ESAIM: PS 25 (2021) 133–203
https://doi.org/10.1051/ps/2021006

ESAIM: Probability and Statistics
www.esaim-ps.org

BACKWARD STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH JUMPS IN A GENERAL FILTRATION

Alexandre Popier*

Abstract. In this paper, we study backward stochastic Volterra integral equations introduced in Lin [*Stochastic Anal. Appl.* **20** (2002) 165–183] and Yong [*Stochastic Process. Appl.* **116** (2006) 779–795] and extend the existence, uniqueness or comparison results for general filtration as in Papapantoleon *et al.* [*Electron. J. Probab.* **23** (2018) EJP240] (not only Brownian-Poisson setting). We also consider L^p-data and explore the time regularity of the solution in the Itô setting, which is also new in this jump setting.

Mathematics Subject Classification. 45D99, 60H20, 60H99.

Received June 6, 2019. Accepted January 26, 2021.

1. Introduction

The aim of this paper is to extend or to adapt some results concerning backward stochastic Volterra integral equations (BSVIEs in short). To the best of our knowledge, [26, 45, 46] were the first papers dealing with BSVIEs and the authors considered the following class of BSVIEs:

$$Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s. \tag{1.1}$$

W is a k-dimensional Brownian motion, f is called the *generator* or the *driver* of the BSVIE and Φ is the *free term* (or sometimes the terminal condition). Filtration $\mathbb{F}$ is the completed filtration generated by the Brownian motion. They proved existence and uniqueness of the solution (Y,Z) (M-solution in [46]) under the natural Lipschitz continuity regularity of f and square integrability condition for the data.

Let us focus on two particular cases. If f and Φ are not dependent on t, we obtain a backward stochastic differential equation (BSDE for short):

$$Y(t) = \xi + \int_t^T f(s,Y(s),Z(s))\,\mathrm{d}s - \int_t^T Z(s)\,\mathrm{d}W_s.$$

Keywords and phrases: Backward Volterra integral equation, general filtration, L^p-solution, jumps, time regularity.

Laboratoire Manceau de Mathématiques, Le Mans Université, Avenue O. Messiaen, 72085 Le Mans cedex 9, France.

* Corresponding author: alexandre.popier@univ-lemans.fr

© The authors. Published by EDP Sciences, SMAI 2021

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Since the seminal paper [32], it has been intensively studied (see among many others [11, 16, 34, 39]). Expanding the paper [15], Papapantoleon *et al.* [31] studied BSDEs of the form:

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s), U(s))\, \mathrm{d}B_s \\ - \int_t^T Z(s)\, \mathrm{d}X^\circ_s - \int_t^T \int_{\mathbb{R}^m} U(s,x) \widetilde{\pi}^\natural(\,\mathrm{d}x,\, \mathrm{d}s) - \int_t^T \,\mathrm{d}M(s). \tag{1.2}$$

Here the underlying filtration $\mathbb{F}$ only satisfies the usual hypotheses (completeness and right-continuity). The exact definition of processes B, X°, $\widetilde{\pi}^\natural$ is given in Section 2. Roughly speaking, X° is a square-integrable martingale, $\widetilde{\pi}^\natural$ is an integer-valued random measure, such that each component of $\langle X^\circ \rangle$ is absolutely continuous w.r.t. B and the disintegration property given B holds for the compensator $\nu^\natural$ of $\widetilde{\pi}^\natural$. Martingale M naturally appears in the martingale representation since no additional assumption on filtration $\mathbb{F}$ is assumed. Their setting contains the particular case where $X^\circ = W$ and $\widetilde{\pi}^\natural$ is a Poisson random measure (Ex. 2.1), but also many others (see the introduction of [31]).

The second particular case of (1.1) is called Type-I BSVIE:

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t,s))\, \mathrm{d}s - \int_t^T Z(t,s)\, \mathrm{d}W_s. \tag{1.3}$$

The extension to L^p-solution ($1 < p < 2$) for the Type-I BSVIE (1.3) has been done in [41]. In the four papers [26, 41, 45, 46], filtration $\mathbb{F}$ is generated by the Brownian motion W. In [44], the authors introduced the jump component $\widetilde{\pi}$. In the filtration generated by W and the Poisson random measure $\widetilde{\pi}$, they consider:

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t,s), U(t,s))\, \mathrm{d}s \\ - \int_t^T Z(t,s)\, \mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x) \widetilde{\pi}(\,\mathrm{d}s,\, \mathrm{d}x)$$

and prove the existence and uniqueness of the solution in the L^2-setting. The result has been extended in [17, 30, 37] (see also [28] for the Lévy case).

BSVIEs were also studied in the Hilbert case [3], with additional perturbation in the Brownian setting [13, 14], in the quadratic case [40], as a probabilistic representation (nonlinear Feynman-Kac formula) for PDEs [43]. Their use for optimal control problem has been well known since the seminal paper [45]; see for example the recent paper [27]. Let us also mention the survey [47].

Combining all these papers, here we want to deal with a BSVIE of the following type[1]:

$$Y(t) = \Phi(t) + \int_t^T f(t, s, Y(s), Z(t,s), Z(s,t), U(t,s), U(s,t))\, \mathrm{d}B_s \\ - \int_t^T Z(t,s)\, \mathrm{d}X^\circ_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x) \widetilde{\pi}^\natural(\,\mathrm{d}x,\, \mathrm{d}s) - \int_t^T \,\mathrm{d}M(t,s). \tag{1.4}$$

Filtration $\mathbb{F}$ and processes B, X° and $\widetilde{\pi}^\natural$ satisfy the same conditions as in [31]. The unknown processes are the quadruplet (Y, Z, U, M) valued in $\mathbb{R}^{d+(d\times k)+d+d}$ such that $Y(\cdot)$ is $\mathbb{F}$-adapted, and for (almost) all $t \in [0,T]$,

[1] In the whole paper, $\mathrm{d}M(t,s)$ is the integration w.r.t. the second time parameter s where t is fixed. In particular $\int_u^v \,\mathrm{d}M(t,s) = M(t,v) - M(t,u)$ is the increment of $M(t,\cdot)$ between the time u and v.

$(Z(t,\cdot),U(t,\cdot))$ are such that the stochastic integrals are well-defined and $M(t,\cdot)$ is a martingale. This BSVIE is called *of Type-II*. We also consider the Type-I BSVIE:

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),U(t,s))\,\mathrm{d}B_s \\ - \int_t^T Z(t,s)\,\mathrm{d}X^\circ_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\,\mathrm{d}x,\,\mathrm{d}s) - \int_t^T \,\mathrm{d}M(t,s). \end{aligned} \tag{1.5}$$

BSDE (1.2) becomes a particular case of the previous BSVIE.

Let us point out that in [31], the authors consider two different types of BSDEs: equation (1.2) and the following one:

$$\begin{aligned} Y(t) = \xi + \int_t^T f(s,Y(s-),Z(s),U(s))\,\mathrm{d}B_s \\ - \int_t^T Z(s)\,\mathrm{d}X^\circ_s - \int_t^T \int_{\mathbb{R}^m} U(s,x)\widetilde{\pi}^\natural(\,\mathrm{d}x,\,\mathrm{d}s) - \int_t^T \,\mathrm{d}M(s). \end{aligned}$$

The only difference concerns the dependence of f w.r.t. Y. Since B is assumed to be random and càdlàg[2], both cases are not equivalent. However the method of resolution in the second case is not adapted for BSVIEs. This case is left for further research.

1.1. Main contributions

Let us outline the main contributions of our paper compared to the existing literature. First we prove the *existence and uniqueness of the adapted solution of the Type-I BSVIE* (1.5) in the L^2-setting (Thm. 3.3 and Prop. 4.3). This first result generalizes the prior results (of course only some of them) since we only assume that the filtration is complete and right-continuous. This is the reason fer the presence of the càdlàg process B and of the additional martingale term M in (1.5).

As explained in the introduction of [46], for Type-II BSVIEs, the notion of M-solution (see Def. 3.6 below) is crucial to ensure the uniqueness of the solution. To define the terms Z and U on the set $\Delta(R,T) = \{(t,s) \in [R,T]^2,\ R \le s \le t \le T\}$, the martingale representation is used. Since B is random, it is not possible to control the terms Z and U obtained by the martingale representation in a tractable way. We detail this point in Section 3.2.1. However *if B is deterministic*, we prove *existence and uniqueness of the M-solution of the Type-II BSVIE* (1.4) in the L^2-setting (Thm. 3.9).

Our proofs are based on a fixed point argument in the suitable space. Thereby we impose some particular integrability conditions on the free term Φ and on process $f(t,0,0,\mathbf{0})$.

In the BSDE theory, many papers deal with L^p-solution (instead of the square integrability condition on the data); see in particular [8, 23, 24, 34] which deal with L^p-solution for BSDE. Insofar as we know, such an extension does not exist for the general BSDE (1.2). This is the reason why we consider the Itô setting where $B_t = t$ in Section 2.2. This denomination comes from [1, 2]. Thus X° is a Brownian motion W and $\widetilde{\pi}^\natural = \widetilde{\pi}$ is a Poisson random measure with intensity measure μ. The Type-I BSVIE (1.5) becomes

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s). \end{aligned} \tag{1.6}$$

[2]French acronym for right continuous with left limit

For this BSVIE, we provide *existence and uniqueness of an adapted solutions in L^p-space* of (1.6) (Thm. 3.10). To the best of our knowledge, there is no existence and uniqueness result for BSVIEs with L^p coefficients in a general filtration.

Another contribution is the study of the regularity of the map $t \mapsto Y(t)$. For the solution of BSDE (1.2), from the càdlàg regularity of all martingales, Y inherits the same time regularity. For BSVIE, we only require that the paths of Y are in $\mathbb{L}^2(0,T)$ (or in $\mathbb{L}^p(0,T)$). Essentially because we assume that Φ and $t \mapsto f(t,\dots)$ are also only in $\mathbb{L}^2(0,T)$. In [46], it is proved that under weak regularity conditions for the data, then the solution of (1.1) $t \mapsto Y(t)$ is continuous from $[0,T]$ to $\mathbb{L}^2(\Omega)$. Let us stress that Malliavin calculus is used to control the $Z(s,t)$ term in the generator. Similarly we show that the paths $t \in [0,T] \mapsto Y(t) \in \mathbb{L}^p(\Omega)$ of the solution of BSVIE (1.6) are càdlàg if roughly speaking Φ and $t \mapsto f(t,\dots)$ satisfy the same property. However this first property does not give a.s. continuity of the paths of Y in general. Getting an almost sure continuity is a more challenging issue and is proved in [44] for BSVIE (1.1) when f does not depend on $Z(s,t)$, assuming a Hölder continuity property of $t \mapsto f(t,\dots)$ for a constant $\Phi(t) = \xi$. To understand the difficulty, let us evoke that if f does not depend on y, the solution Y of BSVIE (1.6) is obtained by the formula: $Y(t) = \lambda(t,t)$ where $\lambda(t,\cdot)$ is the solution of the related BSDE parametrized by t. In the Brownian setting, a.s. $s \mapsto \lambda(t,s)$ is continuous. Using the Kolmogorov continuity criterion, the authors show that $(t,s) \mapsto \lambda(t,s)$ is bi-continuous, which leads to a continuous version of Y. Insofar as we know, there is not an equivalent result to the Kolmogorov criterion for càdlàg paths. Hence we assume that the free term Φ and the generator f are Hölder continuous. Thus we sketch the arguments of [44] to obtain that *a.s. the paths of Y are càdlàg* (Thm. 3.11) if we know that the data Φ and f are Hölder continuous w.r.t. t, meaning that the jumps only come from the martingale parts in the BSVIE. Relaxing the regularity of the data is still an open question.

In the Itô setting, BSVIE (1.4) becomes:

$$\begin{aligned} Y(t) = \Phi(t) &+ \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t),U(t,s),U(s,t))\,\mathrm{d}s \\ &- \int_t^T Z(t,s)\,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s), \end{aligned} \tag{1.7}$$

It would be natural to prove existence and uniqueness of a solution in L^p, as Theorem 3.10 for BSVIE (1.6). Nonetheless there is an issue again. Since Z is integrated w.r.t. the Brownian motion, the natural norm on Z is

$$\mathbb{E} \int_0^T \left[\int_0^T |Z(t,r)|^2 \,\mathrm{d}r \right]^{\frac{p}{2}} \,\mathrm{d}t.$$

But this norm is not symmetric w.r.t. (t,s), except for $p=2$. Despite our efforts, we still cannot overcome this problem. Moreover for $p<2$, it is well known that Burkholder-Davis-Gundy inequality does not apply without continuity. Therefore the extension to $p \neq 2$ seems difficult to prove and is left for further research. We only give the result for $p=2$ (Prop. 3.12), which is a corollary of Theorem 3.9. However the proof can be done following the outline of [46].

A real issue for BSVIEs concerns the comparison principle. In the BSDE theory, the comparison principle holds under quite general conditions (see *e.g.* [23, 24, 31, 34]). Roughly speaking, the comparison result is proved by a linearization procedure and by an explicit form for the solution of a linear BSDE. However in the setting of [31], the comparison is more delicate to handle because of the jumps of the process B. For BSVIEs, these arguments fail and comparison is a difficult question. Paper [42] is the most relevant paper on this topic. It provides comparison results and gives several counter-examples where comparison principle fails. Of course all their counter-examples are still valid in our case; thereby we do not have intrinsically better results. On this topic our contribution is to extend the *comparison results for the Type-I BSVIE* (1.5) (Props. 6.3 and 6.4). Somehow we show that the additional martingale terms do not impair the comparison result. Note that we have to take into account that the driver $f(s, Y(s), Z(s), U(s))$ is *a priori* optional (compared to $f(s, Y(s-), Z(s), U(s))$). Thus the linearization procedure should be handled carefully.

Finally we prove *a duality principle for BSVIE* (1.7) provided we know that the solution X of the forward SVIE is itself càdlàg (see [35]). Note the importance of the time regularity here. This result is the first step for comparison principle for this kind of BSVIEs.

1.2. Breakdown of the paper

The paper is broken down as follows. In the first section, we give the mathematical setting and set out some results concerning the existence and uniqueness of the solution of BSDE (1.2). We also explain what we mean by Itô's setting. In the second part, we present our assumptions in details and the existence and uniqueness results concerning BSVIEs (1.4), (1.5), (1.6) and (1.7).

The proofs of these next two sections are presented in the next two sections. In Section 4, we consider the general case and the proofs are essentially based on a fixed point argument as in [31, 45]. The Itô setting is developed in Section 5 (L^p-solution ($p > 1$) and time regularity); for existence and uniqueness, the arguments are close to those of [46].

The last section is devoted to the comparison results for Type-I BSVIEs with their proofs, together with the duality result for Type-II BSVIE in Itô's framework.

Finally we set out some proofs and auxiliary results in the appendix.

1.3. Remaining open questions

Some open problems are addressed here. First of all, the existence and uniqueness of an M-solution for BSVIE (1.4) is not proved yet, except for a deterministic characteristic B. The second question concerns the comparison principle, at least for Type-I BSVIE (1.5), when B is not continuous. Finally the L^p-theory for BSDE (1.2) and thus for BSVIEs (1.4) or (1.5) is a natural question.

2. Setting, notations and BSDEs

On $\mathbb{R}^d$, $|.|$ denotes the Euclidean norm and $\mathbb{R}^{d\times k}$ is identified with the space of real matrices with d rows and k columns. If $z \in \mathbb{R}^{d\times k}$, we have $|z|^2 = \text{Trace}(zz^*)$. For any metric space G, $\mathcal{B}(G)$ is the Borel σ-field.

Our probabilistic setting is the same as that of Papapantoleon *et al.* [31]. The main notations are set out but the details can be found in this paper, especially in Section 2, and are left to the reader. Throughout this paper, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ such that it is a complete stochastic basis in the sense of Jacod and Shiryaev [19]. Without loss of generality, all semimartingales are supposed to be càdlàg, that is they have a.s. right continuous paths with left limits. Space $\mathcal{H}^2(\mathbb{R}^p)$ denotes the set of $\mathbb{R}^p$-valued, square-integrable $\mathbb{F}$-martingales and $\mathcal{H}^{2,d}(\mathbb{R}^p)$ is the subspace of $\mathcal{H}^2(\mathbb{R}^d)$ consisting of purely discontinuous square-integrable martingales. $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0, T]$ and $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$. On $\widetilde{\Omega} = \Omega \times [0, T] \times \mathbb{R}^n$, a function that is $\widetilde{\mathcal{P}}$-measurable, is called predictable.

The required notions on stochastic integrals are recalled in Section 2.2 of [31]. In particular for $X \in \mathcal{H}^2(\mathbb{R}^p)$, if

$$\langle X\rangle = \int_{(0,\cdot]} \frac{\mathrm{d}\langle X\rangle_s}{\mathrm{d}B_s}\,\mathrm{d}B_s$$

for B predictable non-decreasing and càdlàg, the stochastic integral of Z w.r.t X, denoted $Z \cdot X$ or $\int_0 Z_s\,\mathrm{d}X_s$, is defined on space $\mathbb{H}^2(X)$ of predictable processes $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{d\times k}$ such that

$$\mathbb{E}\int_0^\infty \text{Trace}\left(Z_t \frac{\mathrm{d}\langle X\rangle_t}{\mathrm{d}B_t} Z_t^\top\right)\mathrm{d}B_t < +\infty.$$

Moreover π^X is the $\mathbb{F}$-optional integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^m$ defined by

$$\pi^X(\omega;\, \mathrm{d}t,\, \mathrm{d}x) = \sum_{s>0} \mathbf{1}_{\Delta X_s(\omega)\neq 0} \delta_{(s,\Delta X_s(\omega))}(\,\mathrm{d}t,\, \mathrm{d}x).$$

Stochastic integral $U \star \pi^X$ of U w.r.t. π^X, compensator ν^X of π^X, compensated integer-valued random measure $\widetilde{\pi}^X$ and stochastic integral $U \star \widetilde{\pi}^X$ of U w.r.t. $\widetilde{\pi}^X$ are also defined on space $\mathbb{H}^2(X)$ of $\mathbb{F}$-predictable processes $U : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathbb{E}\left[\int_0^\infty \mathrm{d}\,\mathrm{Trace}\left[\langle U \star \widetilde{\pi}^X\rangle_t\right]\right] < +\infty.$$

Remember that all definitions are only summarized here; all details can be found in [19, 31] for the interested reader.

In the rest of the paper, are fixed:

- An $\mathbb{R}^{k+m}$-valued, $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process $\overline{X} = (X^\circ, X^\natural)$ such that

  $$\overline{X}^T \in \mathcal{H}^2(\mathbb{R}^k) \times \mathcal{H}^{2,d}(\mathbb{R}^m)$$

  with

  $$M_{\pi^{(X^\natural)^T}}\left[\Delta((X^\circ)^T)\big|\widetilde{\mathcal{P}}\right] = 0.$$

  Here X^T is the process stopped at time T: $X^T_t = X_{t\wedge T}$, $t \geq 0$. $M_\pi\left[\cdot\big|\widetilde{\mathcal{P}}\right]$ is the conditional $\mathbb{F}$-predictable projection on π (see [31], Def. 2.1). To simplify the notations, $\pi^\natural = \pi^{(X^\natural)^T}$, $\widetilde{\pi}^\natural = \widetilde{\pi}^{(X^\natural)^T}$. Keep in mind that $\nu^\natural$ is the compensator of $\pi^\natural$, that is the $\mathbb{F}$-predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^m$ for which $\mathbb{E}[U \star \mu^\natural] = \mathbb{E}[U \star \nu^\natural]$ for every non-negative $\mathbb{F}$-predictable function U.
- A non-decreasing predictable and càdlàg B such that each component of $\langle X^\circ\rangle$ is absolutely continuous with respect to B and the disintegration property given B holds for the compensator $\nu^\natural$, that is there exists a transition kernel K such that

  $$\nu^\natural(\omega;\, \mathrm{d}t,\, \mathrm{d}x) = K_t(\omega;\, \mathrm{d}x)\, \mathrm{d}B_t \tag{2.1}$$

  (see [31], Lem. 2.9). This property is called assumption ($\mathbb{C}$) in [31].
- b is the $\mathbb{F}$-predictable process defined in Remark 2.11 of [31] by:

  $$b = \left(\frac{\mathrm{d}\langle X^\circ\rangle}{\mathrm{d}B}\right)^{\frac{1}{2}}.$$

Process B is the first component of the characteristics of semimartingale $\overline{X}$ (see [19], Def. II.2.6).

Example 2.1. In Section 5, X° is the k-dimensional Brownian motion W, $B_t = t$, $b = 1$ and $\widetilde{\pi}^\natural$ is the compensated Poisson random measure $\widetilde{\pi}$, with the compensator $\nu(\,\mathrm{d}t,\, \mathrm{d}x) = \,\mathrm{d}t\mu(\,\mathrm{d}x)$, where μ is σ-finite on $\mathbb{R}^m$ such that

$$\int_{\mathbb{R}^m} (1 \wedge |x|^2)\mu(\,\mathrm{d}x) < +\infty.$$

In this particular case, $K_t(\omega; \mathrm{d}x) = \mu(\mathrm{d}x)$. These spaces are classically used for BSDEs with Poisson jumps (see among others [11, 23]).

Example 2.2. The previous example can be generalized to the case where the compensator of π is random and equivalent to the measure $\mathrm{d}t \otimes \mu(\omega, \mathrm{d}x)$ with a bounded density for example (see the introduction of [4]).

Example 2.3. In Section 3.3 of [31], the authors cite the counterexample of [9]. They also provide two other examples just after their Remark 3.19.

The notion of orthogonal decomposition plays a central role here. Inspired by [19] and defined in Definition 2.2 of [31], if $Y \in \mathcal{H}^2(\mathbb{R}^d)$, then the decomposition

$$Y = Y_0 + Z \cdot X^\circ + U \star \widetilde{\pi}^\natural + N$$

is called *orthogonal decomposition of* Y w.r.t. $\overline{X} = (X^\circ, X^\natural)$ if

- $Z \in \mathbb{H}^2(X^\circ)$ and $U \in \mathbb{H}^2(\pi^\natural)$,
- the martingales $Z \cdot X^\circ$ and $U \star \widetilde{\pi}^\natural$ are orthogonal,
- $N \in \mathcal{H}^2(\mathbb{R}^d)$ with $\langle N, X^\circ \rangle = 0$ and $M_{\pi^\natural}\left[\Delta N \middle| \widetilde{\mathcal{P}}\right] = 0$.

The statements ([31], Props. 2.5 and 2.6, Cor. 2.7) give the existence and the uniqueness of such orthogonal decomposition.

2.1. Setting and known results for BSDE (1.2)

In the rest of the paper, A is the non-increasing, $\mathbb{F}$-predictable and càdlàg process defined by

$$A_t = \int_0^t \alpha_s^2 \,\mathrm{d}B_s. \tag{2.2}$$

The process $\alpha : (\Omega \times \mathbb{R}_+, \mathcal{P}) \to \mathbb{R}_+$ may change through the paper.

For some $\beta \in \mathbb{R}$, let us describe the spaces used to obtain the solution of BSDE (1.2). For ease of notation, the dependence on A is suppressed.

$$\mathbb{L}^2_{\beta,\mathcal{F}_T} = \left\{\xi,\ \mathbb{R}^d\text{-valued, } \mathcal{F}_T\text{-measurable, } \|\xi\|^2_{\mathbb{L}^2_\beta} = \mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] < +\infty\right\},$$

$$\mathcal{H}^2_\beta(R,S) = \left\{M \in \mathcal{H}^2,\ \|M\|^2_{\mathcal{H}^2_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t}\,\mathrm{d}\,\mathrm{Trace}(\langle M\rangle_t)\right] < +\infty\right\},$$

$$\mathbb{H}^2_\beta(R,S) = \Big\{\phi\ \mathbb{R}^d\text{-valued, } \mathbb{F}\text{-optional semimartingale with càdlàg paths and}$$
$$\|\phi\|^2_{\mathbb{H}^2_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t}|\phi(t)|^2\,\mathrm{d}B_t\right] < +\infty\Big\},$$

$$\mathbb{H}^{2,\circ}_\beta(R,S) = \left\{Z \in \mathbb{H}^2(X^\circ),\ \|Z\|^2_{\mathbb{H}^{2,\circ}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t}\,\mathrm{d}\,\mathrm{Trace}(\langle Z\cdot X^\circ\rangle_t)\right] < +\infty\right\},$$

$$\mathbb{H}^{2,\natural}_\beta(R,S) = \left\{U \in \mathbb{H}^2(X^\natural),\ \|U\|^2_{\mathbb{H}^{2,\natural}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t}\,\mathrm{d}\,\mathrm{Trace}(\langle U\star\widetilde{\pi}^\natural\rangle_t)\right] < +\infty\right\},$$

$$\mathcal{H}^{2,\perp}_\beta(R,S) = \left\{M \in \mathcal{H}^2(\overline{X}^\perp),\ \|M\|^2_{\mathcal{H}^{2,\perp}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t}\,\mathrm{d}\,\mathrm{Trace}(\langle M\rangle_t)\right] < +\infty\right\},$$

We try to find the solution of BSDE (1.2) in the product space

$$\mathbb{D}^2_\beta(0,T) = \mathbb{H}^2_\beta(0,T) \times \mathbb{H}^{2,\circ}_\beta(0,T) \times \mathbb{H}^{2,\natural}_\beta(0,T) \times \mathcal{H}^{2,\perp}_\beta(0,T).$$

Let us introduce some additional notations. For an $\mathbb{F}$-predictable function U, we specify

$$\widehat{K}_t(U_t(\omega;\cdot))(\omega) = \int_{\mathbb{R}^m} U_t(\omega;x) K_t(\omega, \mathrm{d}x),$$

where K is the kernel assigned by (2.1). Hence

$$\widehat{U}^\natural_t(\omega) = \int_{\mathbb{R}^m} U_t(\omega,x)\nu^\natural(w;\{t\}\times \mathrm{d}x) = \widehat{K}_t(U_t(\omega;\cdot))(\omega)\Delta B_t(\omega).$$

And

$$\zeta^\natural_t = \int_{\mathbb{R}^m} \nu^\natural(w;\{t\}\times \mathrm{d}x).$$

The justification of the definition of the norms on the previous spaces is given by Lemma 2.12 of [31].

Lemma 2.4. *Let* $(Z,U) \in \mathbb{H}^{2,\circ}_\beta(R,S) \times \mathbb{H}^{2,\natural}_\beta(R,S)$. *Then*

$$\|Z\|^2_{\mathbb{H}^{2,\circ}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t} \left\|b_t Z_t\right\|^2 \mathrm{d}B_t\right], \quad \textit{and} \quad \|U\|^2_{\mathbb{H}^{2,\natural}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t} |\!|\!| U_t(\cdot) |\!|\!|^2_t \,\mathrm{d}B_t\right],$$

where for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$

$$(|\!|\!| U_t(\omega;\cdot)|\!|\!|_t(\omega))^2 = \widehat{K}^{\overline{X}}_t(|U_t(\omega;\cdot) - \widehat{U}^\natural_t(\omega)|^2)(\omega) + (1-\zeta^\natural_t)\Delta B_t(\omega)|\widehat{K}^{\overline{X}}_t(U_t(\omega;\cdot))(\omega)|^2 \geq 0.$$

Furthermore

$$\|Z\cdot X^\circ + U \star \widetilde{\pi}^\natural\|^2_{\mathcal{H}^2_\beta} = \|Z\|^2_{\mathbb{H}^{2,\circ}_\beta} + \|U\|^2_{\mathbb{H}^{2,\natural}_\beta}.$$

As in Lemma 2.13 of [31], we designate

$$\mathfrak{H} = \left\{U : [0,T]\times\Omega\times\mathbb{R}^m \to \mathbb{R}^d,\ U_t(\omega;\cdot) \in \mathfrak{H}_{t,\omega} \text{ for } \mathrm{d}B\otimes \mathrm{d}\mathbb{P} - \text{a.e. } (t,\omega)\right\} \tag{2.3}$$

where

$$\mathfrak{H}_{t,\omega} = \left\{u : \mathbb{R}^m \to \mathbb{R}^d,\ |\!|\!| u(\cdot)|\!|\!|_t < +\infty\right\}.$$

Remark 2.5. In the setting of Example 2.1, $\widehat{U}^\natural \equiv 0$ and

$$|\!|\!| u(\cdot)|\!|\!| = \int_{\mathbb{R}^m} |u(x)|^2 \mu(\mathrm{d}x) = \|u\|^2_{\mathbb{L}^2_\mu(\mathbb{R}^m)},$$

$$\|U\|^2_{\mathbb{H}^{2,\natural}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t} \int_{\mathbb{R}^m} |U_t(x)|^2\mu(\mathrm{d}x)\,\mathrm{d}t\right],$$

$$\|Z\|^2_{\mathbb{H}^{2,\circ}_\beta(R,S)} = \mathbb{E}\left[\int_R^S e^{\beta A_t} \|Z_t\|^2 \, \mathrm{d}t\right].$$

Then the generator can be defined on $\mathbb{L}^2_\mu(\mathbb{R}^m)$ instead of $\mathfrak{H}$ in conditions **(F2)** or **(H2)**.

Now let us describe the conditions on parameters (ξ, f) for BSDE (1.2).

(F1) Terminal condition ξ belongs to $\mathbb{L}^2_{\delta,\mathcal{F}_T}$ for some $\delta > 0$.

(F2) Generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \times \mathfrak{H} \to \mathbb{R}^d$ is such that for any $(y, z, u) \in \mathbb{R}^d \times \mathbb{R}^{d\times m} \times \mathfrak{H}$, the map $(\omega, t) \mapsto f(\omega, t, y, z, u)$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable. Moreover there exist

$$\varpi : (\Omega \times \mathbb{R}_+, \mathcal{P}) \to \mathbb{R}_+, \qquad \vartheta = (\theta^\circ, \theta^\natural) : (\Omega \times \mathbb{R}_+, \mathcal{P}) \to (\mathbb{R}_+)^2$$

such that for $\mathrm{d}B \otimes \mathrm{d}\mathbb{P}$-a.e. (t, ω)

$$\begin{aligned} &|f(\omega, t, y, z, u_t(\omega; \cdot)) - f(\omega, t, y', z', u'_t(\omega; \cdot))|^2 \\ &\quad \le \varpi_t(\omega)|y - y'|^2 + \theta^\circ_t(\omega)\|c_t(\omega)(z - z')\|^2 + \theta^\natural_t(\omega)(|\!|\!|u_t(\omega; \cdot) - u'_t(\omega; \cdot)|\!|\!|_t(\omega))^2 \end{aligned}$$

(F3) Let $\alpha^2 = \max(\sqrt{\varpi}, \theta^\circ, \theta^\natural) > 0$ and define A by (2.2). There exists $\mathfrak{f} > 0$ such that

$$\Delta A_t(\omega) \le \mathfrak{f}, \text{ for } \mathrm{d}B \otimes \mathrm{d}\mathbb{P} - \text{ a.e. } (t, \omega).$$

(F4) For the same δ as in **(F1)**,

$$\mathbb{E}\left[\int_0^T e^{\delta A_t} \frac{|f(t, 0, 0, \mathbf{0})|^2}{\alpha_t^2} \, \mathrm{d}B_t\right] < +\infty.$$

$\mathbf{0}$ denotes the null application from $\mathbb{R}^m$ to $\mathbb{R}$.

Let us evoke the existence and uniqueness result of [31].

