noting that for $y < 0$,

$$\phi_3(0, y, 0) < \infty. \tag{3.57}$$

Case (ii) now follows from (3.54) to (3.57).

Finally, we can use the same ideal as for cases (i) and (ii) to show cases (iii) and (iv). This completes the proof of the lemma. □

For the continuation of the function $\phi_2(0, 0, z)$, we need another technical tool.

Lemma 3.9. *Let $f(x_1, x_2)$ be a probability density function on $\mathbb{R}^2_+$. For a real variable λ, define $\tilde{G}(\lambda) = \int_{\mathbb{R}^2_+} e^{g(\lambda)x_1+\lambda x_2} f(x_1, x_2)\mathrm{d}x$ with $g(\lambda)$ being a bounded and continuously differentiable real function, and*

$$\tau_{\tilde{G}} = \sup\{\lambda \geq 0 : \tilde{G}(\lambda) < \infty\}. \tag{3.58}$$

Then, the complex variable function $\tilde{G}(z)$ is analytic on $\{z \in \mathbb{C} : Re(z) < \tau_{\tilde{G}}\}$.

Proof. We use the Vitali's Theorem to prove it. In fact, we have

$$\int_{\mathbb{R}^2_+} e^{g(\lambda)x_1+\lambda x_2} f(x_1, x_2) dx_1 dx_2 = \int_0^\infty e^{g(\lambda)x_1} dx_1 \Big[\int_0^\infty e^{\lambda x_2} f(x_1, x_2) dx_2 \Big]. \tag{3.59}$$

For convenience, define

$$F(z, x_1) = \int_0^\infty e^{z x_2} f(x_1, x_2) dx_2. \tag{3.60}$$

Since $f(x_1, x_2)$ is a density function, we can get that $F(z, x_1)$ is analytic on the region $\{z \in \mathbb{C} : Re(z) < \tau_{\tilde{G}}\}$ for any $x_1 \in \mathbb{R}_+$. Let

$$\tilde{F}(\lambda, x_1) = e^{g(\lambda)x_1} F(\lambda, x_1). \tag{3.61}$$

Now, it is obvious that $\tilde{F}(\lambda, x_1)$ satisfies the conditions of the Vitali's Theorem (see, for example, Markushevich [26]) on the region $\{z \in \mathbb{C} : Re(z) < \tau_{\tilde{G}}\}$. Then, the lemma holds. □

Remark 3.10. From Lemma 3.9,

(i) From (3.58), one can see that the convergence parameter $\tau_{\tilde{G}}$ is unique;
(ii) If $\tilde{G}(z)$ is singular at some $z_0 \in \mathbb{C}$, then we must have $\tilde{G}(x) = \infty$ for $x \in \big(Re(z_0), \infty\big)$.

Remark 3.11. It follows from Lemmas 3.8 and 3.9 that

(i) $\phi_2(0, 0, z)$ is finite on some region $\{z : Re(z) < \epsilon\}$ with $\epsilon > 0$, which implies that the convergence parameter $\tau_{\phi_2(0,0,z)}$ is greater than 0.

(ii) $\phi_3(0, y, 0)$ is finite on some region $\{y : Re(y) < \epsilon\}$ with $\epsilon > 0$, which implies that the convergence parameter $\tau_{\phi_3(0,y,0)}$ is greater than 0.

The next lemma enables us to express $\phi_2(0, 0, z)$ in terms of the other moment generating functions.

Lemma 3.12. *$\phi_2(0, 0, z)$ can be analytically continued to the region $z \in \{z : Re(z) < \epsilon\}$ with $\epsilon > 0$, and*

$$\begin{aligned}\phi_2(0,0,z) = \; & \frac{\phi_2(k_1 Y_{\max,0}(z), 0, z)}{1-k_1} \\ & + z\frac{\phi_3(k_1 Y_{\max,0}(z), Y_{\max,0}(z), 0)}{(1-k_1)(Y_{\max,0}(z)-z)} - z\frac{\phi_3(0,0,Y_{\max,0}(z))}{(Y_{\max,0}(z)-z)}. \end{aligned} \tag{3.62}$$

Proof. From Corollary 3.5 and (2.12), we get that

$$\begin{aligned}&\phi_1\big(k_1 Y_{\max,0}(z), Y_{\max,0}(z), z\big) \\ &\quad = -\frac{\phi_2\big(k_1 Y_{\max,0}(z), 0, z\big)(Y_{\max,0}(z)-z) + z\phi_3\big(k_1 Y_{\max,0}(z), Y_{\max,0}(z), 0\big)}{(k_1-1)Y_{\max,0}(z)}.\end{aligned} \tag{3.63}$$

On the other hand, equation (3.52) defines an ellipse. For fixed z, there are two solutions to (3.52) for y. Define

$$Y_0(z) = \frac{(c_2-\lambda_2-c_1) - \sqrt{\Delta_{H(0,y,z)}(z)}}{\sigma_2^2}, \tag{3.64}$$

where

$$\Delta_{H(0,y,z)}(z) = (c_2-\lambda_2-c_1)^2 + 2\sigma_2^2\big((c_3-\lambda_3-c_2)z - \frac{1}{2}\sigma_3^2 z^2\big).$$

Using the same method as in the proof of Lemma 3.4, we can get that $Y_0(z)$ is analytic in the cut plane $\mathbb{C} \setminus \{(-\infty, \bar{z}^{\min}] \cup [\bar{z}^{\max}, \infty)\}$, where

$$\Delta_{H(0,y,z)}(\bar{z}^{\min}) = \Delta_{H(0,y,z)}(\bar{z}^{\max}) = 0$$

with

$$\bar{z}^{\min} < 0 < \bar{z}^{\max}.$$

By (3.51) and (3.64), we can find a region such that

$$\phi_1(0, Y_0(z), z) = \frac{\phi_2(0,0,z)(Y_0(z)-z) + z\phi_3(0, Y_0(z), 0)}{Y_0(z)}. \tag{3.65}$$

Next, we study the relationship between $Y_0(z)$ and $Y_{\max,0}(z)$ for $z > 0$. We note that both the two ellipses defined by (3.9) and (3.52), respectively, pass the origin $(0, 0)'$ and

$$\begin{aligned} H(k_1 y, y, z) &= H(0,y,z) - \frac{1}{2}k_1^2\sigma_1^2 y^2 + (c_1-\lambda_1)k_1 y \\ &= H(0,y,z) + \widehat{G}(y), \end{aligned} \tag{3.66}$$

where

$$\widehat{G}(y) = -\frac{1}{2}k_1^2\sigma_1^2 y^2 + (c_1 - \lambda_1)k_1 y.$$

We should note that

$$\widehat{G}(y) > 0 \text{ if and only if } \; y \in \Big[0, \frac{2(c_1-\lambda_1)}{\sigma_2^2 k_1}\Big]. \tag{3.67}$$

From (3.64), we obtain that for $z \in [0, \; \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}]$

$$Y_0(z) \leq 0. \tag{3.68}$$

From (3.66) and (3.68), we get that for $z \in (0, \; \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2})$

$$H(k_1 Y_0(z), Y_0(z), z) < 0. \tag{3.69}$$

On the other hand, from (3.10) and (3.64), we have, for $0 < z < \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}$,

$$Y_0(z) < 0 \text{ and } Y_{\max,0}(z) < 0.$$

Thus,

$$Y_{\max,0}(z) > Y_0(z) \tag{3.70}$$

for $z \in \Big(0, \; \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\Big)$.

It follows from Lemma 3.8, (3.63), (3.65) and (3.70) that

$$\begin{aligned}\phi_1(0, Y_{\max,0}(z), z) &= \frac{\phi_2(0,0,z)(Y_{\max,0}(z) - z) + z\phi_3(0, Y_{\max,0}(z), 0)}{Y_{\max,0}(z)} \\ &= -\frac{\phi_2\big(k_1 Y_{\max,0}(z), 0, z\big)(Y_{\max,0}(z) - z) + z\phi_3(k_1 Y_{\max,0}(z), Y_{\max,0}(z), 0)}{(k_1 - 1)Y_{\max,0}(z)},\end{aligned}$$

where we use the principle of analytic continuation of several complex variables functions (see, for example, Narasimhan [28]). Therefore

$$\begin{aligned}\phi_2(0,0,z) = &\frac{\phi_2(k_1 Y_{\max,0}(z), 0, z)}{1-k_1} \\ &+ z\frac{\phi_3(k_1 Y_{\max,0}(z), Y_{\max,0}(z), 0)}{(1-k_1)(Y_{\max,0}(z) - z)} - z\frac{\phi_3(0, Y_{\max,0}(z), 0)}{(Y_{\max,0}(z) - z)}\end{aligned} \tag{3.71}$$

for $Re\ (z) < \epsilon$ with some $\epsilon > 0$. The proof is completed. □

We continue to address analytic continuation of the function $\phi_2(0,0,z)$. From Lemma 3.9, there exists only one dominant singularity. We denote it by z_{dom}. We first characterize the dominant singularity z_{dom} of $\phi_2(0,0,z)$.

For convenience, let

$$F(y) = \frac{\phi_3(k_1 y, y, 0)}{(1-k_1)} - \phi_3(0, y, 0) \text{ and } D(y,z) = \phi_2(0,y,z) - \frac{\phi_2(k_1 y, y, z)}{1-k_1}.$$

Moreover, let

$$G(z) = D(Y_{\max,0}(z), z) \text{ and } \bar{G}(y) = D(y, Z_{\max,0}(y)). \tag{3.72}$$

From Lemma 3.12, we have:

Lemma 3.13. *$F(y)$ can be analytically continued to a region $\{y : Re(y) < \epsilon\}$ with $\epsilon > 0$, and*

$$F(y) = \frac{Z_{\max,0}(y) - y}{Z_{\max,0}(y)} \bar{G}(y). \tag{3.73}$$

We introduce the following notation.

$$\hat{\phi}_2(0,0,z) := \phi_2(k_1 Y_{\max,0}(z), 0, z). \tag{3.74}$$

Next, we first study the relationship between the convergence parameters of $\phi_2(0,0,z)$, $\hat{\phi}_2(0,0,z)$ and $G(z)$. In fact, we have:

Lemma 3.14. *For the convergence parameters τ_{ϕ_2}, $\tau_{\hat{\phi}_2}$ and τ_G of $\phi_2(0,0,z)$, $\hat{\phi}_2(0,0,z)$ and $G(z)$, respectively, we have*

$$\tau_G = \tau_{\phi_2} = \tau_{\hat{\phi}_2}. \tag{3.75}$$

Proof. We first show

$$\tau_{\phi_2} = \tau_{\hat{\phi}_2}. \tag{3.76}$$

By Lemma 3.8, we just need to focus on $z > 0$. By (2.11), we get that if $Y_{\max,0}(z) \geq 0$, then

$$\tau_{\phi_2} \geq \tau_{\hat{\phi}_2}; \tag{3.77}$$

if $Y_{\max,0}(z) < 0$, then

$$\tau_{\phi_2} \leq \tau_{\hat{\phi}_2}, \tag{3.78}$$

since $L_1(u) \geq 0, L_2(u) \geq 0$, $z > 0$ and $k_1 > 0$.

