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A FRAMEWORK OF BSDES WITH STOCHASTIC LIPSCHITZ COEFFICIENTS

Hun O*, Mun-Chol Kim and Chol-Kyu Pak

Abstract. In this paper, we suggest an effective technique based on random time-change for dealing with a large class of backward stochastic differential equations (BSDEs for short) with stochastic Lipschitz coefficients. By means of random time-change, we show the relation between the BSDEs with stochastic Lipschitz coefficients and the ones with bounded Lipschitz coefficients and stopping terminal time, so they are possible to be exchanged with each other from one type to another. In other words, the stochastic Lipschitz condition is not essential in the context of BSDEs with random terminal time. Using this technique, we obtain a couple of new results of BSDEs with stochastic Lipschitz (or monotone) coefficients.

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1. Introduction

Since their first introduction by Bismut [6] in the linear case and the nonlinear extension by Pardoux and Peng [36], Backward stochastic differential equations (BSDEs for short) have been developed rapidly with various types of generalizations in the last decades.

BSDEs are closely connected to finance, optimal control and partial differential equations [24, 47].

Most of BSDEs are concerned with the case of constant time horizon and the uniform Lipschitz conditions on the driver. In many environments, the Lipschitz condition is too restrictive to be assumed, so much effort has been devoted to relax it.

In this context, El Karoui and Huang [23] studied the BSDEs with stochastic Lipschitz coefficients driven by a general càdlàg martingale and those were developed under weaker conditions in [10]. For the Brownian motion BSDEs, there are some papers going in this direction [2, 5, 39, 44]. Particularly, in [2], Sect. 3, the existence of the measure solution was stated by the way of examining the weak convergence of a sequence of measures which were constructed using the martingale representation and the Girsanov change of measure. Also, the reflected backward stochastic differential equations or backward doubly stochastic differential equations (BDSDEs) driven by a Lévy process with stochastic Lipschitz coefficients were studied in [25, 26, 28, 29, 34, 35, 45].

Although the details are slightly different, the main techniques for BSDEs with stochastic Lipschitz conditions are similar to the procedure of BSDEs with Lipschitz conditions.

Keywords and phrases: Backward stochastic differential equations (BSDEs), time-change, stochastic Lipschitz coefficient, random terminal time, Markov chain.

Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea.

* Corresponding author: cioc13@ryongnamsan.edu.kp

That is, the techniques consist of obtaining some *a priori* estimates and finally using the fixed-point arguments.

Another technique was also used in [16, 39], where the main driver was approximated by Lipschitz drivers, some estimates were obtained for the convergence of approximation sequence and finally it was shown that the limit of this sequence is a unique solution.

In this paper, we approach the problem differently by indirect method. More precisely, we study the BSDEs with stochastic Lipschitz coefficients by studying the BSDEs with random terminal time. The technique is based on time-change represented by stochastic Lipschitz coefficients. This time-change converts the BSDEs with stochastic Lipschitz condition to the ones with uniformly Lipschitz condition and stopping terminal time on another stochastic basis and these two BSDEs are equivalent in some sense. So, if we know the results of BSDEs with random terminal time and uniformly Lipschitz coefficients, then the results are easily extended to the ones with stochastic Lipschitz coefficients through our framework. In other words, the stochastic Lipschitz condition is not a problem in a setting of BSDEs with random terminal time.

We briefly mention that the opposite argument also holds, that is, the randomness of terminal time do not play an essential role under the stochastic Lipschitz condition. During our discussion, if the integrator of the driver is a general continuous increasing process, it is converted to the typically well-known one, that is, the Lebesgue measure by time-change.

Consequently, if we obtain the result of BSDEs with random terminal time and Lebesgue measure integrators under standard Lipschitz conditions, then the research on BSDEs with general measure integrators and stochastic Lipschitz coefficients is just a corollary of that. Motivated by the proposed method, the latter discussion is naturally concerned with the application of the method. First, we study the BSDEs with random terminal time in general space (more precisely, in a separable Hilbert space). We establish both wellposedness and comparison theorems for them. The existence and uniqueness argument is very similar to [19] and we give the more useful results by introducing bounded Lipschitz condition, instead of constant Lipschitz one. However, it is not easy to study the comparison principle for general BSDEs possibly with jumps. Next, we state main results of BSDEs with stochastic Lipschitz coefficients in general space, by using the results of BSDEs with random terminal time which we obtained. Even in the special cases (for example, BSDEs driven by Brownian motion), our results are more innovative than the previous works. Particularly, in a sense, our approach provides a complete theory of the wellposedness of solutions to BSDEs with stochastic Lipschitz coefficients (see the end of Sect. 4). Finally, we give some other results obtained by the applications of time-change. The first subject is a stochastic monotonicity which generalizes the stochastic Lipschitz property. We show a new existence and uniqueness result of L^p-solutions to BSDEs driven by Brownian motions under stochastic monotonicity conditions. The second subject is an undiscounted approach in the stochastic Lipschitz setting. For this purpose, we apply our framework to the Markov chain BSDE. The smart feature is that the discussion on the case of uniform Lipschitz condition is just inherited to the case of stochastic Lipschitz condition under the same conditions on volumes.

In general, for the wellposedness of BSDEs with stochastic Lipschitz condition, the integrability conditions required are stronger than ones with uniformly Lipschitz condition. The main reason is on the discounting property of the terminal time. This discounting property is contributed to the exponential integrability conditions of volumes and these conditions are influenced by the Lipschitz coefficients. In fact, the discounting property is inherited from the monotonicity of the driver. In our framework, the original BSDE with stochastic Lipschitz condition can be seen as the BSDE to stopping time which is time-changed in reverse and the time-independent discounting rate of this BSDE with constant Lipschitz coefficients is preserved while time-change is processed. This means that the stronger integrability conditions are still required if we use the results of BSDEs with random terminal time obtained by using the monotonicity condition as the key tool. But for the Markov chain BSDEs, the results of undiscounted BSDEs to stopping time without assuming the monotonicity which was researched by Samuel N. Cohen [12] make our technique more effective. By means of time-change, we get useful results of Markov chain BSDEs with stochastic Lipschitz coefficients without assuming the stronger integrability assumptions.

The rest of this paper is organized as follows. In Section 2, we propose a general map from the BSDEs with stochastic Lipschitz coefficients to the ones with uniformly Lipschitz coefficients by the technique of time-change.

We discuss this for BSDEs in general space as in [15]. We study the BSDEs with random terminal time in Section 3. Section 4 is devoted to give the main results of BSDEs with stochastic Lipschitz coefficients. In Section 5, we study some other supplementary results and in Section 6, we give some concluding remarks.

We introduce some useful notations below. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. We shall assume that $\mathcal{F} = \mathcal{F}_\infty$ and $\mathcal{F}_0$ is trivial.

- $|\cdot|$ denotes the standard Euclidean norm. If z is a matrix, then we use $\|z\|$. $\|z\|$=Trace$[zz^\mathsf{T}]$, where $[\cdot]^\mathsf{T}$ means the vector transpose.
- $\mathcal{B}(0,\infty)$ denotes the Borel $\sigma-$ field given on $(0,\infty)$.
- $I_A(\cdot)$ means an indicator function of A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.
- $(\overline{\Omega}, \overline{\mathcal{F}})$ means the product measurable space. That is $\overline{\Omega} := \Omega \times (0,\infty)$ and $\overline{\mathcal{F}} := \mathcal{F} \times \mathcal{B}(0,\infty)$.
- $\mathrm{d}Q/\mathrm{d}\mu$ denotes the Radon-Nikodym derivative of Q with respect to μ, where Q is absolutely continuous with respect to μ. If μ is Lebesgue measure and Q is the measure generated by an absolutely continuous function f, then we use f' rather than $\mathrm{d}Q/\mathrm{d}\mu$.
- $\mathbb{E}^Q[\cdot]$ means the expectation under measure Q.
- $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the space of square-integrable random variables.
- Ł and Ł^c are the spaces of local martingales and continuous local martingales, respectively.
- $\mathcal{H}^2$ is the space of square-integrable martingales.
- $\mathcal{H}^2_T$ is the space of square-integrable martingales on $[0,T]$.
- $\mathcal{H}^2_{loc}$ is the space of locally square integrable martingales.
- $L^2(M) := \big\{Z \mid Z \text{ is predictable}, \mathbb{E}\big[\int_0^\infty \|Z_t\|^2 \mathrm{d}\langle M\rangle_t\big] < +\infty\big\}$ where $M \in \mathcal{H}^2$.
- For $M \in \mathcal{H}^2_{loc}$, $L^2_{loc}(M)$ means the space of predictable processes Z for which there exists a localizing sequence (τ_n) such that

$$\mathbb{E}\bigg(\int_0^\infty \|Z\|^2 \mathrm{d}\langle M^{\tau_n}\rangle\bigg) = \mathbb{E}\bigg(\int_0^{\tau_n} \|Z\|^2 \mathrm{d}\langle M\rangle\bigg) < +\infty.$$

- $L^2_{T,loc}(M) := \big\{X \mid X \cdot \mathbf{1}_{[0,T]} \in L_2(M) \text{ for any } T < \infty\big\}$, where $M \in \mathcal{H}^2_{loc}$.
- $\mathcal{E}(X)$ means the Doléans exponential of a semi-martingale X. Hence, $\mathcal{E}(X) = 1 + \mathcal{E}(X)_- \bullet X = 1 + \int_0^\cdot \mathcal{E}(X)_- \mathrm{d}X$. Note that $\mathcal{E}(X) = \exp\big(X - X(0) - [X]^c/2\big) \times \Pi(1 + \Delta X)\exp(-\Delta X)$.
- $\mathcal{V}$ is the space of càdlàg, adapted processes which have finite variation on every finite interval.
- $\mathcal{V}^+ := \{v \in \mathcal{V} \mid v \text{ is increasing}\}$.
- $\mathcal{A} := \big\{A \in \mathcal{V} \mid \mathbb{E}\big[\mathrm{Var}(A)(\infty)\big] < \infty\big\}$, where $\mathrm{Var}(A)$ means the total variation of A.
- $\mathcal{A}_{loc}$ is the space of processes locally belonging to $\mathcal{A}$, that is the space of processes X for which there exists a localizing sequence (τ_n) such that $X^{\tau_n} \in \mathcal{A}$ for all n.
- $\mathcal{A}^+_{loc} := \big\{X \in \mathcal{A}_{loc} \mid X \text{ is increasing}\big\}$.
- Let τ be a stopping time, ϕ, β predictable processes and $M = (M^i)_{i=1}^\infty$ a sequence of $\mathcal{H}^2-$martingales. Set $\phi_t := \int_0^t \phi(s)\mathrm{d}s$.

$$L^2_{\phi(\cdot)}(0,\tau) := \left\{X \;\middle|\; X \text{ is progressive}, \mathbb{E}\left[\int_0^\tau e^{\phi_s}|X(s)|^2\mathrm{d}s\right] < \infty\right\}.$$

$$L^{2,\beta}_{\phi(\cdot)}(0,\tau) := \big\{Y \mid \beta Y \in L^2_{\phi(\cdot)}(0,\tau)\big\}.$$

$$L^2_{\phi(\cdot)}(0,\tau,M) := \left\{Z = (Z^1, Z^2, ...) \;\middle|\; Z \text{ is predictable}, \mathbb{E}\big[\textstyle\sum_i \int_0^\tau e^{\phi_t}\|Z^i(t)\|^2\mathrm{d}\langle M^i\rangle_t\big] < \infty\right\}.$$

$$U^2_{\phi(\cdot)}(0,\tau) := \{Y \mid Y \text{ is càdlàg, adapted and } \mathbb{E}[\sup\{e^{\phi_s}|Y(s)|^2 : 0 \leq s \leq \tau\}] < \infty\}.$$

If we need to stress the Euclidean image space V, we use $L^2_{\phi(\cdot)}(0,\tau,V)$, $L^{2,\beta}_{\phi(\cdot)}(0,\tau,V)$.