Theorem 2.6. *([31], Thm. 3.5, Cor. 3.6) Assume that the parameter (ξ, f) verifies all conditions* **(F1)** *to* **(F4)***, and suppose that*

$$\kappa^{\mathfrak{f}}(\delta) = \frac{9}{\delta} + \frac{\mathfrak{f}^2(2 + 9\delta)}{\sqrt{\delta^2\mathfrak{f}^2 + 4} - 2} \exp\left(\frac{\delta\mathfrak{f} + 2 - \sqrt{\delta^2\mathfrak{f}^2 + 4}}{2}\right) < \frac{1}{2}. \tag{2.4}$$

Then BSDE (1.2) has a unique solution (Y, Z, U, M) such that $(\alpha Y, Z, U, M) \in \mathbb{D}^2_\delta(0, T)$. In particular if δ is sufficiently large and $18e\mathfrak{f} < 1$, then the BSDE has a unique solution in $\mathbb{D}^2_\delta(0, T)$.

2.2. The Itô setting and related BSDEs

In Sections 3.3 and 5, the processes are assumed to be Itô's semimartingales in the sense of Definition 1.16 of [1]. Hence the process B is now deterministic and equal to $B_t = t$. Semimartingale $\bar{X}$ can be represented by the Grigelionis form.[3] Up to some modifications in the generator[4], BSDE (1.2) takes the next form:

$$\begin{aligned} Y(t) = \xi + \int_t^T f(s, Y(s), Z(s), U(s,\cdot))\, \mathrm{d}s \\ - \int_t^T Z(s)\, \mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(s,x)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M(s), \end{aligned} \tag{2.5}$$

where

- W is a Brownian motion
- $\widetilde{\pi}$ is a Poisson random measure on $[0,T] \times \mathbb{R}^m$, with intensity $\mathrm{d}t \otimes \mu(\mathrm{d}x)$.

Let us first evoke some standard notations.

- $\mathbb{S}^p(0,T)$ is the space of all $\mathbb{F}$-adapted càdlàg processes X such that $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty$.
- $\mathbb{H}^p(0,T)$ is the subspace of all predictable processes X such that $\mathbb{E}\left[\left(\int_0^T |X_t|^2\, \mathrm{d}t\right)^{p/2}\right] < +\infty$. In this setting, $\mathbb{H}^{2,\circ}_\beta(0,T)$ (with $\beta = 0$) is equal to $\mathbb{H}^2(0,T)$.
- $\mathcal{H}^p(0,T)$ is the space of all martingales such that $\mathbb{E}\left[(\langle M\rangle_T)^{p/2}\right] < +\infty$. Space $\mathcal{H}^{2,\perp}_\beta(0,T)$, with $\beta = 0$, now becomes $\mathcal{H}^{2,\perp}(0,T)$ and we define in a similar way $\mathcal{H}^{p,\perp}(0,T)$ as a subspace of $\mathcal{H}^p(0,T)$.
- $\mathbb{L}^p_\pi(0,T) = \mathbb{L}^p_\pi(\Omega \times [0,T] \times \mathbb{R}^m)$ is the set of processes ψ such that

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^m} |\psi_s(x)|^2 \pi(\mathrm{d}s,\mathrm{d}x)\right)^{p/2}\right] < +\infty.$$

  For $p = 2$, it corresponds to $\mathbb{H}^{2,\natural}_\beta(0,T)$ with $\beta = 0$ (see Lem. 2.4).
- $\mathbb{L}^p_\mu = \mathbb{L}^p(\mathbb{R}^m, \mu; \mathbb{R}^d)$ is the set of measurable functions $\psi : \mathbb{R}^m \to \mathbb{R}^d$ such that $\|\psi\|^p_{\mathbb{L}^p_\mu} = \int_{\mathbb{R}^m} |\psi(x)|^p \mu(\mathrm{d}x) < +\infty$.
- $\mathbb{D}^p(0,T) = \mathbb{S}^p(0,T) \times \mathbb{H}^p(0,T) \times \mathbb{L}^p_\pi(0,T) \times \mathcal{H}^{p,\perp}(0,T)$.

For BSDE (1.2), the L^p-theory has not been developed yet. But in the Itô case the next result is proved in [24]. Let us reinforce condition **(H3)**:

(H3*) There exists a constant K such that a.s. for any $s \in [0,T]$ and $t \in [0,s]$,

$$K^2 \geq \max(\sqrt{\varpi_t(\omega)}, \theta^\circ_{t,s}(\omega), \theta^\natural_{t,s}(\omega)).$$

Proposition 2.7. *Assume that for any (y,z,ψ), $f(\cdot,y,z,\psi)$ is progressively measurable and that* **(H2)** *and* **(H3*)** *hold. If*

$$\mathbb{E}\left(|\xi|^p + \int_0^T |f(t,0,0,\mathbf{0})|^p \mathrm{d}t\right) < +\infty,$$

[3] In general only on a very good filtered extension of the original probability space. But using the remarks of Section 1.4.3 in [1], we can assume that this form of $\bar{X}$ holds on the original filtered probability space.

[4] In general we should take into account a possible degeneracy of the coefficients. This possibility leads to a non Lipschitz continuous new version of the generator f. We ignore this trouble here.

there exists a unique solution (Y, Z, U, M) in $\mathbb{D}^p(0,T)$ to BSDE (2.5). Moreover for some constant $C = C_{p,K,T}$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^p + \left(\int_0^T |Z_t|^2 \,\mathrm{d}t\right)^{p/2} + \left(\int_0^T \int_{\mathbb{R}^m} |U_s(x)|^2 \pi(\,\mathrm{d}s,\,\mathrm{d}x)\right)^{p/2} + (\langle M\rangle_T)^{p/2}\right] \leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(r,0,0,\mathbf{0})|\,\mathrm{d}r\right)^p\right].$$

Condition **(H3*)** is certainly too strong for this result and could be relaxed. Thereby our results concerning BSVIEs in the Itô setting could be extended to non Lipschitz-continuous w.r.t. y driver f as in [44] (using Mao's condition with a concave function ρ) or if K becomes a function of (ω, s) with a suitable integrability condition (see [46], condition (3.13)). Nonetheless such extensions would increase the length of the paper and are left for further research.

3. Notations, definitions and statement of our main results for BSVIEs

Concerning BSVIEs, the notations of [46] are kept. They are only adapted to our setting and thus some details are skipped (see [46], Sect. 2.1 for interested readers). For $0 \leq R \leq S \leq T$ we denote

$$\Delta[R,S] = \{(t,s) \in [R,S]^2,\ R \leq s \leq t \leq S\},$$
$$\Delta^c[R,S] = \{(t,s) \in [R,S]^2,\ R \leq t < s \leq S\} = [R,S]^2 \setminus \Delta[R,S].$$

This notation should not be confused with the jump of a process.

Recall that A is the non-negative càdlàg non-decreasing and measurable process defined by (2.2). Some $\beta \in \mathbb{R}_+$ and some $0 \leq \delta \leq \beta$ are fixed. Again to simplify the notations, the dependance on A is suppressed. First

$$\mathbb{L}^2_{\beta,\mathcal{F}_T}(0,T) = \left\{\phi : (0,T)\times\Omega \to \mathbb{R}^d,\ \mathcal{B}([0,T])\otimes\mathcal{F}_T - \text{measurable with } \mathbb{E}\left[\int_0^T e^{\beta A_t}\left(e^{\beta A_T}|\phi(t)|^2\right)\,\mathrm{d}B_t\right] < +\infty\right\}.$$

The above space is for the free term $\Phi(\cdot)$ (for which the $\mathbb{F}$-adaptiveness is not required). When $\mathbb{F}$-adaptiveness is required, that is for $Y(\cdot)$:

$$\mathbb{L}^2_{\beta,\mathbb{F}}(0,T) = \left\{\phi : (0,T)\times\Omega \to \mathbb{R}^d,\ \mathcal{B}([0,T])\otimes\mathcal{F}_T - \text{measurable and } \mathbb{F} - \text{adapted with } \mathbb{E}\left[\int_0^T e^{\beta A_t}\left(|\phi(t)|^2\right)\,\mathrm{d}B_t\right] < +\infty\right\}.$$

To control the martingale terms (Z, U, M) and to define the notion of solutions, Type-II BSVIE (1.4) and Type-I BSVIE (1.5) are distinguished.

3.1. Existence and uniqueness for Type-I BSVIEs

To control the martingale terms (Z, U, M) in the Type-I BSVIE (1.5), some other spaces are needed. The set of processes $M(\cdot,\cdot)$ such that for $t \in [R,T]$, $M(t,\cdot) = \{M(t,s),\ s \geq t\}$ belongs to $\mathcal{H}^2(\mathbb{R}^d)$ and

$$\mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle M(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] < +\infty,$$

is denoted by

$$\mathcal{H}^2_{\delta\leq\beta}(\Delta^c(R,T)).$$

In the particular case where $M(t,\cdot) = \int_S^{\cdot} Z(t,s)\,\mathrm{d}X^\circ_s$, then $M \in \mathcal{H}^2_{\gamma,\delta}(\Delta^c(R,T))$ is equivalent to $Z \in \mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(R,T))$ if $Z(t,\cdot)$ belongs to $\mathbb{H}^{2,\circ}_\delta(t,T)$ and from Lemma 2.4

$$\mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle M(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] = \mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \left\|b_s Z(t,s)\right\|^2 \,\mathrm{d}B_s \,\mathrm{d}A_t\right].$$

Note that we strongly rely on the fact that we work on $\Delta^c(R,T)$ and that A_t is $\mathcal{F}_t$-measurable to obtain this equality. For Type-II BSVIEs this point is an issue. Similarly for $U(t,\cdot) \in \mathbb{H}^{2,\natural}_\delta(t,T)$, the martingale M defined by $M(t,\cdot) = U(t,\cdot) \star \widetilde{\pi}^\natural$ is in $\mathcal{H}^2_{\delta\leq\beta}(\Delta^c(R,T))$ if

$$\begin{aligned}&\mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle M(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] \\ &= \mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} |||U(t,\cdot)|||^2_s \,\mathrm{d}B_s \,\mathrm{d}A_t\right] < +\infty.\end{aligned}$$

In this case $U \in \mathbb{H}^{2,\natural}_{\delta\leq\beta}(\Delta^c(R,T))$. To lighten the notations, the martingale for $t \leq u \leq T$

$$\mathrm{M}^\sharp(t,u) = \int_t^u Z(t,s)\,\mathrm{d}X^\circ_s + \int_t^u \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) + M(t,u), \tag{3.1}$$

is introduced, such that BSVIE (1.5) becomes

$$Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),U(t,s))\,\mathrm{d}B_s - \left(\mathrm{M}^\sharp(t,T) - \mathrm{M}^\sharp(t,t)\right). \tag{3.2}$$

Using Corollary 2.7 of [31], $\mathrm{M}^\sharp$ belongs to $\mathcal{H}^2_{\delta\leq\beta}(\Delta^c(R,T))$ if and only if the triplet (Z,U,M) belongs to $\mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(R,T)) \times \mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(R,T)) \times \mathcal{H}^{2,\perp}_{\delta\leq\beta}(\Delta^c(R,T))$:

$$\begin{aligned}\mathbb{E}\Bigg[&\int_0^T e^{(\beta-\delta)A_t} \left(\int_t^T e^{\delta A_r} \,\mathrm{d}\,\mathrm{Trace}[\langle Z(t,\cdot)\cdot X^\circ\rangle_r]\right) \mathrm{d}A_t \\ &+ \int_0^T e^{(\beta-\delta)A_t} \left(\int_t^T e^{\delta A_r} \,\mathrm{d}\,\mathrm{Trace}[\langle U(t,\cdot)\star\widetilde{\pi}^\natural\rangle_r]\right) \mathrm{d}A_t\end{aligned}$$

$$+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle M(t,\cdot)\rangle_r]\right)\mathrm{d}A_t\Bigg] < +\infty.$$

Last the product space

$$\mathfrak{S}^2_{\delta\leq\beta}(\Delta^c(0,T)) = \mathbb{L}^2_{\beta,\mathbb{F}}(0,T)\times\mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(0,T))\times\mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(0,T))\times\mathcal{H}^{2,\perp}_{\delta\leq\beta}(\Delta^c(0,T))$$

is specified, with the naturally induced norm. If $\delta=\delta(\beta)$ is a known function of β, $\mathfrak{S}^2_{\delta\leq\beta}(\Delta^c(0,T))$ is denoted by $\mathfrak{S}^2_{\beta}(\Delta^c(0,T))$.

3.1.1. Adapted solution for Type-I BSVIE

Let us adjust the notion of solution developed in [46] to our setting.

Definition 3.1 (Adapted solution)**.** A quadruple (Y,Z,U,M) is called an adapted solution of the Type-I BSVIE (1.5) if (Y,Z,U,M) belongs to $\mathfrak{S}^2_{\delta\leq\beta}(\Delta^c(0,T))$ for some $\delta\leq\beta$ and if the equation is satisfied a.s. for almost all $t\in[0,T]$.

Note that if (Y,Z,U,M) is a solution of BSDE (1.2) in $\mathbb{D}^2_\beta(0,T)$, then $Y\in\mathbb{L}^2_{\beta,\mathbb{F}}(0,T)$ and

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\left\|b_sZ(s)\right\|^2\mathrm{d}B_s\,\mathrm{d}A_t\right]\\
&=\mathbb{E}\left[\int_0^T\int_0^s e^{(\beta-\delta)A_t}e^{\delta A_s}\left\|b_sZ(s)\right\|^2\mathrm{d}A_t\,\mathrm{d}B_s\right]\\
&\leq\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\mathbb{E}\left[\int_0^T e^{\beta A_s}\left\|b_sZ(s)\right\|^2\mathrm{d}B_s\right]<+\infty.
\end{aligned}$$

In other words (Z,U,M) is in $\mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(0,T))\times\mathbb{H}^{2,\circ}_{\delta\leq\beta}(\Delta^c(0,T))\times\mathcal{H}^{2,\perp}_{\delta\leq\beta}(\Delta^c(0,T))$, and thus (Y,Z,U,M) is an adapted solution of BSVIE (1.5) where $\Phi(t)=\xi$ and f does not depend on t.

Our assumptions and the statement of our results depend on the next quantities. Let us define for $\delta<\gamma\leq\beta$ the function

$$\Pi^{\mathfrak{f}}(\gamma,\delta)=\frac{11}{\delta}+9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}.$$

The next technical lemma is equivalent to Lemma 3.4 in [31]. Its proof is set out in the Appendix.

Lemma 3.2. *The infimum of $\Pi^{\mathfrak{f}}(\gamma,\delta)$ over all $\delta<\gamma\leq\beta$ is given by $M^{\mathfrak{f}}(\beta)=\Pi^{\mathfrak{f}}(\beta,\delta^*(\beta))$, where $\delta^*(\beta)$ is the unique solution on $(0,\beta)$ of the equation:*

$$11(\beta-x)^2-9e^{(\beta-x)\mathfrak{f}}x^2(\mathfrak{f}(\beta-x)-1)=0.$$

Moreover $\lim\limits_{\beta\to+\infty}M^{\mathfrak{f}}(\beta)=9e\mathfrak{f}$.

For $M^{\mathfrak{f}}(\beta)<1/2$, the next quantities are considered

$$\Sigma^{\mathfrak{f}}(\beta)=\frac{2M^{\mathfrak{f}}(\beta)}{1-2M^{\mathfrak{f}}(\beta)},\qquad\widetilde{\Sigma}^{\mathfrak{f}}(\beta)=\Sigma^{\mathfrak{f}}(\beta)\frac{e^{(\beta-\delta^*(\beta))\mathfrak{f}}}{\beta-\delta^*(\beta)}.\tag{3.3}$$

Let us precise the assumptions on the free term Φ and on the generator f of the BSVIE (1.5).

(H1) $\Phi \in \mathbb{L}^2_{\beta,\mathcal{F}_T}(0,T)$.

(H2) The driver f is defined on $\Omega \times \Delta^c(0,T) \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \times \mathfrak{H} \to \mathbb{R}^d$ and for any fixed (t,y,z,u) the process $f(t,\cdot,y,z,u)$ is progressively measurable. Moreover there exist

$$\varpi : (\Omega \times \Delta^c(0,T), \mathcal{P}) \to \mathbb{R}_+, \qquad \vartheta = (\theta^\circ, \theta^\natural) : (\Omega \times \Delta^c(0,T), \mathcal{P}) \to (\mathbb{R}_+)^2$$

such that for $\mathrm{d}B \otimes \mathrm{d}B \otimes \mathrm{d}\mathbb{P}$-a.e. (t,s,ω)

$$\begin{aligned} &|f(\omega,t,s,y,z,u_s(\omega;\cdot)) - f(\omega,t,s,y',z',u'_s(\omega;\cdot))|^2 \\ &\quad \le \varpi(\omega,t,s)|y-y'|^2 + \theta^\circ(\omega,t,s)\|c_s(\omega)(z-z')\|^2 \\ &\quad\quad + \theta^\natural(\omega,t,s)(|||u_s(\omega;\cdot) - u_s(\omega;\cdot)|||_s(\omega))^2. \end{aligned}$$

To simplify the notation in the sequel: $f^0(t,s) = f(t,s,0,0,\mathbf{0})$.

(H3) Hypothesis **(F3)** holds. Namely there exists $\alpha(\omega,s) > 0$ defined on $\Omega \times [0,T]$ such that $\alpha^2_s(\omega) \ge \max(\sqrt{\varpi}(\omega,t,s), \theta^\circ(\omega,t,s), \theta^\natural(\omega,t,s))$ for $(\omega,t,s) \in \Omega \times \Delta^c(0,T)$. Process A is defined by (2.2):

$$A_t = \int_0^t \alpha^2_s \,\mathrm{d}B_s.$$

There exists $\mathfrak{f} > 0$ such that for any t

$$\Delta A_r(\omega) \le \mathfrak{f}, \text{ for } \mathrm{d}B \otimes \mathrm{d}\mathbb{P} - \text{ a.e. } (r,\omega).$$

(H4) For the same β as in **(H1)**, with the constant $\delta^*(\beta)$ of the above Lemma 3.2

$$\mathbb{E}\left[\int_0^T e^{(\beta-\delta^*(\beta))A_t}\left(\int_t^T e^{\delta^*(\beta)A_s}\frac{|f(t,s,0,0,\mathbf{0})|^2}{\alpha^2_s}\,\mathrm{d}B_s\right)\mathrm{d}B_t\right] < +\infty.$$

(H5) The following set

$$\mathfrak{T}^{\Phi,f}_\delta = \left\{ t \in [0,T] : \mathbb{E}\left[e^{\delta^*(\beta)A_T}|\Phi(t)|^2 + \int_t^T e^{\delta^*(\beta)A_s}\frac{|f(t,s,0,0,\mathbf{0})|^2}{\alpha^2_s}\,\mathrm{d}B_s\right] < +\infty \right\} \tag{3.4}$$

is assumed to be equal to $[0,T]$.

Coming back to BSVIE (1.5), our main result on this BSVIE is now stated

Theorem 3.3. *Assume that:*

- *Conditions* **(H1)** *to* **(H5)** *hold.*
- *Constant $\delta = \delta^*(\beta)$ is defined in Lemma 3.2. Constants $\kappa^{\mathfrak{f}}(\delta)$, $M^{\mathfrak{f}}(\beta)$ and $\widetilde{\Sigma}^{\mathfrak{f}}(\beta)$ defined by (2.4), in Lemma 3.2 and by (3.3) verify*

$$\kappa^{\mathfrak{f}}(\delta) < \frac{1}{2}, \qquad M^{\mathfrak{f}}(\beta) < \frac{1}{2}, \qquad \widetilde{\Sigma}^{\mathfrak{f}}(\beta) < 1. \tag{3.5}$$

Then BSVIE (1.5) has a unique adapted solution (Y,Z,U,M) in $\mathfrak{S}^2_\beta(\Delta^c(0,T))$. Moreover there exists a constant $\mathfrak{C}^{\mathfrak{f}}(\beta)$ such that

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle Z(t,\cdot)\cdot X^\circ\rangle_r]\right)\mathrm{d}A_t \right.$$
$$+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle U(t,\cdot)\star\widetilde{\pi}^\natural\rangle_r]\right)\mathrm{d}A_t$$
$$\left.+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle M(t,\cdot)\rangle_r]\right)\mathrm{d}A_t\right]$$
$$\leq \mathfrak{C}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|f^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]. \tag{3.6}$$

Let $(\bar{\Phi},\bar{h})$ be a couple of data each satisfying the above assumptions **(H1)** *to* **(H5)***. Let $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ be a solution of BSVIE (1.5) with data $(\bar{\Phi},\bar{f})$. Define*

$$(\mathfrak{d}Y,\mathfrak{d}Z,\mathfrak{d}U,\mathfrak{d}M)=(Y-\bar{Y},Z-\bar{Z},U-\bar{U},M-\bar{M}).$$

Then using the notation (3.1) the following stability result is verified:

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq \mathfrak{C}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}\Phi(t)|^2\,\mathrm{d}A_t\right]$$
$$+\mathfrak{C}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}f(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]$$

with

$$\mathfrak{d}f(t,r)=f(t,r,Y(r),Z(t,r),U(t,r))-\bar{f}(t,r,Y(r),Z(t,r),U(t,r)).$$

A comment on condition (3.5). The first part $\kappa^{\mathfrak{f}}(\delta)<1/2$ comes from Theorem 2.6 (see [31], Thm. 3.5). The second condition is sufficient to have existence and uniqueness of the solution of the Type-I BSVIE if f does not depend on Y (see the BSVIE (4.4) and Prop. 4.3 below). The last part $\widetilde{\Sigma}^{\mathfrak{f}}(\beta)<1$ ensures the existence and uniqueness thanks to a fixed point argument.

Remark 3.4 (Large values of β)**.** Note that for β large, $\kappa^{\mathfrak{f}}(\delta)<1/2$ and $M^{\mathfrak{f}}(\beta)<1/2$ if and only if $18e\mathfrak{f}<1$ (as in [31]). And since

$$\lim_{\beta\to+\infty}\widetilde{\Sigma}^{\mathfrak{f}}(\beta)=\frac{18(e\mathfrak{f})^2}{1-18e\mathfrak{f}},$$

$\widetilde{\Sigma}^{\mathfrak{f}}(\beta)<1$ if and only if $18e\mathfrak{f}<3(\sqrt{11}-3)<1$. In other words, if $18e\mathfrak{f}<3(\sqrt{11}-3)<1$ (and $3(\sqrt{11}-3)\approx 0.95$), then (3.5) holds for large values of β.

Remark 3.5 (On condition **(H5)**)**.** If assumptions **(H1)** and **(H4)** hold, then

$$\mathbb{E}\int_0^T \left(e^{\delta A_T}|\Phi(t)|^2\right) \mathrm{d}B_t < +\infty, \text{ and } \mathbb{E}\int_0^T \left(\int_t^T e^{\delta A_s} \frac{|f(t,s,0,0,\mathbf{0})|^2}{\alpha_s^2} \mathrm{d}B_s\right) \mathrm{d}B_t < +\infty.$$

For a random process B, the time integral and the expectation cannot be switched and a precise description of the set $\mathfrak{T}_\delta^{\Phi,f}$ is not easy. But if B is deterministic, we deduce that $\mathrm{d}B$-almost every $t \in [0,T]$ belongs to $\mathfrak{T}_\delta^{\Phi,f}$. In the Brownian-Poisson setting (Ex. 2.1), $\mathrm{d}B$ is the Lebesgue measure. If B is piecewise-constant as in the second example after Remark 3.19 of [31] with deterministic jump times, a similar $\mathrm{d}B$-a.e. property is true. In other words **(H5)** is too strong in a lot of cases, but it makes the presentation of our results correct and easier in the general setting.

3.2. Extension for Type-II BSVIEs

Here the objective is to expand some results to Type-II BSVIEs (1.4). As explained in the introduction of [46], for Type-II BSVIEs, the notion of M-solution is crucial to ensure the uniqueness of the solution; uniqueness of an adapted solution fails. Roughly speaking, there is an additional freedom on $\Delta(0,T)$. To avoid this problem, the next definition of M-solution is formulated in [46].

Definition 3.6 (M-solution)**.** Let $S \in [0,T)$. A quadruple (Y,Z,U,M) is called an adapted M-solution of (1.4) on $[S,T]$ if (1.4) holds in the usual Itô sense for almost all $t \in [S,T]$ and, in addition, the following holds: for a.e. $t \in [S,T]$

$$Y(t) = \mathbb{E}\left[Y(t)|\mathcal{F}_S\right] + \int_S^t Z(t,s)\,\mathrm{d}X_s^\circ + \int_S^t \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\mathrm{d}s,\mathrm{d}x) + \int_S^t \mathrm{d}M(t,s). \tag{3.7}$$

Note that the notion of M-solution of [46] is kept, where the letter M stands for "a martingale representation" for $Y(t)$ to determine $Z(\cdot,\cdot)$, $U(\cdot,\cdot)$ and $M(\cdot,\cdot)$ on $\Delta[S,T]$. It should not be confused with the orthogonal martingale part M. As in [46], any M-solution on $[S,T]$ is also an M-solution on $[\bar{S},T]$ with $\bar{S} \in (S,T)$.

3.2.1. The randomness of B is an issue

This notion of M-solution implies that to define the terms Z and U on the set $\Delta(R,T) = \{(t,s) \in [R,T]^2,\ R \leq s \leq t \leq T\}$, the martingale representation is used. In [46] it takes the form: for almost every $t \in [R,T]$,

$$Y(t) = \mathbb{E}\left[Y(t)\middle|\mathcal{F}_R\right] + \int_R^t Z(t,s)\,\mathrm{d}W_s. \tag{3.8}$$

The existence is justified since $Y \in \mathbb{L}^2(\Omega \times [R,T])$. Integrating the relation (3.8) between R and t and using Fubini's theorem leads to

$$\begin{aligned}
\mathcal{Y}(t) = \int_R^t Y(r)\,\mathrm{d}r &= \mathbb{E}\left[\int_R^t Y(r)\,\mathrm{d}r\middle|\mathcal{F}_R\right] + \int_R^t \int_S^r Z(r,u)\,\mathrm{d}W_u\,\mathrm{d}r \\
&= \mathbb{E}\left[\int_R^t Y(r)\,\mathrm{d}r\middle|\mathcal{F}_R\right] + \int_R^t \left(\int_u^t Z(r,u)\,\mathrm{d}r\right) \mathrm{d}W_u \\
&= \mathbb{E}\left[\mathcal{Y}(t)\middle|\mathcal{F}_R\right] + \int_R^t Z^\sharp(t,u)\,\mathrm{d}W_u.
\end{aligned}$$

Moreover

$$\mathbb{E}\int_R^T \left|\int_R^t Z^\sharp(t,s)\,\mathrm{d}W_s\right|^2 \mathrm{d}t = \mathbb{E}\int_R^T \int_R^t |Z^\sharp(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t$$
$$\leq (T-R)^2\mathbb{E}\int_R^T \int_R^r |Z(r,s)|^2\,\mathrm{d}s\,\mathrm{d}r.$$

In other words an M-solution (in the sense of [46]) provides a "integrated martingale"-solution.

For BSVIE (1.4) we only expect

$$\mathbb{E}\left[\int_S^T e^{\beta A_t}|Y(t)|^2\alpha_t^2\,\mathrm{d}B_t\right] < +\infty,$$

where $A = \int \alpha_s^2\,\mathrm{d}B_s$ and B are random processes (Thm. 3.3). To get around this issue, if

$$\mathcal{Y}(t) = \int_S^t Y(s)\alpha_s^2\,\mathrm{d}B_s,$$

the Cauchy-Schwarz inequality yields to

$$|\mathcal{Y}(t)|^2 \leq \frac{1}{\delta}e^{-\delta A_t}\int_t^T e^{\delta A_s}|Y(s)|^2\alpha_s^2\,\mathrm{d}B_s.$$

Hence for almost all $t \in [R,T]$, $\mathcal{Y}(t)$ is in $\mathbb{L}^2(\Omega)$, and the martingale representation leads to

$$\mathcal{Y}(t) = \mathbb{E}\left[\mathcal{Y}(t)\middle|\mathcal{F}_R\right] + \int_R^t Z(t,s)\,\mathrm{d}X_s^\circ + \int_R^t\int_{\mathbb{R}^m} U(t,s,x)\,\mathrm{d}\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) + \int_R^t \mathrm{d}M(t,s). \tag{3.9}$$

The next estimate can easily be obtained: for any $0 < \delta < \beta$

$$\mathbb{E}\int_S^T e^{\delta A_t}\left|\int_S^t \mathrm{dM}^\sharp(t,r)\right|^2 \mathrm{d}A_t \leq 2\frac{e^{(\beta-\delta)\mathfrak{f}}}{\delta(\beta-\delta)}\mathbb{E}\left[\int_S^T e^{\beta A_u}|Y(u)|^2\,\mathrm{d}A_u\right].$$

But this weak norm on $\mathrm{M}^\sharp$ is not sufficient to control Z and U in the generator of BSVIE (1.4). Since process A is supposed to be only predictable, we cannot claim that

$$\mathbb{E}\int_S^T e^{\delta A_t}\left|\int_S^t \mathrm{dM}^\sharp(t,r)\right|^2 \mathrm{d}A_t = \mathbb{E}\int_S^T e^{\delta A_t}\int_S^t \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r\,\mathrm{d}A_t.$$

It leads to a major issue. Moreover the cunning used in [31]:

$$\int_S^t e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r \leq \delta\int_S^t e^{\delta A_u}\int_u^t \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r\,\mathrm{d}A_u$$
$$+e^{\delta A_S}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(\cdot,t)\rangle_t - \mathrm{Trace}\langle \mathrm{M}^\sharp(\cdot,t)\rangle_S\right).$$

is useless here since $e^{\delta A_u}$ is $\mathcal{F}_u$-measurable and not $\mathcal{F}_S$-measurable. *Due to this reason, we cannot address an existence and uniqueness result for BSVIE (1.4) in this general setting, but only for deterministic processes A and B.*

3.2.2. Type-II BSVIE for a deterministic process A

Therefore we restrict ourselves to the case where B is deterministic. Note that from Theorem II.4.15 of [19], semimartingale $\overline{X}$ has deterministic characteristics (that is B and the two other components) if and only if it is a process with independent increments. We also assume that α (see Hyp. **(H3)**) is deterministic.

