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ON THE GENERALIZED KESTEN–MCKAY DISTRIBUTIONS*

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Abstract. We examine the properties of distributions with the density of the form: $\frac{2A_n c^{n-2}\sqrt{c^2-x^2}}{\pi\prod_{j=1}^{n}(c(1+a_j^2)-2a_j x)}$, where $c, a_1, \ldots, a_n$ are some parameters and A_n a suitable constant. We find general forms of A_n, of k-th moment and of k-th polynomial orthogonal with respect to such measures. We also calculate Cauchy transforms of these measures. We indicate connections of such distributions with distributions and polynomials forming the so called Askey–Wilson scheme. On the way we prove several identities concerning rational symmetric functions. Finally, we consider the case of parameters $a_1, \ldots, a_n$ forming conjugate pairs and give some multivariate interpretations based on the obtained distributions at least for the cases $n = 2, 4, 6$.

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1. Introduction

The purpose of this note is to analyze properties of the following family of distributions having densities of the form:

$$f_{KMKn}(x|c, a_1, \ldots, a_n) = \frac{2A_n c^{n-2}\sqrt{c^2-x^2}}{\pi\prod_{j=1}^{n}(c(1+a_j^2)-2a_j x)}, \tag{1.1}$$

defined for $n \geq 0$, $|x| \leq c$, with $c > 0$, $|a_j| < 1$, $j = 1, \ldots, n$. Here A_n is a normalizing constant being the function of parameters $a_1, \ldots, a_n$. We will call this family generalized Kesten–McKay distributions. The name is justified by the fact that the distribution with the following density:

$$f_{KMK2}(x|2/a, a, -a) = \frac{v\sqrt{4(v-1)-x^2}}{2\pi(v^2-x^2)}, \tag{1.2}$$

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where $v = 1 + 1/a^2$ and $|a| < 1$ has been defined, described and what is more, derived in [6] or [8]. Then the name Kesten-McKay distribution has been attributed to this distribution in the literature that appeared after 1981.

Thus, it is justified to call the distribution defined by (1.1) a generalized Kesten-McKay (GKM) distribution.

Note also that for $n = 0$ the distribution with the density $f_{KMK0}(x|c)$ becomes Wigner or semicircle distribution with parameter c

It should be underlined that for $n = 0, 1, 2$ distributions of this kind appear not only in the context of random matrices, random graphs which is a typical application of the Kesten–McKay distributions (see *e.g.* [9–11]), but also in the context of the so-called free probability a part of the non-commutative probability theory recently rapidly developing. One of the first papers where semicircle and related distribution appeared in the non-commutative probability context is [1].

For $n < 5$ distribution f_{KMKn} can be identified as the special case of the Askey–Wilson chain of distributions that make orthogonal 5 families of polynomials of the so-called Askey–Wilson scheme. For the reference see *e.g.* [16]. For $n \geq 5$ the distributions f_{KMKn} were not yet described in detail.

It has to be noted that in 2009 there appeared paper [4]. Although the aim of it was to analyze two-dimensional measures on the plane of the form:

$$\frac{\sqrt{(1-x^2)(1-y^2)}}{g(x,y)},$$

where g was a polynomial in x and y, it also contains some results concerning one-dimensional case. The one-dimensional distribution considered there, is very much alike the distribution we consider in this paper. The authors of the paper call it Bernstein-Szegö distribution (for comparison see [19]). The method they use to analyze these distributions allow them to consider only the case of even degree n of polynomial g. The results of the paper are general and hence rather imprecise. They were obtained by quite complicated integration on the complex plane.

Our results are precise, since we assume the exact form of the polynomial in the denominator. Namely, we assume the knowledge of the roots of this polynomial. Besides, as mentioned above, the order of this polynomial can be also odd. Moreover, due to our assumptions we are able to give precise form of the polynomials orthogonal with respect to this measure.

Our methods are simple and heavily exploit the properties of the Chebyshev polynomials of the second kind. By using them, we are able to obtain some interesting identities concerning Chebyshev polynomials of the second kind as well as symmetric rational functions like for example the formulae given in Remark 1.4 or Corollary 1.7.

It should be mentioned that by considering polynomials of even degree, having pairwise conjugate roots, we are not only able to cover the case of two-dimensional measures on the plane of the form mentioned above, but also generalize the results to 3 or more dimensions defining new distributions and finding they marginals. Compare Section 2 and the remarks at the end of this section.

Hence our results give substance and generalize the results of [4].

We will present a unified approach and recall and collect information on this family that is scattered though literature.

Let us observe first, that if $X \sim f_{KMKn}(x|c, a_1, \ldots, a_n)$ and
$Y \sim f_{KMKn}(x|1, a_1, \ldots, a_n)$, then $X \sim cY$. Hence we will consider further only distributions with the density $f_{KMKn}(x|1, a_1, \ldots, a_n)$ which we will denote for brevity $f_{Kn}(x|a_1, \ldots, a_n)$.

To start our analysis let us recall that for $|a_j| < 1$ we have

$$1/(1 + a_j^2 - 2a_j x) = \sum_{k=0}^{\infty} a_j^k U_k(x), \tag{1.3}$$

where U_k denotes k-th Chebyshev polynomial of the second kind and that:

$$\int_{-1}^{1}\frac{2}{\pi}U_k(x)U_j(x)\sqrt{1-x^2}\mathrm{d}x=\left\{\begin{array}{lll}0 & if & j\neq k\\ 1 & if & j=k\end{array}\right. .$$

Recall also for completeness of the paper that $U_{-1}(x)=0$, $U_0(x)=1$, and for $m\geq 1$ we have

$$2xU_m(x)=U_{m+1}(x)+U_{m-1}(x). \tag{1.4}$$

The above mentioned formula is traditionally extended to all integers m, that leads to the following extended definition of the Chebyshev polynomials:

$$U_{-k}(x)=U_{-k-2}(x), \tag{1.5}$$

for $k\geq 1$.