In order to prove (3.76), we first locate the dominant singularity z_{dom}. From (3.62), we have

$$\begin{aligned} &\phi_2(0,0,z) - \frac{\phi_2(k_1 Y_{\max,0}(z), 0, z)}{1-k_1} \\ &\quad = z \frac{\phi_3(k_1 Y_{\max,0}(z), Y_{\max,0}(z), 0)}{(1-k_1)(Y_{\max,0}(z) - z)} - z \frac{\phi_3(0, Y_{\max,0}(z), 0)}{(Y_{\max,0}(z) - z)}. \end{aligned} \tag{3.79}$$

We observe from (3.10) and (3.12) that

$$Y_{\max,0}\Big(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\Big)=0. \tag{3.80}$$

From Lemma 2.4, we get

$$\phi_2(0,0,\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2})=\phi_3(0,0,0)=c_3-\sum_{i=1}^{3}\lambda_i. \tag{3.81}$$

Hence, from Lemma 3.9 and (3.81), we get

$$\tau_{\phi_2}>\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}.$$

For $z\in(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max})$, one can easily get that

$$Y_{\max,0}(z)>0. \tag{3.82}$$

Therefore,

$$\tau_{\phi_2}\geq\tau_{\hat{\phi}_2}. \tag{3.83}$$

However, from (3.71), we must have (3.76).

Next, we prove (3.75). From (3.62) and (3.76), it is obvious that

$$\tau_{\phi_2}\leq z^{\max}. \tag{3.84}$$

If $\tau_{\phi_2}=z^{\max}$, then, from (3.71), it must be the dominant singularity of $G(z)$. Next, we assume $\tau_{\phi_2}\in(0,\ z^{\max})$. From (2.11) and (3.82), we have that for $\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}<z<z^{\max}$

$$\begin{aligned}\hat{\phi}_2(0,0,z)-\phi_2\big(0,Y_{\max,0}(z),z\big)&=\phi_2(k_1Y_{\max,0}(z),0,z)-\phi_2(0,0,z)\\&=\int_{\mathbb{R}_+^3}\Big(e^{k_1Y_{\max,0}(z)x_1}-1\Big)e^{zx_3}V_2(\mathrm{d}x)\\&\geq\int_{\mathbb{R}_+^3}\Big(e^{k_1Y_{\max,0}(z)x_1}-1\Big)V_2(\mathrm{d}x)>0.\end{aligned} \tag{3.85}$$

It is worth noting that, from Lemma 3.9, (3.76) and (3.85), we have

$$\lim_{z\to\tau_{\phi_2}}\phi_2(0,0,z)=\lim_{z\to\tau_{\phi_2}}\hat{\phi}_2(0,0,z)=\infty. \tag{3.86}$$

If τ_{ϕ_2} is not the dominant singularity of $G(z)$, then $G(z)$ is analytic around τ_{ϕ_2}. So, $G(z)$ is bounded in a neighbourhood of α. By (3.72) and (3.74),

$$\hat{\phi}_2(0,0,z)=(1-k_1)\phi_2(0,Y_{\max,0}(z),z)-(1-k_1)G(z).$$

Hence,

$$\hat{\phi}_2(0,0,z) - \phi_2(0,Y_{\max,0}(z),z) = -k_1\phi_2(0,Y_{\max,0}(z),z) - (1-k_1)G(z). \tag{3.87}$$

From the maximum modulus principle, Lemma 3.9 and (3.87), we obtain that for some region $\{z : 0 < |z-\alpha| \leq \epsilon\}$,

$$\hat{\phi}_2(0,0,z) - \phi_2(0,Y_{\max,0}(z),z) < 0, \tag{3.88}$$

since $k_1 > 0$. It is obvious that (3.85) contradicts to (3.88). Hence the lemma holds. □

Remark 3.15. From the proof of Lemma 3.14, we have the following important fact

$$\tau_{\phi_2} \leq z^{\max}. \tag{3.89}$$

Next, we study the convergence parameter τ_G. In fact, we have:

Lemma 3.16. *If $z^G \in (0,\ z^{\max}]$ is the dominant singularity of $G(z)$, then $\bar{G}(y)$ is analytic at the point $y^0 := Z_{\max,0}(z^G)$.*

Proof. From (3.20), we obtain that the zero y^* of $Z_{\max,0}(y)$ is

$$y^* = 2\frac{\sum_{i=1}^{2}(c_i - \lambda_i - c_{i-1})}{\sum_{i=1}^{2}\sigma_i^2 k_i^2}. \tag{3.90}$$

From (3.10) and Lemma 3.1, we get

$$Y_{\max,0}(z^{\min}) = Y_{\max,0}(z^{\max}) = \frac{\sum_{i=1}^{2}(c_i - \lambda_i - c_{i-1})}{\sum_{i=1}^{2}\sigma_i^2 k_i^2} := \tilde{y}^m. \tag{3.91}$$

Combining (3.90) and (3.91), we obtain that

$$\tilde{y}^m < y^*. \tag{3.92}$$

It follows from (3.18) and (3.20) that for $y \in (0,\ y^*)$

$$Z_{\max,0}(y) < 0. \tag{3.93}$$

From (3.10), one can easily get that $Y_{\max,0}(z)$ is increasing on $[\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max}]$. Hence

$$y^0 \leq \tilde{y}^m < y^*. \tag{3.94}$$

From (3.93) and (3.94), we obtain that

$$Z_{\max,0}(y^0) < 0. \tag{3.95}$$

Therefore $\phi_2(0,0,z)$ is analytic at the point $z^0 := Z_{\max,0}(y^0)$. From (3.72), in order to prove the lemma, we only need to show that $\phi_2\big(k_1 y, 0, Z_{\max,0}(y)\big)$ is analytic at y^0. From (3.95), we must have

$$Z_{\max,1}(y^0) = z^G. \tag{3.96}$$

It follows from Lemma 3.14 that $\phi_2(k_1 Y_{\max,0}(z), 0, z)$ is analytic at z^0. It follows from (3.10) and (3.91) that

$$Z_{\max,0}(y^0) = z^0.$$

From the above arguments, we can get that the lemma holds. □

The zero of $Y_{\max,0}(z) - z$ is critical for us to prove Lemma 3.17 below. Hence, we demonstrate how to evaluate it. Let $f(z) = \big(Y_{\max,0}(z) - z\big)\big(Y_{\max,1}(z) - z\big)$. Then we have

$$f(z) = Y_{\max,0}(z)Y_{\max,1}(z) - z\big(Y_{\max,1}(z) + Y_{\max,0}(z)\big) + z^2. \tag{3.97}$$

It follows from (3.8) and (3.10) that

$$-\frac{1}{2}(\sum_{i=1}^{3}\sigma_i^2 k_i^2)f(z) = \Big(-\frac{1}{2}(\sum_{i=1}^{3}\sigma_i^2 k_i^2)z + \sum_{i=1}^{3}(c_i - \lambda_i - c_{i-1})k_i\Big)z.$$

Hence, the non-zero root of $Y_{\max,0}(z) - z = 0$ is

$$z^* = 2\frac{\sum_{i=1}^{3}(c_i - \lambda_i - c_{i-1})k_i}{\sum_{i=1}^{3}\sigma_i^2 k_i^2}. \tag{3.98}$$

Lemma 3.17. *If $\tau_G \in (\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max})$, then τ_G is the zero z^* of $Y_{\max,0}(z) - z$.*

Proof. From (3.62), we obtain that

$$G(z) = \frac{z}{Y_{\max,0}(z) - z}F(Y_{\max,0}(z)). \tag{3.99}$$

Hence, in order to prove our result, we only need to show $F(Y_{\max,0}(z))$ is analytic on $\{z : Re(z) < z^* + \epsilon\}$ with small enough $\epsilon > 0$. From (3.20), we have

$$Y_{\max,0}(z^*) = z^*. \tag{3.100}$$

Next, we show that

$$Y_{\max,0}(z^*) \neq y^*. \tag{3.101}$$

Since $Y_{\max,0}(z)$ is increasing on $(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max}]$, by (3.92),

$$Y_{\max,0}(z^*) < y^*. \tag{3.102}$$

Finally, it follows from Lemma 3.16 that $\bar{G}(y)$ is analytic at the point $Y_{\max,0}(z^*)$. From the above arguments and (3.73), we complete the proof of the lemma. □

Remark 3.18. From (3.89) and Lemma 3.17, we have that

$$\tau_{\phi_2} > \frac{2(c_3 - \lambda_3 - c_2)}{\sigma_3^2}. \tag{3.103}$$

From Lemma 3.17 and (3.103), we have:

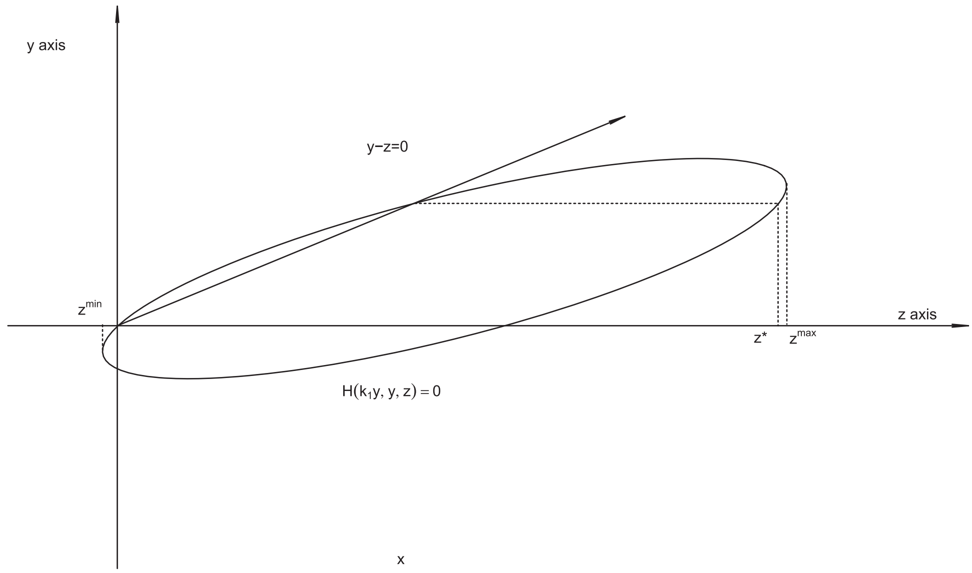

FIGURE 2. Illustration for $z^{\min}$, z^* and $z^{\max}$.

Lemma 3.19. *If the convergence parameter τ_{ϕ_2} is less than $z^{\max}$, then*

$$\tau_{\phi_2} = 2\frac{\sum_{i=1}^{3}(c_i - \lambda_i - c_{i-1})k_i}{\sum_{i=1}^{3}\sigma_i^2 k_i^2}.$$

To apply the Tauberian-like Theorem to connect the asymptotic property of $\phi_2(0,0,z)$ to the corresponding tail asymptotic property of the boundary measure V_2, we need to continue the function $\phi_2(0,0,z)$ further to a larger domain except a neighbourhood of the dominant singularity z_{dom}. By Lemma 3.9, there is exactly one dominant singularity for $\phi_2(0,0,z)$. By Lemma 3.17, there are two candidates for the dominant singularity z_{dom} of $\phi_2(0,0,z)$:

(1) A pole, *i.e.*., the zero z^* of $Y_{\max,0}(z) - z$; or
(2) the branch point $z^{\max}$.

An illustration for $z^{\min}$, z^* and $z^{\max}$ is presented in Fig. 2.

For each of these two cases, we show that the unknown function $\phi_2(0,0,z)$ satisfies the analytic continuation condition required by the Tauberian-like Theorem.

Lemma 3.20. *If $z_{dom} < z^{\max}$, then there exists an $\epsilon > 0$ such that $\phi_2(0,0,z)$ is analytic for $Re(z) < z_{dom} + \epsilon$ except for $z = z_{dom}$ and for each $a > 0$*

$$\sup_{\substack{z \notin B_a(z_{dom}) \\ Re(z) < z_{dom}+\epsilon}} |\phi_2(0,0,z)| < \infty, \tag{3.104}$$

where $B_a(z_{dom}) = \{z \in \mathbb{C} : |z - z_{dom}| < a\}$.

