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LOCATION AND SCALE BEHAVIOUR OF THE QUANTILES OF A NATURAL EXPONENTIAL FAMILY

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Abstract. Let P_0 be a probability on the real line generating a natural exponential family $(P_t)_{t\in\mathbb{R}}$. Fix α in $(0,1)$. We show that the property that $P_t((-\infty,t))\le\alpha\le P_t((-\infty,t])$ for all t implies that there exists a number μ_α such that P_0 is the Gaussian distribution $N(\mu_\alpha,1)$. In other terms, if for all t, the number t is a quantile of P_t associated to some threshold $\alpha\in(0,1)$, then the exponential family must be Gaussian. The case $\alpha=1/2$, *i.e.* when t is always a median of P_t, has been considered in Letac *et al.* [*Statist. Prob. Lett.* **133** (2018) 38–41]. Analogously let Q be a measure on $[0,\infty)$ generating a natural exponential family $(Q_{-t})_{t>0}$. We show that $Q_{-t}([0,t^{-1}))\le\alpha\le Q_{-t}([0,t^{-1}])$ for all $t>0$ implies that there exists a number $p=p_\alpha>0$ such that $Q(\mathrm{d}x)\propto x^{p-1}\mathrm{d}x$, and thus Q_{-t} has to be a gamma law with parameters p and t.

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1. Introduction

Let P_0 be a probability on the real line and assume that its moment generating function

$$M(t)=\int_{-\infty}^{+\infty}e^{tx}P_0(\mathrm{d}x) \tag{1.1}$$

is finite for all real t. Such a probability generates the natural exponential family

$$P_t(\mathrm{d}x)=\frac{e^{tx}}{M(t)}P_0(\mathrm{d}x),\ t\in\mathbb{R}, \tag{1.2}$$

parametrized by the natural parameter t.

For example, the Gaussian probability $P_0(\mathrm{d}x)=(2\pi\sigma^2)^{-1/2}e^{-(x-m)^2/2\sigma^2}$ *i.e.* $P_0=N(m,\sigma^2)$, with $m\in\mathbb{R}$ and $\sigma^2>0$, generates the natural exponential family $(P_t)=(N(m+t\sigma^2,\sigma^2))$. In this case, if $X_t\sim P_t$ for any

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$t \in \mathbb{R}$, then $X_t \sim X_0 + \sigma^2 t$. In other words (P_t) is a location family generated by P_0 with location parameter $\sigma^2 t$. It is easily verified that the property $X_t \sim X_0 + \sigma^2 t$ forces a natural exponential family to be generated by $P_0 = N(m, \sigma^2)$ for some m. A way to see this is to compute the moment generating function of X_t and substitute $X_0 + \sigma^2 t$ to X_t, getting the equation

$$M(t+s) = M(t)M(s)e^{\sigma^2 ts}, \qquad t, s \in \mathbb{R}.$$

Taking logs and deriving with respect to t and s we get that the cumulant generating function $k = \log M$ of P_0 satisfies $k''(u) = \sigma^2$ for all $u \in \mathbb{R}$, from which $k(u) = -mu + u^2/2\sigma^2$, that is precisely the cumulant generating function of $N(m, \sigma^2)$.

A complete characterization of the *general* exponential families which are location families has been given in [2]. However, if we restrict our attention to *natural* exponential families only the Gaussian families mentioned remain.

The following remark is quite natural: the assumption that $X_t \sim X_0 + \sigma^2 t$, for any $t \in \mathbb{R}$, means that the distribution function of $X_t - \sigma^2 t$ is independent of t, and so the same is true for the quantile function. If we make the weaker assumption that for some fixed $\alpha \in (0,1)$ an α–quantile of $X_t - \sigma^2 t$ does not depend on t, does one obtain the same characterization established above? In slightly simplified words, if $X_t \sim P_t$, as defined in (1.2), is such that $\Pr(X_t \leq \sigma^2 t + b) = \alpha$ for any $t \in \mathbb{R}$, for some fixed $b \in \mathbb{R}$, does this imply that P_0 is $N(m, \sigma^2)$ for some m?