Notice that the form of the density (1.1) fits the scheme of distributions and orthogonal polynomials that was considered in [16] and hence we can use ideas and results presented there. First of all, let us observe that the densities considered in this paper are the special cases of the distributions known from the so-called Askey–Wilson scheme of distributions and orthogonal polynomials obtained from the general ones by setting $q=0$. q is a special parameter called the base within the theory of Askey–Wilson polynomials. More precisely, for $n=0$ we deal with the so-called $q-$Hermite polynomials and the so-called q-Normal distribution. Wigner distribution and Chebyshev polynomials of the second kind are their special case for $q=0$. For $n=1$ we deal with the so-called continuous big q-Hermite polynomials and the distribution that makes them orthogonal. When we set $q=0$ and $n=1$ we deal with the distribution $f_{K1}(x|a_1)$. For $n=2$ we deal with the so-called Al-Salam–Chihara polynomials and the distribution that make these polynomials orthogonal. Setting now $q=0$ leads us to the distribution $f_{K2}(x|a_1,a_2)$. Further for $n=3$ we deal with the so-called dual Hahn polynomials and the distribution that makes these polynomials orthogonal. Setting $q=0$ leads us to $f_{K3}(x|a_1,a_2,a_3)$. Finally, for $n=4$ we deal with the so-called Askey–Wilson polynomials and the distribution that make them orthogonal. Setting $q=0$ we get $f_{K4}(x|a_1,a_2,a_3,a_4)$.

Hence in particular, we know the families of orthogonal polynomials that our distributions make orthogonal, for $n=0,\ldots,4$. For the precise definitions and further references see *e.g.* [7] or [16].

In the sequel we will use exchangeably $\mathbf{a}_n$ and $(a_1,\ldots,a_n)$ depending on the required brevity and clarity.

Let us denote $\prod_{j=1,j\neq i}^{n}$ as $\prod_{j\neq i}^{n}$.

We have the following general observation.

Theorem 1.1. *If $a_i\neq a_j$, $i\neq j$, $i,j=1,\ldots,n$, then*
i) the constant A_n in (1.1) is given by:

$$A_n=A_n(\mathbf{a}_n)=1/\sum_{i=1}^{n}\frac{a_i^{n-1}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}. \tag{1.6}$$

ii) Let us define constants $B_{n,k}$ for $n<0$ as $B_{n,k}=0$ and for $n,k\geq 0$ by:

$$B_{n,k}(\mathbf{a}_n)=B_{n,k}=\int_{-1}^{1}U_k(x)f_{Kn}(x|\mathbf{a}_n)\mathrm{d}x,$$

and for $n\geq 0$ and $k\leq -1$ by:

$$B_{n,k}=\left\{\begin{array}{ccc}0 & if & k=-1\\ -B_{n,-k-2} & if & k\leq -2\end{array}\right. . \tag{1.7}$$

Then we have:

$$B_{n,k} = A_n \sum_{i=1}^{n} \frac{a_i^{n+k-1}}{\prod_{j\neq i}^{n}(a_i - a_j)(1 - a_i a_j)}, \tag{1.8}$$

for $k = 0, \ldots$.

Proof. Firstly, notice that the definition of $B_{n,k}$ for the negative integers k follows the fact that we have the identity (1.5), above. We will use the fact that

$$1/\prod_{i=1}^{n}(x - b_i) = \sum_{i=1}^{n} \frac{1}{\prod_{j=1,j\neq i}^{n}(b_i - b_j)} \frac{1}{(x - b_i)}$$

and the fact that $((1 + a^2)/a - (1 + b^2)/b) = (b - a)(1 - ab)/(ab)$. Hence, we have

$$\begin{aligned} f_{Kn}(x|a_1, \ldots, a_n) &= \frac{2A_n(-1)^n}{2^n \pi \prod_{i=1}^{n} a_i} \sqrt{1 - x^2} / \prod_{i=1}^{n}(x - (1 + a_i^2)/(2a_i)) \\ = \frac{2 \times 2^{n-1} A_n(-1)^n}{2^n \pi \prod_{i=1}^{n} a_i} &\sqrt{1 - x^2} \sum_{i=1}^{n} \frac{a_i^{n-1} \prod_{j\neq i}^{n} a_j}{\prod_{j\neq i}^{n}(a_j - a_i)(1 - a_i a_j)} \frac{1}{(x - (1 + a_i^2)/(2a_i))} \\ &= \frac{2A_n}{\pi} \sqrt{1 - x^2} \sum_{i=1}^{n} \frac{a_i^{n-1}}{\prod_{j\neq i}^{n}(a_i - a_j)(1 - a_i a_j)} \frac{1}{((1 + a_i^2) - 2a_i x)}. \end{aligned}$$

Now following (1.3) and orthogonality of polynomials $U's$ with respect to Wigner measure we have

$$\int_{-1}^{1} \frac{2\sqrt{1 - x^2}}{\pi((1 + a_i^2) - 2a_i x)} U_k(x) \mathrm{d}x = a_i^k.$$

□

Remark 1.2. Since coefficients $B_{n,k}$ are the coefficients in the orthogonal expansion in $L_2([-1, 1], w)$, in the basis $\{U_k\}_{k\geq 0}$, where w denotes measure with the density $\frac{2}{\pi}\sqrt{1 - x^2}$, we get the following expansion for free:

$$f_{Kn}(x|\mathbf{a}_n) = \frac{2}{\pi} \sqrt{1 - x^2} \sum_{k=0}^{\infty} B_{n,k} U_k(x). \tag{1.9}$$

For more examples of such expansions see [12].