For the free term Φ and the first component Y of the solution, spaces $\mathbb{L}^2_{\beta,\mathcal{F}_T}(0,T)$ and $\mathbb{L}^2_{\beta,\mathbb{F}}(0,T)$ are conserved. But the definitions of Section 3.1 are adapted, since the martingale terms need to be controlled not only on $\Delta^c(0,T)$, but on the whole set $[0,T]^2$. Thereby, space

$$\mathcal{H}^2_{\delta\leq\gamma}(R,T)$$

is defined as the set of processes $M(\cdot,\cdot)$ such that for $\mathrm{d}B$-a.e. $t\in[R,T]$, $M(t,\cdot)$ belongs to $\mathcal{H}^2$ and

$$\mathbb{E}\left[\int_R^T e^{(\gamma-\delta)A_t}\int_R^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle M(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]<+\infty.$$

Note that this definition implies that $M(t,s)$ is $\mathcal{F}_s$-measurable for any $s\in[R,T]$. In the particular cases where $M(t,\cdot)=\int_S^\cdot Z(t,s)\,\mathrm{d}X^\circ_s$ (resp. $M(t,\cdot)=U(t,\cdot)\star\widetilde{\pi}^\natural$), then $M\in\mathcal{H}^2_{\gamma,\delta}(R,T)$ if

$$\mathbb{E}\left[\int_R^T e^{(\gamma-\delta)A_t}\int_R^T e^{\delta A_s}\left\|b_s Z(t,s)\right\|^2\,\mathrm{d}B_s\,\mathrm{d}A_t\right]<+\infty,$$

(resp.

$$\mathbb{E}\left[\int_R^T e^{(\gamma-\delta)A_t}\int_R^T e^{\delta A_s}\|\!|U(t,\cdot)|\!\|_s^2\,\mathrm{d}B_s\,\mathrm{d}A_t\right]<+\infty.)$$

In this case $Z\in\mathbb{H}^{2,\circ}_{\delta\leq\gamma}(R,T)$ (resp. $U\in\mathbb{H}^{2,\natural}_{\delta\leq\gamma}(R,T)$). Finally we introduce the product space

$$\mathfrak{S}^2_{\delta\leq\gamma}(R,T)=\mathbb{L}^2_{\gamma,\mathbb{F}}(R,T)\times\mathbb{H}^{2,\circ}_{\delta\leq\gamma}(R,T)\times\mathbb{H}^{2,\circ}_{\delta\leq\gamma}(R,T)\times\mathcal{H}^{2,\perp}_{\delta\leq\gamma}(R,T)$$

with the naturally induced norm. If $\delta=\delta(\gamma)$ is a known function of γ, $\mathfrak{S}^2_{\delta\leq\gamma}(R,T)$ is denoted by $\mathfrak{S}^2_\gamma(R,T)$.

Conditions **(H2)** and **(H3)** are modified as follows.

(H2') Generator f is defined on $\Omega\times\Delta^c(0,T)\times\mathbb{R}^d\times(\mathbb{R}^{d\times m})^2\times(\mathfrak{H})^2\to\mathbb{R}^d$ and for any fixed (t,y,z,ζ,u,ν) the process $f(t,\cdot,y,z,\zeta,u,\nu)$ is progressively measurable. Moreover there exist

$$\varpi:(\Omega\times\Delta^c(0,T),\mathcal{P})\to\mathbb{R}_+,\qquad \vartheta=(\theta^\circ,\theta^\natural):(\Omega\times\Delta^c(0,T),\mathcal{P})\to(\mathbb{R}_+)^2$$

such that for $\mathrm{d}B\otimes\mathrm{d}B\otimes\mathrm{d}\mathbb{P}$-a.e. (t,s,ω)

$$\begin{aligned}&|f(\omega,t,s,y,z,\zeta,u_s(\omega;\cdot),\nu_s(\omega;\cdot))-f(\omega,t,s,y',z',\zeta',u'_s(\omega;\cdot),\nu'_s(\omega;\cdot))|^2\\&\quad\leq\varpi_{t,s}(\omega)|y-y'|^2+\theta^\circ_{t,s}(\omega)\left(\|b_s(\omega)(z-z')\|^2+\|b_t(\omega)(\zeta-\zeta')\|^2\right)\\&\qquad+\theta^\natural_{t,s}(\omega)\left[(\|\!|u_s(\omega;\cdot)-u_s(\omega;\cdot)|\!\|_s(\omega))^2+(\|\!|\nu_s(\omega;)-\nu'_s(\omega;)|\!\|_t(\omega))^2\right].\end{aligned}$$

(H3') There exists a deterministic and non-decreasing[5] $\alpha : [0,T] \to \mathbb{R}_+$ such that a.s. for any $s \in [0,T]$ and $t \in [0,s]$,

$$\alpha_s^2 \geq \max(\sqrt{\varpi_t(\omega)}, \theta^\circ_{t,s}(\omega), \theta^\natural_{t,s}(\omega)) > 0.$$

The rest of the condition **(H3)** (or **(F3)**) remains unchanged.

In particular **(H3*)** implies **(H3')**. If **(H3')** holds, then instead of **(H5)**, $\mathrm{d}B$-almost every $t \in [0,T]$ belongs to the set $\mathfrak{T}_\delta^{\Phi,f}$ defined by (3.4). Assumption **(H3')** is true for example for the setting of Example 2.1 if the generator has bounded stochastic Lipschitz coefficients.

Remark 3.7. Let us emphasize that assumption **(H2')** implicitly implies that the process ν_s is in fact $\mathcal{F}_t$-measurable.

Remark 3.8 (On Condition **(H3')**)**.** Denote $L(t,s)^2 = \max(\theta^\circ(\omega,t,s), \theta^\natural(\omega,t,s))$. Assuming that α does not depend on t implies that the $\sup_{t\in[0,s]} L(t,s)^2 \leq \alpha_s^2$ is integrable. If we compare with the conditions imposed in [41, 46], this assumption is stronger:

$$\sup_{t\in[0,T]} \int_t^T L(t,s)^2 \,\mathrm{d}B_s \leq \int_t^T \sup_{t\in[0,T]} L(t,s)^2 \,\mathrm{d}B_s \leq \int_t^T \alpha_s^2 \,\mathrm{d}B_s.$$

Let us study **(H1)** and **(H4)** under the prior conditions. Hypothesis **(H1)** can be rewritten as follows:

$$A_T < +\infty, \quad \text{and} \quad \mathbb{E}\int_0^T e^{\beta A_t} |\Phi(t)|^2 \,\mathrm{d}B_t < +\infty.$$

Moreover if $A_T < +\infty$, the estimate

$$\mathbb{E}\int_0^T \left(|\Phi(t)|^2 + \int_t^T \frac{|f^0(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \right) \mathrm{d}B_t < +\infty$$

leads to **(H1)** and **(H4)**. Let us again highlight that $\delta = \delta^*(\beta)$ is defined in Lemma 3.2.

Theorem 3.9. *If* **(H1)**, **(H2')**, **(H3')** *and* **(H4)** *hold and if constants* $\kappa^{\mathfrak{f}}(\delta)$, $M^{\mathfrak{f}}(\beta)$ *and* $\Sigma^{\mathfrak{f}}(\beta)$ *defined by* (2.4), *in Lemma 3.2 and by* (3.3) *verify*

$$\kappa^{\mathfrak{f}}(\delta) < \frac{1}{2}, \qquad M^{\mathfrak{f}}(\beta) < \frac{1}{2}, \qquad \left(16 + \frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\right) \Sigma^{\mathfrak{f}}(\beta) < 1, \tag{3.10}$$

then the Type-II BSVIE (1.4) *has a unique adapted M-solution* (Y,Z,U,M) *in* $\mathfrak{S}^2_\beta(0,T)$.

Condition (3.10) is much stronger than assumption (3.5). Indeed for large values of β, (3.10) holds if

$$e\mathfrak{f} < \frac{1}{3(51+\sqrt{2603})} \approx 0,0033,$$

whereas $e\mathfrak{f} < 3(\sqrt{11}-3)/18 \approx 0,052$ is sufficient for (3.5) (see Rem. (3.4)). In other words A may be discontinuous, but with very small jumps.

[5]If not, we can replace α by $\sup_{u\in[0,s]} \alpha_u$. This increases the size of A and requires more integrability in assumptions **(H1)** and **(H4)**.

To give an idea of the value of β, let us assume that $\mathfrak{f} = 0$, that is A is continuous. Then $\delta = \delta^*(\beta) = \dfrac{\sqrt{11}}{3+\sqrt{11}}\beta$. The condition (2.4) becomes

$$\frac{9}{\delta} + \frac{4(2+9\delta)}{\delta^2} < \frac{1}{2}.$$

And

$$M^0(\beta) = \frac{(\sqrt{11}+3)^2}{\beta} \approx \frac{40}{\beta}, \quad \widetilde{\Sigma}^0(\beta) = \frac{2(\sqrt{11}+3)^3}{3\beta(\beta - 2(\sqrt{11}+3)^2)}.$$

Some tedious computations show that $\beta > 174$ is sufficient for our condition (3.5). Nonetheless

$$\left(16 + \frac{1}{\beta-\delta}\right)\Sigma^0(\beta) = \left(16 + \frac{\sqrt{11}+3}{3\beta}\right)\frac{2(\sqrt{11}+3)^2}{\beta - 2(\sqrt{11}+3)^2}.$$

This leads to $\beta > 1357$ in order to satisfy (3.10).

3.3. In the Itô setting

In this framework (see Sect. 2.2) the Type-II BSVIE (1.4) becomes BSVIE (1.7):

$$\begin{aligned} Y(t) = \Phi(t) &+ \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t),U(t,s),U(s,t))\,\mathrm{d}s \\ &- \int_t^T Z(t,s)\,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s), \end{aligned}$$

and the Type-I BSVIE (1.5) becomes BSVIE (1.6).

3.3.1. L^p-solution for Type-I BSVIE (1.6)

In the Brownian-Poisson L^2-setting, BSVIE (1.6) was already studied in [44] and if there is only the Brownian component W, for $p \neq 2$, in [41]. These results are extended as follows:

Theorem 3.10. *For $p > 1$, assume that* **(H2)** *and* **(H3*)** *hold. Suppose that*

$$\mathbb{E}\int_0^T |\Phi(t)|^p\,\mathrm{d}t + \mathbb{E}\int_0^T \left(\int_t^T |f^0(t,s)|\,\mathrm{d}s\right)^p \mathrm{d}t < +\infty. \tag{3.11}$$

Then BSVIE (1.6) has a unique adapted M-solution (Y,Z,U,M) such that for any $S \in [0,T]$

$$\begin{aligned} \|(Y,Z,U,M)\|^p_{\mathfrak{S}^p(S,T)} &= \mathbb{E}\left[\int_S^T |Y(t)|^p\,\mathrm{d}t + \int_S^T \left(\int_S^T |Z(t,r)|^2\,\mathrm{d}r\right)^{p/2} \mathrm{d}t \right. \\ &\qquad \left. + \int_S^T \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)}\right)^{p/2} \mathrm{d}t + \int_S^T \left(\langle M(t,\cdot)\rangle_{S,T}\right)^{p/2} \mathrm{d}t\right] \\ &\leq C\mathbb{E}\left[\int_S^T |\Phi(t)|^p\,\mathrm{d}t + \int_S^T \left(\int_t^T |f^0(t,r)|dr\right)^p \mathrm{d}t\right]. \end{aligned} \tag{3.12}$$

Note that

$$\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)} = \int_S^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x).$$

Let us emphasize that no regularity of the paths $t \mapsto Y(t)$ is required; they are *a priori* neither continuous nor càdlàg. Component Y is only supposed to be in $\mathbb{L}^p_{\mathbb{F}}(0,T)$ (or $\mathbb{L}^2_{\beta,\mathbb{F}}(0,T)$ for BSVIE (1.5)). If Y solves BSDEs (1.2) or (2.5), then it has the same regularity as the martingale part (if process B is continuous), thus a.s. it is a càdlàg process. For a BSVIE it is more delicate. In Theorem 4.2 of [46], the author shows that in the Brownian setting the BSVIE

$$Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \tag{3.13}$$

is continuous in $\mathbb{L}^2(\Omega)$, which does not mean that Y has a.s. continuous paths. Of course since t appears in generator f and in the free term Φ, some property on $t \mapsto \Phi(t)$ and $t \mapsto f(t,\ldots)$ has to be added. To obtain the time regularity for BSVIE (3.13), the author uses the Malliavin derivative to control the term $Z(s,t)$ in the generator (see [46], Thm. 4.1 and 4.2). Hence to apply the same arguments, Malliavin calculus in the presence of jumps (see *e.g.* [7, 12]) should be used. This point is left as future research and to avoid this technical machinery, let us study BSVIE (1.6).

Integrability condition (3.11) is replaced by the stronger one:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(\int_t^T |f^0(t,s)|\,\mathrm{d}s\right)^p\right] < +\infty. \tag{3.14}$$

Finally instead of $\Phi \in \mathbb{L}^p_{\mathcal{F}_T}(0,T)$, we also assume that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|\Phi(t)|^p\right] < +\infty. \tag{3.15}$$

Under these two hypotheses and if f satisfies **(H2)** and **(H3*)**, it is possible to deduce that Y is càdlàg from $[0,T]$ to $\mathbb{L}^p_{\mathbb{F}}(\Omega)$, providing some regularity assumption on $t \mapsto \Phi(t)$ and $t \mapsto f(t,s,y,z,\psi)$ holds, as in Theorem 4.2 of [46] in the continuous setting (see Sect. A.3 in the appendix). Let us emphasize again that it does not mean that a.s. the paths of Y are càdlàg.

Now the next regularity result is presented, which is the extension of Theorem 2.4 in [44]. The restriction $p \geq 2$ is due to the dependence of the generator f on U. If it doesn't, the arguments of the proof lead to the same conclusion for $p > 1$. Following [24] it should be possible to extend the theorem for $p > 1$, but this point is left for further research.

Theorem 3.11. *In addition to* **(H2)** *and* **(H3*)***, suppose that the generator satisfies for some* $p \geq 2$ *and* $0 < \alpha < 1$ *such that* $\alpha p > 1$ *and with* $\varrho > 0$*, uniformly in* (ω,s,y,z,ψ)*:*

$$|f(t,s,y,z,\psi) - f(t',s,y,z,\psi)| \leq \varrho|t-t'|^\alpha, \quad \mathbb{E}\left[|\Phi(t)-\Phi(t')|^p\right] \leq \varrho|t-t'|^{\alpha p},$$

for all (y,z,ψ) *and all* $0 \leq t,t' \leq s \leq T$*. Moreover*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_t^T |f^0(t,s)|^p\,\mathrm{d}s\right)\right] < +\infty.$$

Then the solution Y of BSVIE (1.6) *has a càdlàg version, still denoted by Y, such that*

$$\mathbb{E}\left[\sup_{s\in[0,T]} |Y(s)|^p\right] < +\infty.$$

Let us emphasize that our conditions imply that a.s. $t \mapsto \Phi(t)$ is continuous. The irregularity α of f should be compensated by more integrability p for the data. If α is close or equal to 1, the condition $p \geq 2$ is too strong but our arguments are not sufficient in the proof.

3.3.2. L^p-solution for Type-II BSVIEs (1.7) *?*

For Type-II BSVIEs (1.7) we are not able to provide a similar result as Theorem 3.10. In its proof, the generated troubles for $p \neq 2$ are pointed out. But let us detail this issue already.

The reason can be understood just by considering the term Z. Since Z is integrated w.r.t. the Brownian motion W in the BSVIE, the natural norm for Z is: $Z \in \mathbb{L}^p(0,T;\mathbb{H}^p(0,T))$, that is

$$\mathbb{E}\int_0^T \left[\int_0^T |Z(t,s)|^2 \,\mathrm{d}s\right]^{\frac{p}{2}} \mathrm{d}t < +\infty.$$

But it is symmetric w.r.t. (t,s) only for $p = 2$. The two time variables t and s do not play the same role and the integrability property is not the same w.r.t. t or w.r.t. s, except if $p = 2$. Thereby in BSVIE (1.4), we can use both $Z(t,s)$ and $Z(s,t)$ if $p = 2$ (Prop. 3.12). Let us also mention that in the case where the generator depends on the stochastic integrand w.r.t. a Poisson random measure, the case when $p < 2$ has to be handled carefully. Indeed in this case, Burkholder-Davis-Gundy inequality with $p/2 < 1$ does not apply and the $L^{p/2}$-norm of the predictable projection cannot be controlled by the $L^{p/2}$-norm of the quadratic variation (see [25] and the discussion in [24]). The extension to $p \neq 2$ seems difficult to prove and is left for further research. We also stress that $p \geq 2$ implies that $\mathbb{L}^p(S,T;\mathbb{H}^p(R,T)) \subset \mathbb{L}^2(S,T;\mathbb{H}^2(R,T))$ and $\mathbb{L}^p(S,T;\mathbb{L}^p_\pi(R,T)) \subset \mathbb{L}^2(S,T;\mathbb{L}^2_\pi(R,T))$:

$$\mathbb{E}\int_S^T \left(\int_R^T |Z(t,r)|^2 \,\mathrm{d}r\right) \mathrm{d}t \leq C \left(\mathbb{E}\int_S^T \left[\left(\int_R^T |Z(t,r)|^2 \,\mathrm{d}r\right)^{\frac{p}{2}}\right] \mathrm{d}t\right)^{\frac{2}{p}}$$

and

$$\mathbb{E}\int_S^T \|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(R,T)} \,\mathrm{d}t \leq C\left[\mathbb{E}\int_S^T \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(R,T)}\right)^{\frac{p}{2}} \mathrm{d}t\right]^{\frac{2}{p}}.$$

For $1 < p < 2$, this property fails.

However for $p = 2$, coming back to the BSVIE (1.7), $B_t = t$ and if **(H3*)** holds, then $1 \leq e^{\beta A_t} \leq e^{\beta K T}$. Thereby if $\Phi \in \mathbb{L}^2_{\mathcal{F}_T}(0,T)$ and if

$$\mathbb{E}\int_0^T \left(\int_t^T |f^0(t,s)| \,\mathrm{d}s\right)^2 \mathrm{d}t < +\infty, \tag{3.16}$$

then **(H1)**, **(H4)** (and **(H5)**) hold. The next result is a corollary of Theorem 3.9.

Proposition 3.12. *Assume that $\Phi \in \mathbb{L}^2_{\mathcal{F}_T}(0,T)$, that* **(H2)**, **(H3*)** *and* (3.16) *hold*[6]. *Then BSVIE* (1.7) *has a unique adapted M-solution (Y,Z,U,M) in $\mathfrak{S}^2(0,T)$ on $[0,T]$. Moreover for any $S \in [0,T]$*

$$\begin{aligned} \|(Y,Z,U,M)\|^2_{\mathfrak{S}^2(S,T)} &= \mathbb{E}\left[\int_S^T |Y(t)|^2 \,\mathrm{d}t + \int_S^T \left(\int_S^T |Z(t,r)|^2 \,\mathrm{d}r\right) \mathrm{d}t \right. \\ &\qquad \left. + \int_S^T \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)}\right) \mathrm{d}t + \int_S^T \left(\langle M(t,\cdot)\rangle_{S,T}\right) \mathrm{d}t\right] \\ &\leq C\mathbb{E}\left[\int_S^T |\Phi(t)|^2 \,\mathrm{d}t + \int_S^T \left(\int_t^T |f^0(t,r)| \,\mathrm{d}r\right)^2 \mathrm{d}t\right]. \end{aligned} \tag{3.17}$$

Note that from the Bichteler-Jacod inequality (see inequality (5.2) below), concerning U, Estimate (3.17) is completely equivalent to: for any $S \in [0,T]$

$$\mathbb{E}\left[\int_S^T \left(\int_S^T \|U(t,r)\|^2_{\mathbb{L}^2_\mu} \,\mathrm{d}r\right) \mathrm{d}t\right] \leq C\mathbb{E}\left[\int_S^T |\Phi(t)|^2 \mathrm{d}t + \int_S^T \left(\int_t^T |f^0(t,r)| \,\mathrm{d}r\right)^2 \mathrm{d}t\right].$$

4. Existence and uniqueness for BSVIEs (1.4) and (1.5)

The aim of this section is to prove Theorems 3.3 and 3.9.

4.1. Preliminary results

First we consider for any R and S in $[0,T)$ a driver $h : \Omega \times [S,T] \times [R,T] \times \mathbb{R}^{d\times m} \times \mathfrak{H} \to \mathbb{R}^d$ such that **(H2)** holds and:

$$\mathbb{E}\left[\int_S^T e^{(\beta-\delta)A_t} \left(\int_R^T e^{\delta A_s} \frac{|h(t,s,0,\mathbf{0})|^2}{\alpha_s^2} \,\mathrm{d}B_s\right) \mathrm{d}B_t\right] < +\infty. \tag{4.1}$$

Let us recall that the constant $\delta = \delta^*(\beta)$ is defined in Lemma 3.2. For ease of notation, $h^0(t,s) = h(t,s,0,\mathbf{0})$. As for **(H5)**, it is assumed that

$$\left\{t \in [S,T] : \mathbb{E}\left[e^{\delta A_T}|\Phi(t)|^2 + \int_R^T e^{\delta A_s} \frac{|h(t,s,0,\mathbf{0})|^2}{\alpha_s^2} \,\mathrm{d}B_s\right] < +\infty\right\} = [S,T]. \tag{4.2}$$

Then let us introduce the BSDE on $[R,T]$ parameterized by $t \in [S,T]$: for $r \in [R,T]$

$$\begin{aligned} \lambda(t,r) = \Phi(t) &+ \int_r^T h(t,s,z(t,s),u(t,s)) \,\mathrm{d}B_s - \int_r^T z(t,s) \,\mathrm{d}X^\circ_s \\ &- \int_r^T \int_{\mathbb{R}^m} u(t,s,x) \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_r^T \mathrm{d}m(t,s). \end{aligned} \tag{4.3}$$

From Theorem 2.6, if (2.4) holds, that is $\kappa^{\mathfrak{f}}(\delta) < 1/2$, the previous BSDE has a unique solution

$$(\lambda(t,\cdot), z(t,\cdot), u(t,\cdot), m(t,\cdot)).$$

[6] The space $\mathfrak{H}$ in **(H2)** is replaced by $\mathbb{L}^2_\mu$ in this case.

Let us fix $R = S$ and define $Y(t) = \lambda(t,t)$, $t \in [S,T]$, $Z(t,s) = z(t,s)$, $U(t,s,e) = u(t,s,e)$, $M(t,s) = m(t,s)$ for $(t,s) \in \Delta^c[S,T]$. Equation (4.3) becomes

$$Y(t) = \Phi(t) + \int_t^T h(t,s,Z(t,s),U(t,s))\,\mathrm{d}B_s - \int_t^T Z(t,s)\,\mathrm{d}X^\circ_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s), \tag{4.4}$$

which is a special case of (1.5) where f does not depend on y. Using (3.1), this BSVIE can be rewritten for any $t \in [S,T]$

$$Y(t) = \Phi(t) + \int_t^T h(t,s,Z(s,t),U(t,s))\,\mathrm{d}B_s - \int_t^T \mathrm{d}\mathrm{M}^\sharp(t,s).$$

Let us prove the next result.

Lemma 4.1. *Assume that* **(H1)** *holds for* Φ*, that* h *satisfies* **(H2)** *and* **(H3)***, together with conditions* (4.1) *and* (4.2)*. If* (2.4) *holds and if the constant* $M^{\mathfrak{f}}(\beta)$ *defined in Lemma 3.2 verifies:* $M^{\mathfrak{f}}(\beta) < 1/2$*, then the solution* (Y,Z,U,M) *of BSVIE* (4.4) *admits the upper bound:*

$$\mathbb{E}\left[\int_S^T e^{\beta A_t}|Y(t)|^2\,\mathrm{d}A_t + \int_S^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right] \\ \leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_S^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right] \\ +\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_S^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|h^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right], \tag{4.5}$$

where $\Sigma^{\mathfrak{f}}(\beta)$ *is defined by* (3.3).

Proof. Note that for any $t \in [0,T]$

$$Y(t) = \Phi(t) + H(t) - \int_t^T \mathrm{d}\mathrm{M}^\sharp(t,s),$$

where for $t \in [S,T]$, $H(t) = \int_t^T h(t,s,Z(t,s),U(t,s))\,\mathrm{d}B_s$. Therefore

$$\mathbb{E}\left[\int_t^T \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\middle|\mathcal{F}_t\right] = \mathbb{E}\left[|\Phi(t) - Y(t) - H(t)|^2\middle|\mathcal{F}_t\right]$$

For any $\delta > 0$, by the Cauchy-Schwarz inequality and using Lemma B.1 of [31]:

$$|H(t)|^2 \leq \int_t^T e^{-\delta A_s}\,\mathrm{d}A_s \int_t^T e^{\delta A_s}\frac{|h(t,s,Z(t,s),U(t,s))|^2}{\alpha_s^2}\,\mathrm{d}B_s \\ \leq \frac{1}{\delta}e^{-\delta A_t}\int_t^T e^{\delta A_s}\frac{|h(t,s,Z(t,s),U(t,s))|^2}{\alpha_s^2}\,\mathrm{d}B_s.$$

Arguing as in the proof of Lemma 3.8 in [31], we derive that for any $\gamma > 0$ and $\delta > 0$

$$\int_S^T e^{\gamma A_t} |H(t)|^2 \, \mathrm{d}A_t \leq \int_S^T \frac{1}{\delta} e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h(t,s,Z(t,s),U(t,s))|^2}{\alpha_s^2} \, \mathrm{d}B_s \, \mathrm{d}A_t.$$

From our assumption **(H2)** on h, we deduce:

$$|h(t,r,Z(t,r),U(t,r))|^2 \leq 2|h^0(t,r)|^2 + 2\theta_r^\circ |c_r Z(t,r)|^2 + 2\theta_r^\natural \|\!|U(t,r)|\!\|_r^2,$$

thus from the definition of α (hypothesis **(H3)**)

$$\begin{aligned} e^{\delta A_r} \frac{|h(t,r,Z(t,r),U(t,r))|^2}{\alpha_r^2} &\leq 2 e^{\delta A_r} \frac{|h^0(t,r)|^2}{\alpha_r^2} + 2 e^{\delta A_r} \frac{\theta_r^\circ}{\alpha_r^2} |c_r Z(t,r)|^2 \\ &+ 2 e^{\delta A_r} \frac{\theta_r^\natural}{\alpha_r^2} \|\!|U(t,r)|\!\|_r^2 \\ &\leq 2 e^{\delta A_r} \frac{|h^0(t,r)|^2}{\alpha_r^2} + 2 e^{\delta A_r} |c_r Z(t,r)|^2 + 2 e^{\delta A_r} \|\!|U(t,r)|\!\|_r^2. \end{aligned}$$

Thereby for any $t \in [S,T]$

$$\begin{aligned} \int_t^T e^{\delta A_s} \frac{|h(t,s,Z(t,s),U(t,s))|^2}{\alpha_s^2} \, \mathrm{d}B_s &\leq 2 \int_t^T e^{\delta A_r} \frac{|h^0(t,r)|^2}{\alpha_r^2} \, \mathrm{d}B_s \\ &+ 2 \int_t^T e^{\delta A_r} |c_r Z(t,r)|^2 \, \mathrm{d}B_r + 2 \int_t^T e^{\delta A_r} \|\!|U(t,r)|\!\|_r^2 \, \mathrm{d}B_r. \end{aligned}$$

Hence we obtain that

$$\begin{aligned} \mathbb{E}\left[\int_S^T e^{\gamma A_t} |H(t)|^2 \, \mathrm{d}A_t\right] &\leq \mathbb{E}\left[\int_S^T \frac{1}{\delta} e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h(t,s,Z(t,s),U(t,s))|^2}{\alpha_s^2} \, \mathrm{d}B_s \, \mathrm{d}A_t\right] \\ &\leq 2\mathbb{E}\left[\int_S^T \frac{1}{\delta} e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h^0(t,s)|^2}{\alpha_s^2} \, \mathrm{d}B_s \, \mathrm{d}A_t\right] \\ &+ 2\mathbb{E}\left[\int_S^T \frac{1}{\delta} e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \, \mathrm{d}\,\mathrm{Trace}[\langle Z(t,\cdot).X^\circ\rangle_s] \, \mathrm{d}A_t\right] \\ &+ 2\mathbb{E}\left[\int_S^T \frac{1}{\delta} e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \, \mathrm{d}\,\mathrm{Trace}[\langle U(t,\cdot) \star \widetilde{\pi}^\natural\rangle_s] \, \mathrm{d}A_t\right]. \end{aligned}$$

This leads to the next estimate on H:

$$\begin{aligned} \mathbb{E}\left[\int_S^T e^{\gamma A_t} |H(t)|^2 \, \mathrm{d}A_t\right] &\leq \frac{2}{\delta} \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h^0(t,s)|^2}{\alpha_s^2} \, \mathrm{d}B_s \, \mathrm{d}A_t\right] \\ &+ \frac{2}{\delta} \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t} \int_t^T e^{\delta A_s} \, \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s \, \mathrm{d}A_t\right]. \end{aligned} \tag{4.6}$$

Now remark that

$$Y(t) = \mathbb{E}\left[\Phi(t) + \int_t^T h(t,s,Z(t,s),U(t,s))\,\mathrm{d}B_s \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\Phi(t) + H(t) \middle| \mathcal{F}_t\right]. \tag{4.7}$$

Thus for $\gamma > 0$ and $\delta > 0$

$$\begin{aligned}\mathbb{E}\left[\int_S^T e^{\gamma A_t}|Y(t)|^2\,\mathrm{d}A_t\right] &\leq 2\mathbb{E}\left[\int_S^T \mathbb{E}\left[e^{\gamma A_t}|\Phi(t)|^2 + e^{\gamma A_t}|H(t)|^2 \middle| \mathcal{F}_t\right]\,\mathrm{d}A_t\right] \\ &= 2\mathbb{E}\left[\int_S^T \left(e^{\gamma A_t}|\Phi(t)|^2 + e^{\gamma A_t}|H(t)|^2\right)\,\mathrm{d}A_t\right]. \end{aligned} \tag{4.8}$$

Following the proof of Lemma 3.8 in [31], we get

$$\begin{aligned}&\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s \\ &\leq \delta \int_t^T e^{\delta A_u}\int_u^T \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_u + e^{\delta A_t}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_T - \mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_t\right). \end{aligned} \tag{4.9}$$

Then

$$\begin{aligned}&\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}\int_u^T \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\ &= \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}\mathbb{E}\left[\int_u^T \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s \middle| \mathcal{F}_u\right]\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\ &= \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}\mathbb{E}\left[|\Phi(u) - Y(u) - H(u)|^2 \middle| \mathcal{F}_u\right]\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\ &\leq 3\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}\mathbb{E}\left[|\Phi(u)|^2 + |Y(u)|^2 + |H(u)|^2 \middle| \mathcal{F}_u\right]\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\ &\leq 9\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}|\Phi(u)|^2\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\ &\quad +9\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}|H(u)|^2\,\mathrm{d}A_u\,\mathrm{d}A_t\right]. \end{aligned}$$

Note that we used many times that A_t is $\mathcal{F}_u$-measurable and Corollary D.1 in [31], together with (4.7) for the last inequality. For $\gamma > \delta$,

$$\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}|\Phi(u)|^2\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \leq \frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\mathbb{E}\left[\int_S^T e^{\gamma A_u}|\Phi(u)|^2\,\mathrm{d}A_u\right].$$

The same holds for $H(u)$, thus

$$\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_u}\int_u^T \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_u\,\mathrm{d}A_t\right]$$
$$\leq 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\mathbb{E}\left[\int_S^T e^{\gamma A_u}|\Phi(u)|^2\,\mathrm{d}A_u\right]+9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\mathbb{E}\left[\int_S^T e^{\gamma A_u}|H(u)|^2\,\mathrm{d}A_u\right].$$

For the second term in (4.9)

$$\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}e^{\delta A_t}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_T-\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_t\right)\,\mathrm{d}A_t\right]$$
$$\leq \mathbb{E}\left[\int_S^T e^{\gamma A_t}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_T-\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_t\right)\,\mathrm{d}A_t\right]$$
$$\leq 9\mathbb{E}\left[\int_S^T e^{\gamma A_t}\left(|\Phi(t)|^2+|H(t)|^2\right)\,\mathrm{d}A_t\right].$$

Therefore for $\delta<\gamma\leq\beta$

$$\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq \left(9+9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}\left(|\Phi(t)|^2+|H(t)|^2\right)\,\mathrm{d}A_t\right]. \tag{4.10}$$

Combining (4.6), (4.8) and the previous estimate, we deduce for any $\delta<\gamma\leq\beta$:

$$\mathbb{E}\left[\int_S^T e^{\gamma A_t}|Y(t)|^2\,\mathrm{d}A_t\right]+\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq\left(2+9+9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right]$$
$$+\left(2+9+9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|H(t)|^2\,\mathrm{d}A_t\right]$$
$$\leq\left(11+9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right]$$
$$+2\left(\frac{11}{\delta}+9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|h^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]$$
$$+2\left(\frac{11}{\delta}+9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right].$$

From Lemma 3.2, the values $\gamma = \beta$ and $\delta = \delta^*(\beta)$ lead to the infimum $M^{\mathfrak{f}}(\beta)$ of

$$\Pi^{\mathfrak{f}}(\gamma,\delta) = \frac{11}{\delta} + 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}.$$

If $\mathfrak{f}$ and β are such that $M^{\mathfrak{f}}(\beta) < 1/2$, then the conclusion of the lemma follows. □

Moreover a stability result for BSVIE (4.4) holds.