A recent paper [3] gives the answer to this question for $\alpha = 1/2$, $b = 0$ and $\sigma^2 = 1$. Indeed it is proved there that if t is a median of P_t, for any $t \in \mathbb{R}$, then P_0 is the standard Gaussian $N(0,1)$. The first result of the present paper is the extension of this result for any $\alpha \in (0,1)$ (and arbitrary b and σ^2).

Theorem 1.1. *Let P_0 be a probability on the real line which generates the exponential family (1.2). Let $b \in \mathbb{R}$ and suppose that $b + \sigma^2 t$ is an α-quantile of P_t, for $t \in \mathbb{R}$, that is*

$$\int_{(-\infty, b+\sigma^2 t)} e^{tx} P_0(\mathrm{d}x) \leq \alpha M(t) \leq \int_{(-\infty, b+\sigma^2 t]} e^{tx} P_0(\mathrm{d}x), \quad t \in \mathbb{R}. \tag{1.3}$$

Then $P_0 = N(m^*, \sigma^2)$, where $m^* = b - \sigma\Phi^{-1}(\alpha)$, Φ being the standard Gaussian distribution function; moreover $P_t = N(m^* + \sigma^2 t, \sigma^2)$.

For proving Theorem 1.1 we will make use (see (2.3)) of a function proportional to $\alpha x_- + (1-\alpha)x_+$ (where $x_+ = \max(0, x)$ and $x_- = (-x)_+$). Notice that

$$t \mapsto \mathbb{E}(\alpha(Z-t)_- + (1-\alpha)(Z-t)_+)$$

obtains its minimum on the α-quantile of the integrable random variable Z. This last fact implies that the empirical α-quantiles are M-estimators (for a definition, see [5], p. 41).

We describe now our second result: Let Q be a Radon measure on the non-negative real half-line such that its Laplace transform

$$L(t) = \int_{[0,+\infty)} e^{-ty} Q(\mathrm{d}y) \tag{1.4}$$

is finite for all $t > 0$. Such a measure generates the exponential family

$$Q_{-t}(\mathrm{d}y) = \frac{e^{-ty}}{L(t)} Q(\mathrm{d}y), \ t > 0 \tag{1.5}$$

parametrized by the natural parameter t (notice the change of sign w.r.t. the standard usage). In principle this family could be smaller than the natural exponential family generated by Q, but it will turn out not to be the case. As an example, the measure

$$Q^p(\mathrm{d}y) = \frac{1}{\Gamma(p)} y^{p-1}\mathrm{d}y, \tag{1.6}$$

defined for $p > 0$ generates the natural exponential family $Q^p_{-t} = \mathrm{Ga}(p,t)$, with $t > 0$, where $\mathrm{Ga}(p,t)$ is the gamma law with parameters p and t. Now it is immediately verified that if $Y_t \sim Q^p_{-t}$ then $Y_t \sim Y_1/t$, that is (Q^p_{-t}) is a scale family generated by $Q^p_{-1} = \mathrm{Ga}(p,1)$, with scale parameter t^{-1}. It is relatively easy to verify that this property forces Q to be of the form (1.6). However the argument in the scale case is slightly more involved than in the location case and we prefer to give the statement as a proposition.

Proposition 1.2. *Suppose that $(Q_{-t})_{t>0}$ is the natural exponential family defined in (1.5), for some measure Q on the non-negative real half-line. With $Y_t \sim Q_{-t}$, assume that $Y_t \sim \frac{Y_1}{t}$ for any $t > 0$. Then, up to a multiplicative constant, $Q = Q^p$ defined by (1.6), for some $p > 0$.*