In the sequel we will need the following quantities:

$$S_k^{(n)}(\mathbf{a}_n) = \sum_{1\leq j_1 < j_2 < \ldots, < j_k \leq n} \prod_{m=1}^{k} a_{j_m}, \tag{1.10}$$

$$\Delta_m^{(n)}(\mathbf{a}_n) = \sum_{\substack{0\leq j_1,\ldots j_{n-1}\leq m \\ j_1+\ldots+j_{n-1}\leq m}} a_1^{j_1} a_2^{j_2} \ldots a_{n-1}^{j_{n-1}} a_n^{m-(j_1+\ldots+j_{n-1})}, \tag{1.11}$$

for $1 \leq k \leq n \leq m$. Whenever it will be obvious we will drop other than k arguments of the functions S_k and Δ_k. Notice that $S_k^{(n)}$ is the k-th elementary symmetric function of the variables $a_1, \ldots, a_n$. We set $S_0^{(n)}(\mathbf{a}_n) = 1$ and $S_k^{(n)}(\mathbf{a}_n) = 0$ when $k > n > 0$.

Remark 1.3. We have

$$A_1 = 1, A_2 = 1 - a_1 a_2,$$
$$A_3 = \prod_{j=1}^{3} \prod_{k=j+1}^{3} (1 - a_j a_k) = (1 - a_1 a_2)(1 - a_1 a_3)(1 - a_2 a_3),$$
$$A_4 = \prod_{j=1}^{4} \prod_{k=j+1}^{4} (1 - a_j a_k)/(1 - S_4)$$
$$= \frac{(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_2 a_3)(1 - a_1 a_4)(1 - a_2 a_3)(1 - a_2 a_4)(1 - a_3 a_4)}{1 - a_1 a_2 a_3 a_4},$$
$$A_5 = \prod_{j=1}^{5} \prod_{k=j+1}^{5} (1 - a_j a_k)/(1 - S_4 + S_1 S_5 - S_5^2),$$
$$A_6 = \frac{\prod_{j=1}^{6} \prod_{k=j+1}^{6} (1 - a_j a_k)}{(1 - S_4 + S_1 S_5 - S_5^2 - S_6 - S_1^2 S_6 + S_2 S_6 + S_4 S_6 + S_1 S_5 S_6 - S_6^2 - S_2 S_6^2 + S_6^3)}.$$

Notice that for $n = 1, \ldots, 4$ the constant A_n agrees with respective constants given in [16] in formulae (2.4), (2.6) and unnamed formulae on top of 9-th and 10-th pages for $q = 0$, when presenting respectively densities that make big q-Hermite, Al-Salam–Chihara, dual Hahn and Askey-Wilson polynomials orthogonal.

Using this denotations we have $B_{2,k} = \Delta_k^{(2)}$, $B_{3,k} = \Delta_k^{(3)} - S_3 \Delta_{k-1}^{(3)}$. Compare here with formulae (2.7) and (2.10) in [16] for $q = 0$.

Remark 1.4. Let us notice also that following (1.3) we have:

$$1/A_n(\mathbf{a}_n) = \int_{-1}^{1} \frac{2}{\pi} \sqrt{1 - x^2} \prod_{i=1}^{n} \sum_{k_i=1}^{\infty} a_i^{k_i} U_{k_i}(x) \mathrm{d}x$$
$$= \sum_{k_1 \geq 0, \ldots, k_n \geq 0} (\prod_{j=1}^{n} a_i^{k_i}) V_{k_1, \ldots, k_n},$$

where $V_{k_1, \ldots, k_n} = \int_{-1}^{1} \frac{2}{\pi} \sqrt{1 - x^2} \prod_{i=1}^{n} U_{k_i}(x) \mathrm{d}x$. That is, we get the generating function of numbers $V_{k_1, \ldots, k_n}$ for free. This observation gives also an interesting interpretation of the constants A_n.

Corollary 1.5. *i)*

$$\int_{-1}^{1} x^k f_{Kn}(x|\mathbf{a}_n) \mathrm{d}x = \frac{1}{(k+1)2^k} \sum_{j=0}^{\lfloor k/2 \rfloor} (k - 2j + 1) \binom{k+1}{j} B_{n,k-2j}(\mathbf{a}_n),$$

ii)

$$\int_{-1}^{1} U_k(x) U_m(x) f_{Kn}(x|\mathbf{a}_n) \mathrm{d}x = \sum_{j=0}^{\min(m,k)} B_{n,|m-k|+2j}(\mathbf{a}_n).$$

Proof. i) We use well known identity:

$$2^k x^k = \sum_{j=0}^{\lfloor k/2 \rfloor} (\binom{k}{j} - \binom{k}{j-1}) U_{k-2j}(x)$$

(see *e.g.* [18] Proposition 1 with $q = 0$ and the fact that $h_n(x|0) = U_n(x)$) and the fact that

$$\binom{k}{j} - \binom{k}{j-1} = (k - 2j + 1)\binom{k+1}{j}/(k+1).$$

ii) We use identity

$$U_k(x)U_m(x) = \sum_{j=0}^{\min(k,m)} U_{|k-m|+2j}(x),$$

that can be easily derived from

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{|n-m|}(x),$$

where T_n denotes Chebyshev polynomial of the first kind and the formulae relating Chebyshev polynomials of the first and second kind. See also formula (2.13) of [15] with $q = 0$. □