Lemma 4.2. *For $(\bar\Phi, \bar h)$ satisfying the above assumptions* **(H1)**, **(H2)** *and* **(H3)**, *let $(\bar Y, \bar Z, \bar U, \bar M)$ be a solution of BSVIE (4.4) with data $(\bar\Phi, \bar h)$. Define*

$$(\mathfrak{d}Y, \mathfrak{d}Z, \mathfrak{d}U, \mathfrak{d}M) = (Y-\bar Y, Z-\bar Z, U-\bar U, M-\bar M).$$

Under the conditions of Lemma 4.1 on $\mathfrak{f}$ and β, with the same $\delta \in (0,\beta)$, the following stability result holds:

$$\begin{aligned}
&\mathbb{E}\left[\int_S^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2\,\mathrm{d}A_t + \int_S^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]\\
&\le \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_S^T e^{\beta A_t}|\mathfrak{d}\Phi(t)|^2\,\mathrm{d}A_t\right]\\
&+\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_S^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}h(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right] \qquad (4.11)
\end{aligned}$$

where

$$\mathfrak{d}h(t,r) = h(t,r,Z(t,r),U(t,r)) - \bar h(t,r,Z(t,r),U(t,r)).$$

Proof. We outline the proof of Proposition 3.13 in [31]. Note that

$$\begin{aligned}
\mathfrak{d}Y(t) &= \mathfrak{d}\Phi(t) + \int_t^T \left(h(t,r,Z(t,r),U(t,r)) - \bar h(t,r,\bar Z(t,r),\bar U(t,r))\right)\mathrm{d}B_r - \int_t^T \mathrm{d}\mathfrak{d}\mathrm{M}^\sharp(t,r)\\
&= \mathfrak{d}\Phi(t) + \mathfrak{d}H(t) - \int_t^T \mathrm{d}\mathfrak{d}\mathrm{M}^\sharp(t,r).
\end{aligned}$$

If

$$\mathfrak{d}h(t,r) = h(t,r,Z(t,r),U(t,r)) - \bar h(t,r,Z(t,r),U(t,r)),$$

and from the assumptions on $\bar h$ we obtain:

$$\begin{aligned}
&\frac{1}{\alpha_r^2}|h(t,r,Z(t,r),U(t,r)) - \bar h(t,r,\bar Z(t,r),\bar U(t,r))|^2\\
&\le 2\frac{\theta_r^\circ}{\alpha_r^2}|c_r\mathfrak{d}Z(t,r)|^2 + 2\frac{\theta_r^\natural}{\alpha_r^2}|||\mathfrak{d}U(t,r)|||_r^2 + 2\frac{|\mathfrak{d}h(t,r)|^2}{\alpha_r^2}\\
&\le 2|c_r\mathfrak{d}Z(t,r)|^2 + 2|||\mathfrak{d}U(t,r)|||_r^2 + 2\frac{|\mathfrak{d}h(t,r)|^2}{\alpha_r^2}.
\end{aligned}$$

As for the previous lemma, we deduce:

$$\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\mathfrak{d}H(t)|^2\,\mathrm{d}A_t\right] \leq \frac{2}{\delta}\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}h(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right] + \frac{2}{\delta}\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right].$$

Thereby for any $\delta < \gamma \leq \beta$:

$$\begin{aligned}
&\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\mathfrak{d}Y(t)|^2\,\mathrm{d}A_t\right] + \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right] \\
&\leq \left(11 + 9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\mathfrak{d}\Phi(t)|^2\,\mathrm{d}A_t\right] \\
&\quad + \left(11 + 9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\mathfrak{d}H(t)|^2\,\mathrm{d}A_t\right] \\
&\leq \left(11 + 9\delta\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{\gamma A_t}|\mathfrak{d}\Phi(t)|^2\,\mathrm{d}A_t\right] \\
&\quad + 2\left(\frac{11}{\delta} + 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}h(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right] \\
&\quad + 2\left(\frac{11}{\delta} + 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}\right)\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right].
\end{aligned}$$

The conclusion of the lemma follows as for the previous lemma. □

Note that this stability result leads to the uniqueness of the solution of BSVIE (4.4) in the space $\mathfrak{S}^2_{\delta^*(\beta)\leq\beta}(\Delta^c(0,T))$. The following result can be now stated:

Proposition 4.3. *Under conditions* **(H1)** *to* **(H5)**, *if driver f does not depend on y, and if constants $\kappa^{\mathfrak{f}}(\delta)$ and $M^{\mathfrak{f}}(\beta)$ defined by* (2.4) *and in Lemma 3.2 verify*

$$\kappa^{\mathfrak{f}}(\delta) < \frac{1}{2}, \qquad M^{\mathfrak{f}}(\beta) < \frac{1}{2}, \tag{4.12}$$

then BSVIE (1.5) *has a unique adapted solution (Y,Z,U,M) in $\mathfrak{S}^2_\beta(\Delta^c(0,T))$.*

4.2. Proof of Theorem 3.3

Since we only consider a Type-I BSVIE, our arguments are close to those used in [45]. Fix $(y,\zeta,\nu,m) \in \mathfrak{S}^2_\beta(\Delta^c(0,T))$ and consider the BSVIE on $[S,T]$

$$\begin{aligned}
Y(t) = \Phi(t) &+ \int_t^T f(t,s,y(s),Z(t,s),U(t,s))\,\mathrm{d}B_s - \int_t^T Z(t,s)\,\mathrm{d}X_s^\circ \\
&- \int_t^T\int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s),
\end{aligned}\tag{4.13}$$

In order to apply Lemma 4.1, if

$$h(t,s,z,\psi) = f(t,s,y(s),z,\psi),$$

we need to check that

$$\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h(t,s,0,\mathbf{0})|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right] < +\infty.$$

The Lipschitz property **(H2)** of f leads to

$$|h(t,s,0,\mathbf{0})|^2 \leq 2|f^0(t,s)|^2 + 2\varpi_s|y(s)|^2.$$

Using **(H3)**

$$\begin{aligned} e^{\delta A_r}\frac{|h(t,r,0,\mathbf{0})|^2}{\alpha_r^2} &\leq 2e^{\delta A_r}\frac{|f^0(t,r)|^2}{\alpha_r^2} + 2e^{\delta A_r}\frac{\varpi_r}{\alpha_r^2}|y(r)|^2 \\ &\leq 2e^{\delta A_r}\frac{|f^0(t,r)|^2}{\alpha_r^2} + 2e^{\delta A_r}\alpha_r^2|y(r)|^2. \end{aligned}$$

Hence for a.e. $t \in [S,T]$

$$\int_t^T e^{\delta A_r}\frac{|h(t,r,0,\mathbf{0})|^2}{\alpha_r^2}\,\mathrm{d}B_r \leq 2\int_t^T e^{\delta A_r}\frac{|f^0(t,r)|^2}{\alpha_r^2}\,\mathrm{d}B_r + 2\int_t^T e^{\delta A_r}\alpha_r^2|y(r)|^2\,\mathrm{d}B_r.$$

Note that from **(H5)**, the assumption (4.2) holds. Thus BSVIE (4.13) has a unique adapted solution $(Y,Z,U,M) \in \mathfrak{S}^2_\beta(\Delta^c(0,T))$ and from (4.5)

$$\begin{aligned} &\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right] \\ &\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right] \\ &\quad + \Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|h^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]. \end{aligned}$$

Using our estimate on h, we deduce that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right] \\ &\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right] + 2\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|f^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right] \\ &\quad + 2\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}|y(s)|^2\,\mathrm{d}A_s\,\mathrm{d}A_t\right] \end{aligned}$$

$$\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right] + 2\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|f^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]$$
$$+2\Sigma^{\mathfrak{f}}(\beta)\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\mathbb{E}\left[\int_0^T e^{\beta A_s}|y(s)|^2\,\mathrm{d}A_s\right].$$

In other words a map Θ from $\mathfrak{S}^2_\beta(\Delta^c(0,T))$ to $\mathfrak{S}^2_\beta(\Delta^c(0,T))$ is defined by $\Theta(y,\zeta,\nu,m) = (Y,Z,U,M)$.

Now consider (y,ζ,ν,m) and $(\bar{y},\bar{\zeta},\bar{\nu},\bar{m})$ in $\mathfrak{S}^2_\beta(\Delta^c(0,T))$ together with (Y,Z,U,M) and $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ as the solutions of BSVIE (4.13) and

$$(\mathfrak{d}Y,\mathfrak{d}Z,\mathfrak{d}U,\mathfrak{d}M) = (Y-\bar{Y}, Z-\bar{Z}, U-\bar{U}, M-\bar{M}).$$

Then Lemma 4.2 implies that:

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq \Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}h(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right],$$

where

$$\mathfrak{d}h(t,r) = h(t,r,y(r),Z(t,r),U(t,r)) - h(t,r,\bar{y}(r),Z(t,r),U(t,r)).$$

Again **(H2)** leads to:

$$|\mathfrak{d}h(t,r)|^2 \leq \varpi_r(y(r)-\bar{y}(r))^2.$$

Thereby

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq \Sigma^{\mathfrak{f}}(\beta)\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\mathbb{E}\left[\int_0^T e^{\beta A_s}(y(s)-\bar{y}(s))^2\,\mathrm{d}A_s\right].$$

Since it is supposed that

$$\widetilde{\Sigma}^{\mathfrak{f}}(\beta) = \Sigma^{\mathfrak{f}}(\beta)\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta} < 1,$$

this map Θ is a contraction and thus it admits a unique fixed point $(Y,Z,U,M) \in \mathfrak{S}^2_\beta(\Delta^c(0,T))$ which is the unique adapted solution of (1.5) on $[0,T]$.

To prove the upper bound (3.6), apply Lemma 4.1 to the driver $h(t,s,z,u) = f(t,s,Y(s),z,u)$. Note that with **(H2)**:

$$|h(t,s,0,\mathbf{0})|^2 \leq \left(1+\frac{1}{\varepsilon}\right)|f^0(t,s)|^2 + (1+\varepsilon)\varpi_s|Y(s)|^2$$

for any $\varepsilon > 0$. Thus

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y(t)|^2 \,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] \\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2 \,\mathrm{d}A_t\right] \\
&\quad + \Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|h^0(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right] \\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2 \,\mathrm{d}A_t\right] \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|f^0(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right] \\
&\quad + (1+\varepsilon)\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s}|Y(s)|^2 \,\mathrm{d}A_s \,\mathrm{d}A_t\right] \\
&= \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2 \,\mathrm{d}A_t\right] + (1+\varepsilon)\widetilde{\Sigma}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\delta A_s}|Y(s)|^2 \,\mathrm{d}A_s\right] \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|f^0(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right].
\end{aligned}$$

Since $\Sigma^{\mathfrak{f}}(\beta)\dfrac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta} < 1$, choosing ε sufficiently small yields to the desired result.

To finish the proof, the stability estimate should be proved. Using Lemma 4.2 again with

$$h(t,s,z,u) = f(t,s,Y(s),z,u), \quad \bar{h}(t,s,z,u) = \bar{f}(t,s,\bar{Y}(s),z,u),$$

yields to:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2 \,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle \mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] \\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}\Phi(t)|^2 \,\mathrm{d}A_t\right] \\
&\quad + \Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|\mathfrak{d}h(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right]
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{d}h(t,r) &= h(t,r,Z(t,r),U(t,r)) - \bar{h}(t,r,Z(t,r),U(t,r)) \\
&= f(t,r,Y(r),Z(t,r),U(t,r)) - \bar{f}(t,r,Y(r),Z(t,r),U(t,r)) \\
&\quad + \bar{f}(t,r,Y(r),Z(t,r),U(t,r)) - \bar{f}(t,r,\bar{Y}(r),Z(t,r),U(t,r)) \\
&= \mathfrak{d}f(t,r) + \bar{f}(t,r,Y(r),Z(t,r),U(t,r)) - \bar{f}(t,r,\bar{Y}(r),Z(t,r),U(t,r)).
\end{aligned}$$

Since $\bar f$ satisfies **(H2)**,

$$|\bar f(t,r,Y(r),Z(t,r),U(t,r)) - \bar f(t,r,\bar Y(r),Z(t,r),U(t,r))|^2 \leq \varpi_r |\mathfrak{d}Y(r)|^2,$$

we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}Y(t)|^2 \,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathfrak{d}\mathrm{M}^\sharp(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] \\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\mathfrak{d}\Phi(t)|^2\,\mathrm{d}A_t\right] \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\mathfrak{d}f(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right] \\
&\quad + (1+\varepsilon)\widetilde{\Sigma}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_s}|\mathfrak{d}Y(s)|^2\,\mathrm{d}A_s\right].
\end{aligned}$$

The same arguments used previously lead to the conclusion.

4.3. Type-II BSVIE (1.4) for deterministic B

The aim of this section is the study of the Type-II BSVIE (1.4)

$$\begin{aligned}
Y(t) = \Phi(t) &+ \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t),U(t,s),U(s,t))\,\mathrm{d}B_s \\
&- \int_t^T Z(t,s)\,\mathrm{d}X_s^\circ - \int_t^T\int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s)\ .
\end{aligned}$$

Recall that we restrict ourselves to the case where B and α (in assumption **(H2')**) are deterministic and we want to prove Theorem 3.9: under **(H1)**, **(H2')**, **(H3')** and **(H4)** and if constants $\kappa^{\mathfrak{f}}(\delta)$, $M^{\mathfrak{f}}(\beta)$ and $\Sigma^{\mathfrak{f}}(\beta)$ defined by (2.4), in Lemma 3.2 and by (3.3) verify (3.10), then the Type-II BSVIE (1.4) has a unique adapted M-solution (Y,Z,U,M) in $\mathfrak{S}^2_\beta(0,T)$.

From now on, and in the rest of this subsection, A is deterministic (hypothesis **(H3')**). If for some $\gamma\in\mathbb{R}$, $\mathbb{E}\int_S^T e^{\gamma A_s}|Y(s)|^2\alpha_s^2\,\mathrm{d}B_s < +\infty$, then for $\mathrm{d}B$-almost every $t\in[S,T]$, $\mathrm{M}^\sharp$ can be defined by the martingale representation (3.9):

$$Y(t) = \mathbb{E}\left[Y(t)\middle|\mathcal{F}_S\right] + \int_S^t \mathrm{d}\mathrm{M}^\sharp(t,s).$$

Since

$$\begin{aligned}
\int_S^t e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r &\leq \delta\int_S^t e^{\delta A_u}\int_u^t \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r\,\mathrm{d}A_u \\
&\quad + e^{\delta A_S}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_t - \mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_S\right),
\end{aligned}$$

we get

$$\begin{aligned}
&\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_S^t e^{\delta A_u}\int_u^t \mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s \,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\
&\quad = \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_S^t e^{\delta A_u}\mathbb{E}\left[(Y(t)-\mathbb{E}[Y(t)|\mathcal{F}_u])^2|\mathcal{F}_u\right]\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\
&\quad \leq 2\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_S^t e^{\delta A_u} Y(t)^2\,\mathrm{d}A_u\,\mathrm{d}A_t\right] \\
&\quad \leq \frac{2}{\delta}\mathbb{E}\left[\int_S^T e^{\gamma A_t} Y(t)^2\,\mathrm{d}A_t\right].
\end{aligned}$$

Let us emphasize that the first equality only holds because A_t is $\mathcal{F}_u$-measurable even if $t > u$. Similarly since $A_S \leq A_t$

$$\begin{aligned}
&\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}e^{\delta A_S}\left(\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_t - \mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_S\right)\,\mathrm{d}A_t\right] \\
&\quad = \mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}e^{\delta A_S}\mathbb{E}\left[(Y(t)-\mathbb{E}[Y(t)|\mathcal{F}_S])^2|\mathcal{F}_S\right]\,\mathrm{d}A_t\right] \\
&\quad \leq 2\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}e^{\delta A_S} Y(t)^2\,\mathrm{d}A_t\right] \leq 2\mathbb{E}\left[\int_S^T e^{\gamma A_t} Y(t)^2\,\mathrm{d}A_t\right],
\end{aligned}$$

we deduce that

$$\mathbb{E}\left[\int_S^T e^{(\gamma-\delta)A_t}\int_S^t e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_r\,\mathrm{d}A_t\right] \leq 4\mathbb{E}\left[\int_S^T e^{\gamma A_t} Y(t)^2\,\mathrm{d}A_t\right]. \tag{4.14}$$

Let us consider space $\widehat{\mathfrak{S}}^2_{\delta\leq\gamma}(R,T)$, the set of all (y,z,u,m) in $\mathfrak{S}^2_{\delta\leq\gamma}(R,T)$ such that for $\mathrm{d}B$-a.e. $t\in[R,T]$ a.s.

$$\begin{aligned}
\int_R^t y(r)\,\mathrm{d}B_r &= \mathbb{E}\left[\int_R^t y(r)\,\mathrm{d}B_r|\mathcal{F}_R\right] \\
&\qquad + \int_R^t z(t,r)\,\mathrm{d}X^\circ_r + \int_R^t\int_{\mathbb{R}^m} u(t,r,x)\widetilde{\pi}^\natural(\,\mathrm{d}r,\,\mathrm{d}x) + \int_R^t \,\mathrm{d}m(t,r).
\end{aligned}$$

Estimate (4.14) leads to: for any $\delta\in(0,\beta]$

$$\begin{aligned}
&\mathbb{E}\left[\int_R^T e^{(\beta-\delta)A_t}\int_R^t e^{\delta A_r}|b_r z(t,r)|^2\,\mathrm{d}B_r\,\mathrm{d}A_t + \int_R^T e^{(\beta-\delta)A_t}\int_R^t e^{\delta A_r}\|\!|u(t,r)|\!\|_r\,\mathrm{d}B_r\,\mathrm{d}A_t\right. \\
&\qquad \left. + \int_R^T e^{(\beta-\delta)A_t}\int_S^t e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}\langle m(t,\cdot)\rangle_r\,\mathrm{d}A_t\right] \leq 4\mathbb{E}\left[\int_R^T e^{\beta A_t}|y(t)|^2\,\mathrm{d}A_t\right].
\end{aligned}$$

As in [46], the norm of $\mathfrak{S}^2_{\delta\leq\beta}(\Delta^c(0,T))$ is defined on $\widehat{\mathfrak{S}}^2_{\delta\leq\beta}(0,T)$

$$\begin{aligned}\|(y,z,u,m)\|^2_{\widehat{\mathfrak{S}}^2_{\delta\leq\beta}} = \mathbb{E}\Bigg[&\int_0^T e^{\beta A_t}|y(t)|^2\,\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle z(t,\cdot)\cdot X^\circ\rangle_r]\right)\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle u(t,\cdot)\star\widetilde{\pi}^\natural\rangle_r]\right)\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_t^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle m(t,\cdot)\rangle_r]\right)\mathrm{d}A_t\Bigg].\end{aligned}$$

An equivalent norm for $\widehat{\mathfrak{S}}^2_{\delta\leq\beta}(0,T)$ is now proved:

$$\begin{aligned}\|(y,z,u,m)\|^2_{\widehat{\mathfrak{S}}^2_{\delta\leq\beta}} \leq \mathbb{E}\Bigg[&\int_0^T e^{\beta A_t}|y(t)|^2\,\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_0^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle z(t,\cdot)\cdot X^\circ\rangle_r]\right)\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_0^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle u(t,\cdot)\star\widetilde{\pi}^\natural\rangle_r]\right)\mathrm{d}A_t\\&+\int_0^T e^{(\beta-\delta)A_t}\left(\int_0^T e^{\delta A_r}\,\mathrm{d}\,\mathrm{Trace}[\langle m(t,\cdot)\rangle_r]\right)\mathrm{d}A_t\Bigg]\\\leq 5&\|(y,z,u,m)\|^2_{\widehat{\mathfrak{S}}^2_{\delta\leq\beta}}.\end{aligned}\tag{4.15}$$

Fix $\Phi\in\mathbb{L}^2_{\beta,\mathcal{F}_T}(0,T)$ and $(y,\zeta,\nu,m)\in\widehat{\mathfrak{S}}^2_{\delta\leq\beta}(0,T)$ and consider the BSVIE

$$\begin{aligned}Y(t)=\Phi(t)+\int_t^T f(t,s,y(s),Z(t,s),\zeta(s,t),U(t,s),\nu(s,t))\,\mathrm{d}B_s-\int_t^T Z(t,s)\,\mathrm{d}X^\circ_s\\-\int_t^T\int_{\mathbb{R}^m}U(t,s,x)\widetilde{\pi}^\natural(\mathrm{d}s,\mathrm{d}x)-\int_t^T\mathrm{d}M(t,s),\end{aligned}\tag{4.16}$$

This is a particular case of BSVIE (4.4) (and of the Type-I BSVIE (1.5)). To apply Lemma 4.1, if

$$h(t,s,z,\psi)=f(t,s,y(s),z,\zeta(s,t),\psi,\nu(s,t)),$$

condition (4.1) needs to be checked:

$$\mathbb{E}\int_0^T e^{(\beta-\delta)A_t}\left(\int_0^T e^{\delta A_s}\frac{|h(t,s,0,\mathbf{0})|^2}{\alpha_s^2}\,\mathrm{d}B_s\right)\mathrm{d}B_t<+\infty.$$

The Lipschitz property **(H2')** of f leads to

$$|h(t,s,0,\mathbf{0})|^2 \leq 2|f^0(t,s)|^2 + 2\varpi_s|y(s)|^2 + 2\theta^\circ_s|b_t\zeta(s,t)|^2 + 2\theta^\natural_s |||\nu(s,t)|||_t$$

Note that for $s \geq t$, $\zeta(s,t)$ and $\nu(s,t)$ are $\mathcal{F}_t$-measurable. Moreover

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{1}{\alpha_s^2}\theta^\circ_s|b_t\zeta(s,t)|^2\,\mathrm{d}B_s\,\mathrm{d}A_t\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}|b_t\zeta(s,t)|^2\,\mathrm{d}B_s\alpha_t^2\,\mathrm{d}B_t\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\int_0^s e^{(\beta-\delta)A_t}\alpha_t^2|b_t\zeta(s,t)|^2\,\mathrm{d}B_t\,\mathrm{d}B_s\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\alpha_s^2\int_0^s e^{(\beta-\delta)A_t}|b_t\zeta(s,t)|^2\,\mathrm{d}B_t\,\mathrm{d}B_s\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\int_0^s e^{(\beta-\delta)A_t}\,\mathrm{d}\,\mathrm{Trace}[\langle\zeta(s,\cdot)\cdot X^\circ\rangle_t]\,\mathrm{d}A_s\right] \leq 4\mathbb{E}\left[\int_S^T e^{\beta A_t}y(t)^2\,\mathrm{d}A_t\right].
\end{aligned}$$

since α is supposed to be non-decreasing. Similarly

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{1}{\alpha_s^2}\theta^\natural_{t,s}|||\nu(s,t)|||_t^2\,\mathrm{d}B_s\,\mathrm{d}A_t\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\int_0^t e^{(\beta-\delta)A_t}\alpha_t^2|||\nu(s,t)|||_t^2\,\mathrm{d}B_t\,\mathrm{d}B_s\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\alpha_s^2\int_0^s e^{(\beta-\delta)A_t}|||\nu(s,t)|||_t^2\,\mathrm{d}B_t\,\mathrm{d}B_s\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\delta A_s}\int_0^s e^{(\beta-\delta)A_t}\,\mathrm{d}\,\mathrm{Trace}[\langle\nu(s,\cdot)\star\widetilde{\pi}^\natural\rangle_t]\,\mathrm{d}A_s\right] \leq 4\mathbb{E}\left[\int_0^T e^{\beta A_t}y(t)^2\,\mathrm{d}A_t\right].
\end{aligned}$$

Thus BSVIE (4.13) has a unique adapted M-solution $(Y,Z,U,M) \in \mathfrak{S}^2_{\delta\leq\beta}(\Delta^c)$. Moreover from (4.5) and the prior estimates

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y(t)|^2\,\mathrm{d}A_t + \int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle \mathrm{M}^\sharp(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]\\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right]\\
&\quad +\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|h^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]\\
&\leq \frac{\delta}{2}\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\Phi(t)|^2\,\mathrm{d}A_t\right]
\end{aligned}$$

$$+\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|f^0(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]$$
$$+18\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}y(t)^2\,\mathrm{d}A_t\right].$$

Then using (3.9) and (4.14), processes Z, U and M are well defined and controlled on Δ. Therefore $(Y,Z,U,M)\in\mathfrak{S}^2_{\delta\leq\beta}(0,T)$. In other words a map Θ from $\mathfrak{S}^2_{\delta\leq\beta}(0,T)$ into itself is constructed.

Thanks to Lemma 4.2, if $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ is another solution of the BSVIE (4.13) with data $(\bar{y},\bar{\zeta},\bar{\nu},\bar{m})$, then with

$$(\widehat{Y},\widehat{Z},\widehat{Y},\widehat{M})=(Y-\bar{Y},Z-\bar{Z},U-\bar{U},M-\bar{M}),$$

estimate (4.11) becomes

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\widehat{Y}(t)|^2\,\mathrm{d}A_t+\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\widehat{\mathrm{M}^\natural}(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\frac{|\widehat{h}(t,s)|^2}{\alpha_s^2}\,\mathrm{d}B_s\,\mathrm{d}A_t\right]$$

where using **(H2')**

$$\begin{aligned}|\widehat{h}(t,r)|^2&=|f(t,r,y(r),Z(t,r),\zeta(r,t),U(t,r),\nu(r,t))-f(t,r,\bar{y}(r),Z(t,r),\bar{\zeta}(r,t),U(t,r),\bar{\nu}(r,t))|^2\\&\leq\varpi_r|\widehat{y}(r)|^2+\theta^\circ_r|b_t\widehat{\zeta}(r,t)|^2+\theta^\natural_r|\!|\!|\nu(r,t)|\!|\!|^2_t.\end{aligned}$$

The previous computations show that

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}|\widehat{Y}(t)|^2\,\mathrm{d}A_t+\int_0^T e^{(\beta-\delta)A_t}\int_t^T e^{\delta A_s}\,\mathrm{d}\,\mathrm{Trace}\langle\widehat{\mathrm{M}^\natural}(t,\cdot)\rangle_s\,\mathrm{d}A_t\right]$$
$$\leq\left(16+\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\right)\Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t}|\widehat{y}(t)|^2\,\mathrm{d}A_t\right].$$

Hence Θ is a contraction if

$$\left(16+\frac{e^{(\beta-\delta)\mathfrak{f}}}{\beta-\delta}\right)\Sigma^{\mathfrak{f}}(\beta)<1,$$

and the existence and uniqueness of a solution $(Y,Z,U,M)\in\mathfrak{S}^2_\beta(0,T)$ to the Type-II BSVIE (1.4) is obtained.

5. More properties in the Itô framework

In this section, the setting developed in Sections 2.2 and 3.3 is used, and the goal is to prove Theorems 3.10 and 3.11, which are extensions of some results of [46].

In this case, space $\mathfrak{S}^2(\Delta^c(0,T))$ can be more easily defined here and the notations of Sections 2.1, 2.2 and 3.1 (essentially $\beta=0$) are adapted, which leads to the same notations as in [46]. Hence some details are skipped (see [46], Sect. 2.1 for interested readers). For any p, q in $[0, +\infty)$, $H = \mathbb{R}^d$ or $\mathbb{R}^{d\times k}$, and $S \in [0, T]$,

$$\mathbb{L}^p_{\mathcal{F}_S}(\Omega) = \{\xi : \Omega \to H,\ \xi \text{ is } \mathcal{F}_S - \text{measurable}, \mathbb{E}(|\xi|^p) < +\infty\},$$

$$\mathbb{L}^p_{\mathcal{F}_S}(\Omega; \mathbb{L}^q(0,T)) = \Bigg\{\phi : (0,T)\times\Omega \to H,\ \mathcal{B}([0,T])\otimes\mathcal{F}_S - \text{measurable with} \quad \mathbb{E}\left(\int_0^T |\phi(t)|^q \,\mathrm{d}t\right)^{\frac{p}{q}} < +\infty\Bigg\},$$

$$\mathbb{L}^q_{\mathcal{F}_S}(0,T; \mathbb{L}^p_{\mathcal{F}_S}(\Omega)) = \Bigg\{\phi : (0,T)\times\Omega \to H,\ \mathcal{B}([0,T])\otimes\mathcal{F}_S - \text{measurable with} \quad \int_0^T (\mathbb{E}|\phi(t)|^p \,\mathrm{d}t)^{\frac{q}{p}} < +\infty\Bigg\},$$

We identify

$$\mathbb{L}^p_{\mathcal{F}_S}(\Omega; \mathbb{L}^p(0,T)) = \mathbb{L}^p_{\mathcal{F}_S}(0,T; \mathbb{L}^p_{\mathcal{F}_S}(\Omega)) = \mathbb{L}^p_{\mathcal{F}_S}(0,T).$$

For $p = 2$, this space corresponds to $\mathbb{L}^2_{0,\mathcal{F}_S}(0,T)$ of Section 4. When adaptiveness is required, the subscript $\mathcal{F}_S$ is replaced by $\mathbb{F}$. The above spaces are for the free term $\Phi(\cdot)$ (for which $\mathbb{F}$-adaptiveness is not required) and for $Y(\cdot)$ (for which $\mathbb{F}$-adaptiveness is required). Sometimes the subscript $\mathcal{P}$ is also used if predictability is needed.

To control the martingale terms in the BSVIE, other spaces are introduced. For any $p, q \geq 1$

$$\mathbb{L}^q(S,T; \mathcal{H}^p(S,T))$$

is the set of processes $M(\cdot,\cdot)$ such that for almost all $t \in [0,T]$, $M(t,\cdot)$ belongs to $\mathcal{H}^p(S,T)$ and

$$\int_S^T \left[\mathbb{E}\left(\langle M(t,\cdot)\rangle_{S,T}\right)^{\frac{p}{2}}\right]^{\frac{q}{p}} \mathrm{d}t < +\infty.$$

For $p = q = 2$, if M is restricted to $\Delta^c(S,T)$, this space is denoted $\mathcal{H}^2_{0\leq 0}(\Delta^c(S,T))$ in Section 3.1. In the particular case where $M(t,\cdot) = \int_S^\cdot Z(t,s) dW_s$, then $M \in \mathbb{L}^q(S,T; \mathcal{H}^p(S,T))$ is equivalent to

$$Z \in \mathbb{L}^q(S,T; \mathbb{L}^p_{\mathcal{P}}(\Omega; \mathbb{L}^2(S,T))) = \mathbb{L}^q(S,T; \mathbb{H}^p(S,T)),$$

that is Z belongs to the set of all processes $Z : [S,T]^2 \times \Omega \to \mathbb{R}^k$ such that for almost all $t \in [S,T]$, $Z(t,\cdot) \in \mathbb{H}^p(S,T) = \mathbb{L}^p_{\mathcal{P}}(\Omega; \mathbb{L}^2(S,T))$ satisfies

$$\int_S^T \left[\mathbb{E}\left(\int_S^T |Z(t,s)|^2 \,\mathrm{d}s\right)^{\frac{p}{2}}\right]^{\frac{q}{p}} \mathrm{d}t < +\infty.$$

Again for $p = q = 2$, this space is equal to set $\mathcal{H}^{2,\circ}_{0\leq 0}$ of Section 3.1.