1.1. Intdroducing BSDEs in general space

As in [15], we construct the BSDEs assuming only the usual properties of the filtration and that $L^2(\Omega,\mathcal{F},\mathbb{P})$ is a separable Hilbert space. Unless otherwise indicated, we should read all equalities (and inequalities) as "up to a measure-zero set" throughout this paper.

Definition 1.1. For $v \in \mathcal{V}^+$, let us define the measure μ_v on $(\overline{\Omega},\overline{\mathcal{F}})$ as follows.

$$\mu_v(A) := \mathbb{E}\bigg[\int_0^\infty I_A(\omega,t)\mathrm{d}v\bigg], \quad A \in \overline{\mathcal{F}} \tag{1.1}$$

where the integral is taken pathwise in a Stieltjes sense.

This measure μ_v is called the measure induced (or generated) by v.

Note that if $v \in \mathcal{A}^+_{loc}$ then μ_v gives a $\sigma-$ finite measure on $(\overline{\Omega},\overline{\mathcal{F}})$.

We give a simple version of the well-known Martingale representation theorem below (see [20, 21]).

Theorem 1.2 (Martingale representation theorem)**.** *Suppose that $L^2(\Omega,\mathcal{F},\mathbb{P})$ is a separable Hilbert space with an inner product $X\cdot Y = \mathbb{E}[XY]$.*
Then there exists a sequence of $\mathcal{H}^2-$martingales $M = (M^1, M^2, ...)$ such that $\langle M^i, M^j\rangle = 0$ for $i \neq j$ and every $N \in \mathcal{H}^2$ can be represented as

$$N_t = N_0 + \int_0^t Z_u \mathrm{d}M_u = N_0 + \sum_{i=1}^\infty \int_0^t Z^i_u \mathrm{d}M^i_u \tag{1.2}$$

for some sequence of predictable processes $Z = (Z^1, Z^2, ...)$ satisfying $Z \in L^2(M)$.

And the predictable quadratic variation processes of these martingales $\langle M^i\rangle$ satisfy $\langle M^1\rangle \succ \langle M^2\rangle \succ ...$, ($\succ$ denotes absolute continuity of induced measures). If (N^i) is an another such sequence then $\langle N^i\rangle \sim \langle M^i\rangle$, where $\sim$ denotes equivalence of induced measures.

Remark 1.3. If the space is generated by Brownian motion, the martingale representation theorem holds on infinite interval (see *e.g.* [20], Thm. 6 or references therein). This also implies the martingale representation theorem on every finite interval.

Definition 1.4. If every martingale has a representation (1.2) by a sequence of martingales $M = (M^i)_{i=1}^\infty$, then we say that M has the predictable representation property.

For a given $k \in \mathbb{N}$, the general type of BSDE is as follows.

$$Y_t = \xi + \int_t^\tau g(\omega, s, Y_{s-}, Z_s)\mathrm{d}v_s - \sum_{i=1}^\infty \int_t^\tau Z^i_s \mathrm{d}M^i_s, \tag{1.3}$$

where τ is an unbounded $\mathbb{F}$-stopping time which may be infinite with some positive probability, the terminal value ξ is an $\mathcal{F}_\tau$-measurable random variable with values in $\mathbb{R}^k$ such that $\xi = 0$[1] on the set $\{\tau = \infty\}$, the driver $g : \Omega \times (0,\infty) \times \mathbb{R}^k \times \mathbb{R}^{k\times\infty} \to \mathbb{R}^k$ is predictable, $v \in \mathcal{V}$ and the integral of driver is the Lebesgue-Stieltjes integral with respect to the measures generated by the trajectories of v.

A solution of the BSDE (1.3) is a pair of processes (Y, Z) taking values in $\mathbb{R}^k \times \mathbb{R}^{k\times\infty}$, where Y is progressive and Z is predictable.

[1] We have assumed that $\xi = 0$ on the set $\{\tau = \infty\}$, but in fact the value of ξ on that set is irrelevant.

In this paper, we shall make the following assumption on v.

(A0) v is a continuous and increasing process.

It follows from **(A0)** that v is locally bounded and $v \in \mathcal{A}^+_{loc}$.

Noting that the predictable quadratic variation process $\langle M\rangle$ identifies an induced measure on $\overline{\mathcal{F}}$ defined by (1.1), suppose that the induced measure $\mu_{\langle M^i\rangle}$ has the following Lebesgue decomposition.

$$\mu_{\langle M^i\rangle} = \bar{m}^{i,1} + \bar{m}^{i,2}, \quad i \in \mathbb{N}, \tag{1.4}$$

where $\bar{m}^{i,1}$ is absolutely continuous with respect to μ_v and $\bar{m}^{i,2}$ is singular to μ_v.

From the generalized Radon-Nikodym Theorem (see *e.g.* [30], Chap. 3, Prop. 3.49), there exist two processes $m^{i,1}_t, m^{i,2}_t$ such that $\mu_{m^{i,1}} = \bar{m}^{i,1}$ and $\mu_{m^{i,2}} = \bar{m}^{i,2}$.

More precisely $m^{i,j}_t = \mathrm{d}\pi^j_t/\mathrm{d}\mathbb{P}, j = 1, 2$, where $\pi^j_t(B) := \bar{m}^{i,j}\big((0,t] \times B\big), B \in \mathcal{F}$. Thus, we have

$$\langle M^i\rangle_t = m^{i,1}_t + m^{i,2}_t. \tag{1.5}$$

We introduce the stochastic semi-norm $\|\cdot\|_{M_t}$ defined as

$$\|z\|^2_{M_t} := \sum_i \Big[\|z^i\|^2 \cdot (\mathrm{d}\bar{m}^{i,1}_t/\mathrm{d}\mu_v)\Big] = \sum_i \Big[\|z^i\|^2 \cdot (\mathrm{d}\mu_{m^{i,1}}/\mathrm{d}\mu_v)(\cdot, t)\Big], \tag{1.6}$$

for every $z = (z^1, z^2, ...) \in \mathbb{R}^{k\times\infty}$.

2. Time-changes and BSDEs

We begin with the definition of time-change ([40], Chap. V).

Definition 2.1. A time-change C is a family $\{C(s) \mid s > 0\}$ of stopping times such that the maps $s \to C(s)$ are almost surely increasing and right continuous.

Definition 2.2. If C is a time-change, a process X is said to be $C-$continuous if X is constant on each interval $[C_{t-}, C_t]$.

We can define the stopped σ-field $\widetilde{\mathcal{F}}_t := \mathcal{F}_{C(t)}$ and get the new stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t\}_{t\ge 0})$. It can be easily seen that $\widetilde{\mathbb{F}}$ also satisfies the usual conditions from the property of stopped $\sigma-$fields. If X is $\mathcal{F}$-progressive then $\widetilde{X}_t := X_{C_t}$ is $\widetilde{\mathbb{F}}-$adapted and the process $\widetilde{X}_t$ is called the time-changed process of X. We show a typical example of time-change below.

Let us consider an increasing and right-continuous adapted process A (so, progressive) with which we associate

$$C(s) := \inf\{t \mid A(t) > s\}, \tag{2.1}$$

This process $C(s)$ is called the inverse of $A(s)$ and we write it $A^{-1}(s)$.

As the stochastic basis satisfies the usual conditions and A is progressive, $A^{-1}(s)$ which is the hitting time of (s, ∞) is a stopping time for every $s > 0$. And obviously it is increasing and right continuous. Thus $C = A^{-1} = \{A^{-1}(s)|s > 0\}$ is a time-change.

Throughout this section, we suppose that C is almost surely finite, $C_0 = 0$ and for any progressively measurable process X_t, $\widetilde{X}_t$ means the time-changed process of it, unless otherwise indicated. And for the space of processes V with respect to $\mathbb{F}$, $\widetilde{V}$ means the corresponding space with respect to $\widetilde{\mathbb{F}}$. For example, $\widetilde{\mathbb{L}}$ means the space of $\widetilde{\mathbb{F}}-$local martingales. We give some main results concerning the property of time-change under $C-$continuity below.

Lemma 2.3. *([40], Chap. V, Prop. 1.4).*
Let C be a time-change on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. If h is $\mathbb{F}-$progressive, then $\widetilde{h}$ is $\widetilde{\mathbb{F}}-$progressive. And if X is a $C-$continuous process of finite variation, then

$$\int_0^{C_t} h_u \mathrm{d}X_u = \int_0^t \widetilde{h}_u \mathrm{d}\widetilde{X}_u.$$

Lemma 2.4. *([40], Chap. V, Prop. 1.5)*
If C is a time-change on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and $M \in L^c$ satisfies $C-$continuity, then the following hold.

I. $\widetilde{M} \in \widetilde{L}^c$ *and* $\langle \widetilde{M} \rangle = \widetilde{\langle M \rangle}$

II. *If* $h \in L^2_{t,loc}(M)$, *then* $\widetilde{h} \in \widetilde{L}^2_{t,loc}(\widetilde{M})$ *and for each* $t > 0$

$$\int_0^t \widetilde{h}_u \mathrm{d}\widetilde{M}_u = \int_0^{C_t} h_u \mathrm{d}M_u.$$

Now we show the property of time-change for general locally square-integrable martingales.

Lemma 2.5. *If C is a time-change on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and $M \in \mathcal{H}^2_{loc}$ is $C-$continuous, then the followings hold.*

I. $\widetilde{M} \in \widetilde{\mathcal{H}}^2_{loc}$ *and* $\langle \widetilde{M} \rangle = \widetilde{\langle M \rangle}$
II. *If* $h \in L^2_{t,loc}(M)$, $\widetilde{h} \in \widetilde{L}^2_{t,loc}(\widetilde{M})$ *and for each* $t > 0$

$$\int_0^t \widetilde{h}_u \mathrm{d}\widetilde{M}_u = \int_0^{C_t} h_u \mathrm{d}M_u$$

Proof.
I. For any $L \in \text{Ł}$, it is easy to see that $\widetilde{L} \in \widetilde{\text{Ł}}$ from the optional stopping theorem and $C-$continuity of M.

As $M \in \mathcal{H}^2_{loc}$, the predictable quadratic variation $\langle M \rangle$ is in $\mathcal{A}^+_{loc}$ and $M^2 - \langle M \rangle$ is a local martingale from the characterization of $\mathcal{H}^2_{loc}$ martingale (see *e.g.* [30], Chap. 3, Prop. 3.64). Therefore $\widetilde{M^2 - \langle M \rangle} = \widetilde{M}^2 - \widetilde{\langle M \rangle}$ is an $\widetilde{\mathbb{F}}-$local martingale.

Let (τ_n) denote the localizing sequence such that $\langle M \rangle^{\tau_n} \in \mathcal{A}^+$ for every n.

Then $\widetilde{\tau_n} := C^{-1}_{\tau_n} = \inf\{t : C_t \geq \tau_n\}$ is an $\widetilde{\mathbb{F}}-$stopping time for every n and $(\widetilde{\tau_n})$ is a localizing sequence.

Noting that M is $C-$continuous if and only if $\langle M \rangle$ is $C-$continuous (see [40], Chap. IV, Prop. 1.13), $\langle M \rangle$ is constant on $[\tau_n, C_{\widetilde{\tau_n}}]$.

So $\mathbb{E}\big[\widetilde{\langle M \rangle}(\widetilde{\tau_n})\big] = \mathbb{E}\big[\langle M \rangle(C_{\widetilde{\tau_n}})\big] = \mathbb{E}\big[\langle M \rangle(\tau_n)\big] < \infty$. Hence $\widetilde{\langle M \rangle} \in \widetilde{\mathcal{A}}^+_{loc}$.