Proof. Proof of Proposition 1.2 Compute the Laplace transform of Y_t in the point st, where $s, t > 0$. Then using the assumption $Y_t \sim \frac{Y_1}{t}$ one arrives at

$$\frac{L(t+ts)}{L(t)} = \frac{L(1+s)}{L(1)}.$$

Defining $c(t) = \log L(t)$, for $t > 0$ and deriving w.r.t. t and s this implies

$$uc''(u) + c'(u) = 0,$$

where $u = t + ts > 0$. Integrating twice one arrives at $c(u) = -p\log u + \ell$, with $p > 0$ and an arbitrary $\ell \in \mathbb{R}$, from which $L(u) = e^{\ell}/u^p$, the Laplace transform of $e^{\ell}Q^p$. □

Again the statement of the previous proposition is a special case of the general result contained in [2], where all the *general* exponential families which are scale families are determined. However, only the above ones are *natural* exponential families.

The assumption $Y_t \sim Y_1/t$ for any $t > 0$ is equivalent to say that the distribution function of tY_t is independent of t, and so the same is true for the quantile function. If we make the weaker assumption that, for some fixed $\alpha \in (0,1)$, an α-quantile of tY_t does not depend on t, is it enough to obtain the characterization stated in Proposition 1.2? In slightly simplified words, if $Y_t \sim Q_{-t}$ as defined in (1.5), is such that $\Pr(Y_t \le a/t) = \alpha$ for all $t > 0$, for some $a > 0$, does this still imply that Q is proportional to Q^p for some $p > 0$? Our second result gives a positive answer to this conjecture.

Theorem 1.3. *Let Q be a Radon measure on the non-negative real half-line which generates the exponential family (1.5). Let $a > 0$ and suppose that a/t is an α-quantile of Q_{-t}, for $t > 0$, that is*

$$\int_{(0,a/t)} e^{-ty}Q(\mathrm{d}y) \le \alpha L(t) \le \int_{(0,a/t]} e^{-ty}Q(\mathrm{d}y), \;\; t > 0. \tag{1.7}$$

Then Q is proportional to Q^{p^*}, where $p^* = p^*(\alpha)$ is the unique solution in $p > 0$ of the equation $E_p(a) = \alpha$, E_p being the distribution function of $Ga(p,1)$. In addition $Q_{-t} = Ga(p^*, t/a)$.

It is convenient to comment on the existence and the uniqueness of p^*. The family $(\mathrm{Ga}(p,1), p > 0)$ is a convolution semigroup of laws supported by $[0,\infty)$. Hence, for any fixed $a > 0$ the function $p \mapsto E_p(a)$ is strictly decreasing in p and is continuous. From the Markov inequality and the fact that the expectation of $\mathrm{Ga}(p,1)$ is p we have $1 - E_p(a) \le p/a$ and this implies $\lim_{p\downarrow 0} E_p(a) = 1$. The limit of $E_p(a)$ as $p \to \infty$ is zero from the law of large numbers.

The proofs of Theorems 1.1 and 1.3 are given in the next sections. These proofs deduce from (1.3) and (1.7) two convolution equations of type

$$f = f * H$$

in additive and multiplicative forms, respectively. The solutions to these equations have been investigated in [1]. The result for additive convolutions is reported in the final section of [1]. The result for multiplicative convolutions can be obtained with a passage to the additive convolution form by taking logarithms. In the next proposition we report both of them explicitly.

Proposition 1.4. *(1) Suppose H is a probability density on $\mathbb{R}$, and consider the equation*

$$f(t) = \int_{-\infty}^{+\infty} H(t-x) f(x) \mathrm{d}x, \qquad t \in \mathbb{R}, \tag{1.8}$$

where f is a locally integrable, non-negative function. Then f is necessarily a linear combination, with non-negative coefficients, of a constant function with an exponential function of the form $e^{-s^* x}$, where $s^* \neq 0$ is a solution of the following equation in the real unknown s

$$\int_{-\infty}^{+\infty} e^{sx} H(x) \mathrm{d}x = 1. \tag{1.9}$$

If there is no solution of this form then f is necessarily constant.