Let us also consider the case:

$$N(t,s) = \int_S^s \int_{\mathbb{R}^m} \psi(t,u,x)\widetilde{\pi}(\,\mathrm{d}x,\,\mathrm{d}u),\ t \geq S,\ s \geq S.$$

Process N belongs to $\mathbb{L}^q(S,T;\mathcal{H}^p(S,T))$ if and only if $\psi \in \mathbb{L}^q(S,T;\mathbb{L}^p_\pi(S,T))$, namely ψ is in the set of all processes $\psi : [S,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ such that for almost all $t \in [S,T]$, $\psi(t,\cdot,\cdot) \in \mathbb{L}^p_\pi(S,T)$ verifies

$$\int_S^T \left[\mathbb{E}\left(\int_S^T \int_{\mathbb{R}^m} |\psi(t,s,x)|^2 \pi(\,\mathrm{d}s,\,\mathrm{d}x) \right)^{\frac{p}{2}} \right]^{\frac{q}{p}} \mathrm{d}t < +\infty.$$

Let us emphasize that for $p = q = 2$, this space corresponds to $\mathcal{H}^{2,\natural}_{0\leq 0}$ in Section 3.1.

If martingale $\mathrm{M}^\sharp$ is defined by (3.1), due to the orthogonality of the components of $\mathrm{M}^\sharp$, for any $p > 1$, there exist two universal constants c_p and C_p such that

$$\begin{aligned} & c_p \mathbb{E}\left[\left(\int_S^T |Z(t,s)|^2 \,\mathrm{d}s \right)^{\frac{p}{2}} + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,s,x)|^2 \pi(\,\mathrm{d}s,\,\mathrm{d}x) \right)^{\frac{p}{2}} + \left(\langle M(t,\cdot) \rangle_{S,T} \right)^{\frac{p}{2}} \right] \\ & \leq \mathbb{E}\left[\left(\langle \mathrm{M}^\sharp(t,\cdot) \rangle_{S,T} \right)^{\frac{p}{2}} \right] \\ & \leq C_p \mathbb{E}\left[\left(\int_S^T |Z(t,s)|^2 \,\mathrm{d}s \right)^{\frac{p}{2}} + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,s,x)|^2 \pi(\,\mathrm{d}s,\,\mathrm{d}x) \right)^{\frac{p}{2}} + \left(\langle M(t,\cdot) \rangle_{S,T} \right)^{\frac{p}{2}} \right]. \end{aligned}$$

And $\mathrm{M}^\sharp$ belongs to $\mathbb{L}^q(S,T;\mathcal{H}^p(S,T))$ if and only if the triplet (Z,U,M) is in $\mathbb{L}^q(0,T;\mathbb{H}^p(0,T)) \times \mathbb{L}^q(0,T;\mathbb{L}^p_\pi(0,T)) \times \mathbb{L}^q(0,T;\mathcal{H}^{p,\perp}(0,T))$.

Finally the product spaces are denoted:

$$\begin{aligned} \mathcal{M}^p(0,T) &= \mathbb{L}^p(0,T;\mathbb{H}^p(0,T)) \times \mathbb{L}^p(0,T;\mathbb{L}^p_\pi(0,T)) \times \mathbb{L}^p(0,T;\mathbb{M}^{p,\perp}(0,T)) \\ \mathfrak{S}^p(0,T) &= \mathbb{L}^p_\mathbb{F}(0,T) \times \mathbb{L}^p(0,T;\mathbb{H}^p(0,T)) \times \mathbb{L}^p(0,T;\mathbb{L}^p_\pi(0,T)) \times \mathbb{L}^p(0,T;\mathcal{H}^{p,\perp}(0,T)) \\ &= \mathbb{L}^p_\mathbb{F}(0,T) \times \mathcal{M}^p(S,T) \end{aligned}$$

with the naturally induced norm.

Definitions 3.1 (adapted solutions) and 3.6 (M-solutions) remain unchanged, except that (Y,Z,U,M) belongs to $\mathfrak{S}^p(0,T)$ and condition (3.7) becomes:

$$Y(t) = \mathbb{E}\left[Y(t)|\mathcal{F}_S\right] + \int_S^t Z(t,s)\,\mathrm{d}W_s + \int_S^t \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) + \int_S^t \mathrm{d}M(t,s).$$

If (Y,Z,U,M) is a solution of BSDE (2.5) in $\mathbb{D}^p(0,T)$, then by the martingale representation ([19], Lem. III.4.24), we have

$$Y(t) = \mathbb{E}(Y(t)) + \int_0^t \zeta(t,s)\,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^m} \upsilon(t,s,x)\widetilde{\pi}(\,\mathrm{d}x,\,\mathrm{d}s) + m(t,s),$$

where $(\zeta,\upsilon,m) \in \mathcal{M}^p(0,T)$. Thus if

$$(Z(t,s),U(t,s,x),M(t,s)) = \begin{cases} (\zeta(t,s),\upsilon(t,s,x),m(t,s)), & (t,s) \in \Delta(0,T) \\ (Z(s),U(s,x),M(s)), & (t,s) \in \Delta^c(0,T) \end{cases}$$

then (Y,Z,U,M) is an adapted M-solution of BSVIE (1.7) on $[0,T]$, and in fact it is the unique solution (see Thm. 3.10).

To complete this presentation, let us set out some facts concerning the Poisson integral. From the *Burkholder-Davis-Gundy inequality* (see [36], Thm. 48), for all $p \in [1,\infty)$ there exist two universal constants c_p and C_p (not depending on M) such that for any càdlàg $\mathbb{F}$-martingale $M(\cdot)$ and for any $T \geq 0$

$$c_p \mathbb{E}\left(\langle M\rangle_T^{p/2}\right) \leq \mathbb{E}\left[\left(\sup_{t\in[0,T]} |M(t)|\right)^p\right] \leq C_p \mathbb{E}\left(\langle M\rangle_T^{p/2}\right). \tag{5.1}$$

In particular (5.1) means that the Poisson martingale N is well-defined on $[0,T]$ (see Chapter II in [18]) provided we can control the expectation of $\langle N\rangle_T^{p/2}$ for some $p \geq 1$.

From the *Bichteler-Jacod inequality* (see for example [29]), the two cases are distinguished: $p \geq 2$ and $p < 2$.

- Assume that $p \geq 2$. If $\mathbb{E}(\langle N\rangle_T^{p/2}) < +\infty$, then $\mathbb{P}\otimes$Leb-a.s. on $\Omega\times[0,T]$, $\psi(t,\cdot)$ is in $\mathbb{L}^2_\mu$. Hence the generator of our BSVIE can be defined on $\mathbb{L}^2_\mu$.
- But if $p < 2$, $\mathbb{P}\otimes$Leb-a.s. on $\Omega\times[0,T]$, $\psi(t,\cdot)$ is in $\mathbb{L}^p_\mu + \mathbb{L}^2_\mu$ if again $\mathbb{E}(\langle N\rangle_T^{p/2}) < +\infty$. Moreover ψ_t is also in $\mathbb{L}^1_\mu + \mathbb{L}^2_\mu$. Thereby for $p < 2$, our generator is be defined on $\mathbb{L}^1_\mu + \mathbb{L}^2_\mu$ (for the definition of the sum of two Banach spaces, see for example [22]).

See Section 1 of [24] for details on this point. In particular for N defined by

$$N_t = \int_0^t \int_{\mathbb{R}^m} \psi_s(x)\widetilde{\pi}(\mathrm{d}s, \mathrm{d}x), t \geq 0,$$

if $p \geq 2$, there exist two universal constants κ_p and K_p such that

$$\kappa_p \left[\mathbb{E}\left(\langle N\rangle_T^{p/2}\right)\right] \leq \mathbb{E}\left(\int_0^T \|\psi_t\|^2_{\mathbb{L}^2_\mu}\,\mathrm{d}t\right)^{p/2} \leq K_p\left[\mathbb{E}\left(\langle N\rangle_T^{p/2}\right)\right]. \tag{5.2}$$

But if $1 < p < 2$, there only exists a universal constant $K_{p,T}$ such that

$$\mathbb{E}\left[\int_0^T \|\psi_s\|^p_{\mathbb{L}^p_\mu+\mathbb{L}^2_\mu}\,\mathrm{d}s\right] \leq K_{p,T}\mathbb{E}\left(\langle N\rangle_T^{p/2}\right). \tag{5.3}$$

And it holds that $\mathbb{L}^p_\mu + \mathbb{L}^2_\mu \subset \mathbb{L}^1_\mu + \mathbb{L}^2_\mu$.

5.1. Proof of Theorem 3.10

Bear in mind that the aim is the proof of existence and uniqueness in space $\mathfrak{S}^p(0,T)$ of the adapted solution (Y,Z,U,M) of BSVIE (1.6):

$$\begin{aligned} Y(t) = \Phi(t) &+ \int_t^T f(t,s,Y(s),Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \\ &- \int_t^T\int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s), \end{aligned}$$

together with estimate (3.12): for any $S \in [0,T]$

$$\|(Y,Z,U,M)\|^p_{\mathfrak{S}^p(S,T)} = \mathbb{E}\left[\int_S^T |Y(t)|^p\,\mathrm{d}t + \int_S^T\left(\int_S^T |Z(t,r)|^2\,\mathrm{d}r\right)^{p/2}\mathrm{d}t\right.$$

$$+ \int_S^T \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)} \right)^{p/2} \mathrm{d}t + \int_S^T \left(\langle M(t,\cdot) \rangle_{S,T} \right)^{p/2} \mathrm{d}t \Bigg]$$
$$\leq C\mathbb{E}\left[\int_S^T |\Phi(t)|^p \,\mathrm{d}t + \int_S^T \left(\int_t^T |f^0(t,r)| dr \right)^p \mathrm{d}t \right].$$

The proof is based on intermediate results. We consider BSVIE (4.4) which becomes here

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T h(t,s,Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M(t,s), \end{aligned} \tag{5.4}$$

The first result is a modification of Lemmata 4.1 and 4.2, namely

Lemma 5.1. *If*

$$\mathbb{E}\int_S^T \left(\int_S^T |h(t,s,0,\mathbf{0})|\,\mathrm{d}s \right)^p \mathrm{d}t < +\infty \tag{5.5}$$

holds and if $\Phi \in \mathbb{L}^p_{\mathcal{F}_T}(S,T)$, *then BSVIE* (5.4) *has a unique adapted M-solution in* $\mathfrak{S}^p(S,T)$ *and for* $t \in [S,T]$*:*

$$\begin{aligned} \mathbb{E}\left[|Y(t)|^p + \left(\int_S^T |Z(t,r)|^2 \,\mathrm{d}r \right)^{\frac{p}{2}} + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x) \right)^{\frac{p}{2}} \right. \\ \left. + \left(\langle M(t,\cdot) \rangle_{S,T} \right)^{\frac{p}{2}} \right] \leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |h(t,r,0,\mathbf{0})|\,\mathrm{d}r \right)^p \right]. \end{aligned} \tag{5.6}$$

Moreover a stability result for this BSVIE holds. Let $(\bar{\Phi},\bar{h})$ *satisfy the above assumptions* **(H2)** *and* **(H3*)** *and the same integrability conditions. Let* $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ *be the solution of the BSVIE* (5.4) *with data* $(\bar{\Phi},\bar{h})$*. Define*

$$(\mathfrak{d}Y,\mathfrak{d}Z,\mathfrak{d}U,\mathfrak{d}M) = (Y-\bar{Y}, Z-\bar{Z}, U-\bar{U}, M-\bar{M}).$$

Then there exists a constant C *depending on* p, K *and* T, *such that for* $t \in [S,T]$

$$\begin{aligned} \mathbb{E}\left[|\mathfrak{d}Y(t)|^p + \left(\int_S^T |\mathfrak{d}Z(t,r)|^2 \,\mathrm{d}r \right)^{p/2} + \left(\langle \mathfrak{d}M(t,\cdot) \rangle_{S,T} \right)^{p/2} \right. \\ \left. + \left(\int_S^T \int_{\mathbb{R}^m} |\mathfrak{d}U(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x) \right)^{p/2} \right] \leq C\mathbb{E}\Bigg[|\Phi(t) - \bar{\Phi}(t)|^p \\ + \left(\int_S^T |h(t,r,Z(t,r),U(t,r)) - \bar{h}(t,r,Z(t,r),U(t,r))|\,\mathrm{d}r \right)^p \Bigg]. \end{aligned} \tag{5.7}$$

This lemma is a consequence of Proposition 2.7 applied to the parametrized BSDE (4.3). The arguments are similar to those used for Lemmata 4.1 and 4.2 or for Corollary 3.6 of [46]. For completeness the main ideas are evoked here. The parametrized BSDE (4.3) becomes:

$$\lambda(t,r) = \Phi(t) + \int_r^T h(t,s,z(t,s),u(t,s))\,\mathrm{d}s - \int_r^T z(t,s)\,\mathrm{d}W_s \\ - \int_r^T \int_{\mathbb{R}^m} u(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_r^T \,\mathrm{d}m(t,s).$$

From Proposition 2.7, for any $\Phi(\cdot) \in \mathbb{L}^p_{\mathcal{F}_T}(S,T)$ the previous BSDE has a unique solution $(\lambda(t,\cdot), z(t,\cdot), u(t,\cdot), m(t,\cdot))$ in $\mathbb{D}^p(R,T)$ and for a.e. $t \in [S,T]$

$$\mathbb{E}\left[\sup_{r\in[R,T]} |\lambda(t,r)|^p + \left(\int_R^T |z(t,r)|^2\,\mathrm{d}r\right)^{p/2} + (\langle m(t,\cdot)\rangle_{R,T})^{p/2} \right. \\ \left. + \left(\int_R^T \int_{\mathbb{R}^m} |u(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{p/2}\right] \leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_R^T |h(t,r,0,\mathbf{0})|\,\mathrm{d}r\right)^p\right].$$

Moreover the stability property holds for BSDEs ([23], Lem. 5 and proof of Prop. 2 for $p \geq 2$, [24], Prop. 3). Let $(\bar{\Phi}, \bar{h})$ be a couple of data each satisfying the above assumption **(H2)-(H3*)** and the required integrability conditions for the data. Let $(\bar{\lambda}(t,\cdot), \bar{z}(t,\cdot), \bar{u}(t,\cdot), \bar{m}(t,\cdot))$ be the solution of BSDE (4.3) with data $(\bar{\Phi}, \bar{h})$. Denote

$$(\mathfrak{d}\lambda(t,\cdot), \mathfrak{d}z(t,\cdot), \mathfrak{d}u(t,\cdot), \mathfrak{d}m(t,\cdot)) = (\lambda(t,\cdot) - \bar{\lambda}(t,\cdot), z(t,\cdot) - \bar{z}(t,\cdot), u(t,\cdot) - \bar{u}(t,\cdot), m(t,\cdot) - \bar{m}(t,\cdot)).$$

Then there exists a constant C depending on p, K and T, such that

$$\mathbb{E}\left[\sup_{r\in[R,T]} |\mathfrak{d}\lambda(t,r)|^p + \left(\int_R^T |\mathfrak{d}z(t,r)|^2\,\mathrm{d}r\right)^{p/2} + (\langle \mathfrak{d}m(t,\cdot)\rangle_{R,T})^{p/2} \right. \\ \left. + \left(\int_R^T \int_{\mathbb{R}^m} |\mathfrak{d}u(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{p/2}\right] \\ \leq C\mathbb{E}\left[|\Phi(t) - \bar{\Phi}(t)|^p + \left(\int_R^T |h(t,r,z(t,r),u(t,r)) - \bar{h}(t,r,z(t,r),u(t,r))|\,\mathrm{d}r\right)^p\right]. \tag{5.8}$$

Setting $Y(t) = \lambda(t,t)$, $t \in [S,T]$, $Z(t,s) = z(t,s)$, $U(t,s,e) = u(t,s,e)$, $M(t,s) = m(t,s)$ for $(t,s) \in \Delta^c(S,T)$ yields that the equation (4.3) becomes the BSVIE (5.4).

But one result for *stochastic Fredholm integral equation* (SFIE in abbreviated form) can be also deduced. Indeed for BSDE (4.3), let us fix $r = S \in [R,T)$ and define for $t \in [R,S]$ and $s \in [S,T]$:

$$\psi^S(t) = \lambda(t,S),\ Z(t,s) = z(t,s),\ U(t,s) = u(t,s),\ M(t,s) = m(t,s).$$

Then equation (4.3) becomes an SFIE: for $t \in [R,S]$

$$\psi^S(t) = \Phi(t) + \int_S^T h(t,s,Z(t,s),U(t,s))\,\mathrm{d}s - \int_S^T Z(t,s)\,\mathrm{d}W_s \\ - \int_S^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_S^T \,\mathrm{d}M(t,s). \tag{5.9}$$

Lemma 5.2. *If (5.5) holds and if $\Phi \in \mathbb{L}^p_{\mathcal{F}_T}(S,T)$, then SFIE (5.9) has a unique solution such that ψ^S belongs to $\mathbb{L}^p_{\mathcal{F}_S}(R,S)$, (Z,U,M) is in*

$$\mathbb{L}^p(R,S;\mathbb{H}^p(S,T)) \times \mathbb{L}^p(R,S;\mathbb{L}^p_\pi(S,T)) \times \mathbb{L}^p(R,S;\mathcal{H}^{p,\perp}(S,T))$$

and for $t \in [R,S]$

$$\mathbb{E}\left[|\psi^S(t)|^p + \left(\int_S^T |Z(t,r)|^2 \,\mathrm{d}r\right)^{\frac{p}{2}} + + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\mathrm{d}r,\mathrm{d}x)\right)^{\frac{p}{2}} \right.$$
$$\left. (\langle M(t,\cdot)\rangle_{S,T})^{\frac{p}{2}} \right] \leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |h(t,r,0,\mathbf{0})| \,\mathrm{d}r\right)^p \right]. \tag{5.10}$$

Note that here $\psi^S(t)$ is only required to be $\mathcal{F}_S$-measurable for almost all t and not $\mathbb{F}$-adapted. Let us now proceed with the proof of Theorem 3.10.

Proof. The outline of the proof of Theorem 3.7 in [46] is followed.
Step 1. For any $S \in [0,T]$, let us consider the set $\widehat{\mathfrak{S}}^p(S,T)$, the space of all (y,z,u,m) in $\mathfrak{S}^p(S,T)$ such that for a.e. $t \in [S,T]$ a.s.

$$y(t) = \mathbb{E}\left[y(t)|\mathcal{F}_S\right] + \int_S^t z(t,s)\,\mathrm{d}W_s + \int_S^t \int_{\mathbb{R}^m} u(t,s,x)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}x) + \int_S^t \mathrm{d}m(t,s).$$

From this representation, Doob's martingale inequality and the Burkholder-Davis-Gundy inequality, for $t \in [S,T]$, the quantity

$$\mathbb{E}\left[\left(\int_S^t |z(t,r)|^2 \,\mathrm{d}r\right)^{\frac{p}{2}} + (\langle m(t,\cdot)\rangle_{S,t})^{\frac{p}{2}} + \left(\int_S^t \int_{\mathbb{R}^m} |u(t,r,x)|^2 \pi(\mathrm{d}r,\mathrm{d}x)\right)^{\frac{p}{2}}\right]$$

is bounded from above by $C\mathbb{E}|y(t)|^p$. On $\widehat{\mathfrak{S}}^p(S,T)$ the following norm is considered:

$$\|(y,z,u,m)\|^p_{\widehat{\mathfrak{S}}^p(S,T)} = \mathbb{E}\left[\int_S^T |y(t)|^p \,\mathrm{d}t + \int_S^T \left(\int_t^T |z(t,r)|^2 \,\mathrm{d}r\right)^{p/2} \mathrm{d}t \right.$$
$$\left. + \int_S^T \left(\|u(t,r)\|^2_{\mathbb{L}^2_\pi(t,T)}\right)^{p/2} \mathrm{d}t + \int_S^T (\langle m(t,\cdot)\rangle_{t,T})^{p/2} \,\mathrm{d}t\right].$$

The same arguments as inequality (3.48) of [46] show the norm equivalence:

$$\begin{aligned} \|(y,z,u,m)\|^p_{\widehat{\mathfrak{S}}^p(S,T)} &\leq \mathbb{E}\left[\int_S^T |y(t)|^p \,\mathrm{d}t + \int_S^T \left(\int_S^T |z(t,r)|^2 \,\mathrm{d}r\right)^{p/2} \mathrm{d}t \right. \\ &\left. + \int_S^T \left(\|u(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)}\right)^{p/2} \mathrm{d}t + \int_S^T (\langle m(t,\cdot)\rangle_{S,T})^{p/2} \,\mathrm{d}t\right] \\ &\leq (C_p+1)\|(y,z,u,m)\|^p_{\widehat{\mathfrak{S}}^p(S,T)}. \end{aligned} \tag{5.11}$$

Note that using (5.2), if $p \geq 2$, $\|u(t,\cdot)\|^2_{\mathbb{L}^2_\pi(S,T)}$ can be replaced by $\int_S^T \|u(t,r)\|^2_{\mathbb{L}^2_\mu} \,\mathrm{d}r$ in the previous estimates (with the suitable modifications of constant C_p).

If $\Phi \in \mathbb{L}^p_{\mathcal{F}_T}(S,T)$ and $(y,\zeta,\nu,m) \in \widehat{\mathfrak{S}}^p(S,T)$, we consider the BSVIE on $[S,T]$:

$$Y(t) = \Phi(t) + \int_t^T f(t,s,y(s),Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s). \tag{5.12}$$

From conditions **(H2)**, **(H3*)**, it can be easily checked that the generator of this BSVIE satisfies all requirements of Lemma 5.1. Thus this BSVIE has a unique adapted M-solution $(Y,Z,U,M) \in \mathfrak{S}^p(S,T)$ and for any $t \in [S,T]$

$$\mathbb{E}\left[|Y(t)|^p + \left(\int_t^T |Z(t,r)|^2\,\mathrm{d}r\right)^{\frac{p}{2}} + (\langle M(t,\cdot)\rangle_{t,T})^{\frac{p}{2}} + \left(\int_t^T \int_{\mathbb{R}^m} |U(t,r,x)|^2\pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{\frac{p}{2}}\right] \leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_t^T |f^0(t,r)|\,\mathrm{d}r\right)^p + \int_t^T |y(r)|^p\,\mathrm{d}r\right]. \tag{5.13}$$

Therefore $(Y,Z,U,M) \in \widehat{\mathfrak{S}}^p(S,T)$. In other words we construct a map Θ from $\widehat{\mathfrak{S}}^p(S,T)$ to $\widehat{\mathfrak{S}}^p(S,T)$. Moreover arguing as in [46], for (y,ζ,ν,m) and $(\bar{y},\bar{\zeta},\bar{\nu},\bar{m})$ in $\widehat{\mathfrak{S}}^p(S,T)$, if (Y,Z,U,M) and $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ are the solutions of the BSVIE (5.12), then from (5.7) the difference satisfies:

$$\begin{aligned}
&\mathbb{E}\Bigg[|\mathfrak{d}Y(t)|^p + \left(\int_S^T |\mathfrak{d}Z(t,r)|^2\,\mathrm{d}r\right)^{p/2} + (\langle \mathfrak{d}M(t,\cdot)\rangle_{S,T})^{p/2} \\
&\quad + \left(\int_S^T \int_{\mathbb{R}^m} |\mathfrak{d}U(t,r,x)|^2\pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{p/2}\Bigg] \\
&\leq C\mathbb{E}\left[\left(\int_S^T K|\mathfrak{d}y(r)|\,\mathrm{d}r\right)^p\right] \leq CK^p(T-S)^{p-1}\mathbb{E}\left[\left(\int_S^T |\mathfrak{d}y(r)|^p\,\mathrm{d}r\right)\right].
\end{aligned} \tag{5.14}$$

For $T-S$ sufficiently small, this map is a contraction and thus it admits a unique fixed point $(Y,Z,U,M) \in \widehat{\mathfrak{S}}^p(S,T)$ which is the unique adapted M-solution of (1.6) on $[S,T]$. Moreover estimate (3.12) holds. This step determines values $(Y(t),Z(t,s),U(t,s),M(t,s))$ for $(t,s) \in [S,T]\times[S,T]$. Note that at this step, **(H3*)** can be replaced by a weaker condition as in [46].

Let us take a short break in the proof to understand the trouble in the case of a Type-II BSVIE. The driver of the BSVIE (5.12) would be replaced by

$$\int_t^T f(t,s,y(s),Z(t,s),\zeta(s,t),U(t,s),\nu(s,t))\,\mathrm{d}s.$$

Hence in (5.13), the next additional terms would appear:

$$\mathbb{E}\left[\int_S^T \left(\int_t^T |\zeta(r,t)|^2 \,\mathrm{d}r\right)^{p/2} \mathrm{d}t + \int_S^T \left(\int_t^T \|\nu(r,t)\|^2_{\mathbb{L}^2_\pi} \,\mathrm{d}r\right)^{p/2} \mathrm{d}t\right].$$

There are also other extra terms in (5.14). To circumvent this issue, this term could be added in the definition of the norm. However we cannot control this symmetrized version of the norm for (Z,U) in this step, but also in the next one. In other words the map Θ is no longer a contraction.

Step 2. The martingale representation theorem is used to define (Z,U,M) on $[S,T]\times[R,S]$ for any $R\in[0,S)$. Indeed since $\mathbb{E}[Y(t)|\mathcal{F}_S] \in \mathbb{L}^p(S,T;\mathbb{L}^p_{\mathcal{F}_S}(\Omega))$, there exists a unique triple (Z,U,M) in $\mathbb{L}^p(S,T;\mathbb{H}^p(R,S)) \times \mathbb{L}^p(S,T;\mathbb{L}^p_\pi(R,S)) \times \mathbb{L}^p(S,T;\mathcal{H}^{p,\perp}(R,S))$ such that for $t\in[S,T]$:

$$\mathbb{E}[Y(t)|\mathcal{F}_S] = \mathbb{E}\left[Y(t)|\mathcal{F}_R\right] + \int_R^S Z(t,s)\,\mathrm{d}W_s + \int_R^S\int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) + \int_R^S \mathrm{d}M(t,s),$$

and

$$\mathbb{E}\left[\left(\int_R^S |Z(t,r)|^2\,\mathrm{d}r\right)^{\frac{p}{2}} + \left(\int_R^S\int_{\mathbb{R}^m} |U(t,r,x)|^2\pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{\frac{p}{2}} + \left(\langle M(t,\cdot)\rangle_{R,S}\right)^{\frac{p}{2}}\right] \leq C\mathbb{E}|Y(t)|^p.$$

Thus together with the first step, (Z,U,M) is now defined for $(t,s)\in[S,T]\times[R,T]$ and

$$\begin{aligned}
&\mathbb{E}\left[\int_S^T \left(\int_R^T |Z(t,r)|^2\,\mathrm{d}r\right)^{\frac{p}{2}} \mathrm{d}t + \int_S^T\left(\int_R^T\int_{\mathbb{R}^m}|U(t,r,x)|^2\pi(\,\mathrm{d}r,\,\mathrm{d}x)\right)^{\frac{p}{2}}\mathrm{d}t \right. \\
&\qquad \left. + \int_S^T \left(\langle M(t,\cdot)\rangle_{R,T}\right)^{\frac{p}{2}}\,\mathrm{d}t\right] \\
&\quad \leq C\mathbb{E}\left[\int_S^T |\Phi(t)|^p\,\mathrm{d}t + \int_S^T\left(\int_t^T |f^0(t,r)|\,\mathrm{d}r\right)^p \mathrm{d}t\right].
\end{aligned} \tag{5.15}$$

Step 3. From the two previous steps, for $(t,s)\in[R,S]\times[S,T]$, the values of $Y(s)$ and $(Z(s,t),U(s,t))$ are already obtained. Thus let us consider

$$f^S(t,s,z,u) = f(t,s,Y(s),z,u), \quad (t,s,z,u)\in[R,S]\times[S,T]\times\mathbb{R}^{d\times k}\times(\mathbb{L}^1_\mu+\mathbb{L}^2_\mu),$$

and from Lemma 5.2, the SFIE

$$\begin{aligned}
\psi^S(t) = \Phi(t) &+ \int_S^T f^S(t,s,Z(t,s),U(t,s))\,\mathrm{d}s - \int_S^T Z(t,s)\,\mathrm{d}W_s \\
&- \int_S^T\int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_S^T \mathrm{d}M(t,s)
\end{aligned}$$

has a unique solution (ψ^S, Z, U, M) such that for $t \in [R,S]$

$$\begin{aligned}
&\mathbb{E}\left[|\psi^S(t)|^p + \left(\int_S^T |Z(t,r)|^2 \, \mathrm{d}r \right)^{\frac{p}{2}} \right. \\
&\qquad \left. + (\langle M(t,\cdot)\rangle_{S,T})^{\frac{p}{2}} + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\,\mathrm{d}r,\,\mathrm{d}x) \right)^{\frac{p}{2}} \right] \\
&\le C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |f^S(t,r,0,\mathbf{0})| \, \mathrm{d}r \right)^p \right] \\
&= C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |f(t,r,Y(r),0,\mathbf{0})| \, \mathrm{d}r \right)^p \right] \\
&\le C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |f^0(t,r)| \, \mathrm{d}r \right)^p + \int_S^T |Y(r)|^p \, \mathrm{d}r \right].
\end{aligned}$$

Hence using (5.2) for $p \ge 2$ or (5.3) for $p < 2$, and (5.13) and (5.15) yield to:

$$\begin{aligned}
&\mathbb{E}\left[\int_R^S |\psi^S(t)|^p \, \mathrm{d}t + \int_R^S \left(\int_S^T |Z(t,r)|^2 \, \mathrm{d}r \right)^{\frac{p}{2}} \, \mathrm{d}t + \int_R^S (\langle M(t,\cdot)\rangle_{S,T})^{\frac{p}{2}} \, \mathrm{d}t \right. \\
&\qquad \left. + \int_R^S \left(\int_S^T \|U(t,r)\|^2_{\mathbb{L}^1_\mu + \mathbb{L}^2_\mu} \, \mathrm{d}r \right)^{\frac{p}{2}} \, \mathrm{d}t \right] \\
&\le C\mathbb{E}\left[\int_R^S |\Phi(t)|^p \, \mathrm{d}t + \int_R^S \left(\int_S^T |f^0(t,r)| \, \mathrm{d}r \right)^p \mathrm{d}t + \int_S^T |Y(r)|^p \, \mathrm{d}r \right] \\
&\le C\mathbb{E}\left[\int_R^T |\Phi(t)|^p \, \mathrm{d}t + \int_R^T \left(\int_t^T |f^0(t,r)| \, \mathrm{d}r \right)^p \mathrm{d}t \right]. \qquad (5.16)
\end{aligned}$$

Hence (Z, U, M) are defined for $(t,s) \in [R,S] \times [S,T]$, and by the definition of f^S, for $t \in [R,S]$

$$\begin{aligned}
\psi^S(t) &= \Phi(t) + \int_S^T f(t,s,Y(s),Z(t,s),U(t,s)) \, \mathrm{d}s - \int_S^T Z(t,s) \, \mathrm{d}W_s \\
&\quad - \int_S^T \int_{\mathbb{R}^m} U(t,s,x) \widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_S^T \mathrm{d}M(t,s). \qquad (5.17)
\end{aligned}$$

As in the first step and for the same reason, the general driver of the Type-II BSVIE (1.4) cannot be directly handled in equation (5.16).