Proposition 1.6. *For any function $g : \mathbb{R} \longrightarrow \mathbb{R}$ let us denote $\mathbf{g}(\mathbf{a}_n) = (g(a_1), \ldots, g(a_n))$ and also $\mathbf{b}_n^{(i)} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. Let us set $g(x) = (1 + x^2)/(2x)$. Then, if $a_i \neq a_j$, $i \neq j$, $i, j = 1, \ldots, n$, we have*:

$$\sum_{i=1}^{n} \frac{a_i^{n-2} S_k(\mathbf{g}(\mathbf{a}_n^{(i)}))}{\prod_{j\neq i}^{n} (a_j - a_i)(1 - a_i a_j)} = \left\{ \begin{array}{lll} 0 & \textit{for} & k = 0, \ldots, n-2, \\ 1 & \textit{for} & k = n-1 \end{array} \right. . \tag{1.12}$$

Proof. We start from the fact that $1/\prod_{i=1}^{n}(x - b_i) = \sum_{i=1}^{n} D_{n,i} \frac{1}{(x-b_i)}$, where $D_{n,i} = 1/\prod_{j=1, j\neq i}^{n}(b_i - b_j)$. Thus, from the properties of simple fraction decompositions we have the following identity:

$$1/\prod_{i=1}^{n}(x - b_i) = (\sum_{i=1}^{n} D_{n,i} \prod_{j\neq i}(x - b_j))/\prod_{i=1}^{n}(x - b_i).$$

Hence all coefficients in $\sum_{i=1}^{n} D_{n,i} \prod_{j\neq i}(x - b_j)$ by nonzero powers of x must be zero. In particular for $\forall k = 0 \ldots, n-1$ we must have $\sum_{i=1}^{n} D_{n,i} S_k(\mathbf{b}_n^{(i)}) = 0$. Now it remains to substitute $b_i = g(a_i)$ and use the fact that $(b_i - b_j) = (a_j - a_i)(1 - a_i a_j)/(2a_i a_j)$ so that

$$D_{n,i} = 2^{n-1} a_i^{n-2} (\prod_{j=1}^{n} a_j)/\prod_{j\neq i}^{n}((1 - a_i a_j)(a_j - a_i)),$$

since we have $a_i^{n-1} \prod_{j\neq i}^{n} a_j = a_i^{n-2} \prod_{j=1}^{n} a_j$. □

As a corollary we get in particular the following identities.

Corollary 1.7. *$\forall n\geq 2$: i)*

$$\sum_{i=1}^{n}\frac{a_i^{n-2}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}=0, \tag{1.13}$$

from which it follows that (1.8) is valid also for $k=-1$,
ii)

$$\sum_{i=1}^{n}\frac{a_i^{n-2}(1+a_i^2)\sum_{j\neq i}^{n}\prod_{k\neq i}^{n}a_k}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}=0,$$

iii) $\forall m\geq n-1, n\geq 1$:

$$\sum_{j=0}^{n}(-1)^jS_j^{(n)}(\mathbf{a}_n)B_{n,m-j}(\mathbf{a}_n)=0. \tag{1.14}$$

Proof. To prove i) we take $k=0$ in (1.12). To prove ii) we take $k=1$ in (1.12) and notice that

$$a_i^{n-1}(\prod_{j\neq i}^{n}a_j)S_1(\mathbf{g}(\mathbf{a}_n^{(i)})=a_i^{n-1}\sum_{j\neq i}^{n}(1+a_j^2)\prod_{k\neq i,j}^{n}a_k$$
$$=a_i^{n-2}\sum_{j\neq i}^{n}\prod_{k\neq j}a_k+a_i^{n}\sum_{j\neq i}^{n}\prod_{k\neq j}a_k=a_i^{n-2}(1+a_i^2)\sum_{j\neq i}^{n}\prod_{k\neq j}^{n}a_k.$$

To prove iii) we observe first that for $n\geq 1$, $k\geq 0,\forall i=1,\dots n$ we have

$$\sum_{j=0}^{n}(-1)^ja_i^{n+k-j}S_j^{(n)}(\mathbf{a}_n)=0, \tag{1.15}$$

which is elementary to notice. Secondly, using (1.8) for $m\geq n-1$, we get

$$\sum_{k=0}^{n}(-1)^kS_k^{(n)}B_{n,m-k}=A_n\sum_{k=0}^{n}(-1)^kS_k^{(n)}\sum_{i=1}^{n}\frac{a_i^{n+m-k-1}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}$$
$$=A_n\sum_{i=1}^{n}\frac{1}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}\sum_{k=0}^{n}(-1)^kS_k^{(n)}a_i^{n+m-k-1}=0,$$

by (1.15). □

Theorem 1.8. *For every $m\geq 2n-2\geq 0$ the family of polynomials orthogonal with respect to f_{Kn} is of the form:*

$$P_m^{(n)}(x|\mathbf{a}_n)=\sum_{j=0}^{n-1}(-1)^jU_{m-j}(x)S_j^{(n)}(\mathbf{a}_n), \tag{1.16}$$

where $S_j^{(n)}$ is given by (1.10).