(2) Suppose K is a probability density on the positive real half-line and consider the equation

$$g(t) = \int_{0}^{+\infty} K(\frac{t}{y})\, g(y)\, \frac{\mathrm{d}y}{y}, \quad t > 0, \tag{1.10}$$

where g is a locally integrable and non-negative function on $(0,\infty)$. Then $g(t)$ is necessarily a linear combination, with non-negative coefficients, of the function t^{-1} with a power function of the form t^{-1-u^*}, where $u^* \neq 0$ is a solution of the following equation in the real unknown u

$$\int_{0}^{+\infty} y^u K(y) \mathrm{d}y = 1. \tag{1.11}$$

If there is no solution of this form then $g(t) = c/t$, where $c \geq 0$.

Both the equations (1.9) and (1.11) have at most one non zero solution in s and u, respectively. Indeed, the logarithm of the l.h.s of of these equations are convex functions of s and u respectively.

A last section mentions that such a characterization of the normal distribution could be used to design a test of Gaussianity based only on a fixed quantile, and that a similar test could be done for a gamma distribution.

2. Proof of Theorem 1.1

We notice that it is enough to prove the result with $\sigma^2 = 1$. Indeed, it easily proved that if $(P_t, t \in \mathbb{R})$ is a natural exponential family and $X_t \sim P_t$, then, for any $\sigma > 0$ the family $(P'_t, t' \in \mathbb{R})$, where $\sigma^{-1} X_{t/\sigma} \sim P'_t$, is another natural exponential family with natural parameter t. Moreover if (P_t) is a location family with location parameter $\sigma^2 t$, then (P'_t) is a location family with location parameter t.

Let us prove the theorem with $b = 0$. Then we will adjust the solution to take into account an arbitrary value of b. First we prove that P_0 is absolutely continuous. Take $-A \leq s < t \leq A$, for some constant $A > 0$ and compute

$$P_0((s,t)) = \int_{(s,t)} e^{-tx} e^{tx} P_0(\mathrm{d}x) \leq e^{A^2} \int_{(s,t)} e^{tx} P_0(\mathrm{d}x)$$

$$\leq e^{A^2} \left(\int_{(-\infty,t)} e^{tx} P_0(\mathrm{d}x) - \int_{(-\infty,s]} e^{sx} P_0(\mathrm{d}x) + \int_{(-\infty,s]} \left(e^{sx} - e^{tx}\right) P_0(\mathrm{d}x) \right).$$

Using (1.3) and the inequality $|e^u - e^v| \leq |u - v|\, e^w$, for $|u|, |v| \leq w$, this is bounded by

$$e^{A^2} \left(\alpha \left(M(t) - M(s)\right) + |t - s| \int_{\mathbb{R}} |x| e^{A|x|} P_0(\mathrm{d}x) \right) \leq c_A |t - s|$$

since M, being analytic, is locally Lipschitz, and the integral at the l.h.s. is finite by the existence of the moment generating function of P_0 on $\mathbb{R}$.

So we can assume that P_0 has a density p_0. Setting $\alpha = \frac{C}{1+C}$, with $C > 0$, the quantile relation (1.3) leads to

$$\int_{-\infty}^{t} e^{tx} p_0(x)\mathrm{d}x = C \int_{t}^{+\infty} e^{tx} p_0(x)\mathrm{d}x. \tag{2.1}$$

Differentiating w.r.t. t both sides and multiplying by e^{-t^2} one gets

$$p_0(t) + e^{-t^2} \int_{-\infty}^{t} x e^{tx} p_0(x)\mathrm{d}x = -C p_0(t) + C e^{-t^2} \int_{t}^{+\infty} x e^{tx} p_0(x)\mathrm{d}x. \tag{2.2}$$

Introduce the function defined by

$$\mathrm{abs}_C(x) = -Cx 1_{\{x<0\}} + x 1_{\{x>0\}}. \tag{2.3}$$

Multiply both sides of (2.1) by te^{-t^2} and subtract from (2.2). We obtain

$$p_0(t) = \frac{1}{1+C} \int_{-\infty}^{+\infty} \mathrm{abs}_C(t - x) e^{t(x-t)} p_0(x)\mathrm{d}x. \tag{2.4}$$