Step 4. Let us summarize what we have obtained after these three steps. Y is uniquely determined on $[S,T]$ (from Step 1) and (Z,U,M) are uniquely determined on $[S,T] \times [R,T]$ (from Steps 1 and 2) and on $[R,S] \times [S,T]$ (from Steps 1 and 3). Let us now solve (1.6) on $[R,S]^2$. Consider

$$Y(t) = \psi^S(t) + \int_t^S f(t,s,Y(s),Z(t,s),U(t,s)) \, \mathrm{d}s - \int_t^S Z(t,s) \, \mathrm{d}W_s$$

$$- \int_t^S \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^S \,\mathrm{d}M(t,s).$$

It is a BSVIE with terminal condition $\psi^S \in \mathbb{L}^p_{\mathcal{F}_S}(R,S)$ and generator f. As in the first step, this BSVIE has a unique solution in $\mathfrak{S}^p(R,S)$ provided that $S-R>0$ is small enough. Now for $t \in [R,S]$ from the expression (5.17) of ψ^S, we obtain that

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s). \end{aligned}$$

Moreover using the same arguments as in the first step yields to:

$$\begin{aligned} &\mathbb{E}\left[\int_R^S |Y(t)|^p\,\mathrm{d}t + \int_R^S \left(\int_R^S |Z(t,r)|^2\,\mathrm{d}r\right)^{\frac{p}{2}} \,\mathrm{d}t + \int_R^S \left(\langle M(t,\cdot)\rangle_{R,S}\right)^{\frac{p}{2}} \,\mathrm{d}t \right.\\ &\qquad \left. + \int_R^S \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(R,S)}\right)^{\frac{p}{2}} \,\mathrm{d}t\right] \\ &\leq C\mathbb{E}\left[\int_R^S |\Phi(t)|^p\,\mathrm{d}t + \int_R^S \left(\int_t^S |f^0(t,r)|\,\mathrm{d}r\right)^p \,\mathrm{d}t\right] \\ &\leq C\mathbb{E}\left[\int_R^T |\Phi(t)|^p\,\mathrm{d}t + \int_R^T \left(\int_t^T |f^0(t,r)|\,\mathrm{d}r\right)^p \,\mathrm{d}t\right]. \end{aligned}$$

From this inequality together with (3.12) on [S,T], (5.15) and (5.16), it is now proved that BSVIE (1.6) has a unique adapted M-solution (Y,Z,U,M) in $\mathfrak{S}^p(R,T)$ on $[R,T]$ with the estimate (3.12) on $[R,T]$.
Step 5. The conclusion of the proof is done by induction since the time intervals $[S,T]$ (Step 1) and $[R,S]$ (Step 4) are determined by absolute constants depending only on the Lipschitz constant K of f in conditions **(H2)**–**(H3*)** and on the time horizon T. □

The stability result holds in our framework. Let $\bar{\Phi} \in L^p_{\mathcal{F}_T}(0,T)$ and $\bar{f} : \Omega \times [0,T] \times \mathbb{R}^{d+(d\times k)} \times (\mathbb{L}^1_\mu + \mathbb{L}^2_\mu)^2 \to \mathbb{R}^d$ satisfy **(H2)**–**(H3*)** and (3.11), that is $\mathbb{E}\int_0^T \left(\int_t^T |\bar{f}^0(t,s)|\,\mathrm{d}s\right)^p \,\mathrm{d}t < +\infty$. Let $(\bar{Y},\bar{Z},\bar{U},\bar{M})$ in $\mathfrak{S}^p(0,T)$ be the unique adapted M-solution of BSVIE (1.6) with data $\bar{\Phi}$ and $\bar{f}$ (Thm. 3.10). Then for any $S \in [0,T]$

$$\begin{aligned} &\mathbb{E}\left[\int_S^T |Y(t)-\bar{Y}(t)|^p\,\mathrm{d}t + \int_S^T \left(\int_S^T |Z(t,r)-\bar{Z}(t,s)|^2\,\mathrm{d}r\right)^{\frac{p}{2}} \,\mathrm{d}t \right.\\ &\qquad \left. + \int_S^T \left(\langle M(t,\cdot)-\bar{M}(t,\cdot)\rangle_{S,T}\right)^{\frac{p}{2}} \,\mathrm{d}t + \int_S^T \left(\int_S^T \|U(t,r)-\bar{U}(t,r)\|^2_{\mathbb{L}^2_\mu}\,\mathrm{d}r\right)^{\frac{p}{2}} \,\mathrm{d}t\right] \\ &\leq C\mathbb{E}\left[\int_S^T |\Phi(t)-\bar{\Phi}(t)|^p\,\mathrm{d}t\right. \end{aligned}$$

$$+\int_S^T \left(\int_t^T |f(t,r,Y(r),Z(t,r),U(t,r)) - \bar{f}(t,r,Y(r),Z(t,r),U(t,r))| \,\mathrm{d}r\right)^p \mathrm{d}t\Bigg]. \tag{5.18}$$

The proof is based on the same arguments given in [46] (see Eq. (3.71) in particular) and is skipped here.

5.2. Time regularity (Thm. 3.11)

Let us describe several sets for the process Y.

$$D([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega)) = \Big\{\phi \in \mathbb{L}^\infty(0,T;\mathbb{L}^p_{\mathbb{F}}(\Omega)),\ \phi(t) \text{ is } \mathbb{F}-\text{adapted}, \phi(\cdot) \text{ is càdlàg from } [0,T] \text{ to } \mathbb{L}^p_{\mathbb{F}}(\Omega).\Big\},$$

$$D^\sharp([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega)) = \Big\{\phi \in D([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega)),\ \phi(\cdot) \text{ is càdlàg paths a.s.}\Big\},$$

$$\mathbb{D}^p_{\mathbb{F}}(0,T) = \mathbb{D}^p(0,T) = \left\{\phi \in D^\sharp([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega)),\ \mathbb{E}\left[\sup_{t\in[0,T]} |\phi(t)|^p\right] < +\infty\right\}.$$

Again when only measurability is required, subscript $\mathbb{F}$ is replaced by $\mathcal{F}_S$. If we want to deal with continuity, then D (resp. $\mathbb{D}$) is changed to C (resp. $\mathbb{C}$) (see [46], Sect. 2.1). Coming back to a generic martingale $M(t,\cdot)$, the space

$$\mathbb{L}^p(\Omega; D([S,T];\mathbb{M}^p(S,T))$$

is defined as the set of all $M \in \mathbb{L}^\infty(S,T;\mathbb{M}^p(S,T))$ such that $t \mapsto M(t,\cdot)$ is càdlàg from $[S,T]$ to $\mathbb{M}^p(0,T)$ and

$$\mathbb{E}\left(\sup_{t\in[S,T]} \langle M(t,\cdot)\rangle_{S,T}\right)^{\frac{p}{2}} < +\infty.$$

Again if $M(t,\cdot)$ is a Brownian martingale, then $M \in \mathbb{L}^p(\Omega; D([S,T];\mathbb{M}^p(S,T))$ if and only if $Z \in \mathbb{L}^p(\Omega; D([S,T];\mathbb{H}^p(S,T))$ and if $N(t,\cdot)$ is a Poisson martingale, then $N \in \mathbb{L}^p(\Omega; D([S,T];\mathbb{M}^p(S,T))$ is equivalent to $\psi \in \mathbb{L}^p(\Omega; D([S,T];\mathbb{L}^2_\pi(S,T))$.

In Theorem 2.4 of [44], a.s. continuity of Y is proved in the Brownian setting and if the generator of the BSVIE (3.13) is of Type-I, namely for BSVIE (1.3). Our aim now is to extend this property for BSVIE (1.6), assuming that Φ and f are Hölder continuous w.r.t. t. Before the result ([44], Lem. 3.1) is adapted to our setting:

Lemma 5.3. *Let us assume that for $\Phi \in \mathbb{L}^2_{\mathcal{F}_T}(0,T)$, for $f = \{f(t,s),\ 0 \le t \le s \le T\}$ such that*

$$\mathbb{E}\int_0^T \int_0^T |f(t,s)|^2 \mathbf{1}_{s\ge t}\,\mathrm{d}s < +\infty,$$

and for some (Z,U,M) is in $\mathcal{H}^2(0,T)$, the next equality holds for almost all $t \in [0,T]$

$$Y(t) = \Phi(t) + \int_t^T f(t,s)\,\mathrm{d}s + \int_t^T Z(t,s)\,\mathrm{d}W_s + \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) + \int_t^T \mathrm{d}M(t,s).$$

Then

$$e^{\beta t}|Y(t)|^2 \leq e^{\beta T}\mathbb{E}^{\mathcal{F}_t}|\Phi(t)|^2 + \frac{1}{\beta}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|f(t,s)|^2\,\mathrm{d}s$$

and

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z(t,s)|^2\,\mathrm{d}s + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\,\mathrm{d}\langle M(t,\cdot)\rangle_{t,s} + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\int_{\mathbb{R}^m}|U(t,s,x)|^2\pi(\,\mathrm{d}s,\,\mathrm{d}x)\\ \leq e^{\beta T}\mathbb{E}^{\mathcal{F}_t}|\Phi(t)|^2 + \frac{1}{\beta}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|f(t,s)|^2\,\mathrm{d}s.\end{aligned}$$

Proof. The proof is an adaptation of the arguments of [44], together with [23, 24], and is set out in the appendix (see Sect. A.3). □

Now the proof of Theorem 3.11 is presented, following the outline of the proof of Theorem 2.4 in [44].

Proof. From our assumptions, the solution (Y,Z,U,M) belongs to $\mathfrak{S}^p(0,T)$ with $p \geq 2$, thus in $\mathfrak{S}^2(0,T)$.
Step 1. Consider for a fixed t in $[0,T]$:

$$X_t(u) = \mathbb{E}\left[\Phi(t) + \int_t^T f(t,s)\,\mathrm{d}s\middle|\mathcal{F}_u\right], \quad u \in [0,T].$$

From our assumption, $u \mapsto X_t(u)$ is a càdlàg $\mathbb{L}^p$-martingale. For $0 \leq t \leq t' \leq T$, Doob's martingale inequality implies

$$\begin{aligned}\mathbb{E}\left[\sup_{u\in[0,T]}|X_t(u) - X_{t'}(u)|^p\right] &\leq C\mathbb{E}\left[|X_t(T) - X_{t'}(T)|^p\right]\\ &\leq C\mathbb{E}\left[|\Phi(t) - \Phi(t')|^p\right]\\ &\quad + C\mathbb{E}\left[\left|\int_t^{t'} f(t,s)\,\mathrm{d}s\right|^p + \left|\int_{t'}^T |f(t,s) - f(t',s)|\,\mathrm{d}s\right|^p\right].\end{aligned}$$

Hölder's inequality leads to:

$$\mathbb{E}\left[\left|\int_t^{t'} f(t,s)\,\mathrm{d}s\right|^p\right] \leq |t'-t|^{p-1}\mathbb{E}\left[\left|\int_t^T |f(t,s)|^p\,\mathrm{d}s\right|\right].$$

Hence our setting implies:

$$\mathbb{E}\left[\sup_{u\in[0,T]}|X_t(u) - X_{t'}(u)|^p\right] \leq C_\varrho|t-t'|^{\alpha p}.$$

Since $\alpha p > 1$, if $X = (X_t,\ t \in [0,T])$ is considered as a process with values in the Skorohod space $D([0,T];\mathbb{R}^d)$ equipped with the uniform norm, which is a complete metric space, then the Kolmogorov continuity criterion can be applied (see [36], IV.Cor. 1 or [38], Thm. I.2.1): there is a continuous version of $t \in [0,T] \mapsto X_t \in D([0,T];\mathbb{R}^d)$. In particular a.s. $t \mapsto Y(t) := X_t(t)$ is càdlàg:

$$\begin{aligned}|Y(t)-Y(t')| &\leq |X_t(t)-X_t(t')| + |X_t(t')-X_{t'}(t')| \\ &\leq |X_t(t)-X_t(t')| + \sup_{s\in[0,T]} |X_t(s)-X_{t'}(s)|.\end{aligned}$$

Note that

$$\begin{aligned}X_t(u) &= X_t(0) + \int_0^u Z(t,s)\,\mathrm{d}W_s + \int_0^u \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) + M(t,u) \\ &= X_t(0) + \mathrm{M}^\sharp(t,u).\end{aligned}$$

Using the BDG inequality ($p \geq 2$), we have

$$\begin{aligned}\mathbb{E}\left[\left(\left\langle \mathrm{M}^\sharp(t,\cdot)-\mathrm{M}^\sharp(t',\cdot)\right\rangle_{0,T}\right)^{p/2}\right] &\leq C\mathbb{E}\left[\left|\mathrm{M}^\sharp(t,T)-\mathrm{M}^\sharp(t',T)\right|^p\right] \\ &\leq C\left(\mathbb{E}\left[|X_t(0)-X_{t'}(0)|^p\right] + \mathbb{E}\left[|X_t(T)-X_{t'}(T)|^p\right]\right) \\ &\leq C|t-t'|^{\alpha p}.\end{aligned}$$

And

$$\begin{aligned}\mathbb{E}\left[\left(\left\langle \mathrm{M}^\sharp(0,\cdot)\right\rangle_{0,T}\right)^{p/2}\right] &\leq C\mathbb{E}\left[\left|\mathrm{M}^\sharp(0,T)\right|^p\right] \leq C\mathbb{E}\left[|X_0(T)-X_0(0)|^p\right] \\ &\leq C\mathbb{E}\left(|\Phi(0)|^p + \left(\int_0^T |f(0,s)|\,\mathrm{d}s\right)^p\right) \leq C.\end{aligned}$$

Recall that space $\mathcal{H}^2$ is a Banach space (see [10], Sect. VII.3 (98.1)-(98.2) or [36], Sect. V.2). If we consider $t \mapsto \mathrm{M}^\sharp(t,\cdot)$, this map defined on $[0,T]$ takes values in space $\mathcal{H}^2$. Applying the Kolmogorov continuity criterion (see again [38], Thm. I.2.1) leads to:

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} \left\langle \mathrm{M}^\sharp(t,\cdot)\right\rangle_{0,T}\right)^{p/2}\right] \leq C. \tag{5.19}$$

Step 2. Assume that f depends only on z and ψ. Let us define $Z_0(t,s) \equiv 0$, $U_0(t,s) \equiv 0$ and recursively for $n \geq 1$:

$$\begin{aligned}Y_n(t) = \Phi(t) &+ \int_t^T f(t,s,Z_{n-1}(t,s),U_{n-1}(t,s))\,\mathrm{d}s - \int_t^T Z_n(t,s)\,\mathrm{d}W_s \\ &- \int_t^T \int_{\mathbb{R}^m} U_n(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M_n(t,s).\end{aligned}$$

Arguing exactly as in [44] yields to: for any $n \geq 1$

$$\mathbb{E}\left[\left(\left\langle \mathrm{M}_n^\sharp(t,\cdot)-\mathrm{M}_n^\sharp(t',\cdot)\right\rangle_{0,T}\right)^{p/2}\right] \leq C_n|t-t'|^{\alpha p},$$

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}\left\langle \mathrm{M}_n^\sharp(t,\cdot)\right\rangle_{0,T}\right)^{p/2}\right] \leq C_n,$$

$$t\mapsto Y_n(t) \text{ is càdlàg.}$$

Let us now prove the convergence of Y_n. Using Lemma 5.3, conditions **(H2)** and **(H3*)**, we obtain

$$\begin{aligned}
&e^{\beta t}|Y_{n+1}(t)-Y_n(t)|^2+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z_{n+1}(t,s)-Z_n(t,s)|^2\,\mathrm{d}s\\
&\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\|U_{n+1}(t,s,\cdot)-U_n(t,s,\cdot)\|^2_{\mathbb{L}^2_\pi}\,\mathrm{d}s\\
&\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\,\mathrm{d}\langle M_{n+1}(t,\cdot)-M_n(t,\cdot)\rangle_{t,s}\\
&\leq \frac{1}{\beta^2}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|f(t,s,Z_n(t,s),U_n(t,s))-f(t,s,Z_{n-1}(t,s),U_{n-1}(t,s))|^2\,\mathrm{d}s\\
&\leq \frac{2K^2}{\beta^2}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|Z_n(t,s)-Z_{n-1}(t,s)|^2+\|U_n(t,s,\cdot)-U_{n-1}(t,s,\cdot)\|^2_{\mathbb{L}^2_\pi}\right)\,\mathrm{d}s.
\end{aligned}$$

Using inequality (5.2), taking β large enough (greater than $4(K^2K_2^2)$, where K comes from **(H3*)** and K_2 from (5.2)) and iterating the previous inequality leads to:

$$\begin{aligned}
&e^{\beta t}|Y_{n+1}(t)-Y_n(t)|^2+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z_{n+1}(t,s)-Z_n(t,s)|^2\,\mathrm{d}s\\
&\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\|U_{n+1}(t,s,\cdot)-U_n(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\,\mathrm{d}s\\
&\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\,\mathrm{d}\langle M_{n+1}(t,\cdot)-M_n(t,\cdot)\rangle_{t,s}\\
&\leq \frac{1}{2^n}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|Z_1(t,s)-Z_0(t,s)|^2+\|U_1(t,s,\cdot)-U_0(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\right)\,\mathrm{d}s.
\end{aligned}$$

First taking the expectation and integrating w.r.t $t\in[0,T]$, the convergence of (Z_n,U_n,M_n) in $\mathcal{H}^2$ is deduced. Then

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|Y_{n+1}(t)-Y_n(t)|^p\right] &\leq \frac{1}{2^{np/2}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|Z_1(t,s)|^2+\|U_1(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\right)\,\mathrm{d}s\right)^{p/2}\right]\\
&\leq \frac{1}{2^{np/2}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\mathbb{E}^{\mathcal{F}_t}\xi\right)^{p/2}\right],
\end{aligned}$$

where

$$\xi=\sup_{t\in[0,T]}\int_t^T e^{\beta s}\left(|Z_1(t,s)|^2+\|U_1(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\right)\,\mathrm{d}s.$$

From (5.19), $\mathbb{E}(\xi^{p/2}) < +\infty$ and $t \mapsto \mathbb{E}^{\mathcal{F}_t}(\xi)$ is a martingale. By Doob's maximal inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_{n+1}(t) - Y_n(t)|^p\right] \leq \frac{1}{2^{np/2}}\mathbb{E}\left[\sup_{t\in[0,T]} \left(\mathbb{E}^{\mathcal{F}_t}\xi\right)^{p/2}\right]$$
$$\leq C\frac{e^{\beta pT}}{2^{np/2}}\mathbb{E}\left[(\xi)^{p/2}\right] \leq \frac{C}{2^{np/2}},$$

where constant C does not depend on n. Thus there exists a càdlàg adapted process Y such that

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_{n+1}(t) - Y(t)|^p\right] = 0.$$

As an immediate consequence, the limit is the unique solution in $\mathfrak{S}^2(0,T)$ of the BSVIE

$$Y(t) = \Phi(t) + \int_t^T f(t,s,Z(t,s),U(t,s))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s$$
$$- \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s).$$

Step 3. Assume that f now also depends on y. Let us define $Y_0(t) \equiv 0$ and for $n \geq 1$:

$$Y_n(t) = \Phi(t) + \int_t^T f(t,s,Y_{n-1}(s),Z_n(t,s),U_n(t,s))\,\mathrm{d}s - \int_t^T Z_n(t,s)\,\mathrm{d}W_s$$
$$- \int_t^T \int_{\mathbb{R}^m} U_n(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M_n(t,s).$$

We know that $t \mapsto Y_n(t)$ is càdlàg. Using Lemma 5.3 again, we obtain:

$$e^{\beta t}|Y_n(t)|^2 + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}|Z_n(t,s)|^2\,\mathrm{d}s + \mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\|U_n(t,s,\cdot)\|^2_{\mathbb{L}^2_\pi}\,\mathrm{d}s$$
$$+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\,\mathrm{d}\langle M_n(t,\cdot)\rangle_{t,s} \leq e^{\beta T}\mathbb{E}^{\mathcal{F}_t}|\Phi(t)|^2$$
$$+\frac{4K^2}{\beta}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|f^0(t,s)|^2 + |Y_{n-1}(s)|^2 + |Z_n(t,s)|^2 + \|U_n(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\right)\,\mathrm{d}s.$$

Thus for $\beta = 8K^2K_2^2$ (again K_2 coming from (5.2)):

$$e^{\beta t}|Y_n(t)|^2 + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|Z_n(t,s)|^2 + \|U_n(t,s,\cdot)\|^2_{\mathbb{L}^2_\mu}\right)\,\mathrm{d}s$$
$$+\frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\,\mathrm{d}\langle M_n(t,\cdot)\rangle_{t,s}$$
$$\leq e^{\beta T}\mathbb{E}^{\mathcal{F}_t}|\Phi(t)|^2 + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{\beta s}\left(|f^0(t,s)|^2 + |Y_{n-1}(s)|^2\right)\,\mathrm{d}s.$$

Set

$$h_n(t) = \sup_{1\le k\le n} \mathbb{E}\left[\sup_{s\in[t,T]} |Y_k(s)|^p\right].$$

Then

$$\begin{aligned}
h_n(t) &\le \sup_{1\le k\le n} \mathbb{E}\left[\left(\sup_{s\in[t,T]} e^{\beta s}|Y_k(s)|^2\right)^{p/2}\right] \\
&\le C\mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}|\Phi(s)|^2\right)^{p/2}\right] + C\mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\int_s^T e^{\beta u}|f^0(s,u)|^2\,\mathrm{d}u\right)^{p/2}\right] \\
&\quad + C\sup_{1\le k\le n} \mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\int_s^T e^{\beta u}|Y_{k-1}(u)|^2\,\mathrm{d}u\right)^{p/2}\right] \\
&\le C\mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\sup_{r\in[0,T]}|\Phi(r)|^2\right)^{p/2}\right] \\
&\quad + C\mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\sup_{r\in[t,T]}\int_r^T e^{\beta u}|f^0(r,u)|^2\,\mathrm{d}u\right)^{p/2}\right] \\
&\quad + C\sup_{1\le k\le n} \mathbb{E}\left[\left(\sup_{s\in[t,T]} \mathbb{E}^{\mathcal{F}_s}\int_t^T e^{\beta u}|Y_{k-1}(u)|^2\,\mathrm{d}u\right)^{p/2}\right].
\end{aligned}$$

By Doob's maximal inequality

$$\begin{aligned}
h_n(t) &\le C\mathbb{E}\left(\sup_{s\in[t,T]}|\Phi(s)|^p\right) + C\mathbb{E}\left[\left(\sup_{s\in[t,T]}\int_s^T |f^0(s,u)|^2\,\mathrm{d}u\right)^{p/2}\right] \\
&\quad + C\sup_{1\le k\le n}\mathbb{E}\left[\left(\int_t^T |Y_{k-1}(u)|^2\,\mathrm{d}u\right)^{p/2}\right].
\end{aligned}$$

Since $p \ge 2$, by Jensen's inequality,

$$h_n(t) \le C + C\int_t^T |h_n(u)|\,\mathrm{d}u.$$

Gronwall's inequality leads to

$$\sup_{1\le k\le n} \mathbb{E}\left[\sup_{s\in[0,T]} |Y_k(s)|^p\right] \le C$$

for any n, that is

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{s\in[0,T]} |Y_k(s)|^p\right] \leq C.$$

By Lemma 5.3, for almost all $t \in [0,T]$

$$e^{\beta t}|Y_n(t) - Y_m(t)|^2 \leq C\mathbb{E}^{\mathcal{F}_t} \int_t^T e^{\beta s}|Y_{n-1}(s) - Y_{m-1}(s)|^2 \,\mathrm{d}s.$$

Define

$$h(t) = \limsup_{m,n\to+\infty} \mathbb{E}\left[\sup_{s\in[t,T]} |Y_n(s) - Y_m(s)|^2\right].$$

Arguing as above, with Fatou's lemma and the previous uniform (in n and s) estimate, we get

$$h(t) \leq C \int_t^T h(s)\,\mathrm{d}s \Longrightarrow h(t) = 0.$$

Hence there is a càdlàg adapted process Y such that

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{s\in[0,T]} |Y_n(s) - Y(s)|^2\right] = 0.$$

And from the above estimate, $\mathbb{E}\left[\sup_{s\in[0,T]} |Y(s)|^p\right] < \infty$. This achieves the proof of Theorem 3.11. □

5.3. Existence and uniqueness for the Type-II BSVIE (1.7)

Coming back to BSVIE (1.7) and Proposition 3.12, suppose that $\Phi \in \mathbb{L}^2_{\mathcal{F}_T}(0,T)$, that **(H2)**, **(H3*)** and (3.16) hold[7]:

$$\mathbb{E}\int_0^T \left(\int_t^T |f^0(t,s)|\,\mathrm{d}s\right)^2 \mathrm{d}t < +\infty.$$

From Theorem 3.9, BSVIE (1.7) has a unique adapted M-solution (Y,Z,U,M) in $\mathfrak{S}^2(0,T)$ on $[0,T]$. Moreover (3.17) holds.

Nonetheless a direct proof could be given, following the outline of the proof of Theorem 3.10, that is of Theorem 3.7 in [46]. The modifications are quite obvious. In Step 1, fix $\Phi \in \mathbb{L}^2_{\mathcal{F}_T}(S,T)$ and $(y,\zeta,\nu,m) \in \widehat{\mathfrak{S}}^2(S,T)$ and consider the BSVIE on $[S,T]$

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T f(t,s,y(s),Z(t,s),\zeta(s,t),U(t,s),\nu(s,t))\,\mathrm{d}s - \int_t^T Z(t,s)\,\mathrm{d}W_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s). \end{aligned} \tag{5.20}$$

[7]Again the space $\mathfrak{H}$ in **(H2)** is replaced by $\mathbb{L}^2_\mu$ in this case.

We apply Lemma 5.1 and inequalities (5.13) and (5.14) become

$$\begin{aligned}\mathbb{E}\Bigg[&\int_S^T |Y(t)|^2\,\mathrm{d}t + \int_S^T\left(\int_t^T |Z(t,r)|^2\,\mathrm{d}r\right)\mathrm{d}t + \int_S^T \left(\langle M(t,\cdot)\rangle_{t,T}\right)\mathrm{d}t\\ &+\int_S^T\left(\int_t^T \|U(t,r)\|^2_{\mathbb{L}^2_\mu}\,\mathrm{d}r\right)\mathrm{d}t\Bigg]\\ \le C\mathbb{E}&\Bigg[\int_S^T|\Phi(t)|^2\mathrm{d}t + \int_S^T\left(\int_t^T|f^0(t,r)|\,\mathrm{d}r\right)^2\mathrm{d}t + \int_S^T|y(r)|^2\,\mathrm{d}r\\ &+\int_S^T\left(\int_t^T|\zeta(r,t)|^2\,\mathrm{d}r\right)\mathrm{d}t + \int_S^T\left(\int_t^T\|\nu(r,t)\|^2_{\mathbb{L}^2_\mu}\,\mathrm{d}r\right)\mathrm{d}t\Bigg]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[|\mathfrak{d}Y(t)|^2 + \int_S^T|\mathfrak{d}Z(t,r)|^2\,\mathrm{d}r + \langle\mathfrak{d}M(t,\cdot)\rangle_{S,T} + \int_S^T\int_{\mathbb{R}^m}|\mathfrak{d}U(t,r,x)|^2\pi(\,\mathrm{d}r,\,\mathrm{d}x)\right]\\ &\le CK^2(T-S)\mathbb{E}\left[\left(\int_S^T|\mathfrak{d}y(r)|^2\,\mathrm{d}r\right)+\left(\int_S^T\left(|\mathfrak{d}\zeta(t,r)|^2+\|\widehat{\nu}(t,r)\|^2_{\mathbb{L}^2_\pi}\right)\mathrm{d}r\right)\right].\end{aligned}$$

The second step remains unchanged, whereas in the third step, Lemma 5.2 is used, with

$$f^S(t,s,z,u) = f(t,s,Y(s),z,Z(s,t),u,U(s,t)),\quad (t,s,z,u)\in[R,S]\times[S,T]\times\mathbb{R}^k\times\mathbb{L}^2_\mu.$$

The last two steps are almost the same; the modifications are straightforward.

6. Comparison principle

In this section, dimension d is equal to one. Our goal is to extend some results contained in [42]. Note that the comparison principle for BSDEs has been proved in [23, 24] in the quasi left-continuous case (see also [11], Thm. 3.2.1 or [34], Prop. 5.32). In Theorem 3.25 of [31], the comparison principle is established for the BSDE:

$$Y_t = \xi + \int_t^T f(s,Y_{s-},Z_s,U_s(\cdot))\,\mathrm{d}B_s - \int_t^T Z_s\,\mathrm{d}X^\circ_s - \int_t^T U_s(x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M_s.$$

Compared to BSDE (1.2), the difference is that f depends on Y_{s-}, instead of Y_s. This property is crucial in [31], since they have to take into account the discontinuity of B. Before stating the comparison principle, let us recall that a generator f can be "linearized" as follows:

$$\begin{aligned}&f(\omega,s,y,z,u_s(\omega;\cdot)) - f(\omega,s,y',z',u'_s(\omega;\cdot))\\ &\quad=\lambda_s(\omega)(y-y') + \eta_s(\omega)b_s(\omega)(z-z')^\top + f(\omega,s,y,z,u(\cdot)) - f(\omega,s,y,z,u'(\cdot)).\end{aligned}$$

See Remark 3.24 of [31]. Let us emphasize that λ and η also depend on y,y',z,z',u,u',c. In particular λ and η are not predictable if they depend on Y_s. This is the reason why the previous BSDE (and not BSDE (1.2)) is studied for the comparison property in [31]. Nonetheless to simplify the notations and when no confusion may arise, we omit this dependence. If **(F2)** holds, $|\lambda_s(\omega)|^2 \le \varpi_s(\omega)$ and $|\eta_s(\omega)|^2 \le \theta^\circ_s(\omega)$ $\mathrm{d}\mathbb{P}\otimes\mathrm{d}B$-a.e. on $\Omega\times[0,T]$. The comparison principle is the following (similar to [31], Thm. 3.25).

Proposition 6.1. *For $i = 1, 2$, let (Y^i, Z^i, U^i, M^i) be solutions of BSDE (1.2) with standard data $(\bar{X}, B, \xi^i, f^i)$, that is* **(F1)** *to* **(F4)** *hold. Assume that*

(P1) *X° and B are continuous.*
(P2) *Generator f^1 is such that for any (s, y, z, u, u') in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^k \times \mathfrak{H} \times \mathfrak{H}$, there is some map $\kappa \in \mathbb{H}^{2,\natural}$ with $\Delta(\kappa \star \widetilde{\pi}^\natural) > -1$ on $[0, T]$ such that $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$-a.e. (ω, s), denoting $\mathfrak{d}u = u - u'$*

$$f^1(\omega, s, y, z, u(\cdot)) - f^1(\omega, s, y, z, u'(\cdot)) \leq \widehat{K}_s(\mathfrak{d}u_s(\cdot)\kappa_s(\cdot)).$$

If $\xi^1 \leq \xi^2$ a.s., and $f^1(s, Y^2_s, Z^2_s, U^2_s) \leq f^2(s, Y^2_s, Z^2_s, U^2_s)$ $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$-a.e., and if the stochastic exponential $\mathcal{E}(\eta \cdot X^\circ + \kappa \star \widetilde{\pi}^\natural)$ is a uniformly integrable martingale, then we have $\mathbb{P}$-a.s.: $Y^1_t \leq Y^2_t$ for any $t \in [0, T]$.