Proof. The fact that the polynomials $P_m^{(n)}$ can be expressed as linear combination of the last at most $n+1$ polynomials U_m follows directly from [12] (Prop. 1 iii)) or from [4] (Lemma 3.1). Similar fact was noticed for $n = 1, 2$ earlier by Maroni in his papers published in the 90's. Next notice that $P_m^{(n)}$ must be of the form $U_m + \sum_{j=1}^{n} b_j^{(m)} U_{m-j}(x)$. To determine parameters $\left\{b_j^{(m)}\right\}_{j=1}^{n}$ we need n equation of the form $\int_{-1}^{1} x^k P_m^{(n)}(x|\mathbf{a}_n) f_{Kn}(x|\mathbf{a}_n)\mathrm{d}x = 0$, $k = 0, \ldots, n-1$.

Now notice that for $m \geq n-1$ we have

$$\int_{-1}^{1} P_m^{(n)}(x|\mathbf{a}_n) f_{Kn}(x|\mathbf{a}_n)\mathrm{d}x = 0, \tag{1.17}$$

$$2xP_m^{(n)}(x|\mathbf{a}_n) = P_{m+1}^{(n)}(x|\mathbf{a}_n) + P_{m-1}^{(n)}(x|\mathbf{a}_n) \tag{1.18}$$

which in the case of (1.17) follows (1.8), and (1.14) and in case of (1.18) follows directly three term recurrence for the Chebyshev polynomials. More over iterating (1.18) we can express $x^k P_m^{(n)}(x|\mathbf{a}_n)$ as linear combination of $P_{m+l}^{(n)}$, for $l = -k, \ldots, k$. Since we have to have $m - k \geq n-1$, for $k = 0, \ldots, n-1$ we see that polynomials $P_m^{(n)}$ orthogonal for $m \geq 2n-2$. □

Remark 1.9. Notice that polynomials $P_m^{(n)}(x|\mathbf{a}_n)/2^m$ are monic, since $U_n/2^n$ are monic for $n \geq 0$.

Remark 1.10. Recall that the first Askey-Wilson polynomials $aw_k(x|\mathbf{a}_4)$ with $q = 0$ are equal to:

$$\begin{aligned}
aw_1(x|\mathbf{a}_4) &= U_1(x) - \frac{S_1 - S_3}{1 - S_4}, \\
aw_2(x|\mathbf{a}_4) &= U_2(x) - S_1 U_1(x) + S_2 - S_4, \\
aw_3(x|\mathbf{a}_4) &= \sum_{j=0}^{3} (-1)^j U_{k-j}(x) S_j, \\
aw_k(x|\mathbf{a}_4) &= \sum_{j=0}^{4} (-1)^j U_{k-j}(x) S_j,
\end{aligned}$$

$k \geq 4$, where, as agreed above, S_j means in fact $S_j^{(4)}(\mathbf{a}_4)$. From this presentation it follows that may be the formula (1.16) is valid for $m \geq n-1$. In fact, there is a strong argument to support this supposition. Namely numerical simulations suggest that (1.14) might be true for $m \geq 1$. Notice that it is impossible to further extend this formula, i.e. to fit it for the cases $m < n-1$. This is so since immediately we see that $P_1^{(n)}(x|\mathbf{a}_n) = U_1(x) - B_{n,1}(\mathbf{a}_n)$ and $B_{n,1}(\mathbf{a}_n)$ is different from S_1 for $n \geq 3$ since we have:

$$\begin{aligned}
B_{3,1} &= S_1 - S_3, \ B_{4,1} = \frac{S_1 - S_3}{1 - S_4}, \\
B_{5,1} &= \frac{S_1 - S_3 + S_1 S_5 - S_4 S_5}{1 - S_4 + S_1 S_5 - S_5^2}.
\end{aligned}$$

Proposition 1.11. *Let $X \sim f_{Kn}(x|\mathbf{a}_n)$, then for $\forall\,|t| < 1$ we have*

$$E\frac{1}{1 + t^2 - 2tX} = \frac{Q_n(t|\mathbf{a}_n)}{\prod_{i=1}^{n}(1 - ta_i)},$$

where

$$Q_n(t|\mathbf{a}_n) = A_n \sum_{i=1}^{n} a_i^{n-1} \prod_{j\neq i}^{n} \frac{(1-a_jt)}{(a_i-a_j)(1-a_ia_j)}$$

is a polynomial of degree $\max(n-2,0)$ *in* t.

Proof. Recall that $\sum_{j=0}^{\infty} t^j U_j(x) = 1/(1+t^2-2tx)$. Thus

$$\int_{-1}^{1}(\sum_{j=0}^{\infty} t^j U_j(x)) f_{Kn}(x|\mathbf{a}_n)\mathrm{d}x = A_n \sum_{j=0}^{\infty} t^j \sum_{i=1}^{n} \frac{a_i^{n+j-1}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}$$
$$= A_n \sum_{i=1}^{n} \frac{a_i^{n-1}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)} \sum_{j\geq 0} t^j a_i^j = A_n \sum_{i=1}^{n} \frac{1}{1-a_it} \frac{a_i^{n-1}}{\prod_{j\neq i}^{n}(a_i-a_j)(1-a_ia_j)}$$
$$= \frac{1}{\prod_{i=1}^{n}(1-ta_i)} A_n \sum_{i=1}^{n} a_i^{n-1} \prod_{j\neq i}^{n} \frac{(1-a_jt)}{(a_i-a_j)(1-a_ia_j)}.$$

The fact that Q_n is a polynomial of degree $n-2$ follows the fact that $a_i^{n-1}\prod_{j\neq i}^{n}(1-a_jt)$ is a polynomial of degree $n-1$, but the coefficient by t^{n-1} is equal to the sum of $S_n(\mathbf{a}_n)a_i^{n-2.}$ over i. The assertion follows from (1.13). □