As expected, a solution to the equation (2.4) is given by $\varphi(t - m^*)$, where φ is the standard Gaussian density function, and $m^* = -\Phi^{-1}(\alpha)$. Next set

$$p_0(x) = \varphi(x - m) f(x), \tag{2.5}$$

with $m = m^*$. We aim to prove that $f(x)$ has to be constant to solve the equation (2.4), with the substitution (2.5). Rewriting the equation for f, one gets

$$f(t) e^{mt - t^2/2} = \frac{1}{1+C} \int_{-\infty}^{+\infty} \mathrm{abs}_C(t - x) e^{t(x-t) + mx - x^2/2} f(x)\mathrm{d}x$$

which is equivalent to

$$f(t) = \frac{e^{\frac{m^2}{2}}}{1+C} \int_{-\infty}^{+\infty} \mathrm{abs}_C(t - x) e^{-\frac{(t-x+m)^2}{2}} f(x)\mathrm{d}x \tag{2.6}$$

which has the form (1.8) with

$$H(x) = \frac{e^{\frac{m^2}{2}}}{1+C}\mathrm{abs}_C(x)e^{-\frac{(x+m)^2}{2}}. \tag{2.7}$$

The moment generating function of H can be exactly computed

$$\int_{-\infty}^{+\infty} e^{sx}H(x)\mathrm{d}x = 1 + \sqrt{2\pi}e^{(s-m)^2/2}(s-m)\left(\Phi(s-m)-\alpha\right). \tag{2.8}$$

This is clearly equal to 1 only if $s = 0$ (hence H is a density) and if $s = m$. We apply Proposition 1.4, 1) to the equation (2.6). When $m = 0$, that is if $\alpha = \frac{1}{2}$, the r.h.s. of (2.8) is equal to 1 only in 0, hence the only non-negative non trivial solutions of the convolution equation (1.8) with kernel H given by (2.7) are the positive constants. This yields immediately that $p_0(x) = \varphi(x)$, as desired. In the case $\alpha \neq \frac{1}{2}$ the solutions $f(x)$ are linear combinations with non-negative coefficients of the constant 1 and the function e^{-mx}. Coming back to $p_0(x) = f(x)\varphi(x-m)$, this gives density solutions for p_0 which are mixtures of $N(m,1)$ with $N(0,1)$. But only the first one has the distribution function at 0 equal to $\alpha \neq 1/2$, therefore a positive component from $N(0,1)$ is forbidden. This proves that $p_0(x)$ has to be $\varphi(x-m)$.

Finally, to deal with an arbitrary value for b, define $\tau_{-b}(x) = x - b$. Now observe that if P_t has α-quantile $b+t$ then $P_t^* = P_t \circ \tau_{-b}^{-1}$ has α-quantile t and it is still a natural exponential family, for $t \in \mathbb{R}$. So $P_t^* = N(-\Phi^{-1}(\alpha), t)$ and $P_t = N(-\Phi^{-1}(\alpha)+b+t, 1)$, ending the proof of Theorem 1.1. □

3. Proof of Theorem 1.3

First we prove the result with $a = 1$. Assume the relation (1.7) and let $t \to 0+$. From the inequality

$$\int_0^{1/t} e^{-xt}Q(\mathrm{d}x) \leq \alpha \int_0^{\infty} e^{-xt}Q(\mathrm{d}x)$$

where $0 < \alpha < 1$ we see easily that $A = Q(\mathbb{R}^+)$ cannot be finite: for if A was finite the dominated convergence for $f_n(x) = e^{-x/n}1_{[0,n]}(x) \leq 1$ would lead to the contradiction $A \leq \alpha A$. Hence the natural parameter space of the natural exponential family (Q_s) coincides with the negative reals.