Proof. Set out in the appendix. □

Note that the continuity of X° is also supposed in Theorem 3.25 of [31]. Hence if we consider BSVIE (4.4) where generator f does not depend on y, ζ and θ:

$$\begin{aligned} Y(t) = \Phi(t) + \int_t^T f(t, s, Z(t, s), U(t, s))\, \mathrm{d}B_s - \int_t^T Z(t, s)\, \mathrm{d}X^\circ_s \\ - \int_t^T \int_{\mathbb{R}^m} U(t, s, x) \widetilde{\pi}(\,\mathrm{d}x,\, \mathrm{d}s) - \int_t^T \mathrm{d}M(t, s). \end{aligned}$$

the comparison principle holds.

Proposition 6.2. *For $i = 1, 2$, let $f^i : \Omega \times \Delta^c(0, T) \times \mathbb{R}^k \times \mathfrak{H} \to \mathbb{R}$ satisfy* **(H2)** *and* **(H3)**. *Moreover a.s.*

$$f^1(t, s, z, u) \leq f^2(t, s, z, u), \quad \forall (t, s, z, u) \in \Delta^c(0, T) \times \mathbb{R}^k \times \mathfrak{H}.$$

Assume that **(P1)** *and* **(P2)** *hold. Then for any $\Phi^i \in \mathbb{L}^p_{\mathcal{F}_T}(0, T)$ with $\Phi^0(t) \leq \Phi^1(t)$ a.s., $t \in [0, T]$, solutions (Y^i, Z^i, ψ^i, M^i) of (4.4) verify*

$$Y^1(t) \leq Y^2(t), \quad a.s.,\ t \in [0, T].$$

Proof. Let us consider $\lambda^i(t, \cdot)$ solution of the parametrized BSDE (4.3) with data (Φ^i, f^i). From Proposition 6.1, we obtain that a.s. for any $s \in [t, T]$, $\lambda^1(t, s) \leq \lambda^2(t, s)$. Sending s to t, since $Y^i(t) = \lambda^i(t, t)$, the desired result follows. □

Nevertheless to extend this result for generators depending also on y, in Theorem 3.4 of [42], f is supposed to be bounded (from above or from below) by a non-decreasing w.r.t. y generator. The next proposition extends this result to our setting.

Proposition 6.3. *Assume that the setting of Theorem 3.3 holds and consider two generators $f^i : \Omega \times \Delta^c(0, T) \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, satisfying* **(H2)** *and* **(H3)**. *Suppose that a.s. for a.e. $s \in [0, T]$ and for any $0 \leq t \leq s$ and any $(y, z, u) \in \mathbb{R} \times \mathbb{R}^k \times \mathfrak{H}$:*

$$f^1(t, s, y, z, u) \leq \bar{f}(t, s, y, z, u) \leq f^2(t, s, y, z, u). \tag{6.1}$$

Driver $\bar{f}$ also verifies **(H2)-(H3)** *and $y \mapsto \bar{f}(t,s,y,z,u)$ is non-decreasing. If a.s. for $0 \leq t \leq T$, $\Phi^2(t) \geq \Phi^1(t)$, then the corresponding solutions of BSVIEs (1.5) with generator f^i, verify for any $t \in [0,T]$:*

$$Y^2(t) \geq Y^1(t), \quad a.s.$$

Proof. Since the arguments of the proof are almost the same as in [42], the details are referred to the appendix. □

If drivers f^1 and f^2 cannot be "separated" by a non-decreasing generator $\bar{f}$, the restriction to half-linear generators is introduced as in Theorem 3.9 of [42]. Hence suppose that generator f is linear w.r.t. z and ψ:

$$f(t,s,y,z,u) = g(t,s,y) + h(s)b_s z + \widehat{K}_s(u(\cdot)\kappa_s(\cdot)), \tag{6.2}$$

where h is a process bounded by θ° and $\kappa : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}$ is progressively measurable and such that **(P2)** holds. Our comparison result is an extension to the jump case of Theorems 3.8 and 3.9 of [42] for BSVIE (1.5) where f is given by (6.2). The main difference comes from the free terms. Indeed in [42] where $B_t = t$, free term Φ is supposed to be in $C_{\mathcal{F}_T}([0,T], \mathbb{L}^2(\Omega))$ (see the functional spaces defined in Sect. 5.2). Hence for any partition $\Pi = \{t_k,\ 0 \leq k \leq N\}$ of $[0,T]$, if

$$\Phi^\Pi(t) = \sum_{k=1}^N \Phi(t_{k-1}) \mathbf{1}_{(t_{k-1},t_k]}(t), \tag{6.3}$$

then by uniform continuity, there exists a modulus of continuity ρ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|\Phi^\Pi(t) - \Phi(t)\right|^2\right] \leq \sup_{|t-t'|\leq\|\Pi\|} \mathbb{E}\left[\left|\Phi(t') - \Phi(t)\right|^2\right] \leq \rho(\|\Pi\|).$$

$\|\Pi\|$ is the mesh of the partition. In our setting, we cannot separate t and Ω, since B is random.

Proposition 6.4. *Consider two drivers $g^i : \Omega \times \Delta^c(0,T) \times \mathbb{R} \to \mathbb{R}$ satisfying* **(H2)**. *Suppose that for $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$-a.e. $(\omega, s) \in \Omega \times [0,T]$ and for any $0 \leq t \leq \tau \leq s$ and any $y \in \mathbb{R}$:*

$$g^2(t,s,y) - g^1(t,s,y) \geq g^2(\tau,s,y) - g^1(\tau,s,y) \geq 0. \tag{6.4}$$

Moreover for either $i = 1$ or $i = 2$

$$(g^i(t,s,y) - g^i(t,s,y'))(y-y') \geq (g^i(\tau,s,y) - g^i(\tau,s,y'))(y-y') \tag{6.5}$$

again for $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$-a.e. $(\omega,s) \in \Omega \times [0,T]$ and for any $0 \leq t \leq \tau \leq s$ and any y, y' in $\mathbb{R}$. Furthermore there exists a continuous non-decreasing function $\rho : [0,T] \to [0,+\infty)$ with $\rho(0) = 0$ such that a.s. for a.e. $s \in [0,T]$ and for any $0 \leq t, t' \leq s$

$$|g^i(t,s,y) - g^i(t,s,y') - g^i(t',s,y) + g^i(t',s,y')| \leq \rho(|t-t'|) \times |y-y'|. \tag{6.6}$$

Suppose that the difference of the free terms $\mathfrak{d}\Phi = \Phi^2 - \Phi^1$ satisfies:

$$\lim_{\|\Pi\|\to 0} \mathbb{E}\left[\int_0^T e^{\beta A_t} \left(\mathfrak{d}\Phi^\Pi(t) - \mathfrak{d}\Phi(t)\right) \mathrm{d}A_t\right] = 0, \tag{6.7}$$

where $\mathfrak{d}\Phi^\Pi$ *is defined by* (6.3)*. If a.s. for* $0 \leq t \leq \tau \leq T$*,*

$$\Phi^2(t) - \Phi^1(t) \geq \Phi^2(\tau) - \Phi^1(\tau) \geq 0, \tag{6.8}$$

then the corresponding solutions of BSVIEs (1.5) *with generator* f^i *given by* (6.2) *with* g^i *instead of* g*, verify for* $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$*-a.e.* $(\omega, t) \in \Omega \times [0, T]$

$$Y^2(t) \geq Y^1(t).$$

Proof. Let us first copy the arguments of the proof of Theorem 3.9 in [42]. Suppose that g^1 is differentiable and (6.5) holds for $i = 1$. Then

$$\begin{aligned} Y^2(t) - Y^1(t) &= \Phi^2(t) - \Phi^1(t) + \int_t^T \left[g^2(t,s,Y^2(s)) - g^1(t,s,Y^2(s)) \right] \mathrm{d}B_s \\ &+ \int_t^T L(t,s)(Y^2(s) - Y^1(s))\,\mathrm{d}B_s + \int_t^T h(s) b_s (Z^2(t,s) - Z^1(t,s))\,\mathrm{d}X^\circ_s \\ &+ \int_t^T \int_{\mathbb{R}^m} \kappa(s,x)(U^2(t,s,x) - U^1(t,s,x)) K_s(\,\mathrm{d}x)\,\mathrm{d}B_s \\ &- \int_t^T (Z^2(t,s) - Z^1(t,s))\,\mathrm{d}X^\circ_s - \int_t^T \int_{\mathbb{R}^m} (\psi^2(t,s,x) - \psi^1(t,s,x)) \widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) \\ &- \int_t^T \mathrm{d}(M^2(t,s) - M^1(t,s)) \end{aligned} \tag{6.9}$$

where

$$L(t,s) = \frac{g^1(t,s,Y^2(s)) - g^1(t,s,Y^1(s))}{Y^2(s) - Y^1(s)} \mathbf{1}_{Y^2(s) \neq Y^1(s)}, \quad 0 \leq t \leq s \leq T.$$

From **(H2)**, L is bounded by $\varpi(\omega, t, s)$. In other words we need to prove that the solution of the BSVIE

$$\begin{aligned} \mathfrak{d}Y(t) &= \mathfrak{d}\Phi(t) + \int_t^T L(t,s) \mathfrak{d}Y(s)\,\mathrm{d}B_s - \int_t^T \mathrm{d}\mathfrak{d}M(t,s) \\ &+ \int_t^T h(s) b_s \mathfrak{d}Z(t,s)\,\mathrm{d}B_s - \int_t^T \mathfrak{d}Z(t,s)\,\mathrm{d}X^\circ_s \\ &+ \int_t^T \widehat{K}_s(\kappa(s,\cdot)\mathfrak{d}U(t,s,\cdot))\,\mathrm{d}B_s - \int_t^T \int_{\mathbb{R}^m} \mathfrak{d}U(t,s,x) \widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) \end{aligned}$$

satisfies: for any $t \in [0,T]$, a.s. $\mathfrak{d}Y(t) \geq 0$. Here

$$\mathfrak{d}\Phi(t) = \Phi^2(t) - \Phi^1(t) + \int_t^T \left[g^2(t,s,Y^2(s)) - g^1(t,s,Y^2(s)) \right] \mathrm{d}B_s.$$

From our assumptions (6.4), (6.5) and (6.8), for $0 \leq t \leq \tau \leq T$, $\mathfrak{d}\Phi(t) \geq \mathfrak{d}\Phi(\tau) \geq 0$, $L(t,s) - L(\tau,s) \geq 0$ and from (6.6)

$$|L(t,s) - L(t',s)| \leq \rho(|t - t'|).$$

Now following the proof of Theorem 3.8 in [42], consider a partition $\Pi = \{t_k,\ 0 \le k \le N\}$ of $[0,T]$ and assume first that

$$L^\Pi(t,s) = \sum_{k=1}^N L(t_{k-1},s)\mathbf{1}_{(t_{k-1},t_k\wedge s]}(t), \quad \Phi^\Pi(t) = \sum_{k=1}^N \phi_k \mathbf{1}_{(t_{k-1},t_k]}(t)$$

where L^Π still satisfies $L^\Pi(t,s) - L^\Pi(\tau,s) \ge 0$ and ϕ_k are $\mathcal{F}_T$-measurable r.v. such that

$$\phi_1 \ge \phi_2 \ge \ldots \ge \phi_{N-1} \ge \phi_N \ge 0.$$

Let $(Y^\Pi(\cdot), Z^\Pi(\cdot,\cdot), U^\Pi(\cdot,\cdot), M^\Pi(\cdot,\cdot))$ be the solution of the BSVIE:

$$\begin{aligned} Y^\Pi(t) = \Phi^\Pi(t) &+ \int_t^T \Big[L^\Pi(t,s)Y^\Pi(s) + h(s)b_s Z^\Pi(t,s) + \widehat{K}_s(\kappa(s,\cdot)U^\Pi(t,s,\cdot))\Big]\,\mathrm{d}B_s \\ &- \int_t^T Z^\Pi(t,s)\,\mathrm{d}X^\circ_s - \int_t^T\int_{\mathbb{R}^m} U^\Pi(t,s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M^\Pi(t,s). \end{aligned} \tag{6.10}$$

Introduce the BSDE

$$\begin{aligned} Y_N(t) = \phi_N &+ \int_t^T \Big[L(t_{N-1},s)Y_N(s) + h(s)b_s Z_N(s) + \widehat{K}_s(\kappa(s,\cdot)U_N(s,\cdot))\Big]\,\mathrm{d}B_s \\ &- \int_t^T Z_N(s)\,\mathrm{d}X^\circ_s - \int_t^T\int_{\mathbb{R}^m} U_N(s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M_N(s). \end{aligned}$$

Then for $t_{N-1} < t \le s \le T$,

$$(Y_N(s), Z_N(s), U_N(s,e), M_N(s)) = (Y^\Pi(s), Z^\Pi(t,s), U^\Pi(t,s,e), M^\Pi(t,s))$$

solves BSVIE (6.10) on the interval $(t_{N-1},t_N]$. By the uniqueness of the solution and the comparison principle for BSDE, we obtain a.s. for any $s \in (t_{N-1},t_N]$, $Y^\Pi(s) = Y_N(s) \ge 0$. Since all martingales are càdlàg processes and since B is continuous (Hyp. **(P1)**),

$$\begin{aligned} Y^\Pi(t^+_{N-1}) = \phi_N &+ \int_{t_{N-1}}^T \Big[L(t_{N-1},s)Y^\Pi(s) + h(s)b_s Z_N(s) + \widehat{K}_s(\kappa(s,\cdot)U_N(s,\cdot))\Big]\,\mathrm{d}B_s \\ &- \int_{t_{N-1}}^T Z_N(s)\,\mathrm{d}X^\circ_s - \int_{t_{N-1}}^T\int_{\mathbb{R}^m} U_N(s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_{t_{N-1}}^T \mathrm{d}M_N(s) \ge 0. \end{aligned}$$

Now the BSVIE on $(t_{N-2},t_{N-1}]$ can be written as follows:

$$\begin{aligned} Y^\Pi(t) &= \phi_{N-1} + \int_t^T \Big[L(t_{N-2},s)Y^\Pi(s) + h(s)b_s Z^\Pi(t,s) + \widehat{K}_s(\kappa(s,\cdot)U^\Pi(t,s,\cdot))\Big]\,\mathrm{d}B_s \\ &\quad - \int_t^T Z^\Pi(t,s)\,\mathrm{d}X^\circ_s - \int_t^T\int_{\mathbb{R}^m} U^\Pi(t,s,x)\widetilde{\pi}^\natural(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \mathrm{d}M^\Pi(t,s) \\ &= \phi_{N-1} - \phi_N + Y^\Pi(t^+_{N-1}) + \int_{t_{N-1}}^T \left[L(t_{N-2},s) - L(t_{N-1},s)\right] Y^\Pi(s)\,\mathrm{d}B_s \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_{N-1}}^{T} h(s) b_s \left[Z^\Pi(t,s) - Z_N(s) \right] \mathrm{d}B_s - \int_{t_{N-1}}^{T} (Z^\Pi(t,s) - Z_N(s)) \, \mathrm{d}X^\circ_s \\
&+ \int_{t_{N-1}}^{T} \widehat{K}_s \left(\kappa(s,\cdot)(U^\Pi(t,s,\cdot) - U_N(s,\cdot)) \right) \mathrm{d}B_s \\
&- \int_{t_{N-1}}^{T} \int_{\mathbb{R}^m} \left[U^\Pi(t,s,x) - U_N(s,x) \right] \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_{t_{N-1}}^{T} \mathrm{d}(M^\Pi(t,s) - M_N(s)) \\
&+ \int_{t}^{t_{N-1}} \left[L(t_{N-2}, s) Y^\Pi(s) + h(s) b_s Z^\Pi(t,s) + \widehat{K}_s(\kappa(s,\cdot) U^\Pi(t,s,\cdot)) \right] \mathrm{d}B_s \\
&- \int_{t}^{t_{N-1}} Z^\Pi(t,s) \, \mathrm{d}X^\circ_s - \int_{t}^{t_{N-1}} \int_{\mathbb{R}^m} U^\Pi(t,s,x) \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_{t}^{t_{N-1}} \mathrm{d}M^\Pi(t,s).
\end{aligned}
$$

Now consider the terminal condition

$$
\xi_N = \phi_{N-1} - \phi_N + Y^\Pi(t^+_{N-1}) + \int_{t_{N-1}}^{T} \left[L(t_{N-2}, s) - L(t_{N-1}, s) \right] Y^\Pi(s) \, \mathrm{d}B_s
$$

and solution $(\widetilde{Y}_N, \widetilde{Z}_N, \widetilde{U}_N, \widetilde{M}_N)$ of the linear BSDE on $[t_{N-1}, T]$:

$$
\begin{aligned}
\widetilde{Y}_N(t) = \xi_N &+ \int_{t}^{T} \left[h(s) b_s \widetilde{Z}_N(s) + \widehat{K}_s(\kappa(s,\cdot) \widetilde{U}_N(t,s,\cdot)) \right] \mathrm{d}B_s \\
&- \int_{t}^{T} \widetilde{Z}_N(s) \, \mathrm{d}X^\circ_s - \int_{t}^{T} \int_{\mathbb{R}^m} \widetilde{U}_N(t,s,x) \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_{t}^{T} \mathrm{d}\widetilde{M}_N(s).
\end{aligned}
$$

By our conditions, ξ_N is non-negative and thus a.s. $\widetilde{Y}_N(t) \geq 0$ on $[t_{N-1}, T]$. The uniqueness of adapted solutions to the BSVIE leads to:

$$
Z^\Pi(t,s) = Z_N(s) + \widetilde{Z}_N(s), \quad U^\Pi(t,s) = U_N(s) + \widetilde{U}_N(s), \quad M^\Pi(t,s) = M_N(s) + \widetilde{M}_N(s)
$$

for $(t,s) \in (t_{N-2}, t_{N-1}] \times (t_{N-1}, t_N]$ and our previous BSVIE becomes

$$
\begin{aligned}
Y^\Pi(t) = \widetilde{Y}_N(t_{N-1}) &+ \int_{t}^{t_{N-1}} \left[L(t_{N-2}, s) Y^\Pi(s) + h(s) b_s Z^\Pi(t,s) + \widehat{K}_s(\kappa(s,\cdot) U^\Pi(t,s,\cdot)) \right] \mathrm{d}B_s \\
&- \int_{t}^{t_{N-1}} Z^\Pi(t,s) \, \mathrm{d}X^\circ_s - \int_{t}^{t_{N-1}} \int_{\mathbb{R}^m} U^\Pi(t,s,x) \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_{t}^{t_{N-1}} \mathrm{d}M^\Pi(t,s).
\end{aligned}
$$

Again we solve the BSDE:

$$
\begin{aligned}
&Y_{N-1}(t) = \widetilde{Y}_N(t_{N-1}) \\
&\quad + \int_{t}^{t_{N-1}} \left[L(t_{N-2}, s) Y_{N-1}(s) + h(s) b_s Z_{N-1}(t,s) + \widehat{K}_s(\kappa(s,\cdot) U_{N-1}(t,s,\cdot)) \right] \mathrm{d}B_s \\
&\quad - \int_{t}^{t_{N-1}} Z_{N-1}(t,s) \, \mathrm{d}X^\circ_s - \int_{t}^{t_{N-1}} \int_{\mathbb{R}^m} U_{N-1}(t,s,x) \widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_{t}^{t_{N-1}} \mathrm{d}M_{N-1}(t,s)
\end{aligned}
$$

on $[t_{N-2}, t_{N-1}]$ and by the uniqueness and the comparison principle for BSDE, we deduce that

$$
Y^\Pi(t) = Y_{N-1}(t) \geq 0, \quad t \in [t_{N-2}, t_{N-1}].
$$

By induction we obtain that $Y^\Pi(t) \geq 0$, $t \in [0,T]$.

From Theorem 3.3, the stability estimate for BSVIE yields to:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A_t} |Y^\Pi(t) - Y(t)|^2 \, \mathrm{d}A_t\right] \\
&\leq \mathfrak{C}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t} |\Phi^\Pi(t) - \Phi(t)|^2 \, \mathrm{d}A_t\right] \\
&\quad + \mathfrak{C}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|L^\Pi(t,s) - L(t,s)|^2 |Y(s)|^2}{\alpha_s^2} \, \mathrm{d}B_s \, \mathrm{d}A_t\right].
\end{aligned}$$

From the time regularity condition for L, $|L^\Pi(t,s) - L(t,s)|^2 \leq \rho(\|\Pi\|)^2$. Condition (6.7) implies that

$$\lim_{\|\Pi\| \to 0} \mathbb{E}\left[\int_0^T e^{\beta A_t} |Y^\Pi(t) - Y(t)|^2 \, \mathrm{d}A_t\right] = 0.$$

The conclusion of the proposition follows and this achieves the proof. □

Comparison principle in the Itô setting

In the framework of Section 3.3, the results of Propositions 6.3 and 6.4 remain true here. Condition (6.7) holds in this case if free terms Φ^i belong to $C([0,T], \mathbb{L}^2(\Omega))$, as in [42].

The *duality principle* of linear stochastic integral equations (see [45], Sect. 4) plays an important role for comparison principle or optimal control problem (see [46], Sect. 5). This result is based on the notion of FSVIE (see among many others [5, 6, 20, 21, 33, 35]). In [45, 46], the next FSVIE is considered: for $t \in [0,T]$

$$X(t) = \Psi(t) + \int_0^t \Upsilon_0(t,s) X(s) \, \mathrm{d}s + \int_0^t \sum_{i=1}^k \Upsilon_i(t,s) X(s) \, \mathrm{d}W_i(s),$$

where $\Upsilon_i(\cdot,\cdot) \in \mathbb{L}^\infty([0,T]; \mathbb{L}^\infty_{\mathbb{F}}(0,T;\mathbb{R}^{d\times d}))$ for $i = 0, 1, \ldots, k$. It means that $\Upsilon_i : \Omega \times [0,T]^2 \to \mathbb{R}^{d\times d}$ is bounded, $\mathcal{F}_T \otimes \mathcal{B}([0,T]^2)$-measurable and for almost all $t \in [0,T]$, $\Upsilon_i(t,\cdot)$ is $\mathcal{F}$-adapted. Then for any $\Psi \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, there exists a unique solution X in $\mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

Here the next extension is studied: for $t \in [0,T]$

$$\begin{aligned}
X(t) = \Psi(t) &+ \int_0^t \Upsilon_0(t,s) X(s) \, \mathrm{d}s + \int_0^t \sum_{i=1}^k \Upsilon_i(t,s) X(s-) \, \mathrm{d}W_i(s) \\
&+ \int_0^t \int_{\mathbb{R}^m} \Xi(t,s,x) X(s-) \widetilde{\pi}(\mathrm{d}s, \mathrm{d}x).
\end{aligned} \tag{6.11}$$

The same conditions on the Υ_i are kept. We assume that $\Xi : \Omega \times [0,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ is bounded and such that for almost all $t \in [0,T]$, $\Xi(t,\cdot,\cdot)X(\cdot) \star \widetilde{\pi}$ is well defined. Since we are interested in càdlàg processes X, we use the setting of Condition 4.1 in [35]. Hence we also suppose that Υ_i and Ξ are differentiable w.r.t. t with a bounded derivative (uniformly in (ω,t,s)). Thus we can apply ([35], Thm. 4.3): if Ψ is a càdlàg process, then there exists a unique càdlàg solution X of the previous FSVIE. The key point is that X is a càdlàg process, hence for a.e. $t \in [0,T]$, $X(t) = X(t-)$.

Lemma 6.5. *Let $\Psi(\cdot) \in \mathbb{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d) \cap \mathbb{D}^2(0,T)$ and $\Phi(\cdot) \in \mathbb{L}^2((0,T)\times\Omega;\mathbb{R}^d)$. Let $X \in \mathbb{L}^2(0,T;\mathbb{R}^d)$ be the càdlàg solution of the linear FSVIE (6.11). We also consider the BSVIE:*

$$Y(t) = \Phi(t) - \int_t^T Z(t,s)\,\mathrm{d}W_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}(\,\mathrm{d}s,\,\mathrm{d}x) - \int_t^T \,\mathrm{d}M(t,s)$$
$$+ \int_t^T \left[\Upsilon_0(s,t)^\top Y(s) + \sum_{i=1}^k \Upsilon_i(s,t)^\top Z_i(s,t) + \int_{\mathbb{R}^m} \Xi(s,t,x)^\top U(s,t,x)\mu(\,\mathrm{d}x) \right] \mathrm{d}s.$$

Then

$$\mathbb{E}\int_0^T \langle \Psi(t), Y(t)\rangle \,\mathrm{d}t = \mathbb{E}\int_0^T \langle X(t), \Phi(t)\rangle \,\mathrm{d}t.$$

Proof. The arguments are the same as [30, 46] and are based on the orthogonality of W, $\widetilde{\pi}$ and M. Details are skipped here. □

Let us emphasize that the role of the càdlàg property of X is important here. Thus it should be possible to relax the regularity assumption on the coeffcients Υ_i or Ξ of the FSVIE. But as for a BSVIE, the regularity of the paths of X is neither a direct property nor an easy stuff.

Note that the extension of the duality result to the setting of Section 4 is an issue. Indeed a solution for the Type-II BSVIE (1.4) is first needed. But the orthogonality between B, X°, $\widetilde{\pi}^\natural$ and M is much more delicate and several simplifications are not true anymore with these processes.

With the previous duality result, it is possible to extend the comparison principle for M-solution of a Type-II BSVIE of the form:

$$Y(t) = \Phi(t) + \int_t^T (g(t,s,Y(s)) + C(s)Z(s,t))\,\mathrm{d}s - \int_t^T \,\mathrm{d}\mathrm{M}^\sharp(t,s).$$

Up to some technical conditions, one can follow the scheme of Theorems 3.12 and 3.13 in [42].

Appendix A.

A.1 Proof of Lemma 3.2

Recall that for $\delta < \gamma \leq \beta$:

$$\Pi^{\mathfrak{f}}(\gamma,\delta) = \frac{11}{\delta} + 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)}.$$

The lemma states that the infimum of $\Pi^{\mathfrak{f}}(\gamma,\delta)$ over all $\delta < \gamma \leq \beta$ is given by $M^{\mathfrak{f}}(\beta) = \widetilde{\Pi}^{\mathfrak{f}}(\beta, \delta^*(\beta))$, where $\delta^*(\beta)$ is the unique solution on $(0,\beta)$ of the equation:

$$11(\beta - x)^2 - 9e^{(\beta-x)\mathfrak{f}}x^2(\mathfrak{f}(\beta-x) - 1) = 0.$$

For any fixed $\gamma \in (0, \beta]$, since

$$\lim_{\delta \to 0} \Pi^{\mathfrak{f}}(\gamma, \delta) = \lim_{\delta \to \gamma} \Pi^{\mathfrak{f}}(\gamma, \delta) = +\infty,$$

there exists a $\delta^*(\gamma) \in (0, \gamma)$ such that

$$\Pi^{\mathfrak{f}}(\delta^*(\gamma), \delta) = \inf_{0<\delta<\gamma} \Pi^{\mathfrak{f}}(\gamma, \delta),$$

and $\delta^*(\gamma)$ is the critical point of

$$\frac{\partial \Pi^{\mathfrak{f}}}{\partial \delta}(\gamma, \delta) = -\frac{11}{\delta^2} + 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)^2}\left(-\mathfrak{f}(\gamma-\delta)+1\right),$$

that is

$$-\frac{11}{(\delta^*(\gamma))^2} = 9\frac{e^{(\gamma-\delta^*(\gamma))\mathfrak{f}}}{(\gamma-\delta^*(\gamma))^2}\left(\mathfrak{f}(\gamma-\delta^*(\gamma))-1\right) < 0.$$

Note that necessarily

$$\left(\gamma - \frac{1}{\mathfrak{f}}\right) \vee 0 < \delta^*(\gamma) < \gamma.$$

Let us differentiate w.r.t. γ:

$$\frac{\partial \Pi^{\mathfrak{f}}}{\partial \gamma}(\gamma, \delta) = 9e^{(\gamma-\delta)\mathfrak{f}}\frac{1}{\gamma-\delta}\left(\mathfrak{f} - \frac{1}{\gamma-\delta}\right) = 9\frac{e^{(\gamma-\delta)\mathfrak{f}}}{(\gamma-\delta)^2}\left(\mathfrak{f}(\gamma-\delta)-1\right).$$

Hence a critical point should satisfy:

$$\gamma^*(\delta) = \delta + \frac{1}{\mathfrak{f}}.$$

It is admissible if and only $\beta > 1/\mathfrak{f}$.

- Assume that $\beta > 1/\mathfrak{f}$. Then

$$\Pi^{\mathfrak{f}}(\gamma^*(\delta), \delta) = \frac{11}{\delta} + 9e\mathfrak{f}.$$

  This quantity has no critical point on $0 < \delta < \gamma < \beta$. Hence the infimum is attained on its boundary.
- If $\beta\mathfrak{f} \leq 1$, the partial derivative w.r.t. γ remains non-positive. There is no critical point in this case. And again the infimum is attained at its boundary.

The cases where one among δ or γ goes to zero or where their difference goes to 0, lead to the value $+\infty$. The only remaining case is therefore $0 < \delta < \gamma = \beta$. Then the minimum is attained at $\delta^*(\beta)$ and the infimum is equal to

$$M^{\mathfrak{f}}(\beta) = \frac{11}{\delta^*(\beta)} + 9\frac{e^{(\beta-\delta^*(\beta))\mathfrak{f}}}{(\beta-\delta^*(\beta))}.$$

Let us recall that $\delta^*(\beta)$ is the unique solution in the interval $((\beta - 1/\mathfrak{f}) \vee 0, \beta)$ of the equation

$$11(\beta - x)^2 - 9e^{(\beta - x)\mathfrak{f}} x^2 (\mathfrak{f}(\beta - x) - 1) = 0.$$

Now let us compute the limit as β goes to ∞. Since $\delta^*(\beta) \geq \beta - 1/\mathfrak{f}$, $\lim_{\beta \to +\infty} \delta^*(\beta) = +\infty$. Hence since $\delta^*(\beta)$ solves the previous equation, dividing by x^2, we obtain that

$$\frac{11}{9} \left(\frac{\beta - \delta^*(\beta)}{\delta^*(\beta)} \right)^2 e^{-(\beta - \delta^*(\beta))\mathfrak{f}} + 1 = \mathfrak{f}(\beta - \delta^*(\beta)).$$

Thus $\lim_{\beta \to +\infty} (\beta - \delta^*(\beta)) = \dfrac{1}{\mathfrak{f}}$. Thereby $\lim_{\beta \to +\infty} M^{\mathfrak{f}}(\beta) = 9e\mathfrak{f}$.

A.2 Comparison principle (Sect. 6)

A.2.1 Proof of Proposition 6.1

This result is a comparison principle for the BSDE (1.2), where X° and B are continuous (assumption **(P1)**). The proof follows the arguments of Theorem 3.25 in [31]. Nonetheless since B is supposed to be continuous, there are some important simplifications.