And $\widetilde{\langle M \rangle}$ is also $\widetilde{\mathbb{F}}-$predictable from the $C-$continuity. Accordingly, using again the characterization of $\mathcal{H}^2_{loc}$ martingale, $\widetilde{M} \in \widetilde{\mathcal{H}}^2_{loc}$ and $\widetilde{\langle M \rangle} = \langle \widetilde{M} \rangle$.
II. This is a simple consequence of **I** and Lemma 2.3 together with the relation between stochastic integral and quadratic variation. □

Remark 2.6. Lemma 2.5 still holds for $\mathcal{H}^2-$martingales under $C-$continuity. That is, if M is $\mathcal{H}^2-$martingale satisfying $C-$continuity, then $\widetilde{M} \in \widetilde{\mathcal{H}}^2$. In this case we use the characterization of $\mathcal{H}^2-$martingales (see *e.g.* [30], Chap. II, Prop. 2.84) and the same procedure is used for the proof.

Remark 2.7. We note that the Lemmas 2.3–2.5 still hold in random time horizon. For example, in Lemma 2.5, if ξ is a non-negative random variable and $\int_0^\infty h_u \mathrm{d}\langle M\rangle_u < \infty$ on $\{\xi = \infty\}$, then

$$\int_0^\xi \widetilde{h}_u \mathrm{d}\widetilde{M}_u = \int_0^{C_\xi} h_u \mathrm{d}M_u \quad \mathbb{P}-\text{a.s.}$$

Now we return to the discussion on BSDE. For the BSDE on which we discuss, the sequence of $\mathcal{H}^2-$martingales M^i $(i = 1, 2, ...)$ has the predictable representation property on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$.

At this point, the martingale representation on $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{F})$ is naturally expected whereas the time-changed processes of $M^i(i = 1, 2, ...)$ are $\widetilde{\mathcal{H}}^2-$martingales under $C-$continuity by Lemma 2.5.

Lemma 2.8. *Let C be a time-change and $\mathcal{H}^2-$martingales M^i $(i = 1, 2, ...)$ be $C-$continuous. Then, the sequence of $\widetilde{\mathcal{H}}^2-$martingales $(\widetilde{M}^i)$ has the predictable representation property for any $\widetilde{\mathcal{H}}^2-$martingale satisfying $C^{-1}-$continuity such as in Theorem 1.2.*

Proof. Let $\widetilde{N}$ be an $\widetilde{\mathcal{H}}^2-$martingale satisfying $C^{-1}-$continuity. Then $\widetilde{N}_t = \widetilde{N}_{C^{-1}(C_t)} = N_{C_t}$, where $N_t := \widetilde{N}_{C_t^{-1}}$. Obviously, $N \in \mathcal{H}^2$ by Lemma 2.5. Therefore using Theorem 1.2 and Lemma 2.5,

$$\widetilde{N}_t = N_{C_t} = N_0 + \sum_{i=1}^\infty \int_0^{C_t} Z_u^i \mathrm{d}M_u^i = \widetilde{N}_0 + \sum_{i=1}^\infty \int_0^t \widetilde{Z}_u^i \mathrm{d}\widetilde{M}_u^i,$$

for some sequence of $\mathbb{F}-$predictable processes, (Z^i) satisfying

$$\mathbb{E}\left[\sum_{i=1}^\infty \int_0^\infty (Z_u^i)^2 \mathrm{d}\langle M^i\rangle_u\right] < +\infty.$$

Using Lemmas 2.3 and 2.5 again,

$$\mathbb{E}\left[\sum_{i=1}^\infty \int_0^\infty (Z_u^i)^2 \mathrm{d}\langle M^i\rangle_u\right] = \mathbb{E}\left[\sum_{i=1}^\infty \int_0^\infty (\widetilde{Z}_u^i)^2 \mathrm{d}\langle \widetilde{M}^i\rangle_u\right].$$

This leads to

$$\mathbb{E}\left[\sum_{i=1}^\infty \int_0^\infty (\widetilde{Z}_u^i)^2 \mathrm{d}\langle \widetilde{M}^i\rangle_u\right] < +\infty. \tag{2.2}$$

Hence for any $\widetilde{N} \in \widetilde{H}^2$, there exists a sequence of $\widetilde{\mathbb{F}}-$predictable processes, $\widetilde{Z}^i(i = 1, 2, ...)$ satisfying (2.2) such that

$$\widetilde{N}_t = \widetilde{N}_0 + \sum_{i=1}^\infty \int_0^t \widetilde{Z}_u^i \mathrm{d}\widetilde{M}_u^i.$$

Then by using Lemma 2.5, we can easily deduce that the martingales $\widetilde{M}^i, i = 1, 2, ...$ are mutually orthogonal. The absolute continuity of the induced measures and the uniqueness of the representation are similarly proved. □

If we know the results for the BSDE (1.3) with uniformly Lipschitz condition, it is possible to extend to the case where the driver has the stochastic Lipschitz coefficients. This is the main argument in this section.

Conveniently, we rewrite the BSDE (1.3) omitting the index i as follows.

$$Y_t = \xi + \int_t^\tau g(\omega, s, Y_{s-}, Z_s)\mathrm{d}v_s - \int_t^\tau Z_s \mathrm{d}M_s, \quad 0 \le t \le \tau, \tag{2.3}$$

where $Z = (Z^1, Z^2, ...)$ and $M = (M^1, M^2, ...)^\mathsf{T}$.

Assume that the driver of (2.3) satisfies the following stochastic Lipschitz condition.

(A1) There exist predictable processes $r(t)$ and $u(t)$ such that

$$\|g(\omega, t, y, z) - g(\omega, t, y', z')\| \le r(t)|y - y'| + u(t)\|z - z'\|_{M_t}, \quad d\mu_v - \text{a.s.}$$

for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times\infty}$, where $\alpha^2(t) := r(t) + u^2(t) > \varepsilon$ for some $\varepsilon > 0$ and $\alpha^2(t)$ is assumed to be (pathwise) Stieltjes-integrable with respect to v on every finite interval in $\mathbb{R}^+$.

Now we define the following process.

$$\phi(t) := \int_0^t \alpha^2(s)\mathrm{d}v_s. \tag{2.4}$$

The remarkable point is that ϕ^{-1} *i.e.* the inverse of ϕ_t defined by (2.1) is a time-change. We shall make a good use of this process in the view of time-change. Since α_t^2 is Stieltjes-integrable with respect to v on every finite interval in $\mathbb{R}^+$, it is clear that ϕ^{-1} is a.s. finite and $\phi^{-1}(0) = \phi(0) = 0$. From now, the symbol C which has meant time-change will be replaced by ϕ^{-1}. The focus of this section is on the technique, so we do not have detailed discussion on the space of solutions. The main result in this section is as follows.

Theorem 2.9. *Let $C = \{C_s | s \ge 0\}$ be an a.s. finite time-change. Suppose that the driving martingale M is $C-$continuous.*

1. *Let us assume that $\widetilde{v} = v_{C(s)}$ is absolutely continuous. If (Y_t, Z_t) is a solution of BSDE (2.3) satisfying on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, then $(y_t, z_t) := (Y_{C(t)}, Z_{C(t)})$ is a solution of the following BSDE on $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{\mathbb{F}})$.*

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{g}(\omega, s, y_{s-}, z_s)\mathrm{d}ss - \int_t^{\widetilde{\tau}} z_s \mathrm{d}\widetilde{M}_s, \quad 0 \le t \le \widetilde{\tau}, \tag{2.5}$$

   where

$$\widetilde{g}(\omega, s, y, z) := g(\omega, C(s), y, z) \cdot \frac{\mathrm{d}\widetilde{v}_s}{\mathrm{d}s}, \quad \widetilde{\tau} := C^{-1}(\tau), \quad \widetilde{M}_s := M_{C(s)}. \tag{2.6}$$

   The converse is also true, that is if (y_t, z_t) is a solution of the BSDE (2.5), then $(Y_t, Z_t) := (y_{C^{-1}(t)}, z_{C^{-1}(t)})$ is a solution of the BSDE (2.3).

2. *Let us assume that the driver satisfies the stochastic Lipschitz condition **(A1)**. If we put $C = \phi^{-1}$ (where ϕ is defined by (2.4)), then $\mathrm{d}\widetilde{v}_s/\mathrm{d}s = \alpha^{-2}(\phi^{-1}(s)) = \widetilde{\alpha}^{-2}(s)$ and the new driver $\widetilde{g}$ satisfies bounded Lipschitz continuity such that for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times\infty}$,*

$$|\widetilde{g}(\omega, t, y, z) - \widetilde{g}(\omega, t, y', z')| \le \frac{\widetilde{r}(t)}{\widetilde{\alpha}(t)}|y - y'| + \frac{\widetilde{u}(t)}{\sqrt{\widetilde{\alpha}(t)}}\|z - z'\|_{M_t}, \quad \mathrm{d}t \times \mathrm{d}\mathbb{P} - a.s.$$

Proof.
(1) As (Y_t, Z_t) is the solution of (2.3), we get

$$y_t := Y_{C(t)} = \xi + \int_{C(t)}^{\tau} g(\omega, s, Y_{s-}, Z_s)\mathrm{d}v_s - \int_{C(t)}^{\tau} Z_s \mathrm{d}M_s, \quad 0 \le t \le \widetilde{\tau}.$$

By Lemma 2.3 and Remark 2.7, we see that

$$\begin{aligned}\int_{C(t)}^{\tau} g(\omega, s, Y_{s-}, Z_s)\mathrm{d}v_s &= \int_t^{\widetilde{\tau}} g(\omega, C(s), Y_{C(s)-}, Z_{C(s)})\mathrm{d}\widetilde{v}_s/\mathrm{d}s \cdot \mathrm{d}s \\ &= \int_t^{\widetilde{\tau}} \widetilde{g}(\omega, s, y_{s-}, z_s)\mathrm{d}s.\end{aligned}$$

By Lemma 2.5 and $C-$continuity of M,

$$\int_{C(t)}^{\tau} Z_s \mathrm{d}M_s = \int_t^{C^{-1}(\tau)} Z_{C(s)}\mathrm{d}M_{C(s)} = \int_t^{\widetilde{\tau}} z_s \mathrm{d}\widetilde{M}_s.$$

So we have

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{g}(\omega, s, y_{s-}, z_s) - \int_t^{\widetilde{\tau}} z_s \mathrm{d}\widetilde{M}_s, \quad 0 \le t \le \widetilde{\tau}.$$

As Y_t is $\mathbb{F}-$progressive, y_t is $\widetilde{\mathbb{F}}-$progressive. Due to the fact that all stochastic integrals are indistinguishable from the stochastic integrals of predictable processes, we can consider z_t is predictable. Accordingly, (y_t, z_t) is a solution of BSDE (2.5) on $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{\mathbb{F}})$. Passing back through the above procedure, the converse argument is trivial.
(2) From now, C is replaced by ϕ^{-1}. We split the proof into three steps.