Next we prove that Q is absolutely continuous. Take $0 < s < t < +\infty$ and compute

$$Q((t^{-1}, s^{-1})) = \int_{(t^{-1},s^{-1})} e^{sx}e^{-sx}Q(\mathrm{d}x) \leq e\int_{(t^{-1},s^{-1})} e^{-sx}Q(\mathrm{d}x)$$

$$= e\left(\int_{(0,s^{-1})} e^{-sx}Q(\mathrm{d}x) - \int_{(0,t^{-1}]} e^{-tx}Q(\mathrm{d}x) + \int_{(0,t^{-1}]} \left(e^{-tx} - e^{-sx}\right)Q(\mathrm{d}x)\right).$$

By (1.7) the difference between the first two integrals at the r.h.s. is bounded by $\alpha\left(L(s) - L(t)\right)$, whereas the remaining integral is non positive. Again since L is analytic in the positive real half-line it is locally Lipschitz and this proves the absolute continuity of Q, that is $Q(\mathrm{d}x) = q(x)\mathrm{d}x$, with q non-negative and locally integrable.

Now we can write (1.7) in the form of an equality, setting again $\alpha = \frac{C}{1+C}$, namely

$$\int_0^{t^{-1}} e^{-ty}q(y)\mathrm{d}y = C\int_{t^{-1}}^{+\infty} e^{-ty}q(y)\mathrm{d}y, t > 0. \tag{3.1}$$

Differentiating both sides w.r.t. t, one gets

$$\frac{1+C}{t^2}e^{-1}q(t^{-1}) = C\int_{t^{-1}}^{+\infty} ye^{-ty}q(y)\mathrm{d}y - \int_0^{t^{-1}} ye^{-ty}q(y)\mathrm{d}y.$$

Adding the l.h.s. of (3.1) and subtracting the r.h.s., both multiplied by t^{-1}, to the r.h.s. of the above equality, we get for any $t > 0$

$$q\left(t^{-1}\right) = \frac{et^2}{1+C}\left\{\int_0^{t^{-1}} (t^{-1}-y)e^{-yt}q(y)\mathrm{d}y + C\int_{t^{-1}}^{+\infty} (y-t^{-1})e^{-yt}q(y)\mathrm{d}y\right\}. \tag{3.2}$$

With the help of the function abs_C defined in (2.3) , equality (3.2) is rewritten as

$$q(t^{-1}) = \frac{et}{(1+C)}\int_0^{\infty} \mathrm{abs}_C\,(1-ty)\,e^{-ty}q(y)\mathrm{d}y,\ \ t>0. \tag{3.3}$$

Next, for any $p > 0$, define $q_p(x) = \frac{1}{\Gamma(p)}x^{p-1}1_{(0,+\infty)}(x)$. Recall from the introduction that for $p = p^*(\alpha)$ one has

$$\int_0^1 y^{p^*-1}e^{-y}\mathrm{d}y = C\int_1^{+\infty} y^{p^*-1}e^{-y}\mathrm{d}y. \tag{3.4}$$

Now multiply both sides of (3.3) by t^{p^*-2} and change the variable of integration at the r.h.s. to be $z = y^{-1}$. One gets

$$t^{p^*-2}q(t^{-1}) = \frac{et^{p^*-1}}{1+C}\int_0^{\infty} \mathrm{abs}_C\left(1-tz^{-1}\right)e^{-t/z}g\left(z^{-1}\right)\frac{dz}{z^2}. \tag{3.5}$$

Defining the l.h.s. of the above equality to be $g(t)$, one has

$$q(t) = g(t^{-1})t^{p^*-2}, \tag{3.6}$$

and turns the equation (3.5) into an equation of the form (1.10) in g with

$$K(y) = \frac{e}{1+C}\mathrm{abs}_C(1-y)e^{-y}y^{p^*-1}1_{(0,+\infty)}(y). \tag{3.7}$$

The Mellin transform of K can be easily computed

$$\int_0^{\infty} y^u K(y)\mathrm{d}y = 1 + e\Gamma(p^*+u)(p^*+u-1)\{\alpha - E_{p^*+u}(1)\}. \tag{3.8}$$