Proof. From Lemma 3.17, we see that if $z_{dom} < z^{\max}$, then z_{dom} is a pole of the function $\phi_2(0,0,z)$. Hence, $\phi_2(0,0,z)$ is analytic for $Re(z) < z_{dom} + \epsilon$ except for $z = z_{dom}$. It remains to show (3.104) for each $a > 0$. In such a case, z_{dom} is a pole of $\phi_2(0,0,z)$. It follows from Lemma 3.17 that z_{dom} is a zero of $Y_{\max,0}(z)-z$. So

$$\sup_{\substack{z\notin B_a(z_{dom})\\ Re(z)<z_{dom}+\epsilon}} \left|\frac{1}{Y_{\max,0}(z)-z}\right| < \infty. \tag{3.105}$$

On the other hand, from (3.10), we get that

$$Y_{\max,1}(z^*)-z^* \neq 0. \tag{3.106}$$

From (3.97),

$$f(z)=\frac{2(z^*-z)z}{\sum_{i=1}^3 \sigma_i^2 k_i^2}. \tag{3.107}$$

From (3.99) and (3.107), we obtain that

$$G(z)=\frac{2z(Y_{\max,1}(z)-z)}{f(z)\sum_{i=1}^3\sigma_i^2k_i^2}F(Y_{\max,0}(z))=\frac{2(Y_{\max,1}(z)-z)}{\sum_{i=1}^3\sigma_i^2k_i^2(z^*-z)}F(Y_{\max,0}(z)). \tag{3.108}$$

From (3.102) and (3.108), we have that for $Re(z)<z_{dom}+\epsilon$,

$$F\big(Y_{\max,0}(z)\big)<\infty. \tag{3.109}$$

Finally, we can easily get that

$$k_1-1<\infty. \tag{3.110}$$

Equations (3.105) to (3.110) yield (3.104). The proof is completed. □

Lemma 3.21. *If $z_{dom}=z^{\max}$, then $\phi_2(0,0,z)$ is analytic in $\mathbb{G}_{\delta_0}(z^{\max})$, where δ_0 is chosen in Lemma 3.7 and $\mathbb{G}_\delta$ is defined by (2.24). Moreover, for each $a>0$,*

$$\sup_{\substack{z\in\mathbb{G}_{\delta_0}(z^{\max})\\ z\notin B_a(z^{\max})}} |\phi_2(0,0,z)|<\infty.$$

Proof. We first show that $\phi_2(0,0,z)$ is analytic on $z\in\mathbb{G}_{\delta_0}(z^{\max})$. It follows from Lemma 3.9 that $\phi_2(0,0,z)$ is analytic for $Re(z)<z_{dom}$. Furthermore, by (3.81), we have $z_{dom}>0$. Hence, in order to prove the lemma, it suffices to show that $\phi_2(0,0,z)$ is analytic on $z\in\mathbb{G}_{\delta_0}(z^{\max})\cap\{z\in\mathbb{C}: Re(z)>0\}$.

Since $z_{dom}=z^{\max}$, from Lemma 3.17, we must have $z^*\geq z^{\max}$. We first assume that

$$z^{\max}<z^*. \tag{3.111}$$

Combining (3.92) and Lemma 3.16, we have that $F(Y_{\max,0}(z))$ is analytic at $\mathbb{G}_{\delta_0}(z^{\max})$. Hence, from (3.99) and Corollary 3.5, we can get the lemma.

Next, we assume that $z^{\max}=z^*$. The proof of this case is the combination of the proof of Lemma 3.20 and that of the case (3.111). So, we omit the details of the proof here. □

4. Singularity Analysis of $\phi(0,0,z)$ and Exact Tail Asymptotics for Marginal Distributions

From the arguments in the previous section, we are ready to present asymptotic properties of the function $\phi_2(0,0,z)$. Before stating these properties, we first present a technical lemma, which plays an important role in finding the tail asymptotics of the marginal distribution $\mathbb{P}(L_3 < z)$.

Lemma 4.1. *$\phi(0,0,z)$ and $\phi_2(0,0,z)$ have the same singularities.*

Proof. Let $w = (0,0,z)'$. Then,

$$H(0,0,z) = -\frac{1}{2}\sigma_3^2 z^2 + (c_3 - \lambda_3 - c_2)z. \tag{4.1}$$

Since $Y_3(t)$ only increases at times t for which $L_3(t) = 0$, we have that for any $(x,y,z)' \in \mathbb{R}^3$,

$$\phi_3(x,y,z) = \phi_3(x,y,0).$$

Then, by (2.12) and (2.19),

$$H(0,0,z)\phi(0,0,z) = -z\phi_2(0,0,z) + z\big(c_3 - \sum_{i=1}^{3}\lambda_i\big). \tag{4.2}$$

From (3.103), we get that

$$\phi_2\left(0,0,\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\right) < +\infty.$$

Letting $z = \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}$ in (4.2), we obtain

$$\phi_2\left(0,0,\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\right) = c_3 - \sum_{i=1}^{3}\lambda_i. \tag{4.3}$$

Therefore, by (4.2) and (4.3),

$$\phi(0,0,z) = \frac{\phi_2\big(0,0,\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\big) - \phi_2(0,0,z)}{-\frac{1}{2}\sigma_3^2 z + (c_3-\lambda_3-c_2)}. \tag{4.4}$$

By (4.3) and (4.4), one can easily conclude that $z = \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}$ is a removable singularity of $\phi(0,0,z)$. The proof of this lemma is completed. □

Based on Lemma 4.1, the asymptotic properties of $\phi_2(0,0,z)$ lead to corresponding properties of the function $\phi(0,0,z)$. Then, the application of the Tauberian-like Theorem gives the exact tail asymptotics for the marginal distribution of L_3.

From Lemmas 3.14 and 3.17, we can get that z_{dom} is either z^* or $z^{\max}$. In order to obtain tail asymptotics for the marginal L_3, we need to study asymptotic properties of the moment generating function ϕ_2 at the point z_{dom}. We first present asymptotic properties of $G(z)$ defined in (3.72) at the point z_{dom}.

Lemma 4.2. *For the function $G(z)$, we have:*

(i) If $z_{dom} = z^* < z^{\max}$, then z_{dom} is a simple pole of $G(z)$, and

$$\lim_{z\to z_{dom}} (z_{dom} - z)G(z) = \frac{2\big(Y_{\max,1}(z_{dom}) - z_{dom}\big)F\big(Y_{\max,0}(z_{dom})\big)}{\sum_{i=1}^{3}\sigma_i^2 k_i^2}; \tag{4.5}$$

(ii) If $z_{dom} = z^{\max} < z^*$, then z_{dom} is a branch point of $G(z)$. Moreover

$$\lim_{z\to z_{dom}} \frac{G(z_{dom}) - G(z)}{\sqrt{z_{dom} - z}} = z_{dom}\sqrt{\frac{2}{-Z''_{\max,1}(\tilde{y}^m)}}\frac{F(\tilde{y}^m) + F'(\tilde{y}^m)}{z_{dom} - \tilde{y}^m}; \tag{4.6}$$

(iii) If $z_{dom} = z^{\max} = z^*$, then z_{dom} is a pole of $G(z)$, and

$$\lim_{z\to z_{dom}} \sqrt{z_{dom} - z}G(z) = \frac{F\big(Y_{\max,0}(z_{dom})\big)z_{dom}(\sum_{i=1}^{2}\sigma_i^2 k_i^2)}{\sqrt{(\sum_{i=1}^{2}\sigma_i^2 k_i^2)\sigma_3^2(z_{dom} - z^{\min})}}. \tag{4.7}$$

Proof. We first prove case (i). From Lemma 3.17, in such a situation, z_{dom} is the zero of $H_2(Y_{\max,0}(z), z) = Y_{\max,0}(z) - z$. (4.5) follows from (3.108), since $z_{dom} = z^*$.

Next, we prove case (ii). From Lemma 3.16, (3.91) and (3.92), we get that $F(z)$ is analytic at the point $\tilde{y}^m$. Hence,

$$F\big(Y_{\max,0}(z)\big) = F(\tilde{y}^m) + F'(\tilde{y}^m)(Y_{\max,0}(z) - \tilde{y}^m) + o(|Y_{\max,0}(z) - \tilde{y}^m|). \tag{4.8}$$

It follows from (3.99) that

$$\begin{aligned} G(z) &= \frac{z}{\big(Y_{\max,0}(z) - \tilde{y}^m + \tilde{y}^m - z\big)}F(Y_{\max,0}(z)) \\ &= \frac{z\big((Y_{\max,0}(z) - \tilde{y}^m) - (\tilde{y}^m - z)\big)}{(Y_{\max,0}(z) - \tilde{y}^m)^2 - (\tilde{y}^m - z)^2}F\big(Y_{\max,0}(z)\big). \end{aligned} \tag{4.9}$$

From (4.8) and (4.9),

$$\begin{aligned} G(z) = {}& \frac{F(\tilde{y}^m)z(Y_{\max,0}(z) - \tilde{y}^m)}{(Y_{\max,0}(z) - \tilde{y}^m)^2 - (\tilde{y}^m - z)^2} \\ & -(\tilde{y}^m - z)z\frac{F(\tilde{y}^m) + F'(\tilde{y}^m)(Y_{\max,0}(z) - \tilde{y}^m)}{\big(Y_{\max,0}(z) - \tilde{y}^m\big)^2 - \big(\tilde{y}^m - z\big)^2} + o\big(|Y_{\max,0}(z) - \tilde{y}^m|\big). \end{aligned} \tag{4.10}$$

Next, we consider the term $Y_{\max,0}(z) - \tilde{y}^m$. From Lemma 3.6 and (3.92) that $Z_{\max,1}(y)$ is analytic at the point $\tilde{y}^m$. Hence

$$\begin{aligned} Z_{\max,1}(y) = {}& Z_{\max,1}(\tilde{y}^m) + Z'_{\max,1}(\tilde{y}^m)(y - \tilde{y}^m) \\ & + \frac{1}{2}Z''_{\max,1}(\tilde{y}^m)(y - \tilde{y}^m)^2 + o(|y - \tilde{y}^m|^2). \end{aligned} \tag{4.11}$$

Since $Z_{\max,1}(y)$ takes the maximum at the point $\tilde{y}^m$ on $[y^{\min}, y^{\max}]$,

$$Z'_{\max,1}(\tilde{y}^m) = 0. \tag{4.12}$$

On the other hand, from (3.20) and (3.92),

$$Z_{\max,1}(y) - Z_{\max,1}(y^m) < 0. \tag{4.13}$$

From (4.11), (4.12) and (4.13),

$$\tilde{y}^m - y = \sqrt{\frac{2\big(Z_{\max,1}(y^m) - Z_{\max,1}(y)\big)}{-Z''_{\max,1}(\tilde{y}^m)}} + o(|y - \tilde{y}^m|). \tag{4.14}$$

Similar to (3.96), we have that for z close to $z^{\max}$

$$Y_{\max,0}(z) = y \text{ and } Z_{\max,1}(y) = z. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain that

$$\tilde{y}^m - Y_{\max,0}(z) = \sqrt{\frac{2(z^{\max} - z)}{-Z''_{\max,1}(\tilde{y}^m)}} + o(|z - z^{\max}|^{\frac{1}{2}}).$$

Hence,

$$\tilde{y}^m - Y_{\max,0}(z) = (z^{\max} - z)^{\frac{1}{2}} \sqrt{\frac{2}{-Z''_{\max,1}(\tilde{y}^m)}} + o(|z - z^{\max}|^{\frac{1}{2}}). \tag{4.16}$$