Remark 1.12. Recall that the Cauchy transform of a measure μ is defined by:

$$\mathcal{C}_\mu(y) = \int \frac{d\mu(x)}{y-x},$$

where the integral is understood as a principal value. Note that the names Hilbert or Stjeltjes transform are also used. It has been intensively studied recently in connection with free probability or signal processing. For the reference see [3] and also papers of D. Voiculescu and his fellow researchers as well as of B. Shapiro and his fellow researchers. Notice also that Proposition 1.11 helps to get values of the Cauchy transform of $f_{Kn}(x|\mathbf{a}_n)$ for real $|y|>1$ by taking $y=(1+t^2)/2t$, $|t|<1$. More precisely, we have

$$\mathcal{C}_{Kn}((1+t^2)/2t) = 2t\frac{Q_n(t|\mathbf{a}_n)}{\prod_{i=1}^{n}(1-ta_i)}.$$

Remark 1.13. By direct calculations we have

$$Q_1(t|a)=1,\ Q_2(t|\mathbf{a}_2)=1,\ Q_3(t|\mathbf{a}_3)=1-tS_3(\mathbf{a}_3),$$
$$Q_4(t|\mathbf{a}_4)=((1-S_4)-t(S_3-S_1S_4)+t^2S_4(1-S_4))/(1-S_4).$$

As an immediate consequence of this formula (1.3) and the definition of $B_{n,k}$ we get the characteristic functions of numbers $B_{n,k}(\mathbf{a}_n)$.

Corollary 1.14. *For* $\forall n\geq 0$ *we have*

$$\sum_{k\geq 0} t^k B_{n,k}(\mathbf{a}_n) = Q_n(t|\mathbf{a}_n)/\prod_{i=1}^{n}(1-ta_i).$$

2. Complex parameters

In this section we will study properties of the generalized Kesten-MacKay distributions for even $n = 2k$ and parameters a_i , $i = 1, \ldots, 2k$ being complex and forming conjugate pairs. The new parameters will have new names, namely for the conjugate the pair for example $a_i = \rho_i \exp(i\theta_i)$ and $a_{k+i} = \rho_i \exp(-i\theta_i)$ we will denote $y_i = \cos\theta_i$ for $i = 1, \ldots, k$. Besides notice that we have

$$\begin{aligned}(1 + \rho_i^2 \exp(2i\theta_i) - 2x\rho_i \exp(i\theta_i))(1 + \rho_i^2 \exp(-2i\theta_i) - 2x\rho_i \exp(-i\theta_i))\\ = 1 + \rho_i^4 + 4x^2\rho_i^2 - 4x\rho_i(1 + \rho_i^2)\cos\theta_i + 2\rho_i^2 \cos 2\theta_i = w(x, y_i|\rho_i),\end{aligned}$$

where we denoted for simplicity:

$$w(x, y|\rho) = (1 - \rho^2)^2 - 4xy\rho(1 + \rho^2) + 4\rho^2(x^2 + y^2). \tag{2.1}$$

Hence now the density $f_{K2k}(x|a_1, \ldots, a_{2k})$ will have the following form that we will denote by $f_{Mk}(x|\mathbf{y}_k, \rho_k)$:

$$f_{Mk}(x|\mathbf{y}_k, \rho_k) = A_{2k} \frac{2\sqrt{1 - x^2}}{\pi \prod_{j=1}^{k} w(x, y_i|\rho_i)}, \tag{2.2}$$

with $|\rho_i| < 1$, $|y_i| \leq 1$.

Following Remark 1.3 we have

Lemma 2.1. *i)* $S_1^{(2)} = 2\rho y$, $S_2^{(2)} = \rho^2$, $A_2 = 1 - \rho^2$,

ii)

$$\begin{aligned}S_1^{(4)}(\mathbf{a}_4) &= y_1\rho_1 + y_2\rho_2,\\ S_2^{(4)}(\mathbf{a}_4) &= \rho_1^2 + \rho_2^2 + 4y_1y_2\rho_1\rho_2,\\ S_3^{(4)}(\mathbf{a}_4) &= 2\rho_1\rho_2(\rho_1 y_1 + \rho_2 y_2),\\ A_4 &= (1 - \rho_1^2)(1 - \rho_2^2)w(y_1, y_2|\rho_1\rho_2)/(1 - \rho_1^2\rho_2^2).\end{aligned}$$

iii)

$$\begin{aligned}S_1^{(6)}(\mathbf{a}_6) &= \rho_1 y_1 + \rho_2 y_2 + \rho_3 y_3,\\ S_2^{(6)}(\mathbf{a}_6) &= \rho_1^2 + \rho_2^2 + \rho_3^2 + 4(\rho_1\rho_2 y_1 y_2 + \rho_1\rho_3 y_1 y_3 + \rho_2\rho_3 y_2 y_3),\end{aligned}$$

$$\begin{aligned}S_3^{(6)}(\mathbf{a}_6) = {}& 2(\rho_1^2 + \rho_3^2)\rho_2 y_2 + 2(\rho_2^2 + \rho_3^2)\rho_1 y_1\\ &+ 2(\rho_1^2 + \rho_2^2)\rho_3 y_3 + 8\rho_1\rho_2\rho_3 y_1 y_2 y_3,\end{aligned}$$

$$S_4^{(6)}(\mathbf{a}_6) = \rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_1^2\rho_3^2 + 4\rho_1\rho_2\rho_3(\rho_3 y_1 y_2 + \rho_2 y_1 y_3 + \rho_1 y_2 y_3),$$