Step 1. We first show that $\widetilde{v}(\cdot, \omega) := v\big(\phi^{-1}(\cdot, \omega), \omega\big)$ is absolutely continuous for each $\omega \in \Omega$. As v is increasing and continuous, v^{-1} defined by (2.1) is a time-change and v is $v^{-1}-$continuous. Therefore by Lemma 2.3, we can see that

$$\begin{aligned}\phi(t) = \int_0^t \alpha^2(s)\mathrm{d}v_s &= \int_0^{v^{-1}(v_t)} \alpha^2(s)\mathrm{d}v_s - \int_t^{v^{-1}(v_t)} \alpha^2(s)\mathrm{d}v_s \\ &= \int_0^{v^{-1}(v_t)} \alpha^2(s)\mathrm{d}v_s = \int_0^{v_t} \alpha^2(v_s^{-1})\mathrm{d}s = \Big[\int_0^{\cdot} \alpha^2(v_s^{-1})\mathrm{d}s \circ v\Big](t).\end{aligned}$$

Thus, $\widetilde{v}_t = [v \circ \phi^{-1}](t) = \left[v \circ v^{-1} \circ \left(\int_0^{\cdot} \alpha^2(v_s^{-1})\mathrm{d}s\right)^{-1}\right](t) = \left(\int_0^{\cdot} \alpha^2(v_s^{-1})\mathrm{d}s\right)^{-1}(t).$
Noting that $\alpha^2(s) > \varepsilon$, $\widetilde{v}^{-1}(\cdot) = \int_0^{\cdot} \alpha^2(v_s^{-1})\mathrm{d}s$ is strictly increasing and absolutely continuous for each $\omega \in \Omega$ and so is the reversed process. Hence $\widetilde{v}_t$ (resp. $\mu_{\widetilde{v}}$) is absolutely continuous with respect to Lebesgue measure (resp $\mathrm{d}t \times \mathrm{d}\mathbb{P}$) and

$$\mathrm{d}\mu_{\widetilde{v}}/(\mathrm{d}t \times \mathrm{d}\mathbb{P}) = \mathrm{d}\widetilde{v}_t/\mathrm{d}t = 1/[\alpha^2 \circ v^{-1} \circ v \circ \phi^{-1}](t) = \alpha^{-2}(\phi_t^{-1}) = \widetilde{\alpha}^{-2}(t).$$

In fact, we can see that $\widetilde{v}_t$ (resp. $\mu_{\widetilde{v}}$) is equivalent to Lebesgue measure (resp. $\mathrm{d}t \times \mathrm{d}\mathbb{P}$). We also mention that v is $\phi^{-1}-$continuous.

Step 2. We derive the Lebesgue decomposition of the measure induced by $\langle \widetilde{M} \rangle$. First, we show that m^1 is a.s. $\phi^{-1}-$continuous. Suppose that v is a constant on $[a,b]$ $(0 \le a < b)$. Then for any $c \in [a,b]$ and $B \in \mathcal{F}$, $\bar{m}([c,b]\times B) = \mathbb{E}[\int_c^b I_B(\omega)\cdot(\mathrm{d}\bar{m}^1/\mathrm{d}\mu_v)(\omega,t)\mathrm{d}v_t] = 0$. Noting that $\bar{m}([0,t]\times B) = \int_B m_t^1 \mathrm{d}\mathbb{P}$,

$$0 = \bar{m}([c,b]\times B) = \bar{m}([0,b]\times B) - \bar{m}([0,c]\times B) = \int_B (m_b^1 - m_c^1)\mathrm{d}\mathbb{P}.$$

Hence m^1 is a.s. constant on $[a,b]$. Because v is $\phi^{-1}-$continuous from **Step 1**, we can see that m^1 is a.s. $\phi^{-1}-$continuous. Recalling (1.5) and using Lemma 2.5, we obtain (omitting the index i)

$$\langle \widetilde{M} \rangle_t = \widetilde{\langle M \rangle}_t = \widetilde{m}_t^1 + \widetilde{m}_t^2. \tag{2.7}$$

And the continuity of ϕ which comes from the continuity of v implies $\phi(\phi_t^{-1}) = t$. Now we can use Lemma 2.3 to show

$$\begin{aligned}\mu_{\widetilde{m}^1}(A) &= \mathbb{E}\left[\int_0^\infty I_A(\omega,t)\mathrm{d}\widetilde{M}_t^1\right] = \mathbb{E}\left[\int_0^\infty I_A(\omega,\phi(\omega,t))\mathrm{d}m_t^1\right] \\ &= \mathbb{E}\left[\int_0^\infty I_A(\omega,\phi(\omega,t))(\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v)(\omega,t)\cdot \mathrm{d}v_t\right] \\ &= \mathbb{E}\left[\int_0^\infty I_A(\omega,t)[\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\phi_t^{-1})\mathrm{d}\widetilde{v}_t\right]\end{aligned}$$

for any $A \in \overline{\mathcal{F}}$, where $[\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\phi_t^{-1}) := [\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\omega,\phi^{-1}(\omega,t))$.
Thus $\mu_{\widetilde{m}^1} \prec \mu_{\widetilde{v}}$ and $\mathrm{d}\mu_{\widetilde{m}^1}/\mathrm{d}\mu_{\widetilde{v}} = [\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\phi_t^{-1})$.
Noting that $\mu_{\widetilde{v}} \prec \mathrm{d}t\times\mathrm{d}\mathbb{P}$ by **Step 1**, we can deduce $\mu_{\widetilde{m}^1} \prec \mathrm{d}t\times\mathrm{d}\mathbb{P}$ and

$$\mathrm{d}\mu_{\widetilde{m}^1}/(\mathrm{d}t\times\mathrm{d}\mathbb{P}) = [\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\phi_t^{-1})\cdot \mathrm{d}\widetilde{v}_t/\mathrm{d}t. \tag{2.8}$$

Similarly, $\mu_{\widetilde{m}^2}$ is orthogonal to $\mathrm{d}t\times\mathrm{d}\mathbb{P}$. This shows that (2.7) is the Lebesgue decomposition of $\langle \widetilde{M} \rangle$ with respect to $\mathrm{d}t\times\mathrm{d}\mathbb{P}$.

Step 3. Finally, we show that $\widetilde{g}$ satisfies bounded Lipschitz continuity. It follows from the results in **Step 2** that

$$\begin{aligned}\|z\|^2_{\widetilde{M}_t} = \|z\|^2\cdot[\mathrm{d}\mu_{\widetilde{m}^1}/(\mathrm{d}t\times\mathrm{d}\mathbb{P})] &= \|z\|^2\cdot[\mathrm{d}\overline{m}^1/\mathrm{d}\mu_v](\phi_t^{-1})\mathrm{d}\widetilde{v}_t/\mathrm{d}t \\ &= \alpha^{-2}(\phi_t^{-1})\|z\|^2_{M_u}\big|_{u=\phi^{-1}(t)}.\end{aligned} \tag{2.9}$$

From the stochastic Lipschitz condition on g,

$$\begin{aligned}|\widetilde{g}(\omega,t,y,z) - \widetilde{g}(\omega,t,y',z')| &= |\widetilde{g}(\omega,\phi^{-1}(t),y,z) - \widetilde{g}(\omega,\phi^{-1}(t),y',z')|\alpha^{-2}(\phi_t^{-1}) \\ &\le \alpha^{-2}(\phi^{-1}(t))\big[r_{\phi^{-1}(t)}|y-y'| + u_{\phi^{-1}(t)}\big(\|z-z'\|_{M_s}\big|_{s=\phi^{-1}(t)}\big)\big] \\ &= \alpha^{-2}(\phi^{-1}(t))\big[r_{\phi^{-1}(t)}|y-y'| + u_{\phi^{-1}(t)}\alpha(\phi^{-1}(t))\|z-z'\|_{\widetilde{M}_t}\big] \\ &= \frac{\widetilde{r}(t)}{\widetilde{r}(t)+\widetilde{u}(t)^2}|y-y'| + \frac{\widetilde{u}(t)}{\sqrt{\widetilde{r}(t)+\widetilde{u}(t)^2}}\|z-z'\|_{\widetilde{M}_t}, \quad \mathrm{d}\mu_{\widetilde{v}} - a.s.\end{aligned}$$

for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times\infty}$. From **Step 1**, we know that $\mu_{\widetilde{v}}$ is equivalent to $\mathrm{d}t\times\mathrm{d}\mathbb{P}$. So Lipschitz property on $\widetilde{g}$ holds $\mathrm{d}t\times\mathrm{d}\mathbb{P}-$a.s. □

Remark 2.10. If the trajectories of C are strictly increasing, then M is $C-$continuous as one sees easily. In the second assertion, if the trajectories of v are strictly increasing then ϕ^{-1} is strictly increasing and continuous (that is $\phi^{-1}(\phi(t)) = \phi(\phi^{-1}(t)) = t$), so we do not have to assume that M is $\phi^{-1}-$continuous. We also note that $C-$ continuity of M is equivalent to $C-$ continuity of m^2.

Remark 2.11. In the proof of second assertion, the continuity of v which leads to the continuity of ϕ, plays an important role. This guarantees $v(v^{-1}(t)) = \phi(\phi^{-1}(t)) = t$. If v is a finite variation process possibly with jumps, it may be needed to decompose the Stieltjes measures generated by the trajectories of v as the continuous part and the discontinuous one. More generally, the case where C is discontinuous, is of particular interest. Perhaps such generalizations may be non-trivial.

Remark 2.12. Note that the terminal time τ is allowed to take infinite values on some event with positive probability in Theorem 2.9. It is sufficient to assume that only the time-change is a.s. finite to guarantee our results (see Rem. 2.7).

Remark 2.13. If we aim to simplify only the continuous integrator of driver, it is sufficient to use v^{-1} as the time-change.

Remark 2.14. In second assertion of Theorem 2.9, it is also possible to discuss a stochastic monotonicity condition. Hence, the stochastic monotone condition can be replaced by a bounded monotonicity condition through the time-change.

We conclude this section with the following statement.

The terminal time of BSDEs with stochastic Lipschitz coefficients

In [3], the BSDE with random terminal time was studied separately, after the discussion on finite time BSDE. In fact, the discussion on the case of random terminal time is not necessary. When we study the BSDEs with stochastic Lipschitz coefficients, the randomness of terminal time does not play an important role. This is illustrated as follows. Due to Remark 2.13, we suppose that the BSDE is given in the following type.

$$Y_t = \xi + \int_t^\tau g(\omega, s, Y_{s-}, Z_s)\mathrm{d}s - \int_t^\tau Z_s \mathrm{d}M_s. \tag{2.10}$$

We introduce the following process.

$$\Phi(\omega, t) := \frac{t}{1 + \tau \wedge t}, \quad t \geq 0.$$

After the simple calculation, we get

$$\Phi^{-1}(t) = t/(1-t), \quad [\Phi^{-1}(t)]' = -(t-1)^{-2}, \quad 0 \leq t \leq \widetilde{\tau} = \Phi(\tau) < 1.$$

Obviously Φ^{-1} is time-change. By Theorem 2.9, we get the following BSDE on $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{\mathbb{F}})$ equivalent to (2.10) in some sense.

$$y_t = \xi + \int_t^1 G(\omega, s, y_{s-}, z_s)\mathrm{d}s - \int_t^1 z_s \mathrm{d}\widetilde{M}_s, \tag{2.11}$$

where $G(\omega, s, y, z) := I_{s \leq \widetilde{\tau}} g(\omega, \Phi^{-1}(s), y, z)[\Phi^{-1}(s)]'$ and $\widetilde{M}_s := M_{\Phi^{-1}(s)}$. We mention that the new driver G is stochastic Lipschitz even though the original driver g is uniform Lipschitz. In fact, if we suppose that g has constants r, u as the Lipschitz coefficients, for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times\infty}$,

$$\begin{aligned}|G(\omega,s,y,z) - G(\omega,s,y',z')| &= I_{s\le\widetilde{\tau}}|(\Phi^{-1})'(s)|\cdot|g(\omega,\Phi_s^{-1},y,z) - g(\omega,\Phi_s^{-1},y',z')|\\ &\le I_{s\le\widetilde{\tau}}|(\Phi^{-1})'(s)|[r|y-y'| + u\|z-z'\|_{\widetilde{M}_s}|(\Phi^{-1})'(s)|^{-1/2}]\\ &= I_{s\le\widetilde{\tau}}[r(1-s)^{-2}|y-y'| + u(1-s)^{-1}\|z-z'\|_{\widetilde{M}_s}]\\ &\le r(1-\widetilde{\tau})^{-2}|y-y'| + u(1-\widetilde{\tau})^{-1}\|z-z'\|_{\widetilde{M}_s}\\ &= r(1+\tau)^2|y-y'| + u(1+\tau)\|z-z'\|_{\widetilde{M}_s}.\end{aligned}$$

This means that the stopping terminal time of BSDEs can be converted to constant and this operation is adapted to the class of BSDEs with stochastic Lipschitz condition. Consequently, when we study the BSDEs with random Lipschitz coefficients and random terminal time (two randomness), it is sufficient to consider only one randomness.