From (4.10) and (4.16), we obtain that

$$\begin{aligned} G(z) \ &= (z^{\max} - z)^{\frac{1}{2}} \frac{F(y^m) z^{\max}}{-(\tilde{y}^m - z^{\max})^2} \sqrt{\frac{2}{-Z''_{3,1}(\tilde{y}^m)}} + \frac{z^{\max} F(\tilde{y}^m)}{(\tilde{y}^m - z^{\max})} \\ &+ (z^{\max} - z)^{\frac{1}{2}} \frac{F'(\tilde{y}^m) z^{\max} \sqrt{\frac{2}{-Z''_{\max,1}(\tilde{y}^m)}}}{-(\tilde{y}^m - z^{\max})} + o(|z - \tilde{y}^m|^{\frac{1}{2}}). \end{aligned} \tag{4.17}$$

Combining (4.10) and (4.16), we obtain that

$$\lim_{z \to z^{\max}} \frac{G(z^{\max}) - G(z)}{\sqrt{y^{\max} - z}} = z^{\max} \sqrt{\frac{2}{-Z''_{3,1}(\tilde{y}^m)}} \frac{F(\tilde{y}^m) + F'(\tilde{y}^m)}{z^{\max} - \tilde{y}^m}. \tag{4.18}$$

Finally, we prove case (iii). Due to Lemmas 3.2 and 3.17, we obtain that

$$H_2\big(Y_{\max,0}(z^{\max}), z^{\max}\big) = 0, \tag{4.19}$$

and

$$\Delta(z^{\max}) = 0. \tag{4.20}$$

Hence,

$$\sqrt{z^{\max} - z} \frac{z}{Y_{\max,0}(z) - z} F(Y_{\max,0}(z))$$

$$= \frac{z\sqrt{z^{\max} - z}}{H_2(Y_{\max,0}(z), z) - H_2(Y_{\max,0}(z^{\max}), z^{\max})} F(Y_{\max,0}(z)). \tag{4.21}$$

From (3.91), (3.92) and (4.19), $F(z)$ is analytic at $\tilde{y}^m = Y_{\max,0}(z^{\max})$. Therefore,

$$\lim_{z \to z^{\max}} \sqrt{z^{\max} - z} \frac{z}{Y_{\max,0}(z) - z} F(Y_{\max,0}(z)) = \frac{F\big(Y_{\max,0}(z^{\max})\big) z^{\max} (\sum_{i=1}^{2} \sigma_i^2 k_i^2)}{\sqrt{(\sum_{i=1}^{2} \sigma_i^2 k_i^2) \sigma_3^2 (z^{\max} - z^{\min})}}. \tag{4.22}$$

□

We are now in the position to obtain asymptotic properties of $\phi_2(0,0,z)$ and $\hat{\phi}_2(0,0,z)$ around the dominant singularity z_{dom}.

Lemma 4.3. *For the asymptotic behavior of $\phi_2(0,0,z)$ and $\hat{\phi}_2(0,0,z)$ around the dominant singularity z_{dom}, we have:*

(i) *If $z_{dom} = z^* < z^{\max}$, then*

$$\lim_{z \to z_{dom}} (z_{dom} - z)\phi_2(0,0,z) = C_1(z_{dom}), \tag{4.23}$$

$$\lim_{z \to z_{dom}} (z_{dom} - z)\hat{\phi}_2(0,0,z) = C_2(z_{dom}); \tag{4.24}$$

(ii) *If $z_{dom} = z^{\max} < z^*$, then*

$$\lim_{z \to z_{dom}} \frac{\phi_2(0,0,z_{dom}) - \phi_2(0,0,z)}{\sqrt{z_{dom} - z}} = C_3(z_{dom}), \tag{4.25}$$

$$\lim_{z \to z_{dom}} \frac{\hat{\phi}_2(0,0,z_{dom}) - \hat{\phi}_2(0,0,z)}{\sqrt{(z_{dom} - z)}} = C_4(z_{dom}); \tag{4.26}$$

(iii) *If $z_{dom} = z^{\max} = z^*$, then*

$$\lim_{z \to z_{dom}} \sqrt{(z_{dom} - z)}\phi_2(0,0,z) = C_5(z_{dom}), \tag{4.27}$$

$$\lim_{z \to z_{dom}} \sqrt{(z_{dom} - z)}\hat{\phi}_2(0,0,z) = C_6(z_{dom}). \tag{4.28}$$

Here $C_i(z_{dom}), i = 1, \dots, 6,$ are non-zero constants.

Proof. Here, we only prove case (i), other cases can be proved in the same fashion. It follows from (3.103) that we only need focus on $z \in \big(2\frac{(c_3 - \lambda_3 - c_2)}{\sigma_3}, z^{\max}\big)$. From (3.10), we get that

$$Y_{\max}(z) \geq 0 \text{ for all } z \in \Big(2\frac{(c_3 - \lambda_3 - c_2)}{\sigma_3},\ z^{\max}\Big). \tag{4.29}$$

Combining (2.11) and (4.29), we get

$$\hat{\phi}_2(0,0,z) \geq \phi_2(0,0,z) \tag{4.30}$$

for any $z \in \big(2\frac{(c_3-\lambda_3-c_2)}{\sigma_3}, z^{\max}\big)$. If case (i) would not hold, then, from Lemmas 3.9, 3.14 and 4.2, we should have

$$C_1(z^*) = C_2(z^*) = \infty. \tag{4.31}$$

If $k_1 > 1$, then from (4.30) we have

$$G(z) = \phi_2(0,0,z) + \frac{1}{k_1-1}\hat{\phi}_2(0,0,z) \geq \frac{k_1}{k_1-1}\phi_2(0,0,z). \tag{4.32}$$

From (4.31) and (4.32), we get that

$$\lim_{z\to z^*}(z^*-z)G(z) = \infty, \tag{4.33}$$

which contradicts to Lemma 4.2.

On the other hand, if $0 < k_1 < 1$, then from (4.30), we have

$$G(z) = \phi_2(0,0,z) + \frac{1}{k_1-1}\hat{\phi}_2(0,0,z) \leq \frac{k_1}{k_1-1}\hat{\phi}_2(0,0,z). \tag{4.34}$$

Under this, it is easy to check that

$$G(z) < 0 \text{ for } z \in \big(2\frac{(c_3-\lambda_3-c_2)}{\sigma_3},\ z^{\max}\big). \tag{4.35}$$

Hence, from Lemma 4.2, we get that

$$-\infty < \lim_{z\to z^*}(z^*-z)G(z) < 0. \tag{4.36}$$

However, from (4.30) and (4.34), we have

$$\lim_{z\to z^*}(z^*-z)G(z) = -\infty, \tag{4.37}$$

which contradicts to (4.36). From above arguments, (4.23) and (4.24) are proved.

Now we show that $C_i(z^*)$, $i = 1,2$, are non-zero. It follows from (4.5), (4.30) and (4.32) that $C_2(z^*) \neq 0$. Now we assume that $C_1(z^*) = 0$. Then from (3.74) and the definition of $V_2(\cdot)$ given in (2.10), we have

$$\begin{aligned}
\hat{\phi}_2(0,0,z) &= \int_{\mathbb{R}^3_+} \exp\big\{k_1 Y_{\max,0}(z)x_1 + zx_3\big\} V_2(\mathrm{d}x) \\
&< \frac{1}{2}\Big(\int_{\mathbb{R}^3_+} \exp\big\{2k_1 Y_{\max,0}(z)x_1\big\} V_2(\mathrm{d}x) + \int_{\mathbb{R}^3_+} \exp\big\{2zx_3\big\} V_2(\mathrm{d}x)\Big) \\
&\leq \frac{1}{2}\Big(\int_{\mathbb{R}^3_+} \exp\big\{2k_1 Y_{\max,0}(z)x_1\big\} V_2\big(d(2x)\big) + \int_{\mathbb{R}^3_+} \exp\big\{2zx_3\big\} V_2\big(d(2x)\big)\Big).
\end{aligned} \tag{4.38}$$

Hence, as $z \to z^*$, from (4.24) and (4.38), we have

$$C_2(z^*) \leq \frac{1}{2}\tilde{C}_2(z^*) + \frac{1}{2}C_1(z^*), \tag{4.39}$$

where

$$\tilde{C}_2(z^*) = \lim_{z\to z^*}(z^*-z)\int_{\mathbb{R}^3_+}\exp\left\{2k_1Y_{\max,0}(z)x_1\right\}V_2\big(d(2x)\big).$$

On the other hand, it is obvious that

$$\int_{\mathbb{R}^3_+}\exp\left\{2k_1Y_{\max,0}(z)x_1\right\}V_2\big(d(2x)\big) < \int_{\mathbb{R}^3_+}\exp\left\{2k_1Y_{\max,0}(z)x_1+2zx_3\right\}V_2\big(d(2x)\big). \tag{4.40}$$

By (4.40), letting $z\to z^*$ yields

$$\tilde{C}_2(z^*) \le C_2(z^*). \tag{4.41}$$

Hence, (4.39) and (4.41) contradict to $C_1(z^*)=0$ and $C_2(z^*)\neq 0$. From above arguments, we get that $C_i(z^*)>0$, $i=1,2$. The proof of this lemma is completed. □

From Lemmas 4.1 and 4.3, we can easily obtain asymptotic behavior of $\phi(0,0,z)$. In fact, we have:

Lemma 4.4. *For the moment generating function $\phi(0,0,z)$, a total of three types of asymptotics exists as z approaches to z_{dom}, based on the detailed property of z_{dom}:*

Case 1: If $z_{dom}=z^*<z^{\max}$, then

$$\lim_{z\to z_{dom}}(z_{dom}-z)\phi(0,0,z)=\bar{K}_1(z_{dom}); \tag{4.42}$$

Case 2: If $z_{dom}=z^{\max}<z^*$, then

$$\lim_{z\to z_{dom}}\frac{\phi(0,0,z_{dom})-\phi(0,0,z)}{\sqrt{z_{dom}-z}}=\bar{K}_2(z_{dom}); \tag{4.43}$$

Case 3: If $z_{dom}=z^*=z^{\max}$, then

$$\lim_{z\to z_{dom}}\sqrt{z_{dom}-z}\,\phi(0,0,z)=\bar{K}_3(z_{dom}), \tag{4.44}$$

where $\bar{K}_i(z_{dom})$, $i=1,2,3$, are non-zero constants depending on z_{dom}.

It is clear that the asymptotic behavior of the function $\phi(0,0,z)$ at its dominant singularity z_{dom} depends on the value of z_{dom}, which is equal to z^* or/and $z^{\max}$. In practice, it is important to compare the values of z^* and $z^{\max}$. In fact, we have the following lemma.

Lemma 4.5. *z^* exists in $(0,\ z^{\max}]$ if and only if $Y_{\max,0}(z^{\max})\ge z^{\max}$.*

Proof. If $z^{\max}=z^*$, one can easily see that the lemma holds. Next, we assume $z^{\max}\neq z^*$. From (3.10), we obtain that $Y_{\max,0}(z)$ is increasing on $(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max}]$. We first assume that z^* exists in $(0,\ z^{\max})$. Since

$$0<z^*=Y_{\max,0}(z^*),$$

we have

$$z^* > \frac{2(c_3 - \lambda_3 - c_2)}{\sigma_3^2}. \tag{4.45}$$

Therefore

$$\tilde{y}^{\max} := Y_{\max,0}(z^{\max}) > Y_{\max,0}(z^*). \tag{4.46}$$

On the other hand, we note that the line $H_2(y, z) = z - y = 0$ intersects the ellipse $H(ky, y, z) = 0$ at one point except for the point $(0,\ 0)'$. From (3.98), we know that the point $(Y_{\max,0}(z^*), z^*)'$ is the other intersection point of $H_2(y, z) = 0$ and $H(k_1 y, y, z) = 0$. Hence, we must have

$$\tilde{y}^{\max} > z^{\max}. \tag{4.47}$$

Next, we assume

$$\tilde{y}^{\max} > z^{\max}. \tag{4.48}$$

We prove that z^* belongs to $(0,\ z^{\max})$. From (4.48), we obtain that the point $(\tilde{y}^{\max}, z^{\max})'$ is above the line $H_2(y, z) = 0$. From (3.10), we get that the point $\left(Y_{\max,0}(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}), \frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2}\right)'$ is below the line $H_2(y, z) = 0$. On the other hand, $Y_{\max,0}(z)$ is continuous on $\left(\frac{2(c_3-\lambda_3-c_2)}{\sigma_3^2},\ z^{\max}\right)$. By the above arguments, one can get that the lemma holds. □

Remark 4.6. From the proof to Lemma 4.5, we can conclude that if z^* exists, it is unique, which can be evaluated according to Lemma 3.19.