$$\begin{aligned}S_5^{(6)}(\mathbf{a}_6) &= 2\rho_1\rho_2\rho_3(\rho_1\rho_2 y_3 + \rho_1\rho_3 y_2 + \rho_2\rho_3 y_1),\ S_6^{(6)}(\mathbf{a}_6) = \rho_1^2\rho_2^2\rho_3^2,\\ A_6 &= (1 - \rho_1^2)(1 - \rho_2^2)(1 - \rho_3^2)\frac{w(y_1, y_2|\rho_1\rho_2)w(y_2, y_3|\rho_2\rho_3)w(y_1, y_3|\rho_1\rho_3)}{w3(y_1, y_2, y_3|\rho_1, \rho_2, \rho_3)},\end{aligned}$$

where we denoted

$$w3(y_1,y_2,y_3|\rho_1,\rho_2,\rho_3)=(1-\rho_1^2\rho_2^2)(1-\rho_2^2\rho_3^3)(1-\rho_1^2\rho_3^2)(1-\rho_1^2\rho_2^2\rho_3^2)\\ -4\rho_1\rho_2\rho_3(1+\rho_1^2\rho_2^2\rho_3^2)(\rho_1(1-\rho_2^2)(1-\rho_3^2)y_2y_3\\ +\rho_2(1-\rho_1^2)(1-\rho_3^2)y_1y_3+\rho_3(1-\rho_1^2)(1-\rho_2^2)y_1y_2)\\ +4\rho_1^2\rho_2^2\rho_3^2((1-\rho_1^2)(1-\rho_2^2\rho_3^2)y_1^2+(1-\rho_2^2)(1-\rho_1^2\rho_3^2)y_2^2+(1-\rho_3^2)(1-\rho_1^2\rho_2^2)y_3^2). \tag{2.3}$$

Proof. All calculations were done using Mathematica 10. □

Remark 2.2. Recall that for all $|x|,|y|\leq 1$ and $|\rho|<1$ we have the following useful expansion:

$$\frac{1-\rho^2}{w(x,y|\rho)}=\sum_{j=0}^{\infty}\rho^j U_j(x)U_j(y), \tag{2.4}$$

which is nothing else but the famous Poisson–Mehler formula for $q=0$ (for the reference, see *e.g.* [5] (13.1.24) or for alternative proof [12]).

Following the above mentioned remark we have

$$f_{Mk}(x|\mathbf{y}_k,\rho_k)=\frac{2A_{2k}}{\pi\prod_{j=1}^{k}(1-\rho_j^2)}\sqrt{1-x^2}\sum_{m_1,m_2,\dots m_k=0}^{\infty}\prod_{j=1}^{k}\rho_j^{m_j}U_{m_j}(x)U_{m_j}(y_j), \tag{2.5}$$

$$\prod_{j=1}^{k}(1-\rho_j^2)/A_{2k}=\sum_{m_1,m_2,\dots m_k=0}^{\infty}V_{m_1,\dots,m_k}\prod_{j=1}^{k}\rho_j^{m_j}U_{m_j}(y_j), \tag{2.6}$$

where as before, above $V_{k_1,\dots,k_n}=\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-x^2}\prod_{i=1}^{n}U_{k_i}(x)\mathrm{d}x$.

Since each density f_{Mk} can be presented as a linear combination of $f_{M1}(x|y_i,\rho_i)$, $k=1,\dots,k$ (by simple fraction decomposition) we will analyze f_{M1} first. We have the following result:

Theorem 2.3. *i)* $\forall y\in[-1,1]:\int_{-1}^{1}f_{M1}(x|y,\rho)\mathrm{d}x=1$,

ii) $\int_{-1}^{1}f_{M1}(x|y,\rho)\frac{2}{\pi}\sqrt{1-y^2}dy=\frac{2}{\pi}\sqrt{1-x^2}$,

iii) $\int_{-1}^{1}f_{M1}(x|y_1,\rho_1)f_{M1}(y_1|y_2,\rho_2)dy_1=f_{M1}(x|y_2,\rho_1\rho_2)$,

iv) Polynomials orthogonal with respect to f_{M1} *are as follows:* $P_{-1}(x|y,\rho)=0$, $P_0(x|y,\rho)=1$, $P_1(x|y,\rho)=U_1(x)-2\rho y$, *and*

$$P_m(x|y,\rho)=U_m(x)-2\rho yU_{m-1}(x)+\rho^2U_{m-2}(x)$$

for $m\geq 2$.

Proof. i) Either we use directly properties of $f_{K2}(x|a,b)$ and the fact that in our case $ab=\rho^2$, or we apply (2.4). ii) follows directly from i) and the fact that $w(x,y|\rho)=w(y,x|\rho)$. iii) we have:

$$\int_{-1}^{1}f_{M1}(x|y_1,\rho_1)f_{M1}(y_1|y_2,\rho_2)\mathrm{d}y_1\\ =\frac{2}{\pi}\sqrt{1-x^2}\int_{-1}^{1}\frac{2}{\pi}\frac{(1-\rho_1^2)(1-\rho_2^2)\sqrt{1-y_1^2}}{w(y_1,x|\rho_1)w(y_1,y_2|\rho_2)}\mathrm{d}y_1\\ =\frac{2}{\pi}(1-\rho_1^2)(1-\rho_2^2)\sqrt{1-x^2}/A_4=f_{M1}(x|y_2,\rho_1\rho_2),$$

since $A_4 = (1-\rho_1^2)(1-\rho_2^2)w(x, y_2|\rho_1\rho_2)/(1-\rho_1^2\rho_2^2)$. iv) We use assertions of Theorem 1.8 and Lemma 2.1 i). ☐