Let us define

$$(\mathfrak{d}Y, \mathfrak{d}Z, \mathfrak{d}Y, \mathfrak{d}M, \mathfrak{d}\xi) = (Y^1 - Y^2, Z^1 - Z^2, U^1 - U^2, M^1 - M^2, \xi^1 - \xi^2)$$

and

$$\mathfrak{d}f^1_s = f^1(s, Y^1_{s-}, Z^1_s, U^1_s) - f^1(s, Y^2_{s-}, Z^2_s, U^2_s), \quad \mathfrak{d}f^{1,2}_s = f^1(s, Y^2_s, Z^2_s, U^2_s) - f^2(s, Y^2_s, Z^2_s, U^2_s).$$

Let us stress that in $\mathfrak{d}f^1$, we consider Y^i_{s-} and not Y^i_s. For a non-negative predictable process γ, define $v = \int_0^\cdot \gamma_s \, \mathrm{d}B_s$. Define $\mathcal{E}(v)_t = \exp(v_t)$. The Itô formula gives:

$$\mathcal{E}(v)_t \mathfrak{d}Y_t = \mathcal{E}(v)_T \mathfrak{d}\xi + \int_t^T \left[\mathcal{E}(v)_{s-} (\mathfrak{d}f^{1,2}_s + \mathfrak{d}f^1_s) - \gamma_s \mathfrak{d}Y_{s-} \right] \mathrm{d}B_s - \int_t^T \mathcal{E}(v)_{s-} \mathfrak{d}Z_s \, \mathrm{d}X^\circ_s$$
$$- \int_t^T \int_{\mathbb{R}^m} \mathcal{E}(v)_{s-} \mathfrak{d}U_s(x) \, \mathrm{d}\widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathcal{E}(v)_{s-} \, \mathrm{d}\mathfrak{d}M_s.$$

See Theorem 3.20 of [31] and note the modification due to the continuity of B. Let us also emphasize that the continuity of B also leads to:

$$\int_t^T f^1(s, Y^1_s, Z^1_s, U^1_s) \, \mathrm{d}B_s = \int_t^T f^1(s, Y^1_{s-}, Z^1_s, U^1_s) \, \mathrm{d}B_s.$$

Namely we can use this predictable version of the driver in BSDE (1.2). This property fails in general and this is the reason of the Section 3.6 in [31]. Anyway the linearization procedure for δf^1 implies that

$$\mathfrak{d}f^1_s = \lambda_s \mathfrak{d}Y_{s-} + \eta_s b_s \mathfrak{d}Z^\top_s + f^1(s, Y^2_{s-}, Z^2_s, U^1_s) - f^1(s, Y^2_{s-}, Z^2_s, U^2_s)$$

where λ is a one-dimensional predictable process s.t. $|\lambda_s(\omega)| \leq \varpi^1_s(\omega)$ and η is a m-dimensional predictable process such that $|\eta|^2 \leq \theta^{1,\circ}$, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}B$-a.e. (using **(H2)** for f^1). Let us choose $\gamma = \lambda$ and from **(P2)**

$$\begin{aligned}
\mathcal{E}(v)_t \mathfrak{d}Y_t \leq \mathcal{E}(v)_T \mathfrak{d}\xi &+ \int_t^T \left[\mathcal{E}(v)_{s-} \mathfrak{d}f^{1,2}_s\right] \mathrm{d}B_s \\
&+ \int_t^T \mathcal{E}(v)_{s-} \eta_s b_s \mathfrak{d}Z_s^\top \, \mathrm{d}B_s - \int_t^T \mathcal{E}(v)_{s-} \mathfrak{d}Z_s \, \mathrm{d}X^\circ_s \\
&+ \int_t^T \mathcal{E}(v)_{s-} \widehat{K}_s(\mathfrak{d}U_s(\cdot)\kappa_s(\cdot)) \, \mathrm{d}B_s - \int_t^T \int_{\mathbb{R}^m} \mathcal{E}(v)_{s-} \mathfrak{d}U_s(x) \, \mathrm{d}\widetilde{\pi}^\natural(\mathrm{d}s, \mathrm{d}x) \\
&- \int_t^T \mathcal{E}(v)_{s-} \, \mathrm{d}\mathfrak{d}M_s.
\end{aligned}$$

Now by the Girsanov transform, if $\mathbb{Q}$ is the probability measure defined by:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\eta \cdot X^\circ + \kappa \star \widetilde{\pi}^\natural\right),$$

where $\mathcal{E}(v)$ stands for the stochastic exponential operator, then the assumptions on κ imply that $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Taking the conditional expectation under $\mathbb{Q}$ in the previous inequality yields to

$$\mathcal{E}(v)_t \mathfrak{d}Y_t \leq \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(v)_T \mathfrak{d}\xi + \int_t^T \left[\mathcal{E}(v)_{s-} \mathfrak{d}f^{1,2}_s\right] \mathrm{d}B_s \middle| \mathcal{F}_t\right] \leq 0.$$

The details concerning the disappeared martingale terms can be found in the proof of Theorem 3.25 in [31]. Hence the conclusion of the Proposition follows.

A.2.2 Proof of Proposition 6.3

The arguments are the same as for Theorem 3.4 in [42]. Let us consider the unique solution in $\mathfrak{S}^2_\beta(\Delta^c)$ of the BSVIE:

$$\begin{aligned}
\bar{Y}(t) = \overline{\Phi}(t) &+ \int_t^T \bar{f}(t, s, \bar{Y}(s), \bar{Z}(t,s), \bar{U}(t,s)) \, \mathrm{d}B_s - \int_t^T \bar{Z}(t,s) \, \mathrm{d}X^\circ_s \\
&- \int_t^T \int_{\mathbb{R}^m} \bar{U}(t,s,x) \widetilde{\pi}(\mathrm{d}x, \mathrm{d}s) - \int_t^T \mathrm{d}\bar{M}(t,s),
\end{aligned} \tag{A.1}$$

together with the solution of:

$$\begin{aligned}
\widetilde{Y}_1(t) = \overline{\Phi}(t) &+ \int_t^T \bar{f}(t, s, \widetilde{Y}_0(s), \widetilde{Z}_1(t,s), \widetilde{U}_1(t,s)) \, \mathrm{d}B_s - \int_t^T \widetilde{Z}_1(t,s) \, \mathrm{d}X^\circ_s \\
&- \int_t^T \int_{\mathbb{R}^m} \widetilde{U}_1(t,s,x) \widetilde{\pi}(\mathrm{d}x, \mathrm{d}s) - \int_t^T \mathrm{d}\widetilde{M}_1(t,s),
\end{aligned}$$

where $\widetilde{Y}_0 = Y^2$. Since

$$\bar{f}(t, s, \widetilde{Y}_0(s), z, u) \leq f^2(t, s, \widetilde{Y}_0(s), z, u), \qquad \overline{\Phi}(t) \leq \Phi^2(t),$$

from Proposition 6.2, there exists a measurable set Ω_t^1 verifying $\mathbb{P}(\Omega_t^1) = 0$ such that

$$\widetilde{Y}_1(\omega,t) \leq \widetilde{Y}_0(\omega,t), \quad \omega \in \Omega \setminus \Omega_t^1, \quad t \in [0,T].$$

Then consider the BSVIE

$$\widetilde{Y}_2(t) = \overline{\Phi}(t) + \int_t^T \bar{f}(t,s,\widetilde{Y}_1(s),\widetilde{Z}_2(t,s),\widetilde{U}_2(t,s))\,\mathrm{d}B_s - \int_t^T \widetilde{Z}_2(t,s)\,\mathrm{d}X_s^\circ - \int_t^T \int_{\mathbb{R}^m} \widetilde{U}_2(t,s,x)\widetilde{\pi}(\,\mathrm{d}x,\,\mathrm{d}s) - \int_t^T \,\mathrm{d}\widetilde{M}_2(t,s).$$

The previous arguments show that for any $t \in [0,T]$, there exists a measurable set Ω_t^2 verifying $\mathbb{P}(\Omega_t^2) = 0$ such that

$$\widetilde{Y}_2(\omega,t) \leq \widetilde{Y}_1(\omega,t), \quad \omega \in \Omega \setminus \Omega_t^2, \quad t \in [0,T].$$

By induction, a sequence $(\widetilde{Y}_k, \widetilde{Z}_k, \widetilde{U}_k, \widetilde{Y}_k) \in \mathfrak{S}^2_\beta(\Delta^c)$ and Ω_t^k satisfying $\mathbb{P}(\Omega_t^k) = 0$ can be constructed

$$\widetilde{Y}_k(t) = \overline{\Phi}(t) + \int_t^T \bar{f}(t,s,\widetilde{Y}_{k-1}(s),\widetilde{Z}_k(t,s),\widetilde{U}_k(t,s))\,\mathrm{d}B_s - \int_t^T \widetilde{Z}_k(t,s)\,\mathrm{d}X_s^\circ - \int_t^T \int_{\mathbb{R}^m} \widetilde{U}_k(t,s,x)\widetilde{\pi}(\,\mathrm{d}x,\,\mathrm{d}s) - \int_t^T \,\mathrm{d}\widetilde{M}_k(t,s),$$

and for any $t \in [0,T]$ and $\omega \in \Omega \setminus \left(\bigcup_{k\geq 1} \Omega_t^k\right)$:

$$Y^2(\omega,t) \geq \widetilde{Y}_1(\omega,t) \geq \widetilde{Y}_2(\omega,t) \geq \cdots$$

The set $\bigcup_{k\geq 1} \Omega_t^k$ is a $\mathbb{P}$-null set. And using Lemma 4.2 and inequality (4.11) yields to:

$$\mathbb{E}\left[\int_0^T e^{\beta A_t} |\widetilde{Y}_k(t) - \widetilde{Y}_\ell(t)|^2 \,\mathrm{d}A_t + \int_S^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \,\mathrm{d}\,\mathrm{Trace}\langle \widehat{\mathrm{M}^\sharp}(t,\cdot)\rangle_s \,\mathrm{d}A_t\right] \leq \Sigma^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{(\beta-\delta)A_t} \int_t^T e^{\delta A_s} \frac{|\widehat{f}(t,s)|^2}{\alpha_s^2} \,\mathrm{d}B_s \,\mathrm{d}A_t\right],$$

with

$$\widehat{f}(t,s) = \bar{f}(t,s,\widetilde{Y}_{k-1}(s),\widetilde{Z}_k(t,s),\widetilde{U}_k(t,s)) - \bar{f}(t,s,\widetilde{Y}_{\ell-1}(s),\widetilde{Z}_k(t,s),\widetilde{U}_k(t,s)).$$

Condition **(H2)** on $\bar{f}$ leads to

$$\mathbb{E}\left[\int_0^T e^{\beta A_t} |\widetilde{Y}_k(t) - \widetilde{Y}_\ell(t)|^2 \,\mathrm{d}A_t\right] \leq \widetilde{\Sigma}^{\mathfrak{f}}(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t} |\widetilde{Y}_{k-1}(t) - \widetilde{Y}_{\ell-1}(t)|^2 \,\mathrm{d}A_t\right].$$

From hypothesis (3.5) of Theorem 3.3, sequence $(\widetilde{Y}_k, \widetilde{Z}_k, \widetilde{U}_k, \widetilde{Y}_k)$ is a Cauchy sequence in $\mathfrak{S}^2_\beta(\Delta^c)$, converging to solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{M})$ of BSVIE (A.1). Thereby for any $t \in [0,T]$

$$\bar{Y}(t) \leq Y^2(t), \quad \text{a.s..}$$

Similar arguments imply that $Y^1(t) \leq \bar{Y}(t)$ and achieve the proof of the Proposition.

A.3 L^p-continuity

If (Y, Z, U, M) solves BSVIE (1.6), then taking $h(t, s, z, \psi) = f(t, s, Y(s), z, \psi)$ and using estimate (5.6) of Lemma 4.1, we have:

$$\begin{aligned}
&\mathbb{E}\left[|Y(t)|^p + \left(\int_S^T |Z(t,r)|^2 \,\mathrm{d}r \right)^{\frac{p}{2}} + (\langle M(t,\cdot)\rangle_{S,T})^{\frac{p}{2}} \right. \\
&\left. + \left(\int_S^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\,\mathrm{d}x,\,\mathrm{d}r) \right)^{\frac{p}{2}} \right] \leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_S^T |h(t,r,0,0)| \,\mathrm{d}r \right)^p \right].
\end{aligned}$$

Since f is Lipschitz continuous, taking $S = t$, the Gronwall inequality leads to

$$\begin{aligned}
&\mathbb{E}\left[|Y(t)|^p + \left(\int_t^T |Z(t,r)|^2 \,\mathrm{d}r \right)^{\frac{p}{2}} + (\langle M(t,\cdot)\rangle_{t,T})^{\frac{p}{2}} + \left(\int_t^T \int_{\mathbb{R}^m} |U(t,r,x)|^2 \pi(\,\mathrm{d}x,\,\mathrm{d}r) \right)^{\frac{p}{2}} \right] \\
&\leq C\mathbb{E}\left[|\Phi(t)|^p + \left(\int_t^T |\Phi(r)| \,\mathrm{d}r \right)^p + \left(\int_t^T |f^0(t,r)| \,\mathrm{d}r \right)^p \right].
\end{aligned}$$

Under our stronger integrability conditions (3.14) (3.15) for f^0 and Φ, a stronger estimate on (Z, U, M) is derived:

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(\int_t^T |Z(t,r)|^2 \,\mathrm{d}r \right)^{\frac{p}{2}} + (\langle M(t,\cdot)\rangle_{t,T})^{\frac{p}{2}} + \left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(t,T)} \right)^{\frac{p}{2}} \right] < +\infty.$$

This property is important to get the càdlàg in mean property of Y.

Lemma A.1. *Assume that* **(H2)** *and* **(H3*)** *hold. Then the solution of BSVIE* (1.6) *satisfies: for any* $(t, t') \in [S, T]$ *and if* $t_\star = t \wedge t'$ *and* $t^\star = t \vee t'$*:*

$$\begin{aligned}
&\mathbb{E}\left[|Y(t) - Y(t')|^p\right] \\
&\quad + \mathbb{E}\left[\left(\int_S^{t_\star} |Z(t,r) - Z(t',r)|^2 dr \right)^{p/2} + \left(\int_{t^\star}^T |Z(t,r) - Z(t',r)|^2 dr \right)^{p/2} \right] \\
&\quad + \mathbb{E}\left[\left([M(t,\cdot) - M(t',\cdot)]_{S,t_\star} \right)^{p/2} + ([M(t,\cdot) - M(t',\cdot)]_{t^\star,T})^{p/2} \right] \\
&\quad + \mathbb{E}\left[\left(\|U(t,\cdot) - U(t',\cdot)\|^2_{\mathbb{L}^2_\pi(S,t_\star)} \right)^{p/2} + \left(\|U(t,\cdot) - U(t',\cdot)\|^2_{\mathbb{L}^2_\pi(t^\star,T)} \right)^{p/2} \right]
\end{aligned}$$

$$\leq C\mathbb{E}\left[|\Phi(t)-\Phi(t')|^p\right] + C\mathbb{E}\left[\left(\int_{t_\star}^{t^\star} |h(t,r,Z(t,r),\psi(t,r))|\, dr\right)^p\right]$$
$$+C\mathbb{E}\left[\left(\int_{t_\star}^{t^\star} |Z(t,r)|^2\, dr\right)^{p/2}\right] + C\mathbb{E}\left[\left(\|U(t,\cdot)\|^2_{\mathbb{L}^2_\pi(t_\star,t^\star)}\right)^{p/2}\right]$$
$$+C\mathbb{E}\left[\left([M(t,\cdot)]_{t_\star,t^\star}\right)^{p/2}\right]$$
$$+C\mathbb{E}\left[\left(\int_{t^\star}^{T} |h(t,r,Z(t,r),\psi(t,r)) - h(t',r,Z(t,r),\psi(t,r))|dr\right)^p\right].$$

Proof. We consider BSVIE (4.4):

$$Y(t) = \Phi(t) + \int_t^T h(t,s,Z(t,s),U(t,s))ds - \int_t^T Z(t,s)dW_s - \int_t^T \int_{\mathcal{E}} U(t,s,e)\widetilde{\pi}(de,ds) - \int_t^T dM(t,s).$$

We take t, t' in $[S,T]$ and w.l.o.g. let $S \leq t \leq t' \leq T$. Applying (5.8) to the solution of the BSDE with parameter t, we obtain:

$$\mathbb{E}\left[\sup_{r\in[t',T]} |\lambda(t,r)-\lambda(t',r)|^p + \left(\int_{t'}^T |z(t,r)-z(t',r)|^2 dr\right)^{p/2}\right.$$
$$\left. + \left([m(t,\cdot)-m(t',\cdot)]_{t',T}\right)^{p/2} + \left(\int_{t'}^T\int_{\mathcal{E}} |u(t,r,e)-u(t',r,e)|^2\pi(de,dr)\right)^{p/2}\right]$$
$$\leq C\mathbb{E}\left[|\Phi(t)-\Phi(t')|^p + \left(\int_{t'}^T |h(t,r,z(t,r),u(t,r)) - h(t',r,z(t,r),u(t,r))|dr\right)^p\right]$$
$$= C\mathbb{E}\left[|\Phi(t)-\Phi(t')|^p + \left(\int_{t'}^T |h(t,r,Z(t,r),\psi(t,r)) - h(t',r,Z(t,r),\psi(t,r))|dr\right)^p\right]$$

Remark that

$$\mathbb{E}\left[|Y(t)-Y(t')|^p\right] = \mathbb{E}\left[|\lambda(t,t)-\lambda(t',t')|^p\right]$$
$$\leq C\mathbb{E}\left[|\lambda(t,t)-\lambda(t,t')|^p\right] + C\mathbb{E}\left[\sup_{r\in[t',T]} |\lambda(t,r)-\lambda(t',r)|^p\right]$$
$$\leq C\mathbb{E}\left[\left(\int_t^{t'} |h(t,r,Z(t,r),\psi(t,r))|\, dr\right)^p\right] + C\mathbb{E}\left[\left(\int_t^{t'} |Z(t,r)|^2\, dr\right)^{p/2}\right]$$
$$+C\mathbb{E}\left[\left(\int_t^{t'} |\psi(t,r)|^2_{\mathbb{L}^2_\pi}\, dr\right)^{p/2}\right] + C\mathbb{E}\left[\left([M(t,\cdot)]_{t,t'}\right)^{p/2}\right]$$
$$+C\mathbb{E}\left[\sup_{r\in[t',T]} |\lambda(t,r)-\lambda(t',r)|^p\right].$$

Moreover the notion of M-solution (Eq. (3.7)) implies that

$$
\begin{aligned}
Y(t) - Y(t') - \mathbb{E}\left[Y(t) - Y(t')|\mathcal{F}_S\right] &= \int_S^t (Z(t,r) - Z(t',r))dW_r \\
&+ \int_S^t \int_{\mathcal{E}} (U(t,r,e) - U(t',r,e))\widetilde{\pi}(de,dr) + \int_S^t d(M(t,r) - M(t',r)) \\
&+ \int_t^{t'} Z(t',r)dW_r + \int_t^{t'} \int_{\mathcal{E}} U(t',r,e)\widetilde{\pi}(de,dr) + \int_t^{t'} dM(t',r).
\end{aligned}
$$

Using BDG's inequality, we get that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_S^t |Z(t,r) - Z(t',r)|^2 dr\right)^{p/2}\right] + \mathbb{E}\left[\left([M(t,\cdot) - M(t',\cdot)]_{S,t}\right)^{p/2}\right] \\
&\quad + \mathbb{E}\left[\left(\int_S^t \int_{\mathcal{E}} |U(t,r,e) - U(t',r,e)|^2 \pi(de,dr)\right)^{p/2}\right] \\
&\quad + \mathbb{E}\left[\left(\int_t^{t'} |Z(t',r)|^2 dr\right)^{p/2}\right] + \mathbb{E}\left[\left([M(t',\cdot)]_{t,t'}\right)^{p/2}\right] \\
&\quad + \mathbb{E}\left[\left(\int_t^{t'} \int_{\mathcal{E}} |U(t',r,e)|^2 \pi(de,dr)\right)^{p/2}\right] \\
&\leq C\mathbb{E}\left[|Y(t) - Y(t') - \mathbb{E}\left[Y(t) - Y(t')|\mathcal{F}_S\right]|^p\right] \leq C\mathbb{E}\left[|Y(t) - Y(t')|^p\right].
\end{aligned}
$$

Combining the previous inequalities, we obtain the desired result for BSVIE (4.4). For BSVIE (1.6), we apply the prior arguments using the generator $h(t,s,z,\psi) = f(t,s,Y(s),z,\psi)$. □

From this lemma, it is possible to deduce that Y belongs to $D([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega))$, provided that we have regularity assumption on $t \mapsto \Phi(t)$ and $t \mapsto f(t,s,y,z,\psi)$, as in Theorem 4.2 of [46] in the continuous setting. Note that the estimate on (Z,U,M) derived before Lemma A.1 is crucial here. Let us emphasize again that it does not mean that Y is in $D^\sharp([0,T];\mathbb{L}^p_{\mathbb{F}}(\Omega))$; in other words we do not deduce that a.s. the paths are càdlàg.

Let us now prove Lemma 5.3.

Proof. Fix one $t \in [0,T]$ such that the equation is satisfied and define on $[t,T]$

$$
\begin{aligned}
X_t(u) &= Y(t) - \int_t^u f(t,s)\,\mathrm{d}s - (\mathrm{M}^\sharp(t,u) - \mathrm{M}^\sharp(t,t)) \\
&= \mathbb{E}\left[Y(t) + \int_t^T f(t,s)\,\mathrm{d}s \middle| \mathcal{F}_u\right] + \int_u^T f(t,s)\,\mathrm{d}s.
\end{aligned}
$$

The process $X_t = \{X_t(u),\ u \in [t,T]\}$ is a càdlàg semimartingale. And by Doob's martingale inequality:

$$
\mathbb{E}\left[\sup_{u\in[t,T]} |X_t(u)|^2\right] \leq C\mathbb{E}\left[|X_t(T)|^2 + \left(\int_t^T |f(t,s)|\,\mathrm{d}s\right)^2\right] \leq C\mathbb{E}\left[|Y(t)|^2 + \left(\int_t^T |f(t,s)|\,\mathrm{d}s\right)^2\right].
$$

Using Itô's formula for $u \mapsto |X_t(u)|^2 e^{\beta(u-t)}$, on $[t,T]$ we obtain that:

$$\begin{aligned}
&|Y(t)|^2 + \int_t^T e^{\beta(s-t)} |Z(t,s)|^2 \,\mathrm{d}s + \int_t^T e^{\beta(s-t)} \int_{\mathbb{R}^m} |U(t,s,x)|^2 \,\pi(\,\mathrm{d}x,\,\mathrm{d}s) + \int_t^T e^{\beta(s-t)} \,\mathrm{d}\langle M(t,\cdot)\rangle_{0,s} \\
&\leq |X_t(T)|^2 e^{\beta(T-t)} + 2\int_t^T e^{\beta(s-t)} X_t(s) f(t,s) \,\mathrm{d}s - \beta \int_t^T |X_t(s)|^2 e^{\beta(s-t)} \,\mathrm{d}s \\
&\quad + 2\int_t^T e^{\beta(s-t)} X_t(s) \,\mathrm{d}\mathrm{M}^\sharp(t,s). \qquad (A.2)
\end{aligned}$$

From our hypotheses and the control of $u \mapsto X_t(u)$, the martingale terms are true martingales. By the Young inequality we obtain:

$$2\int_t^T e^{\beta(s-t)} X_t(s) f(t,s) \,\mathrm{d}s \leq \beta \int_t^T e^{\beta(s-t)} |X_t(s)|^2 \,\mathrm{d}s + \frac{1}{\beta} \int_t^T e^{\beta(s-t)} |f(t,s)|^2 \,\mathrm{d}s$$

Thus, since $X_t(T) = \Phi(t)$, taking the conditional expectation w.r.t. $\mathcal{F}_t$ in (A.2) gives the desired control. □

Acknowledgements. We would like to thank sincerely the anonymous referee for helpful comments and suggestions. We are very thankful to Alison and Christophe for their proofreading and their English text correction.

References

[1] Y. Ait-Sahalia and J. Jacod, High-Frequency Financial Econometrics. Princeton University Press, 1 edition (2014).

[2] Y. Aït-Sahalia and J. Jacod, Semimartingale: Itô or not? *Stochastic Process. Appl.* **128** (2018) 233–254.

[3] V. Anh and J. Yong, Backward stochastic Volterra integral equations in Hilbert spaces. In Differential & difference equations and applications. Hindawi Publ. Corp., New York (2006) 57–66.

[4] D. Becherer, Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.* **16** (2006) 2027–2054.

[5] M.A. Berger and V.J. Mizel, Volterra equations with Itô integrals. I. *J. Integral Equ.* **2** (1980) 187–245.

[6] M.A. Berger and V.J. Mizel, Volterra equations with Itô integrals. II. *J. Integr. Equ.* **2** (1980) 319–337.

[7] N. Bouleau and L. Denis, Dirichlet forms methods for Poisson point measures and Lévy processes. With emphasis on the creation-annihilation techniques. Vol. 76 of *Probability Theory and Stochastic Modelling.* Springer, Cham (2015).

[8] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** (2003) 109–129.

[9] F. Confortola, M. Fuhrman and J. Jacod, Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control. *Ann. Appl. Probab.* **26** (2016) 1743–1773.

[10] C. Dellacherie and P.-A. Meyer, Probabilités et potentiel. Théorie des martingales. Chapitres V à VIII. Hermann, Paris (1980).

[11] Ł. Delong, Backward stochastic differential equations with jumps and their actuarial and financial applications. BSDEs with jumps. *European Actuarial Academy (EAA) Series.* Springer, London (2013).

[12] G. Di Nunno, B. Øksendal and F. Proske, Malliavin calculus for Lévy processes with applications to finance. *Universitext.* Springer-Verlag, Berlin (2009).

[13] J. Djordjević and S. Janković, On a class of backward stochastic Volterra integral equations. *Appl. Math. Lett.* **26** (2013) 1192–1197.

[14] J. Djordjević and S. Janković, Backward stochastic Volterra integral equations with additive perturbations. *Appl. Math. Comput.* **265** (2015) 903–910.

[15] N. El Karoui and S.-J. Huang, A general result of existence and uniqueness of backward stochastic differential equations. In Backward stochastic differential equations (Paris, 1995–1996). Volume 364 of *Pitman Res. Notes Math. Ser.* Longman, Harlow (1997) 27–36.

[16] N. El Karoui, S.G. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* **7** (1997) 1–71.

[17] Y. Hu and B. Øksendal, Linear Volterra backward stochastic integral equations. *Stochastic Process. Appl.* **129** (2019) 626–633.

[18] J. Jacod, Calcul stochastique et problèmes de martingales. Vol. 714 of *Lecture Notes in Mathematics.* Springer, Berlin (1979).

[19] J. Jacod and A.N. Shiryaev, Limit theorems for stochastic processes. Vol. 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition (2003).

[20] A.M. Kolodiĭ, Existence of solutions of stochastic Volterra integral equations. In Theory of random processes, No. 11. “Naukova Dumka”, Kiev (1983) 51–57.

[21] A.M. Kolodiĭ, Existence of solutions of stochastic integral equations of Itô-Volterra type with locally integrable and continuous trajectories. In Theory of random processes, No. 12. “Naukova Dumka”, Kiev (1984) 32–40.

[22] S.G. Kreĭn, Yu. Ī. Petunīn and E.M. Semënov, Interpolation of linear operators. Translated from the Russian by J. Szűcs. Vol. 54 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, R.I. (1982).

[23] T. Kruse and A. Popier, Bsdes with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics* **88** (2016) 491–539.

[24] T. Kruse and A. Popier, L^p-solution for BSDEs with jumps in the case $p < 2$: corrections to the paper ‘BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics* **89** (2017) 1201–1227.

[25] E. Lenglart, D. Lépingle and M. Pratelli, Présentation unifiée de certaines inégalités de la théorie des martingales. With an appendix by Lenglart. In *Seminar on Probability, XIV (Paris, 1978/1979) (French).* Vol. 784 of *Lecture Notes in Math.* Springer, Berlin (1980) 26–52.

[26] J. Lin, Adapted solution of a backward stochastic nonlinear Volterra integral equation. *Stochastic Anal. Appl.* **20** (2002) 165–183.

[27] P. Lin and J. Yong, Controlled singular volterra integral equations and pontryagin maximum principle. *SIAM J. Control Optim.* **58** (2020) 136–164.

[28] W. Lu, Backward stochastic Volterra integral equations associated with a Levy process and applications. Preprint arXiv:1106.6129 (2011).

[29] C. Marinelli and M. Röckner, On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI.* Vol. 2123 of *Lecture Notes in Math.* Springer, Cham (2014) 293–315.

[30] L. Overbeck and J.A.L. Röder, Path-dependent backward stochastic Volterra integral equations with jumps, differentiability and duality principle. *Probab. Uncertain. Quant. Risk* **3** (2018) 4.

[31] A. Papapantoleon, D. Possamaï and A. Saplaouras, Existence and uniqueness results for BSDE with jumps: the whole nine yards. *Electr. J. Probab.* **23** (2018) EJP240.

[32] E. Pardoux and S.G. Peng, Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** (1990) 55–61.

[33] E. Pardoux and P. Protter, Stochastic Volterra equations with anticipating coefficients. *Ann. Probab.* **18** (1990) 1635–1655.

[34] E. Pardoux and A. Rascanu, Stochastic Differential Equations, Backward SDEs, Partial Differential Equations. Vol. 69 of *Stochastic Modelling and Applied Probability.* Springer-Verlag (2014).

[35] P.E. Protter, Volterra equations driven by semimartingales. *Ann. Probab.* **13** (1985) 519–530.

[36] P.E. Protter, Stochastic integration and differential equations. Vol. 21 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, second edition (2004).

[37] Y. Ren, On solutions of backward stochastic Volterra integral equations with jumps in Hilbert spaces. *J. Optim. Theory Appl.* **144** (2010) 319–333.

[38] D. Revuz and M. Yor, Continuous martingales and Brownian motion. Vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, third edition (1999).

[39] R. Situ, Theory of stochastic differential equations with jumps and applications. *Mathematical and Analytical Techniques with Applications to Engineering.* Springer, New York (2005).

[40] H. Wang, J. Sun and J. Yong, Recursive Utility Processes, Dynamic Risk Measures and Quadratic Backward Stochastic Volterra Integral Equations. Preprint arXiv:1810.10149 (2018).

[41] T. Wang, L^p solutions of backward stochastic Volterra integral equations. *Acta Math. Sin. (Engl. Ser.)* **28** (2012) 1875–1882.

[42] T. Wang and J. Yong, Comparison theorems for some backward stochastic Volterra integral equations. *Stochastic Process. Appl.* **125** (2015) 1756–1798.

[43] T. Wang and J. Yong, Backward stochastic Volterra integral equations—representation of adapted solutions. *Stochastic Process. Appl.* **129** (2019) 4926–4964.

[44] Z. Wang and X. Zhang, Non-Lipschitz backward stochastic Volterra type equations with jumps. *Stoch. Dyn.* **7** (2007) 479–496.

[45] J. Yong, Backward stochastic Volterra integral equations and some related problems. *Stochastic Process. Appl.* **116** (2006) 779–795.

[46] J. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations. *Probab. Theory Related Fields* **142** (2008) 21–77.

[47] J. Yong, Backward stochastic Volterra integral equations—a brief survey. *Appl. Math. J. Chinese Univ. Ser. B* **28** (2013) 383–394.