From Lemmas 2.6, 3.20, 3.21, 4.1 and 4.4, we have the following main result of this section:

Theorem 4.7. *For the marginal stationary distribution $\mathbb{P}\{L_3 > z\}$, we have the following tail asymptotic properties for large z:*

Case 1: *If $z_{dom} = z^* < z^{\max}$, then*

$$\mathbb{P}\{L_3 > z\} \sim \hat{K}_1 e^{-z_{dom} z};$$

Case 2: *If $z_{dom} = z^{\max} < z^*$, then*

$$\mathbb{P}\{L_3 > z\} \sim \hat{K}_2 e^{-z_{dom} z} z^{-\frac{3}{2}};$$

Case 3: *If $z_{dom} = z^* = z^{\max}$, then*

$$\mathbb{P}\{L_3 > z\} \sim \hat{K}_3 e^{-z_{dom} z} z^{-\frac{1}{2}},$$

where $\hat{K}_i$, $i = 1, 2, 3$, are non-zero constants.

Proof. Cases (1) and (3) are direct consequences of Lemmas 3.20, 3.21, 4.1, 4.4 and 2.6.

Next, we prove case (2). From (4.43), we have

$$\lim_{z \to z_{dom}} \sqrt{z_{dom} - z}\, \frac{\phi(0, 0, z_{dom}) - \phi(0, 0, z)}{z_{dom} - z} = \bar{K}_2(z_{dom}). \tag{4.49}$$

On the other hand, it follows from Dai and Harrison [6] that the density function $f(x)$ of the marginal distribution $\mathbb{P}(L_3 < x)$ exists. From Dai and Miyazawa [7], we get that

$$\frac{\phi(0,0,z_{dom}) - \phi(0,0,z)}{z_{dom} - z}$$

is the moment generating function of the density function

$$\bar{f}(x) = e^{-z_{dom}x} \int_x^{\infty} e^{z_{dom}u} f(u) \mathrm{d}u. \tag{4.50}$$

Therefore, from Lemma 2.6 and (4.49), we have

$$\bar{f}(u) \sim \check{K}(z_{dom}) u^{-\frac{1}{2}} e^{-z_{dom}u}, \tag{4.51}$$

where $\check{K}$ is a constant depending on z_{dom}.

From (4.50) and (4.51), we obtain that as $x \to \infty$

$$\int_x^{\infty} e^{z_{dom}u} f(u) \mathrm{d}u \sim \hat{K}(z_{dom}) x^{-\frac{1}{2}}. \tag{4.52}$$

That is

$$\lim_{x\to\infty} \frac{\int_x^{\infty} e^{z_{dom}u} f(u) \mathrm{d}u}{\hat{K}(z_{dom}) x^{-\frac{1}{2}}} = 1. \tag{4.53}$$

At the same time, we note that

$$\lim_{x\to\infty} \int_x^{\infty} e^{z_{dom}u} f(u) \mathrm{d}u = 0, \text{ and } \lim_{x\to\infty} \hat{K}(z_{dom}) x^{-\frac{1}{2}} = 0. \tag{4.54}$$

By (4.53), (4.54) and L'Hôspital's Rule, we obtain that

$$\lim_{x\to\infty} \frac{\int_x^{\infty} e^{z_{dom}u} f(u) \mathrm{d}u}{\hat{K}(z_{dom}) x^{-\frac{1}{2}}} = \lim_{x\to\infty} \frac{2e^{z_{dom}x} f(x)}{\hat{K}(z_{dom}) x^{-\frac{3}{2}}} = 1. \tag{4.55}$$

That is

$$f(x) \sim K_1(z_{dom}) e^{-z_{dom}x} x^{-\frac{3}{2}}, \tag{4.56}$$

where K_1 is a constant. From (4.56), we conclude that case (2) holds. □

5. Tail behaviours of joint stationary distributions

In this section, we study the tail behavior of the joint stationary distribution π. It should be pointed out that the extension of the kernel method presented in previous sections for tail asymptotics of the marginal distribution of L_3 is not valid for the tail asympotics of the joint distribution. Instead, we propose a new idea for the main result in this section based on extreme value theory and copula. Before stating the main result, we first introduce the domain of attraction of some extreme value distribution function $G_{DA}(\cdot)$.

Definition 5.1. (*Domain of Attraction*) Assume that $\{X_n = (X_n^{(1)}, \dots, X_n^{(d)})'\}$ are i.i.d. multivariate random vectors with common distribution $\tilde{F}(\cdot)$ and the marginal distributions $\tilde{F}_i(\cdot)$, $i = 1, \dots, d$. If there exist normalizing constants $a_n^{(i)} > 0$ and $b_n^{(i)} \in \mathbb{R}$, $1 \le i \le d$, $n \ge 1$ such that as $n \to \infty$

$$\mathbb{P}\Big\{\frac{M_n^{(i)} - b_n^{(i)}}{a_n^{(i)}} \le x^{(i)}, 1 \le i \le d\Big\} = \tilde{F}^n\Big(a_n^{(1)}x^{(1)} + b_n^{(1)}, \dots, a_n^{(d)}x^{(d)} + b_n^{(d)}\Big)$$
$$\to G_{DA}(x^{(1)}, \dots, x^{(d)}),$$

where the maximum $M_n^{(i)} = \bigvee_{k=1}^n X_k^{(i)}$ are the componentwise maxima, then we call the distribution function $G_{DA}(\cdot)$ a multivariate extreme value distribution function, and $\tilde{F}$ is in the domain of attraction of $G_{DA}(\cdot)$. We denote this by $\tilde{F} \in D(G_{DA})$.

For convenience, we let $F_\pi(x, y, z)$ denote the joint stationary distribution function of $\{L(t)\}$ and F_i, $i = 1, 2, 3$, denote the stationary distribution function of the i-th buffer content process. From Dai and Miyazawa ([7], Thms. 2.2 and 2.3), and Theorem 4.7, we can easily get the following lemma.

Lemma 5.2. *For any $i \in \{1, 2, 3\}$, we have*

$$1 - F_i(x) \sim C_i \exp\{-\alpha_i x\} x^{\mu_i}, \tag{5.1}$$

where α_i is the dominant singularity of the moment generating function of the marginal distribution F_i, C_i is a non-zero constant, and $\mu_i \in \{0, -\frac{1}{2}, -\frac{3}{2}\}$.

From Lemma 5.2, we can get that

Lemma 5.3. *For any $i \in \{1, 2, 3\}$, we have*

$$F_i(x) \in D\big(G_1(x)\big),$$

where

$$G_1(x) = \exp\{-e^{-x}\}. \tag{5.2}$$

Proof. From Dai and Harrison [6], F_i, $i = 1, 2, 3$, have continuous densities. It straightforwardly follows from (5.1) and L'Hôspital's Rule that as $x \to \infty$,

$$F_i'(x) \sim \alpha_i C_i \exp\{-\alpha_i x\} x^{\mu_i}, \tag{5.3}$$

since $1 - F_i(x) \to 0$ and $C_i \exp\{-\alpha_i x\} x^{\mu_i} \to 0$, as $x \to \infty$. Moreover, it is obvious that

$$C_i > 0, \text{ for all } i = 1, 2, 3. \tag{5.4}$$

For large enough $x > 0$, due to (5.3), we have

$$F_i'(x) = \alpha_i C_i \exp\{-\alpha_i x\} x^{\mu_i} + o\big(\alpha_i C_i \exp\{-\alpha_i x\} x^{\mu_i}\big), \text{ as } x \to \infty, \tag{5.5}$$

where $o(\cdot)$ denotes small oh as $x \to \infty$. Now, we consider the existence of the second-order derivative $F_i''(x)$ of the function $F_i(x)$ and the asymptotic equivalence of $F_i''(x)$, as $x \to \infty$. Let $g_i(x) = \alpha_i C_i \exp\{-\alpha_i x\} x^{\mu_i}$ for convenience. Then we can rewrite the equation (5.5) as

$$F_i'(x) = g_i(x) + o\big(g_i(x)\big), \ x \to \infty. \tag{5.6}$$

To reach our aim, we first discuss some properties of the function $g_i(x)$. For any large enough $u > \mathbb{R}_+$ fixed, we first show that

$$o\big(g_i(x)\big) - o\big(g_i(u)\big) \sim o_u\big(g_i(x) - g_i(u)\big), \text{ as } x \to u, \tag{5.7}$$

where $o_u(\cdot)$ denotes the small oh as $x \to u$.

Note that

$$\lim_{x \to u} g_i(x) = g_i(u). \tag{5.8}$$

It is obvious that as $x \to u$

$$g_i(x)/g_i(u) \to 1 \text{ and } g_i(u)/g_i(x) \to 1. \tag{5.9}$$

Moreover, since $F_i'(x)$ and $g_i(x)$ are both continuous, from (5.6) and (5.8), as $x \to u$

$$o\big(g_i(x)\big) - o\big(g_i(u)\big)/g(u) \to 0 \text{ and } o(g_i(x)) - o(g_i(u))/g_i(x) \to 0. \tag{5.10}$$

We first assume that $x > u$. From (5.9) and (5.10), for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x - u < \delta$

$$(1 - \epsilon)g_i(u) \le g_i(x) \le (1 + \epsilon)g_i(u), \tag{5.11}$$

and

$$o(g_i(x)) - o(g_i(u)) < \epsilon^2 g_i(u). \tag{5.12}$$

From (5.10), (5.11) and (5.12), we have

$$\frac{o(g_i(u)) - o(g_i(x))}{g_i(u) - g_i(x)} \le \frac{o(g_i(u))}{\epsilon g_i(u)} \le \frac{\epsilon^2 g_i(u)}{\epsilon g_i(u)} < \epsilon. \tag{5.13}$$

Below we assume that $x < u$, then from (5.9), for any small enough $\epsilon > 0$, there exists $\delta > 0$ such that for all $u - x < \delta$

$$(1 - \epsilon)g_i(x) \le g_i(u) \le (1 + \epsilon)g_i(x), \tag{5.14}$$

and

$$o\big(g_i(u)\big) - o\big(g_i(x)\big) < \epsilon^2 g_i(x). \tag{5.15}$$

Hence, from (5.15) and (5.14),

$$\frac{o(g_i(x)) - o(g_i(u))}{g_i(x) - g_i(u)} \le \frac{o(g_i(x))}{\epsilon g_i(x)} \le \epsilon. \tag{5.16}$$

From (5.13) and (5.16), we can get (5.7).