Remark 2.15. In [2], the measure solution was only studied for finite time BSDE. The above statement shows that the finite time can be just extended to the random terminal time by means of time-change. Other results (*e.g.* [25, 26, 28, 34, 35]), which were restricted to only finite time interval, may be also directly extended to the random time interval.

Due to the main interest of Theorem 2.9, it is important to give general results of the BSDE (2.5) under bounded Lipschitz condition. The next section deals with a class of BSDEs with random terminal time. For the BSDE (2.5), the integral is respect to Lebesgue measure and $g(\omega,t,Y_{t-},Z_t) = g(\omega,t,Y_t,Z_t), \mathrm{d}t\times\mathrm{d}\mathbb{P}-a.s.$ for the solution (Y,Z), so we shall use Y_t in the driver term for notational simplicity.

3. BSDEs with random terminal time in general space

Consider the following BSDE with random terminal time.

$$Y_t = \xi + \int_t^\tau g(\omega,s,Y_s,Z_s)\mathrm{d}s - \int_t^\tau Z_s\mathrm{d}M_s. \tag{3.1}$$

where $\int_t^\tau Z_s\mathrm{d}M_s = \sum_i\int_t^\tau Z_s^i\mathrm{d}M_s^i$.

Proposition 3.1. *Let τ be an almost finite stopping time. Suppose that the following conditions hold.*

1. ***Bounded Lipschitz continuity.***
   There exist positive, bounded processes $a(t)$ and $b(t)$ such that for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times\infty}$,

$$|g(\omega,t,y,z) - g(\omega,t,y',z')| \le a(t)|y-y'| + b(t)\|z-z'\|^2_{M_t}.$$

2. ***Integrability.***
   There exists a process $\rho(t) > b^2(t) + 2a(t) + \varepsilon$ $(\varepsilon > 0)$ such that

$$\mathbb{E}\left[e^{\rho_\tau}|\xi|^2 + \int_0^\tau e^{\rho_s}|g(\omega,s,0,0)|^2\mathrm{d}s\right] < \infty.$$

   where $\rho_t := \int_0^t \rho(s)\mathrm{d}s$.

Then, the BSDE (3.1) has a unique solution (Y,Z) in $L^2_{\rho(\cdot)}(0,\tau)\times L^2_{\rho(\cdot)}(0,\tau,M)$. Moreover Y belongs to $U^2_{\rho(\cdot)}(0,\tau)$.

The proof is very similar to the proof of [19], Theorem 3.4. For the completeness, we give the proof in Appendix.

Remark 3.2. We can easily extend this result under monotonicity condition using standard argument by a sequence of Lipschitz drivers. In this case, the growth condition on driver should be added. Also, it is not difficult to study (3.1) for any stopping time τ, which may have infinite values with a positive probability as in [38]. This allows us to deal with the BSDE with stochastic monotone coefficients and general stopping time. On the other hand, we discussed the solution in L^2-setting whereas many results were established in L^p-setting for any $p > 1$. However it is difficult to study L^p-solutions ($p > 1$) of BSDEs possibly with jumps (see [46]). So, we need to restrict the study of L^p-solution to the case where the driving martingale is continuous. The BSDE driven by Brownian motion was well-studied in such various settings which we mentioned above. To avoid the complexity, and to focus on the main idea, we study the L^p-solutions of BSDEs with stochastic monotone coefficients and general stopping time, in a Brownian setting (see Sect. 5).

Now we discuss the comparison theorem for BSDE (3.1) under bounded Lipschitz condition. We shall restrict our discussion to the scalar BSDE. A comparison theorem for BSDEs driven by martingales was first studied by Carbone *et al.* [10]. They studied the following BSDE, which is a special case of (2.3)

$$Y_t = \xi + \int_t^T g(\omega, s, Y_s, Z_s)\mathrm{d}\langle M\rangle_s - \int_t^T Z_s \mathrm{d}M_s - N_T + N_t, t \in [0, T]. \tag{3.2}$$

The solution of (3.2) is a triple (Y, Z, N) and N is a martingale orthogonal to M. Let (Y^1, Z^1, N^1) and (Y^2, Z^2, N^2) be the solutions of (3.2) corresponding to (ξ^1, g^1) and (ξ^2, g^2), respectively. Let us define

$$\Delta_Y(g_t^1) := \frac{g^1(\omega, t, Y_t^1, Z_t^1) - g^1(\omega, t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2}, \quad \Delta_Z(g_t^1) := \frac{g^1(\omega, t, Y_t^2, Z_t^1) - g^1(\omega, t, Y_t^2, Z_t^2)}{Z_t^1 - Z_t^2}.$$

They showed the comparison theorem for (3.2) under the following assumptions using linear BSDE.

(Comp 1) $\Delta_Y(g_t^1)$ and $\Delta_Z(g_t^1)$ are both bounded (that is, the driver is bounded Lipschitz).

(Comp 2) $\mathcal{E}(\Delta_Z(g^1) \bullet M)$ is a positive uniformly integrable martingale.

(Comp 3) $\mathbb{E}\big[\big(\sup_t \exp\big(\int_0^t \Delta_Y(g_s^1)\mathrm{d}\langle M\rangle_s\big)\big)^2[\mathcal{E}(\Delta_Z(g^1) \bullet M)_T]^2\big] < \infty$.

Afterwards, Cohen *et al.* [17] showed a general comparison theorem for (3.2) with $\mathrm{d}t$ instead of $\mathrm{d}\langle M\rangle_t$, by means of super-martingale measures which corresponds to the "no-arbitrage" condition in financial sense.

They introduced the following assumptions.

(Comp 4) There exists a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that

$$\int_0^t [g^1(\omega, s, Y_s^1, Z_s^1) - g^1(\omega, s, Y_s^1, Z_s^2)]\mathrm{d}s - \int_0^t (Z_s^1 - Z_s^2)\mathrm{d}M_s - N_t.$$

is a $\widetilde{\mathbb{P}}-$supermartingale.

(Comp 5) For all $t \in [0, T]$, if

$$Y_t^1 + \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^t g^1(\omega, s, Y_s^1, Z_s^2)\mathrm{d}s \middle| \mathcal{F}_t\right] \le Y_t^2 + \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^t g^1(\omega, s, Y_s^2, Z_s^2)\mathrm{d}t \middle| \mathcal{F}_t\right]$$

then $Y_t^1 \ge Y_t^2$.

Furthermore, they extended this result to the BSDE (1.3) with deterministic Stieltjes measure integrator in [15]. Hence, the comparison theorem for BSDE (1.3) holds if the following assumptions are satisfied.

(Comp 4') There exists a measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that

$$\int_0^t [g^1(\omega, s, Y_s^1, Z_s^1) - g^1(\omega, s, Y_s^1, Z_s^2)]\mathrm{d}v_s - \int_0^t (Z_s^1 - Z_s^2)\mathrm{d}M_s.$$

is a $\widetilde{\mathbb{P}}-$supermartingale.

(Comp 5') For all $t \in [0, T]$, if

$$Y_t^1 + \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^t g^1(\omega, s, Y_s^1, Z_s^2)\mathrm{d}v_s \middle| \mathcal{F}_t\right] \leq Y_t^2 + \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^t g^1(\omega, s, Y_s^2, Z_s^2)\mathrm{d}t \middle| \mathcal{F}_t\right]$$

then $Y_t^1 \geq Y_t^2$.

In [17], Theorem 2 (resp. [15], Thm. 7.2), it was proved that the assumption **(Comp 5)** (resp. **(Comp 5')**) holds if g^1 is bounded Lipschitz in y. Let us define the process $X(\omega, t, y, z, z') := \frac{g^1(\omega,t,y,z) - g(\omega,t,y,z')}{\|z-z'\|^2_{M_t}}(z - z')$. In [11], the following sufficient assumption to guarantee **(Comp 4')** was given (see Lem. 1 therein).

(Comp 4") g^1 is bounded Lipschitz in z and for all $y \in \mathbb{R}$ and $z, z' \in \mathbb{R}^{1\times\infty}$,

$$\frac{|g^1(\omega, t, y, z) - g^1(\omega, t, y, z')|}{\|z - z'\|^2_{M_t}}|(z - z') \cdot \Delta M_t| < 1.$$

In [18], the following weaker assumption than **(Comp 4")** was introduced.

(Comp 4"') g^1 is bounded Lipschitz in z and for all $y \in \mathbb{R}$ and $z, z' \in \mathbb{R}^{1\times\infty}$,

$$\frac{g^1(\omega, t, y, z) - g^1(\omega, t, y, z')}{\|z - z'\|^2_{M_t}}(z - z') \cdot \Delta M_t > -1.$$

Indeed, if **(Comp 4")** (or **(Comp 4"')**) is satisfied, then the measure $\widetilde{\mathbb{P}}$ is obtained by

$$\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P} = \mathcal{E}(X \bullet M)_T, \quad X_t := \frac{g^1(\omega, t, Y_t^1, Z_t^1) - g^1(\omega, t, Y_t^1, Z_t^2)}{\|Z_t^1 - Z_t^2\|^2_{M_t}}(Z_t^1 - Z_t^2)$$

for $(Y^i, Z^i) \in U_0^2(0, T) \times L_0^2(0, T, M)$ (see the proof of [11], Lem. 1 or [18], Lem. 1). Also, $\mathcal{E}(X \bullet M)$ is a martingale such that $\mathbb{E}[|\mathcal{E}(X \bullet M)_T|^p] < \infty$ for any $p > 0$ if g^1 is Lipschitz in z (see [11], Lem. 2). In addition to that, if **(Comp 4"')** is satisfied, then $\mathcal{E}(X \bullet M)$ is positive as one sees easily.

Consequently, we can consider that assumption **(Comp 4"')** is the most appropriate to guarantee the comparison principle for BSDE with Lebesgue integrator and Lipschitz coefficients.

Proposition 3.3. *Let us assume that the underlying space is a Kolmogorov type filtered space, that is, whenever* $(\Omega, \mathcal{F}_t, \mathbb{Q}_t)$ *are consistent probability spaces for* $0 \leq t < \infty$*, then there is a probability measure on* $\mathcal{F} = \mathcal{F}_\infty$ *such that every* $\mathbb{Q}_t$ *is a restriction of* $\mathbb{Q}$ *to* $\mathcal{F}_t$*. Suppose* (ξ^i, g^i) *is a pair for which the conditions of Proposition 3.1* *are satisfied with the same coefficients $a(t), b(t)$ and $\rho(t)$ for each $i = 1, 2$. Let (Y^i, Z^i) be a solution of BSDE (3.1) corresponding to (ξ^i, g^i) for each $i = 1, 2$. We further assume that* ***(Comp 4"')*** *holds for any $t \geq 0$. If $\xi^1 \geq \xi^2$ and $g^1(\omega, t, Y_t^2, Z_t^2) \geq g^2(\omega, t, Y_t^2, Z_t^2)$, then we have $Y_t^1 \geq Y_t^2$.*

Proof. Let us define $(\hat{Y}_t^i, \hat{Z}_t^i) := (e^{a_t} Y_t^i, e^{a_t} Z_t^i)$ for $i = 1, 2$, where $a_t := \int_0^t a(s)\mathrm{d}s$. Obviously, $Y_t^1 \geq Y_t^2$ if and only if $\hat{Y}_t^1 \geq \hat{Y}_t^2$. And $(\hat{Y}_t^i, \hat{Z}_t^i) \in U^2_{b^2(\cdot)}(0, \tau) \times L^2_{b^2(\cdot)}(0, \tau, M)$ as one sees easily. By Itô's formula, $(\hat{Y}_t^i, \hat{Z}_t^i)$ satisfies

$$\hat{Y}_t^i = e^{a_\tau} \xi^i + \int_t^\tau \hat{g}^i(\omega, s, \hat{Y}_s^i, \hat{Z}_s^i)\mathrm{d}s - \int_t^\tau \hat{Z}_s^i \mathrm{d}M_s. \tag{3.3}$$

where $\hat{g}^i(\omega, s, y, z) := e^{a_s} g^i(\omega, s, e^{-a_s} y, e^{-a_s} z) - a(s)y$. From the definition, we obtain

$$\frac{\hat{g}^1(\omega, t, y, z) - \hat{g}^1(\omega, t, y, z')}{\|z - z'\|^2_{M_t}}(z - z') = \frac{g^1(\omega, t, e^{-a_t} y, e^{-a_t} z) - g^1(\omega, t, e^{-a_t} y, e^{-a_t} z')}{\|e^{-a_t}(z - z')\|^2_{M_t}}[e^{-a_t}(z - z')]. \tag{3.4}$$

from which $\hat{g}^1$ also satisfies **(Comp 4"')**. Also, we see that

$$r_t := \frac{\hat{g}^1(\omega, t, \hat{Y}_t^1, \hat{Z}_t^2) - \hat{g}^1(\omega, t, \hat{Y}_t^2, \hat{Z}_t^2)}{\hat{Y}_t^1 - \hat{Y}_t^2} \leq 0. \tag{3.5}$$