Now observe that the quantity inside the brackets of (3.8) at the r. h. s. is always increasing in u; moreover it is equal to 0 for $u = 0$, due to (3.4). Hence for any value of $C > 0$ the function K is always a density. When $p^* = 1$ (equivalently, $C = e-1$, or $\alpha = 1-e^{-1}$), $u = 0$ is the unique global minimum point of the r. h. s. of (3.8). Then, by Proposition 1.4, 2), the only non negative non trivial solutions to the equation (1.10), with K given by (3.7), have necessarily the form $g_0(t) = c_0t^{-1}$, with $c_0 > 0$. Thus $q(y) = c_0q_{p^*}(y) = c_0'y^{p^*-1}$. Moreover, for $p^* \neq 1$ (equivalently, $C \neq e-1$, or $\alpha \neq 1-e^{-1}$) the value $u = 1-p^* \neq 0$ makes the expression at the r. h. s. of (3.8) equal to 1, too. As a consequence $g_1(t) = t^{p^*-2}$ is also a solution of the multiplicative convolution equation with K given by (3.7). Applying again Proposition 1.4, 2) all the non negative non trivial solutions are linear combinations

$$g_0(t) + c_1 g_1(t) = c_0 t^{-1} + c_1 t^{p^*-2},$$

with $c_0, c_1 \geq 0$ not both zero. Substituting in (3.6) we get that the solutions to (3.2) have the form

$$q(y) = c_0' y^{p^*-1} + c_1.$$

But for $c_1 > 0$ the condition (3.1) is violated for $t = 1$. Indeed, in this case the difference between the l.h.s. and the r.h.s. of (3.1) is equal to $c_1(1 - \frac{1+C}{e})$ and this is different from 0 as soon as $C \neq e - 1$. So as desired $q(y) = c_0' y^{p^*-1}$.

Finally, to deal with an arbitrary value of $a > 0$, first define $\sigma_{a^{-1}}$ to be the multiplication by a^{-1}. Now observe that, if Q_{-t} has α-quantile at^{-1} then $Q^*_{-t} = Q_{-t} \circ \sigma^{-1}_{a^{-1}}$ has α-quantile t^{-1} and it is still a natural exponential family, for $t > 0$. So $Q^*_{-t} = \mathrm{Ga}(p^*, t)$ and $Q_{-t} = \mathrm{Ga}(p^*, at^{-1})$, ending the proof of Theorem 1.3. ☐

4. Statistical considerations

A statistical application of the above characterization consists in the following exploratory testing procedure for normality. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a probability measure P_0 on the real line. The obvious estimate of the α-quantile of P_0 is the empirical α-quantile, computed on the sample $X_1, \ldots, X_n$ and a graph of the different α-quantiles gives an indication about the Gaussianity of P_0: this is what a Q-Q plot does (see [6]). However, Theorem 1.1 justifies a different procedure which consists in estimating the α-quantiles of P_t, as given by (1.2), for the *same* α and various values of t, and fitting to them a line.

The general principle is this: observing a sample drawn from the unknown distribution P_0, we can approximate $P_t(\mathrm{d}x) = e^{tx} P_0(\mathrm{d}x)/M(t)$ with the random probability measure

$$P_t^{(n)}(\mathrm{d}x) = \frac{n}{\sum_{j=1}^n e^{tX_j}} \times \frac{1}{n} \sum_{i=1}^n e^{tX_i} \delta_{X_i},$$

which, by the law of large numbers, converges weakly towards P_t a.s. The α-quantile of P_t can be estimated with the α-quantile $q_t^{(n)}$ of $P_t^{(n)}$ and a good fitting of a line to the points $(t_i, q_{t_i}^{(n)})$ for some choice of $t_i, i = 1, \ldots, m$, is an indication favouring the hypothesis that P_0 is Gaussian. In a similar fashion, the characterization of Theorem 1.3 can be used for an exploratory test of the hypothesis that a sample is drawn from a Gamma distribution: in this case the estimated α-quantiles are plotted against the natural parameter t on a log–log scale and fitted with a line of slope -1.

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