From (5.6) and (5.7), we have that as $u \to \infty$

$$\lim_{x \to u} \frac{F_i'(x) - F_i'(u)}{x - u} = \lim_{x \to u} \frac{g_i(x) - g_i(u)}{x - u} + \lim_{x \to u} \frac{o_u(g_i(x) - g_i(u))}{x - u}. \tag{5.17}$$

Hence, for larg enough $x \in \mathbb{R}_+$, $F_i''(x)$ exists. Furthermore, from (5.17), one can easily verify that

$$\lim_{x\to\infty} F_i''(x) = 0. \tag{5.18}$$

Then, by (5.3) and L'Hôspital's Rule

$$F_i''(x) \sim -\alpha_i^2 C_i \exp\{-\alpha_i x\} x^{\mu_i}. \tag{5.19}$$

It follows from the asymptotic equivalence (5.3) and (5.19) that

$$\lim_{x\to\infty} \frac{F_i''(x)\big(1 - F_i(x)\big)}{\Big(F_i'(x)\Big)^2} = -1. \tag{5.20}$$

Then, it follows from Proposition 1.1 in Resnick [30] that $F_i \in D(G_1)$. □

In the previous section, we obtained exact tail asymptotic properties of the marginal distributions. Now, based on these results, we can study the upper tail dependence for the joint stationary distribution. Before stating the tail dependent result for the joint stationary distribution, we introduce a technical lemma.

Lemma 5.4. *Suppose that* $\Big\{\hat{X}_n = \big(\hat{X}_n^{(1)}, \hat{X}_n^{(2)}, \hat{X}_n^{(3)}\big)'\Big\}_{n\in\mathbb{N}}$ *are i.i.d. random vectors in* $\mathbb{R}^3$ *with a common joint continuous distribution* $\tilde{F}(\cdot)$*, and the marginal distributions* $\tilde{F}_i(\cdot)$*,* $i = 1, 2, 3$*. Moreover, we assume that* $\tilde{F}_i(\cdot)$*,* $i = 1, 2, 3$*, are all in the domain of attraction of some univariate extreme value distribution* $\hat{G}_1(\cdot)$*, i.e.., there exist* $a_n^{(i)}$ *and* $b_n^{(i)}$ *such that as* $n \to \infty$

$$\tilde{F}_i^n\Big(a_n^{(i)} x + b_n^{(i)}\Big) \to \hat{G}_1(x).$$

Then, the following are equivalent:

(1) $\tilde{F}$ *is in the domain of attraction of a product measure, that is,*

$$\tilde{F}^n\Big\{a_n^{(i)} x^{(i)} + b_n^{(i)}, i = 1, 2, 3\Big\} \to \Pi_{i=1}^3 \hat{G}_1\big(x^{(i)}\big) := \hat{G}(x_1, x_2, x_3); \tag{5.21}$$

(2) For any $1 \le i < j \le 3$*,*

$$\lim_{t\to\infty} \mathbb{P}\Big\{\hat{X}^{(i)} > t, \hat{X}^{(j)} > t\Big\} / \big(1 - \tilde{F}_q(t)\big) \to 0, \tag{5.22}$$

where $q \in \{i, j\}$*.*

By a slight modification of the proof of Proposition 5.27 in Resnick [30], we can prove the above lemma. Hence, we omit the detail here.

Remark 5.5. By Proposition 5.24 in Resnick [30], the asymptotic independence in (5.21) can be reduced to two-dimensional case. Moreover, the bivariate asymptotic independence could be seen from the tail behaviour of $\hat{X}^{(i)}$ and $\hat{X}^{(j)}$. Hence, one can see the equivalence between (5.21) and (5.22) intuitively.

For the joint stationary distribution function $F_\pi(\cdot)$, we have the following tail dependence.

Lemma 5.6. *The joint stationary distribution function $F_\pi(\cdot)$ is asymptotically independent, that is, there exist $a_n(\mu_i, \alpha_i)$ and $b_n(\mu_i, \alpha_i)$, $i = 1, 2, 3$, such that*

$$F_\pi^n\big(a_n(\mu_i, \alpha_i)x^{(i)} + b_n(\mu_i, \alpha_i), i = 1, 2, 3\big) \to \Pi_{i=1}^3 G_1(x^{(i)}), \ \text{as } n \to \infty,$$

where $G_1(x)$ is given by (5.2).

To prove Lemma 5.6, we first introduce the inverse (left continuous) $\widetilde{H}^{\leftarrow}$ of a function $\widetilde{H}$ by

$$\widetilde{H}^{\leftarrow}(y) = \inf\{s : \widetilde{H}(s) \geq y\}.$$

Proof. Here, we will use (5.22), an equivalent statement, to prove this lemma. Without loss of generality, we assume that $L(0)$ follows the stationary distribution π. Hence for any $t \in \mathbb{R}_+$ and $(x, y, z)' \in \mathbb{R}_+^3$,

$$\mathbb{P}\{L(t) \geq (x, y, z)'\} = \mathbb{P}\{L \geq (x, y, z)'\}. \tag{5.23}$$

At the same time, we note that, to prove (5.22), it suffices to show the following upper tail dependence:

$$\lim_{u \to 1-} \frac{\bar{C}_{ij}(u, u)}{1 - u} = 0, \tag{5.24}$$

where $\bar{C}_{ij}(\cdot)$ is defined by

$$\bar{C}_{ij}(u, v) = \mathbb{P}\Big\{L_i \geq (F_i)^{\leftarrow}(u), L_j \geq (F_j)^{\leftarrow}(v)\Big\}.$$

Below, we let $i = 1$ and $j = 2$ for simplicity. Other cases can be proved in the same fashion. At the same time, from (5.23), we get that for any $t \in \mathbb{R}_+$

$$\bar{C}_{12}(u, u) = \mathbb{P}\Big\{L_1(t) \geq (F_1)^{\leftarrow}(u), L_2(t) \geq (F_2)^{\leftarrow}(u)\Big\}. \tag{5.25}$$

Next, we introduce the last exit time τ_t before t, out of the boundaries by

$$\tau_t = \inf\Big\{s : L_i(u) > 0 \text{ for all } i = 1, 2, 3, \ s \leq u \leq t\Big\}.$$

As a convention, let $\tau_0(t) = t$ if $L_i(t) = 0$, for some $i \in \{1, 2, 3\}$. Hence, we have

$$\tau_t \leq t, \quad \text{a.s.} \tag{5.26}$$

From (5.56), it is obvious that

$$L(t) = L(\tau_t) + \tilde{X}(t) - \tilde{X}(\tau_t) + R\Big(Y(t) - Y(\tau_t)\Big), \tag{5.27}$$

where $\tilde{X}(t)$ is given by (2.4). Moreover, noting that $\{\tilde{X}(t)\}$ is a 3-dimensional Brownian motion, we get that for large enough $t \in \mathbb{R}_+$

$$\tau_t > 0, \ \text{a.s.} \tag{5.28}$$

From Proposition 1 in Konstantopoulous, Last and Lin [18], (5.27) and (5.26), we have that if $L(t) > \mathbf{0}$, where $\mathbf{0} = (0,0,0)'$, then

$$L(t) = L(\tau_t) + \tilde{X}(t) - \tilde{X}(\tau_t). \tag{5.29}$$

On the other hand, we note that as $u \to 1-$

$$(F_i)^{\leftarrow}(u) \to \infty,\ i = 1,2. \tag{5.30}$$

Hence, from (5.25), (5.29) and (5.30),

$$\begin{aligned}&\mathbb{P}\Big\{L_1(t) \geq (F_1)^{\leftarrow}(u), L_2(t) \geq (F_2)^{\leftarrow}(u)\Big\}\\ &\leq \lim_{n\to\infty}\int_0^\infty \mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n,\ i = 1,2\,|\tau_t = s\Big\}\mathrm{d}F_{\tau_t}(s),\end{aligned} \tag{5.31}$$

where $F_{\tau_t}(\cdot)$ is the distribution function of τ_t. From the above arguments, we have that

$$\begin{aligned}\lim_{u\to 1-}\frac{\bar{C}_{12}(u,u)}{1-u} &\leq \lim_{u\to 1-}\lim_{n\to\infty}\frac{\int_0^\infty \mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{1-u}\mathrm{d}F_{\tau_t}(s)\\ &\leq \lim_{n\to\infty}\lim_{u\to 1-}\int_0^\infty \frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{1-u}\mathrm{d}F_{\tau_t}(s)\\ &\leq \lim_{n\to\infty}\int_0^\infty \lim_{u\to 1-}\frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{1-u}\mathrm{d}F_{\tau_t}(s).\end{aligned} \tag{5.32}$$

Moreover, for any $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$, we have

$$\begin{aligned}&\frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{1-u}\\ &\qquad = \frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{\mathbb{P}\Big\{\tilde{X}_1(t) - \tilde{X}_1(s) \geq \big(F_{(s,t,n)}\big)^{\leftarrow}(u) - n\Big\}},\end{aligned} \tag{5.33}$$

where $F_{(s,t,n)}(\cdot)$ is the distribution function of the Gaussian random variable $\tilde{X}_1(t) - \tilde{X}_1(s) - n$. Finally, from Theorem 4.7, we can get that for u close to $1-$,

$$\big(F_{(s,t,n)}\big)^{\leftarrow}(u) \leq (F_i)^{\leftarrow}(u),\ i = 1,2. \tag{5.34}$$

Combining (5.33) and (5.34) yields

$$\begin{aligned}&\frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq (F_i)^{\leftarrow}(u) - n, i = 1,2\Big\}}{\mathbb{P}\Big\{\tilde{X}_1(t) - \tilde{X}_1(s) \geq \big(F_{(s,t,n)}\big)^{\leftarrow}(u) - n\Big\}}\\ &\qquad \leq \frac{\mathbb{P}\Big\{\tilde{X}_i(t) - \tilde{X}_i(s) \geq \big(F_{(s,t,n)}\big)^{\leftarrow}(u) - n, i = 1,2\Big\}}{\mathbb{P}\Big\{\tilde{X}_1(t) - \tilde{X}_1(s) \geq \big(F_{(s,t,n)}\big)^{\leftarrow}(u) - n\Big\}} \to 0,\end{aligned} \tag{5.35}$$

as $u \to 1-$, since for a Gaussian random vector $\hat{X} = (\hat{X}_1, \hat{X}_2)'$ with the coefficient being less than 1, for any $j \in \{1, 2\}$,

$$\frac{\mathbb{P}\{\hat{X}_i \geq x, i = 1, 2\}}{\mathbb{P}\{\hat{X}_j \geq x\}} \to 0, \text{ as } x \to \infty.$$

It follows from (5.32) and (5.35) that (5.24) holds. From the above discussion, the lemma is proved. □

Remark 5.7. For $a_n(\mu_i, \alpha_i)$ and $b_n(\mu_i, \alpha_i)$, $i = 1, 2, 3$, in Lemma 5.6, we can use tail equivalence to obtain their explicit expressions. Since they are not the focus of this paper, we will not elaborate them here.

Now, we present the main result of this section.