And $g^1(\omega, t, Y_t^2, Z_t^2) \geq g^2(\omega, t, Y_t^2, Z_t^2)$ implies that $\hat{g}^1(\omega, t, \hat{Y}_t^2, \hat{Z}_t^2) \geq \hat{g}^2(\omega, t, \hat{Y}_t^2, \hat{Z}_t^2)$. Define

$$\hat{X}_t := \frac{\hat{g}^1(\omega, t, \hat{Y}_t^1, \hat{Z}_t^1) - \hat{g}^1(\omega, t, \hat{Y}_t^1, \hat{Z}_t^2)}{\|\hat{Z}_t^1 - \hat{Z}_t^2\|^2_{M_t}}(\hat{Z}_t^1 - \hat{Z}_t^2).$$

It follows from (3.4) that $\hat{X}_t = X_t$. Let $T \geq 0$ be a finite time and $\mathbb{Q}^T := \mathcal{E}(\hat{X} \bullet M)_T \cdot \mathbb{P} = \mathcal{E}(X \bullet M)_T \cdot \mathbb{P}$. Then, as we mentioned before,

$$H_t := \int_0^t [\hat{g}^1(\omega, s, \hat{Y}_s^1, \hat{Z}_s^1) - \hat{g}^1(\omega, s, \hat{Y}_s^1, \hat{Z}_s^2)]\mathrm{d}s - \int_0^t (\hat{Z}_s^1 - \hat{Z}_s^2)\mathrm{d}M_s.$$

is a $\mathbb{Q}^T-$martingale up to time T. Then, by a simple rearrangement,

$$\hat{Y}_t^1 - \hat{Y}_t^2 + \int_0^t [\hat{g}^1(\omega, s, \hat{Y}_s^1, \hat{Z}_s^2) - \hat{g}^2(\omega, s, \hat{Y}_s^2, \hat{Z}_s^2)]\mathrm{d}s$$

is a $\mathbb{Q}^T-$martingale up to T. Since $\mathbb{Q}^T := \mathcal{E}(X \bullet M)_T \cdot \mathbb{P}$, we notice that the measures $\mathbb{Q}^T$ are consistent, that is, $\mathbb{Q}^T|_{\mathcal{F}_t} = \mathbb{Q}^t|\mathcal{F}_t$, for any $t \leq T$. As the underlying space is a Kolmogorov type filtered space, there exists a measure $\mathbb{Q}$ such that $\mathbb{Q}|_{\mathcal{F}_T} = \mathbb{Q}^T|_{\mathcal{F}_T}$ for all T. For any $T < \infty$, it follows from $\mathbb{Q}^T|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$ that $\mathbb{Q}|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}$, where $\sim$ means the equivalence of measures. By [12], Lemma 5, it follows that $\mathbb{Q}|_{\mathcal{F}_\tau} \sim \mathbb{P}|_{\mathcal{F}_\tau}$. Let $A \in \mathcal{F}_t$ be a set such that $\hat{Y}_t^1 - \hat{Y}_t^2 < 0$ on A. Define a stopping time $\tau^* := \inf\{s \geq t |\ \hat{Y}_s^1 - \hat{Y}_s^2 \geq 0\} \leq \tau$. Since H_t is a $\mathbb{Q}-$martingale in $\mathbb{R}^+$, $r_t \leq 0$ and $\hat{g}^1(\omega, t, \hat{Y}_t^2, \hat{Z}_t^2) - \hat{g}^2(\omega, t, \hat{Y}_t^2, \hat{Z}_t^2) \geq 0$, we obtain

$$\begin{aligned} -\mathbf{1}_A|\hat{Y}_t^1 - \hat{Y}_t^2| = \mathbf{1}_A(\hat{Y}_t^1 - \hat{Y}_t^2) &= \mathbb{E}^{\mathbb{Q}}\left[\left(\hat{Y}_{\tau^*}^1 - \hat{Y}_{\tau^*}^2 + \int_t^{\tau^*} \mathbf{1}_A[\hat{g}^1(\omega, s, \hat{Y}_s^1, \hat{Z}_s^2) - \hat{g}^2(\omega, s, \hat{Y}_s^2, \hat{Z}_s^2)]\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \\ &\geq \mathbb{E}^{\mathbb{Q}}\left[\left(\hat{Y}_{\tau^*}^1 - \hat{Y}_{\tau^*}^2 - \int_t^{\tau^*} \mathbf{1}_A r_t|\hat{Y}_s^1 - \hat{Y}_s^2|\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \geq 0 \end{aligned} \tag{3.6}$$

which implies $\mathbf{1}_A(\hat{Y}_t^1 - \hat{Y}_t^2) = 0$. That is, $\hat{Y}_t^1 \geq \hat{Y}_t^2$, $\mathbb{Q} - a.s.$ Since $\mathbb{Q}|_{\mathcal{F}_\tau} \sim \mathbb{P}|_{\mathcal{F}_\tau}$, we get $\hat{Y}_t^1 \geq \hat{Y}_t^2$, $\mathbb{P} - a.s.$ $\square$

Remark 3.4. Let $(\Omega, \mathcal{F}) := ((\mathbb{R}^d)^{[0,\infty]}, \mathcal{B}((\mathbb{R}^d)^{[0,\infty]}))$ and $\{\mathcal{F}_t\}_{t\geq 0}$ be a natural filtration generated by the canonical process. Then, $(\Omega, \mathcal{F})$ is a Kolmogorov type filtered space by Kolmogorov's Extension Theorem (see *e.g.* [16], Thm. A.2.6 or A.2.7).

4. The main results

From the Theorem 2.9 and Proposition 3.1, we can get the general results for BSDE (2.3) with stochastic Lipschitz coefficients. Recall that $\alpha^2(t) = r(t) + u^2(t)$, $\phi(t) = \int_0^t \alpha^2(s)\mathrm{d}s$ and $\widetilde{X}(t) = X(\phi^{-1}(t))$ for any progressively measurable process X.

Theorem 4.1. *Suppose that the following conditions hold.*

1. ***Stochastic Lipschitz continuity:*** *The driver g satisfies the assumption **(A1)**.*
2. ***Integrability:*** *There exists a predictable process $\beta(t)$ satisfying $\beta(t) > (1+\varepsilon)(2r(t) + u^2(t))$, $\varepsilon > 0$ (or equivalently $\beta(t) = \theta(2r(t) + u^2(t))$ for some $\theta > 1$) such that*

$$\mathbb{E}\left[\exp\left(\int_0^\tau \beta(t)\mathrm{d}t\right)|\xi|^2 + \int_0^\tau \exp\left(\int_0^t \beta(u)\mathrm{d}u\right)\left|\frac{g(\omega,t,0,0)}{\alpha(t)}\right|^2 \mathrm{d}v_t\right] < \infty. \tag{4.1}$$

Then, BSDE (2.3) has a unique solution (Y,Z) in $L^2_{\beta(\cdot)}(0,\tau) \times L^2_{\beta(\cdot)}(0,\tau,M)$. Moreover $Y \in U^2_{\beta(\cdot)}(0,\tau)$.

Proof. By Theorem 2.9, we obtain the time-changed BSDE corresponding to (2.3)

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{g}(\omega,s,y_s,z_s)\mathrm{d}s - \int_t^{\widetilde{\tau}} z_s \mathrm{d}\widetilde{M}_s, \quad 0 \leq t \leq \widetilde{\tau} \tag{4.2}$$

and $\widetilde{g}$ is bounded Lipschitz with Lipschitz coefficients $a(t) := \frac{\widetilde{r}(t)}{\widetilde{r}(t)+\widetilde{u}^2(t)}$ and $b(t) := \frac{\widetilde{u}(t)}{\sqrt{\widetilde{r}(t)+\widetilde{u}^2(t)}}$.
Let us define $\rho(t) := \widetilde{\beta}(t)\cdot\widetilde{\alpha}^{-2}(t)$. From $\beta(t) > (1+\varepsilon)(2r(t)+u^2(t))$, it follows that

$$\rho(t) > \frac{(1+\varepsilon)(2\widetilde{r}(t) + \widetilde{u}^2(t))}{\widetilde{r}(t)+\widetilde{u}^2(t)} \geq 2a(t) + b^2(t) + \varepsilon.$$

Now, we can check that the integrability condition for (4.2) as follows.

$$\begin{aligned}
\infty &> \mathbb{E}\left[\exp\left(\int_0^\tau \beta(t)\mathrm{d}t\right)|\xi|^2 + \int_0^\tau \exp\left(\int_0^t \beta(u)\mathrm{d}u\right)\left|\frac{g(\omega,t,0,0)}{\alpha(t)}\right|^2 \mathrm{d}v_t\right] \\
&= \mathbb{E}\left[\exp\left(\int_0^{\widetilde{\tau}} \widetilde{\beta}(t)\widetilde{\alpha}^{-2}(t)\mathrm{d}t\right)|\xi|^2 + \int_0^{\widetilde{\tau}} \exp\left(\int_0^t \widetilde{\beta}(u)\widetilde{\alpha}^{-2}(u)\mathrm{d}u\right)\left|\frac{g(\omega,\phi^{-1}(t),0,0)}{\widetilde{\alpha}(t)}\right|^2 \widetilde{\alpha}^{-2}(t)\mathrm{d}t\right] \\
&= \mathbb{E}\left[\exp\left(\int_0^{\widetilde{\tau}} \rho(t)\mathrm{d}t\right)|\xi|^2 + \int_0^{\widetilde{\tau}} \exp\left(\int_0^t \rho(u)\mathrm{d}u\right)|\widetilde{g}(\omega,t,0,0)|^2\mathrm{d}t\right]
\end{aligned}$$

Therefore, by Proposition 3.1, BSDE (4.2) has a unique solution (y,z) in $L^2_{\rho(\cdot)}(0,\widetilde{\tau}) \times L^2_{\rho(\cdot)}(0,\widetilde{\tau},\widetilde{M})$ and $y \in U^2_{\rho(\cdot)}(0,\widetilde{\tau})$. The pair $(Y_t, Z_t) := (y_{\phi(t)}, z_{\phi(t)})$ satisfies (2.3) by Theorem 2.9. It remains to prove that $(Y,Z) \in \left(L^{2,\alpha}_{\beta(\cdot)}(0,\tau) \cap U^2_{\beta(\cdot)}(0,\tau)\right) \times L^2_{\beta(\cdot)}(0,\tau,M)$. But this just follows by observing that

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left[\exp\left(\int_0^t \beta(u)\mathrm{d}u\right)|Y(t)|^2\right]\right) = \mathbb{E}\left(\sup_{0\leq t\leq\widetilde{\tau}}\left[\exp\left(\int_0^t \rho(u)\mathrm{d}u\right)|y(t)|^2\right]\right) < \infty,$$

$$\mathbb{E}\bigg(\int_0^{\tau}\exp\bigg(\int_0^t\beta(u)\mathrm{d}u\bigg)|\alpha(t)Y(t)|^2\mathrm{d}t\bigg)=\mathbb{E}\bigg(\int_0^{\widetilde{\tau}}\exp\bigg(\int_0^t\rho(u)\mathrm{d}u\bigg)|y(t)|^2\mathrm{d}t\bigg)<\infty,$$

$$\mathbb{E}\bigg(\int_0^{\tau}\exp\bigg(\int_0^t\beta(u)\mathrm{d}u\bigg)\|Z(t)\|^2\mathrm{d}\langle M\rangle_t\bigg)=\mathbb{E}\bigg(\int_0^{\widetilde{\tau}}\exp\bigg(\int_0^t\rho(u)\mathrm{d}u\bigg)\|z(t)\|^2\mathrm{d}\langle\widetilde{M}\rangle_t\bigg)<\infty.$$

So, the proof is complete. □

Next, we study the comparison theorem for (2.3). To the best of our knowledge, this is the first time that the comparison theorem for BSDEs possibly with jumps under stochastic Lipschitz coefficients has been proved. We need first to establish the following result (the proof is given in appendix).