Remark 2.4. Results of the Theorem 2.3 indicate possible applications of the distributions f_{M1} and Wigner in multivariate analysis and stochastic processes. More precisely assertion i) shows that $f_{M1}(x|y,\rho)$ is in fact a conditional distribution. ii) shows that $f_{M1}(x|y,\rho)f_{M0}(y)$ can be treated as a density of certain bivariate distribution with f_{M0}, that is Wigner distribution, as its marginals. Finally iii) is nothing else but the so-called Chapman–Kolmogorov property. These properties are known and applied in stochastic processes, see, *e.g.* [2] and [13]. We quoted them for the sake of completeness of the paper and also in order to present new proofs of these properties directly basing on the general properties of generalized Kesten distributions discussed in the first part of this paper.

Following the above mentioned remark let us denote by $f_2(x, y|\rho)$ the two-dimensional measure defined by:

$$f_2(x, y|\rho) = f_{M1}(x|y,\rho)f_{M0}(y) = \frac{(1-\rho^2)\sqrt{(1-x^2)(1-y^2)}}{4\pi^2 w(x, y|\rho)}. \tag{2.7}$$

Remark 2.5. Notice also that

$$\begin{aligned} f_{M2}(x|y_1, y_2, \rho_1, \rho_2) &= \frac{2}{\pi}\frac{\sqrt{1-x^2}(1-\rho_1^2)(1-\rho_2^2)w(y_1, y_2|\rho_1\rho_2)}{w(x, y_1|\rho_1)w(x, y_2|\rho_2)(1-\rho_1^2\rho_2^2)} \\ &= \frac{f_{M1}(y_1|x,\rho_1)f_{M1}(x|y_2,\rho_2)f_{M0}(y_2)}{f_{M1}(y_1|y_2,\rho_1\rho_2)f_{M0}(y_2)}, \end{aligned}$$

which can be interpreted in the following way. Let us consider 3 element discrete Markov chain X_1, X_2, X_3 such that transition density $X_2|X_3$ is $f_{M1}(x|y_2,\rho_2)$, transition $X_1|X_2$ is $f_{M1}(x|y_1,\rho_1)$ while marginal density of X_3 is $f_{M0}(y_2)$ then the conditional density of $X_2|X_1, X_3$ is $f_{M2}(x|y_1,\rho_1, y_2,\rho_2)$.

Lemma 2.6. $\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-y_1^2}\frac{w3(y_1,y_2,y_3|\rho_1,\rho_2,\rho_3)}{w(y_1,y_2|\rho_1\rho_2)w(y_2,y_3|\rho_2\rho_3)w(y_1,y_3|\rho_1\rho_3)}\mathrm{d}y_1 = \frac{1-\rho_2^2\rho_3^2}{w(y_2,y_{3,}|\rho_2\rho_3)}$.

Proof. We start from (2.5) considered for $k = 3$ and get

$$\begin{aligned} \prod_{j=1}^{3}(1-\rho_i^2)/A_6 &= \frac{w3(y_1, y_2, y_3|\rho_1,\rho_2,\rho_3)}{w(y_1, y_2|\rho_1\rho_2)w(y_2, y_3|\rho_2\rho_3)w(y_1, y_3|\rho_1\rho_3)} \\ &= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\rho_1^{m_1}\rho_2^{m_2}\rho_3^{m_3}V_{m_1,m_2,m_3}U_{m_1}(y_1)U_{m_2}(y_2)U_{m_3}(y_3) \end{aligned}$$

basing on Lemma 2.1. Now using (2.6) and again Lemma 2.1 we get:

$$\begin{aligned} &\int_{-}^{1}\frac{2}{\pi}\sqrt{1-y_1^2}\frac{w3(y_1, y_2, y_3|\rho_1,\rho_2,\rho_3)}{w(y_1, y_2|\rho_1\rho_2)w(y_2, y_3|\rho_2\rho_3)w(y_1, y_3|\rho_1\rho_3)}\mathrm{d}y_1 \\ &= \sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\rho_2^{m_2}\rho_3^{m_3}V_{0,m_2,m_3}U_{m_2}(y_2)U_{m_3}(y_3) \\ &= \sum_{m_2=0}^{\infty}(\rho_2\rho_3)^{m_2}U_{m_2}(y_2)U_{m_2}(y_3) = \frac{1-\rho_2^2\rho_3^2}{w(y_2, y_{3,}|\rho_2\rho_3)}. \end{aligned}$$

☐

From this Lemma we derive the following important conclusion. Namely that the following function:

$$\begin{aligned} & g(y_1, y_2, y_3|\rho_1, \rho_2, \rho_3) \\ &= \frac{8}{\pi^3}\sqrt{1-y_1^2}\sqrt{1-y_2^2}\sqrt{1-y_3^2}\frac{w3(y_1, y_2, y_3|\rho_1, \rho_2, \rho_3)}{w(y_1, y_2|\rho_1\rho_2)w(y_2, y_3|\rho_2\rho_3)w(y_1, y_3|\rho_1\rho_3)} \end{aligned} \tag{2.8}$$

can be treated as the density of some $3D$ distribution with $2D$ marginals equal to $f_{2M}(y_1, y_2|\rho_1\rho_2)$, $f_{2M}(y_1, y_3|\rho_1\rho_3)$, $f_{2M}(y_2, y_3|\rho_2\rho_3)$.

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