Theorem 5.8. *As* $(x, y, z)' \to (\infty, \infty, \infty)'$,

$$\mathbb{P}\{L_1 \geq x, L_2 \geq y, L_3 \geq z\} / \Big(K x^{\mu_1} y^{\mu_2} z^{\mu_3} \exp\big\{ -(\alpha_1 x + \alpha_2 y + \alpha_3 z) \big\} \Big) \to 1, \tag{5.36}$$

where α_i *is the dominant singularity of* L_i, $\mu_i \in \{0, -\frac{1}{2}, -\frac{3}{2}\}$ *is the exponent corresponding to* α_i *in Lemma 5.2, and* K *is a constant.*

Proof. To prove this theorem, we use a version of L'Hôspital's rule for multivariate functions introduced by Lawor [19] (see, Thms. 4 and 5 in Lawor [19]). To apply this result, we need the following transformation. Let $\bar{X} = (\bar{X}_1, \bar{X}_2, \bar{X}_3)'$ be a random vector with the joint distribution $\tilde{F}$ and marginal distributions $\tilde{F}_i$, $i = 1, 2, 3$. Then, we can make the following transformation:

$$X_i^* = \frac{-1}{\log\big(\tilde{F}_i(\bar{X}_i)\big)}, \text{ for } i = 1, 2, 3. \tag{5.37}$$

By the transformation in (5.37), we transform each marginal $\bar{X}_i$ of a random vector $\bar{X}$ to a unit Fréchet variable X_i^*, that is,

$$\mathbb{P}\{X_i^* < x\} = \exp\{-\frac{1}{x}\}, \quad \text{for } x \in \mathbb{R}_+. \tag{5.38}$$

For the trivariate extreme value distribution $\tilde{G}(x, y, z) = G_1(x) G_1(y) G_1(z)$, define

$$G^*(x, y, z) = \tilde{G}\left(\Big(\frac{-1}{\log G_1}\Big)^{\leftarrow}(x), \Big(\frac{-1}{\log G_1}\Big)^{\leftarrow}(y), \Big(\frac{-1}{\log G_1}\Big)^{\leftarrow}(z) \right). \tag{5.39}$$

Hence, from (5.37) and (5.38), we know that $G^*(\cdot)$ is the joint distribution function with the common marginal Fréchet distribution $\Phi(x) = \exp\{-x^{-1}\}$. Furthermore, for the stationary random vector L, define

$$Y_i^* = \frac{1}{1 - F_i(L_i)}. \tag{5.40}$$

Let $F^*(y_1, y_2, y_3)$ be the joint distribution function of $Y^* = (Y_1^*, Y_2^*, Y_3^*)'$. Then, it follows from Proposition 5.10 in [30] and Lemma 5.6 that

$$F^*(y_1, y_2, y_3) \in D\big(G^*(y_1, y_2, y_3)\big). \tag{5.41}$$

By (5.41), we have that for any $\vec{y} = (y_1, y_2, y_3)' \in \mathbb{R}^3_+$, as $n \to \infty$,

$$\big(F^*(n\vec{y})\big)^n \to G^*(\vec{y}). \tag{5.42}$$

It follows from (5.42) that as $n \to \infty$

$$F^*(n\vec{y}) \sim \big(G^*(\vec{y})\big)^{\frac{1}{n}}.$$

By a simple monotonicity argument, we can replace n in the above equation by t. Then we have that as $t \to \infty$,

$$F^*(t\vec{y}) \sim \big(G^*(\vec{y})\big)^{\frac{1}{t}}. \tag{5.43}$$

On the other hand, by Lemma 5.3, for any $y \in \mathbb{R}_+$, as $t \to \infty$

$$F_i^*(ty) \sim \big(G_1^*(y)\big)^{\frac{1}{t}}, \text{ for any } i = 1, 2, 3. \tag{5.44}$$

Combining (5.43) and (5.44), we get that as $t \to \infty$

$$F^*(t\vec{y}) \sim F_1^*(ty_1) \cdot F_2^*(ty_2) \cdot F_3^*(ty_3). \tag{5.45}$$

Let $C^*(u_1, u_2, u_3)$ be the copula of the random vector $(Y_1^*, Y_2^*, Y_3^*)'$, *i.e.*.,

$$C^*\big(F_1^*(x), F_2^*(y), F_3^*(z)\big) = F^*(x, y, z). \tag{5.46}$$

Furthermore, let $\hat{C}(u_1, u_2, u_3)$ be the corresponding survival copula of Y^*. Then, we have (see, for example, Eq. (2.46) in Schmitz [31]):

$$\hat{C}(u_1, u_2, u_3) = \sum_{i=1}^{3} u_i + \sum_{1\le i<j\le 3} C_{ij}^*(1-u_i, 1-u_i) - C^*(1-u_1, 1-u_2, 1-u_3) - 2. \tag{5.47}$$

For convenience, for any $(x_1, x_2, x_3)' \in \mathbb{R}^3_+$, let $u_i(t) = \bar{F}_i^*(tx_i)$. Hence for any $t \in \mathbb{R}_+$,

$$\begin{aligned} \hat{C}\big(u_1(t), u_2(t), u_3(t)\big) &= \bar{F}^*(tx_1, tx_2, tx_3), \\ C^*\big(1-u_1(t), 1-u_2(t), 1-u_3(t)\big) &= F^*(tx_1, tx_2, tx_3). \end{aligned} \tag{5.48}$$

Moreover, from (5.45), we get that as $t \to \infty$,

$$C^*\big(1-u_1(t), 1-u_2(t), 1-u_3(t)\big) \sim \big(1-u_1(t)\big)\cdot\big(1-u_2(t)\big)\cdot\big(1-u_3(t)\big), \tag{5.49}$$

and for any $1 \le i < j \le 3$,

$$C_{ij}^*\big(1-u_i(t), 1-u_j(t)\big) \sim \big(1-u_i(t)\big)\cdot\big(1-u_j(t)\big). \tag{5.50}$$

From (5.47), (5.49) and (5.50), we get that as $t \to \infty$,

$$\hat{C}\big(u_1(t), u_2(t), u_3(t)\big) \sim u_1(t)\cdot u_2(t)\cdot u_3(t), \tag{5.51}$$

which is equivalent to, for any $(x,y,z)' \in \mathbb{R}_+^3$,

$$\lim_{t\to\infty} \frac{\bar{F}^*(tx,ty,tz)}{\bar{F}_1^*(tx)\cdot \bar{F}_2^*(ty)\cdot \bar{F}_3^*(tz)} = 1. \tag{5.52}$$

To complete the proof, we need to show

$$\lim_{(x,y,z)'\to(\infty,\infty,\infty)'} \frac{\bar{F}^*(x,y,z)}{\bar{F}_1^*(x)\cdot \bar{F}_2^*(y)\cdot \bar{F}_3^*(z)} = 1. \tag{5.53}$$

From (5.46) and (5.51), to prove (5.53), we only need to show

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)' \text{ and } (u_1,u_2,u_3)'\in I^3} \frac{\hat{C}(u_1,u_2,u_3)}{u_1u_2u_3} = 1, \tag{5.54}$$

where $I = [0,\ 1]$. Note that

$$\lim_{x\to 0} \frac{1-\exp\{-x\}}{x} = 1. \tag{5.55}$$

Hence, from (5.52) we get that

$$\lim_{t\to 0} \frac{\hat{C}(tu_1,tu_2,tu_3)}{t^3u_1u_2u_3} = 1. \tag{5.56}$$

It is worthwhile to point out that the limit (5.54) has the form of $\frac{0}{0}$. Hence, we apply the multivariate L'Hôpital's rule (see, for example, Thms. 4 and 5 in Lawor [19]) to prove it. Without much effort, we can construct a multivariate differentiable function $\tilde{C}(u_1,u_2,u_3)$ such that

$$\hat{C}(u_1,u_2,u_3) = \tilde{C}(u_1,u_2,u_3) \text{ for all } (u_1,u_2,u_3)' \in I^3,$$

and

$$\tilde{C}(tu_1,tu_2,tu_3) \sim t^3u_1u_2u_3,\ \text{ as } t\to 0.$$

Hence, it suffices to show that

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)' \text{ and } (u_1,u_2,u_3)'\in I^3} \frac{\hat{C}(u_1,u_2,u_3)}{u_1u_2u_3} = \lim_{(u_1,u_2,u_3)'\to(0,0,0)'} \frac{\tilde{C}(u_1,u_2,u_3)}{u_1u_2u_3} = 1. \tag{5.57}$$

Near the origin $(0,0,0)'$, the zero sets of both $\tilde{C}(u_1,u_2,u_3)$ and $u_1u_2u_3$ consist of the hypersurfaces $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$. By the multivariate L'Hôpital's rule to prove (5.57), we need to show that for each component E_i of $\mathbb{R}^3 \setminus \mathcal{C}$, where $\mathcal{C} = \{u_1 = 0\} \cup \{u_2 = 0\} \cup \{u_3 = 0\}$, we can find a vector $\vec{z}$, not tangent to $(0,0,0)'$, such that $D_{\vec{z}}(u_1u_2u_3) \neq 0$ on E_i, and

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)' \text{ and } (u_1,u_2,u_3)'\in E_i} \frac{D_{\vec{z}}\tilde{C}(u_1,u_2,u_3)}{D_{\vec{z}}(u_1u_2u_3)} = 1.$$

For the component E_1 bounded by the hypersurfaces of $\mathcal{H}_i = \{(u_1,u_2,u_3)' : (u_1,u_2,u_3)' \in \mathbb{R}^3_+ \text{ and } u_i = 0\}$, $i = 1,2,3$, choose a vector, say $\vec{z} = (1,1,1)'$, then $\vec{z}$ is not tangent to any hypersurfaces $u_i = 0$, $i = 1,2,3$ at the point $(0,0,0)'$. Next, we take the limits along the direction $\vec{z} = (1,1,1)'$. It follows from (5.55) and (5.56) that

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)' \text{ and } (u_1,u_2,u_3)'\in E_1} \frac{D_{\vec{z}}\tilde{C}(u_1,u_2,u_3)}{D_{\vec{z}}(u_1u_2u_3)} = 1. \tag{5.58}$$

Similar to (5.58), for any other components E_i, $i = 2,\dots,8$, we can find a vector $\vec{z}$ such that $\vec{z}$ is not tangent to any hypersurfaces $u_i = 0$, $i = 1,2,3$, at the point $(0,0,0)'$. Moreover, we have

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)' \text{ and } (u_1,u_2,u_3)'\in E_i} \frac{D_{\vec{z}}\tilde{C}(u_1,u_2,u_3)}{D_{\vec{z}}(u_1u_2u_3)} = 1. \tag{5.59}$$

From (5.57) to (5.59) and [19],

$$\lim_{(u_1,u_2,u_3)'\to(0,0,0)'} \frac{\hat{C}(u_1,u_2,u_3)}{u_1u_2u_3} = 1. \tag{5.60}$$

It follows from (5.40) that for any $(x,y,z)' \in \mathbb{R}^3_+$,

$$\begin{aligned}\mathbb{P}\{L_1 \ge x, L_2 \ge y, L_3 \ge z\} &= \mathbb{P}\big\{Y_1^* \ge \frac{1}{1-F_1(x)}, Y_2^* \ge \frac{1}{1-F_2(y)}, Y_3^* \ge \frac{1}{1-F_3(z)}\big\} \\ &= F^*\Big(\frac{1}{\bar{F}_1(x)}, \frac{1}{\bar{F}_2(y)}, \frac{1}{\bar{F}_3(z)}\Big).\end{aligned} \tag{5.61}$$

Combining (5.60) and (5.61), we get that, as $(x,y,z)' \to (\infty,\infty,\infty)'$,

$$\mathbb{P}\{L_1 \ge x, L_2 \ge y, L_3 \ge z\}\Big/\left(\bar{F}_1^*\left(\frac{1}{\bar{F}_1(x)}\right)\cdot \bar{F}_2^*\left(\frac{1}{\bar{F}_2(y)}\right)\cdot \bar{F}_3^*\left(\frac{1}{\bar{F}_3(z)}\right)\right) \to 1. \tag{5.62}$$

By (5.55) and (5.62), we get that, as $(x,y,z)' \to (\infty,\infty,\infty)'$,

$$\mathbb{P}\{L_1 \ge x, L_2 \ge y, L_3 \ge z\}\Big/\Big(\bar{F}_1(x)\cdot\bar{F}_2(y)\cdot\bar{F}_3(z)\Big) \to 1. \tag{5.63}$$

Finally, it follows from Lemma 5.2 and (5.63) that

$$\bar{F}_i(x) \sim K_i x^{\mu_i}\exp\{-\alpha_i x\},\ i = 1,2,3, \tag{5.64}$$

where K_i is a constant. From (5.63) and (5.64), we have now proved the theorem. □

6. Concluding remarks

In previous sections, we obtained exact tail asymptotics for L_3, see Theorem 4.7, and asymptotic independence for L, see Theorem 5.8. From the discussion in Section 2, we know that $\{L(t)\}$ is an SRBM with the reflection matrix R given by (2.6). Recall that a general d-dimensional SRBM, denoted as $\tilde{Z} = \{\tilde{Z}(t), t \ge 0\}$, is defined as follows:

$$\tilde{Z}(t) = \tilde{B}(t) + \tilde{R}M(t),\ \text{for } t \ge 0, \tag{6.1}$$

where $\tilde{Z}(0) = \tilde{B}(0) \in \mathbb{R}_+^d$, $\tilde{B} = \{\tilde{B}(t)\}$ is an unconstrained Brownian motion with drift vector $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_d)'$ and covariance matrix $\Sigma = (\Sigma_{i,j})_{d\times d}$, the reflection matrix $\tilde{R}$ is a $d \times d$ matrix specifying the reflection behavior at the boundaries, and $M = \{M(t)\}$ is a d-dimensional process with the local times $M_1, \dots, M_d$ such that:

(i) the local time $M_i(t)$ is continuous and non-decreasing with $M_i(0) = 0$;
(ii) $M_i(t)$ only increases at times t for which $\tilde{Z}_i(t) = 0$, $i = 1, \dots, d$;
(iii) $\tilde{Z}(t) \in \mathbb{R}_+^d$, $t \geq 0$.