Proposition 4.2. *If $(\Omega,\mathcal{F})$ is a Kolmogorov type filtered space with respect to $\mathbb{F}$, then it is still a Kolmogorov type filtered space with respect to $\widetilde{\mathbb{F}}$.*

Theorem 4.3 (Comparison Theorem)**.** *Let us assume that the underlying space is a Kolmogorov type filtered space. Suppose that the pair (ξ^i,g^i) satisfies the conditions of Theorem 4.1 for each $i=1,2$. Let (Y^i,Z^i) be the solution of BSDE (2.3), corresponding to (ξ^i,g^i) for each $i=1,2$. We further assume that **(Comp 4”')** is satisfied for any $t\geq 0$. If $\xi^1\geq\xi^2$ and $g^1(\omega,t,Y_t^2,Z_t^2)\geq g^2(\omega,t,Y_t^2,Z_t^2)$, then we have $Y_t^1\geq Y_t^2$.*

Proof. By Theorem 2.9, it is sufficient to show that the comparison holds for BSDE (4.2) with bounded Lipschitz condition. Hence we should prove $y_t^1\geq y_t^2$ for $(y_t^i,z_t^i)=(Y^i_{\phi^{-1}(t)},Z^i_{\phi^{-1}(t)})$. As $g^1(\omega,t,Y_t^2,Z_t^2)\geq g^2(\omega,t,Y_t^2,Z_t^2)$, we get $\widetilde{g}^1(\omega,t,y_t^2,z_t^2)\geq\widetilde{g}^2(\omega,t,y_t^2,z_t^2)$. Using M is $\phi^{-1}-$continuous, we obtain $\Delta\widetilde{M}_t=(\Delta M)_{\phi^{-1}(t)}$.

We recall that $\|\cdot\|_{\widetilde{M}_t^2}=\|\cdot\|_{M_s^2}|_{s=\phi^{-1}(t)}\cdot\widetilde{\alpha}^{-2}(t)$ (see (2.9)). Therefore, we have

$$\frac{\widetilde{g}^1(\omega,t,y,z)-\widetilde{g}^1(\omega,t,y,z')}{\|z-z'\|^2_{\widetilde{M}_t}}(z-z')\cdot\Delta\widetilde{M}_t=\frac{g^1(\omega,\phi^{-1}(t),y,z)-g^1(\omega,\phi^{-1}(t),y,z')}{\|z-z'\|^2_{\widetilde{M}_s}|_{s=\phi^{-1}(t)}}(z-z')\cdot\Delta M_{\phi^{-1}(t)}>-1.$$

By Propositions 3.3 and 4.2, it follows that $y^1\geq y^2$. □

Comparison with the existing literature

Let us consider the following BSDE driven by a d-dimensional Brownian motion W, as a special case.

$$Y_t=\xi+\int_t^{\tau}g(\omega,s,Y_s,Z_s)\mathrm{d}Q_s-\int_t^{\tau}Z_s\mathrm{d}W_s,\quad 0\leq t\leq\tau,\tag{4.3}$$

where Q is an increasing, continuous process such that $\mathrm{d}t=q_t\mathrm{d}Q_t$ for some progressively measurable process q. Due to $\langle W^i\rangle_t=t,i\in\{1,...,d\}$, the stochastic semi-norm defined by (1.6) is obtained as $\|z_t\|^2_{W_t}=\|z\|^2\cdot q_t$ for $z\in\mathbb{R}^{k\times d}$. The above BSDE (4.3) was studied by Pardoux and Rasscanu [39]. They studied the L^p-solution of (4.3) up to time T, under the stochastic Lipschitz condition with $u_t=q_t\cdot u'_t$ for some process $u'(t)$. In the case of $p=2$, one of their assumptions is given as follows (see [39], Thm. 5.21).

$$\mathbb{E}\bigg[\exp\bigg(\int_0^T2[r(t)+u^2(t)]\mathrm{d}t\bigg)|\xi|^2+\bigg(\int_0^T\exp\bigg(\int_0^t[r(s)+u^2(s)]\mathrm{d}s\bigg)|g(\omega,t,0,0)|\mathrm{d}v_t\bigg)^2\bigg]<\infty.\tag{4.4}$$

This is stronger than our integrability assumption (4.1).

The integrability assumptions in Cohen and Elliott [16] (see Thm. A.9.20 therein) is different from ours. But their result was proved only for bounded terminal value. Moreover, their approach requires the additional integrability assumptions related to jumps because they use the comparison principle as a key tool.

If the filtration is generated by a Lévy process, we can take $M = (M^i)_{i=1}^{\infty}$ to be the Teugels martingales associated with a Lévy process from the corresponding predictable representation property in [31]. Hence, the BSDE driven by a Lévy process which was introduced in [32] is a special case of (2.3). So, Theorem (4.1) just gives the existence and uniqueness results of the BSDE driven by a Lévy process as a corollary. This is weaker than those of [26] where the parameter in the integrability assumptions was assumed to be large enough (see Thm. 3.2 therein). In most papers related to BSDEs with stochastic Lipschitz conditions, the following integrability assumption with a parameter β plays a crucial role.

$$\mathbb{E}\bigg[\exp\bigg(\int_0^\tau \beta[r(t)+u^2(t)]\mathrm{d}t\bigg)|\xi|^2+\bigg(\int_0^\tau \exp\bigg(\int_0^t \beta[r(s)+u^2(s)]\mathrm{d}s\bigg)\bigg|\frac{g(\omega,t,0,0)}{\alpha(t)}\bigg|^2\mathrm{d}v_t\bigg)\bigg]<\infty. \tag{4.5}$$

If assumption (4.5) holds with $\beta = 2$, then our assumption (4.1) is just satisfied. So, assumption (4.5) is stronger than ours in the case of $\beta \geq 2$. In [3, 5, 23], β needs to be large enough with at least larger than 90. In [10], β needs to be larger than only 3. The assumption (4.4) in [39] seems to imply $\beta = 2$. If the driver g does not depend on y (hence $r(t) = 0$), then (4.1) is equivalent to (4.5) with $\beta > 1$. In this special case, Ankirchner *et al.* [2] studied the existence of the measure solution under the following assumption.

$$\mathbb{E}\bigg[\exp\bigg(\int_0^T \beta u^2(t)\mathrm{d}t\bigg)|\xi|^\delta+\int_0^T |g(\omega,t,0)|^\gamma\mathrm{d}t\bigg]<\infty,\ \text{for some } \Psi>1,\gamma>1 \text{ and } \beta,\delta>\Psi. \tag{4.6}$$

The above assumption (4.6) seems to be weaker than ours although the integrability parameter β on terminal value needs to be larger than Ψ for some $\Psi > 1$. However, (4.6) is only suitable for the existence of the measure solution. Under this assumption, we cannot guarantee the existence and uniqueness of strong solution, which is the objective of this paper.

For other types of BSDEs such as backward doubly stochastic differential equations or reflected BSDEs, the basic integrability assumption was similar to (4.5) and β needs to be large enough (see [25, 26, 28, 34, 35, 44, 45]).

Now, we describe the main advantage of our result. Suppose that we study the BSDEs with random terminal time. The stochastic Lipschitz condition is an extension of uniform Lipschitz condition. In principle, the results obtained under Lipschitz condition should coincide with ones obtained as a special case of BSDEs with stochastic Lipschitz condition. Our main result (Thm. 4.1) satisfies such nice property. In fact, when we discuss the BSDEs with Lipschitz coefficients a, b and random terminal time, the parameter γ in the integrability assumption need to satisfy $\gamma > b^2 + 2a$ or more generally $\gamma > \frac{b^2}{p-1} + 2a$ for L^p−solutions (see [1, 7–9, 19, 37, 38, 44]). If $r(t) = a$, $u(t) = b$ and $\beta(t) = \beta > b^2 + 2a$ in Theorem 4.1, then $\beta > (1 + \epsilon)(2a + b^2)$ is just satisfied for sufficiently small $\varepsilon > 0$. As one occasional (but typical) example, we note that [19], Theorem 3.4 can be seen as a direct consequence of Theorem 4.1 in this section.

5. Some other results

5.1. Wiener-type BSDEs with stochastic monotone coefficients

For $\xi : \Omega \to \mathbb{R}^k$, $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ and $\tau : \Omega \to [0, \infty]$, we consider the BSDE:

$$Y_t = \xi + \int_t^\tau f(\omega, s, Y_s, Z_s)\mathrm{d}s - \int_t^\tau Z_s \mathrm{d}W_s, \tag{5.1}$$

where W is a d-dimensional Brownian motion. First, we present the following Proposition which is a minor version of Theorem 5.2 in [9].

Proposition 5.1. *Suppose that the following conditions hold.*

1. *There exist bounded, progressively processes $a(t)$ and $b(t)$ such that for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k\times d}$,*

(1.1) $(y-y')(f(\omega,t,y,z)-f(\omega,t,y',z)) \le a(t)|y-y'|^2$,

(1.2) $|f(\omega,t,y,z')-f(\omega,t,y,z')| \le b(t)\|z-z'\|$.

2. *For any* t, z, $f(\omega,t,\cdot,z)$ *is a.s. continuous.*
3. *For all* $r>0$ *and* $n \in N$, $\Psi_r(t) := \sup_{|y|\le r}|f(\omega,t,y,0)-f(\omega,t,0,0)|$ *belongs to* $L^1((0,n)\times\Omega, \mathrm{d}t\times\mathrm{d}\mathbb{P})$ *(or* $L^1((0,n)\times\Omega)$ *for the case of* $k=1$*).*
4. *There exists a process* $\rho(t) \ge \nu(t) := a(t) + \frac{b^2(t)}{2(p-1)} + \varepsilon (\varepsilon > 0)$ *such that*

$$\mathbb{E}\left[e^{p\rho_\tau}|\xi|^p + \int_0^\tau e^{p\rho_s}|f(\omega,s,0,0)|^p\mathrm{d}s\right] < \infty,$$

and

$$\mathbb{E}\left[\int_0^\tau e^{p\rho_s}|f(\omega,s,e^{-\nu_s}\bar{\xi}(s),e^{-\nu_s}\bar{\eta}(s))|^p\mathrm{d}s\right] < \infty.$$

where $\rho_t := \int_0^t \rho(s)\mathrm{d}s$, $\nu_t := \int_0^t \nu(s)\mathrm{d}s$, $\bar{\xi} = e^{\nu_\tau}\xi$, $\bar{\xi}(t) = \mathbb{E}(\bar{\xi}|\mathcal{F}_t)$ *and* $\bar{\eta}$ *is predictable and such that*

$$\bar{\xi} = \mathbb{E}(\bar{\xi}) + \int_0^{+\infty}\bar{\eta}(t)\mathrm{d}W_t, \quad \mathbb{E}\left[\left(\int_0^{+\infty}\|\bar{\eta}(t)\|^2\mathrm{d}t\right)^{p/2}\right] < \infty.$$