It is known that a necessary condition for the existence of the stationary distribution for $\tilde{Z}$ is

$$\tilde{R} \text{ is non-singular and } \tilde{R}^{-1}\tilde{\mu} < \mathbf{0}, \tag{6.2}$$

where $\tilde{R}^{-1}$ is the inverse matrix of $\tilde{R}$ and $\mathbf{0} = (0, \dots, 0)'$. For more information about SRBMs, we refer to Harrison and Hasenbein [14], Harrison and Reiman [15, 16], Varadhan and Williams [32], Williams [33, 34] among others.

SRBM with triangular reflection matrix. A natural extension of our model is the d-dimensional Brownian-driven tandem queue with intermediate inputs, which is an SRBM $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_d)'$, whose reflection matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{d\times d}$ satisfies

$$\tilde{r}_{ij} = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

An interesting problem is to obtain exact tail asymptotics for the marginal stationary distributions of $\tilde{Z}$. For this model, the kernel equation in (3.1) becomes a d-dimensional ellipsoid $H(z_1, \dots, z_d) = 0$. The exact asymptotic analysis seems to be analogous to that for the 3-dimensional case. However, there are still some new technical challenges we need to address. For example, to study exact tail asymptotics for the marginal stationary distribution of $\tilde{Z}_d$, the main challenge comes from the construction of the ellipses which locate the maximum point $z_d^{\max}$ and the interlace between the moment generating functions of boundary measures and marginal stationary distributions. The detailed analysis is beyond the scope of the current paper.

SRBM with general reflection matrix. An immediate question is: Can we generalize our study to a general d-dimensional SRBM $\tilde{Z}$ with $d \geq 3$? To answer this question, we first recall the key components in our analysis for the 3-dimensional model L:

(1) The fundamental form, or the functional equation satisfied by the (unknown) moment generating functions of the joint stationary distribution and boundary measures, see the equation in (2.12).
Similar to Lemma 2.3, by using the Itô's formula, such a relationship can be obtained for a d-dimensional SRBM $\tilde{Z}$.
(2) The kernel method, including analytic continuation of the unknown moment generating functions and asymptotic analysis.
We first briefly review why the kernel method can be applied to our case. Inspired by the kernel method for 2-dimensional queueing systems, a binary alternative equation in (3.9) is constructed based on the ternary kernel equation in (3.1), and then the kernel method is employed to study the binary alternative equation in (3.9) instead of the kernel equation in (3.1). Finally the analytic continuation of the unknown moment generating functions and asymptotic analysis are obtained. However, for the general d-dimensional SRBM $\tilde{Z}$, it is not possible to find binary alternatives for applying the kernel method. Hence, analytic continuation could not be carried out by the kernel method at this moment. This looks to be the main challenge for studying a general model $\tilde{Z}$. It is our conjecture that the counterpart analytic continuation property (to

Lemma 3.12) still holds for $\tilde{Z}$. If this is true, the asymptotic analysis on the dominant singularity should prevail.

(3) Based on the asymptotic analysis of the unknown moment generating functions, the Tauberian-like Theorem leads to exact tail asymptotic properties for the boundary measures and marginal stationary distributions, see Theorem 4.7.
If Step (2) works well for a general $\tilde{Z}$, then we can get exact tail asymptotic properties for the boundary measures and marginal stationary distributions of $\tilde{Z}$, the counterpart to Theorem 4.7, by using the same Tauberian-like Theorem (see Lemma 2.6).

(4) Furthermore, by extreme value theory and copula, asymptotic independence for the joint stationary distribution can be obtained, see Theorem 5.8.
If we can obtain exact tail asymptotics for marginal stationary distributions of a general $\tilde{Z}$, then, similar to Theorem 5.8, we can study exact tail asymptotics and dependence structure of the joint stationary distribution of $\tilde{Z}$.

From above discussions, we know that, due to the challenge arisen in Step (2), we do not have a complete study on exact tail asymptotics for a general model $\tilde{Z}$ by using our method at this moment. However, the methods developed in this paper could be applied to discuss rough tail asymptotics for a general d-dimensional SRBM $\tilde{Z}$. The large deviations for $\tilde{Z}$ have been studied intensively. Under some mild conditions (see, for example, Conditions 2.1 and 2.5 in Dupuis and Ramanan [9]), the large deviations principle for $\tilde{Z}$ has been established, see, for example, Avram, Dai and Hasenbein [1], Dupuis and Ramanan [9] and Majewski [25]. Therefore, based on the large deviations for $\tilde{Z}$, we expect to establish rough asymptotitcs for the marginal stationary distributions of $\tilde{Z}$, and we can then discuss rough asymptotics and dependence structure of the joint stationary distribution of $\tilde{Z}$ by the method developed in Section 5. In our ongoing work, we discuss this topic in detail.

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References

[1] F. Avram, J.G. Dai and J.J. Hasenbein, Explicit solutions for variational problems in the quadrant. *Queueing Syst.* **37** (2001) 259–289.
[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions of generating trees. *Discrete Math.* **246** (2002) 29–55.
[3] M. Bousquet-Mélou, Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.* **15** (2005) 1451–1491.
[4] V. Bhaskar and P. Lallement, Modeling a supply chain using a network of queues. *Appl. Math. Model.* **34** (2010) 2074–2088.
[5] H.S. Dai, D.A. Dawson and Y.Q. Zhao, Kernel method for stationary tails: from discrete to continuous, in *Asymptotic Laws and Methods in Stochastics,* edited by D.A. Dawson, R. Kulik, M. Ould Haye, B. Szyszkowicz, Y.Q. Zhao (2015) 297–327.
[6] J.G. Dai and J.M. Harrison, Reflecting Brownian motion in an orthant: Numerical methods for steady-state analysis. *Ann. Appl. Prob.* **2** (1992) 65–86.
[7] J.G. Dai and M. Miyazawa, Reflecting Brownian motion in two dimensions: exact asymptotics for the stationary distribution. *Stoch. Syst.* **1** (2011) 146–208.
[8] K. Debicki, M. Mandjes and M. van Uitert, A tandem queue with Lévy input: a new representation of the downstream queue length. *Prob. Eng. Inform. Sci.* **21** (2007) 83–107.
[9] P. Dupuis and K. Ramanan, A time-reversed representation for the tail probabilities of stationary reflected Brownian motion. Stoch. Process. Appl. **98** (2002) 253–287.
[10] G. Fayolle, R. Iasnogorodski and V. Malyshev, *Random Walks in the Quarter-Plane*, second ed. Springer, New York (2017).
[11] S. Franceschi and I. Kurkova, Asymptotic expansion of stationary distribution for reflected Brownian motion in the quarter plane via analytic approach. *Stoch. Syst.* **7** (2017) 32–94.
[12] G. Giambene, Queuing Theory and Telecommunications. Springer, Boston (2014).
[13] M.K. Govil and M. Fu, Queueing theory in manufacturing: a survey. *J. Manufactur. Syst.* **18** (1999) 214–240.
[14] J.M. Harrison and J.J. Hasenbein, Reflected Brownian motion in the quadrant: Tail behavior of the stationary distribution. Queu. Syst. **61** (2009) 113–138.
[15] J. Harrison and M. Reiman, On the distribution of multidimensional reflected Brownian motion. *SIAM J. Appl. Math.* **41** (1981) 345–361.

[16] J. Harrison and M. Reiman, Reflected Brownian motion on an orthant. *Ann. Probab.* **9** (1981) 302–308.
[17] J.M. Harrison and R.J. Williams, Brownian models of open queueing networks with homogeneous customer populations. *Stochastic* **22** (1987) 77–115.
[18] T. Konstantopoulous, G. Last and S.J. Lin, On a class of Lévy stochastic networks. *Queu. Syst.* **46** (2004) 409–437.
[19] G.R. Lawlor, L'Hôspital's rule for multivariable functions. *Am. Math. Monthly* **127** (2020) 717–725.
[20] H. Li and Y.Q. Zhao, Tail asymptotics for a generalized two-dimensional queueing model – a kernel method. *Queu. Syst.* **69** (2011) 77–100.
[21] H. Li and Y.Q. Zhao, A kernel method for exact tail asymptotics-random walks in the quarter plane. *Queu. Models Serv. Manag.* **1** (2018) 95–129.
[22] P. Lieshout and M. Mandjes, Tandem Brownian queues. *Math. Methods Oper. Res.* **66** (2007) 275–298.
[23] P. Lieshout and M. Mandjes, Asymptotic analysis of Lévy-driven tandem queues. *Queu. Syst.* **60** (2008) 203–226.
[24] M. Mandjes, Packet models revisited: tandem and priority systems. *Queu. Syst.* **47** (2004) 363–377.
[25] K. Majewski, Large deviations of the steady state distribution of reflected processes with applications to queueing systems. *Queu. Syst.* **29** (1998) 351–381.
[26] A.I. Markushevich, Theory of Functions of A Complex Variable. Vol. I.II.III, English ed. Chelsea Publishing Co., New York (1977).
[27] M. Miyazawa and T. Rolski, Tail asymptotics for a Lévy-driven tandem queue with an intermediate input. Queu. Syst. **63** (2009) 323–353.
[28] R. Narasimhan, Several Complex Variables. The University of Chicago Press, Chicago and London (1964).
[29] T. Robertazzi, Computer Networks and Systems — Queueing Theory and Performance Evaluation, third ed. Springer, New York (2000).
[30] S.I. Resnick, Extreme Values, Regular Variation, and Point Processes. Springer, New York (1987).
[31] V. Schmitz, *Copulas and Stochastic Processes.* Ph.D thesis: Achen University, 2003.
[32] S. Varadhan and R. Williams, Brownian motion in a wedge with oblique reflection. *Commun. Pure Appl. Math.* **38** (1985) 405–443.
[33] R. Williams, Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.* **13** (1985) 758–778.
[34] R.J. Williams, Semimartingale reflecting Brownian motions in the orthant. In *"IMA Volumes in Mathematics and Its Applications, Volume 71",* edited by F.R. Kelly, R.J. Williams (1995) 125–137.
[35] R.J. Williams, On the approximation of queueing networks in heavy traffic. In "*Stochastic Networks: Theory and Applications,*" edited by F.P. Kelly, S. Zachary, I. Ziedins (1996) 35–56.
[36] W. Whitt, Stochastic-Process Limits. Springer, New York (2002).
[37] Y.Q. Zhao, Kernel method-an analytic approach for tail asymptotics in stationary probabilities of 2-dimensional queueing systems. Preprint arXiv:2101.11661 (2021).