Then, (5.1) *has a unique* L^p*-solution* (Y,Z) *such that*

$$\mathbb{E}\left[\sup_{0\le t\le\tau} e^{p\rho_t}|Y_t|^p + \int_0^\tau e^{p\rho_t}|Y_t|^p\mathrm{d}t + \int_0^\tau e^{p\rho_t}|Y_t|^{p-2}\|Z_t\|^2\mathrm{d}t\right] < \infty.$$

Proof. We can adopt the same strategy such as in [9], Theorem 5.2 with the use of bounded processes $a(t)$ and $b(t)$, instead of constant coefficients. See also the proof of Proposition 3.1. □

Theorem 5.2. *Suppose that the following conditions hold for BSDE* (5.1)*.*

1. *There exist non-negative progressive process* $u(t)$ *and progressive process* $r(t)$ *such that for any* $y, y' \in \mathbb{R}^k$ *and* $z, z' \in \mathbb{R}^{k\times d}$*;*
   (1.1) $\alpha^2(t) := r(t)^+ + u^2(t) > \varepsilon$ *for some* $\varepsilon > 0$,
   (1.2) $(y-y')(f(\omega,t,y,z)-f(\omega,t,y',z')) \le r(t)|y-y'|^2$,
   (1.3) $|f(\omega,t,y,z)-f(\omega,t,y',z')| \le u(t)\|z-z'\|$.
2. *For all* t, z, $f(\omega,t,\cdot,z)$ *is a.s. continuous.*
3. *For all* $r>0$ *and* $n\in N$, $\Psi_r(t) := \sup_{|y|\le r}|f(\omega,t,y,0)-f(\omega,t,0,0)|$ *belongs to* $L^1((0,n)\times\Omega,\mathrm{d}t\times d\mathbb{P})$ *(or* $L^1((0,n)\times\Omega)$ *for the case of* $k=1$*).*
4. *There exists a process* $\beta(t) \ge \pi(t)(1+\varepsilon)$, $\pi(t) := r(t) + \frac{u^2(t)}{2(p-1)}$, $\varepsilon > 0$ *(or equivalently* $\beta(t) = \theta\pi(t)$ *for some* $\theta > 1$*) such that*

$$\mathbb{E}\left[e^{p\beta_\tau}|\xi|^p + \int_0^\tau e^{p\beta_s}\alpha(s)^{-2(p-1)}|f(\omega,s,0,0)|^p\mathrm{d}s\right] < \infty. \tag{5.2}$$

and

$$\mathbb{E}\left[\int_0^\tau e^{p\beta_s}|f(\omega,s,e^{-\pi_s}\bar{\xi}(s),e^{-\pi_s}\bar{\eta}(s))|^p\mathrm{d}s\right] < \infty.$$

where $\beta_t := \int_0^t \beta(s)\mathrm{d}s$, $\pi_t := \int_0^t \pi(s)\mathrm{d}s$, $\bar{\xi} = e^{\pi_\tau}\xi$, $\bar{\xi}(t) = \mathbb{E}(\bar{\xi}|\mathcal{F}_t)$ *and* $\bar{\eta}$ *is predictable and such that*

$$\bar{\xi} = \mathbb{E}(\bar{\xi}) + \int_0^{+\infty} \bar{\eta}(t)\mathrm{d}W_t, \quad \mathbb{E}\bigg[\bigg(\int_0^{+\infty} \|\bar{\eta}(t)\|^2\mathrm{d}t\bigg)^{p/2}\bigg] < \infty.$$

Then, the BSDE (5.1) *has a unique solution such that*

$$\mathbb{E}\bigg[\sup_{0\le t\le\tau} e^{p\beta_t}|Y_t|^p + \int_0^\tau \alpha(t)e^{p\beta_t}|Y_t|^p\mathrm{d}t + \int_0^\tau e^{p\beta_t}|Y_t|^{p-2}\|Z_t\|^2\mathrm{d}t\bigg] < \infty.$$

Proof. We consider the following time-changed BSDE corresponding to (5.1).

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{f}(\omega, s, y_s, z_s)\mathrm{d}s - \int_t^{\widetilde{\tau}} z_s\mathrm{d}\widetilde{W}_s,\ t \le s \le \widetilde{\tau}.$$

We note that the process related to time-change is given as $\phi(t) = \int_0^t \alpha^2(s)\mathrm{d}s,\ \alpha^2(s) = r(s)^+ + u^2(s)$. The process $\widetilde{W}$ is an $\widetilde{\mathbb{F}}$-continuous martingale, but not a $\widetilde{\mathbb{F}}$-Brownian motion. So we define $\overline{W}_t := \int_0^t [(\phi^{-1})'(s)]^{-1/2}\mathrm{d}\widetilde{W}_s = \int_0^t \widetilde{\alpha}(s)\mathrm{d}\widetilde{W}_s$. One can easily check that $\langle \overline{W}^i, \overline{W}^j\rangle_t = \mathbf{1}_{\{i=j\}}t$. Thus, $\overline{W}$ is an $\widetilde{\mathbb{F}}-$Brownian motion by Lèvy's characterization theorem. By the Theorem 2.9, we have

$$|\widetilde{f}(\omega, t, y, \widetilde{\alpha}(t)z) - \widetilde{f}(\omega, t, y, \widetilde{\alpha}(t)z')| \le b(t)\|z - z'\|,$$

$$(y - y')(\widetilde{f}(\omega, t, y, z) - \widetilde{f}(\omega, t, y', z)) \le a(t)|y - y'|^2.$$

where $a(t) := \frac{\widetilde{r}^+(t)}{\widetilde{r}^+(t)+\widetilde{u}^2(t)}$ and $b(t) = \frac{\widetilde{u}(t)}{\sqrt{\widetilde{r}^+(t)+\widetilde{u}^2(t)}}$. On the other hand, for all $r > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}\int_0^n \sup_{|y|\le r} |\widetilde{f}(\omega, t, y, 0) - \widetilde{f}(\omega, t, 0, 0)|\mathrm{d}t &= \int_0^{\phi^{-1}(n)} \sup_{|y|\le r} |f(\omega, t, y, 0) - f(\omega, t, 0, 0)|\mathrm{d}t \\ &\le \int_0^{n/\varepsilon} \sup_{|y|\le r} |f(\omega, t, y, 0) - f(\omega, t, 0, 0)|\mathrm{d}t\end{aligned}$$

where we used an inequality $\phi(t) \ge \varepsilon t$. The pair $(y_t, \bar{z}_t) := (y_t, z_t/\widetilde{\alpha}(t))$ yields the equation:

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{f}(\omega, s, y_s, \widetilde{\alpha}(s)\bar{z}_s)\mathrm{d}s - \int_t^{\widetilde{\tau}} \bar{z}_s\mathrm{d}\overline{W}_s,\ t \le s \le \widetilde{\tau}.$$

The rest of the proof is almost the same as the proof of Theorem 4.1, thanks to Proposition 5.1. □

Remark 5.3. For the comparison theorem of L^p-solutions to (5.1) under stochastic monotonicity condition, the readers may refer to [33].

Comparison with the existing literature

Theorem 5.2 is a generalized version of some known results. In [3], the stochastic monotonicity was considered in L^2-setting. Wang *et al.* [44] studied the L^p-solution of BSDEs with bounded terminal value and stochastic

Lipschitz coefficients satisfying $\phi(\tau) \le L$ for some $L \ge 0$ in the case of $1 < p < 2$. They introduced the following integrability assumptions (see also [35]).

$$\mathbb{E}\Big[e^{p\theta\phi(\tau)}|\xi|^p + \Big(\int_0^\tau e^{2\theta\phi(s)}\big|\frac{f(\omega,s,0,0)}{\alpha(s)}\big|^2\Big)^{p/2}\Big] < \infty.$$

When $\phi(\tau) \le L$ and $1 < p < 2$, by Hölder's inequality, it follows that

$$\mathbb{E}\Big[\int_0^\tau e^{p\theta\pi_s}\alpha(s)^{-2(p-1)}|f(\omega,s,0,0)|^p \mathrm{d}s\Big] = \mathbb{E}\Big[\int_0^\tau e^{p\theta\pi_s}\big|\frac{f(\omega,s,0,0)}{\alpha(s)}\big|^p \cdot \alpha(s)^{2-p}\mathrm{d}s\Big]$$
$$\le \mathbb{E}\Big[\Big(\int_0^\tau e^{2\theta\pi_s}\big|\frac{f(\omega,s,0,0)}{\alpha(s)}\big|^2\mathrm{d}s\Big)^{p/2}\cdot\Big(\int_0^\tau \alpha^2(s)\mathrm{d}s\Big)^{\frac{2-p}{2}}\Big] \le L^{\frac{2-p}{2}}\cdot\mathbb{E}\Big[\Big(\int_0^\tau e^{2\theta\pi_s}\big|\frac{f(\omega,s,0,0)}{\alpha(s)}\big|^2\mathrm{d}s\Big)^{p/2}\Big].$$

So our integrability assumptions are clearly weaker than those of [44]. Moreover, the integrability parameter θ can be taken as $\theta < 2$ in our assumptions, whereas θ needs to be sufficiently large in [44]. Pardoux and Răşcanu [39] studied the L^p-solution of BSDEs under stochastic Lipschitz condition, too. One of their assumptions was given as follows (see Thm. 5.21 therein).

$$\mathbb{E}\Big[e^{p\psi_T}|\xi|^p + \Big(\int_0^T e^{\psi_t}|g(\omega,t,0,0)|\mathrm{d}t\Big)^p\Big] < \infty. \tag{5.3}$$

where $\psi(t) := r(t) + \frac{u^2(t)}{1\wedge(p-1)},\ \psi_t := \int_0^t \psi(s)\mathrm{d}s$. For $\theta \approx 1$, we see that

$$p\theta\pi_t = p\theta\left(r(t) + \frac{u^2(t)}{2(p-1)}\right) \le p\left(\theta r(t) + \frac{u^2(t)}{p-1}\right) \approx p\left(r(t) + \frac{u^2(t)}{1\wedge(p-1)}\right) = p\psi_t.$$

So our Assumption (5.2) seems to be weaker than (5.3). The remarkable result in this context can be found again in Pardoux and Răşcanu [39] (see Cor. 5.59 therein). They studied the L^p-solution of BSDEs with stochastic monotone coefficients. But in their result, the Lipschitz coefficient in z needs to be only a deterministic process. On the other hand, their definition of solution to BSDE (5.1) is different from ours, so their integrability assumptions, which are suitable for the study of such solution, are given with a different nature. Thus, the relation between our assumption and that of [39] is not clear. But the term $p(r(t) + \theta\frac{u^2(t)}{2(1\wedge(p-1)}),\ \theta > 1$ in their integrability assumption (see [39], p. 443) is very similar to the term $p\theta\pi_t$ in ours. For $\theta \approx 1$, it is seen that

$$p\theta\pi_t = p\theta\left(r(t) + \frac{u^2(t)}{2(p-1)}\right) \le p\theta\left(r(t) + \frac{u^2(t)}{2\wedge 2(p-1)}\right) \approx p\left(r(t) + \theta\frac{u^2(t)}{2\wedge 2(p-1)}\right).$$

So, our result may be more general in a sense.

Now, let us consider the BSDE with a strict monotone driver (hence $r(t) \le 0$ and the driver is stochastic Lipschitz only in z). In this case, we can get a useful result by referring to [41].

Theorem 5.4. *Suppose that the assumptions (1.2), (1.3) and (2) in Theorem 5.2 hold with $r(t) \le 0$. Let us assume that the driver satisfies the growth condition such that $|f(\omega,t,y,z)| \le |f(\omega,t,0,z)| + l_t\psi(|y|)$, for some continuous, increasing function ψ. We further assume that $\forall t \ge 0, f(\omega,t,0,0) = 0$ and $|\xi| \le M$ for some $M \ge 0$. Then there exists a solution (Y_t, Z_t) of (5.1) such that $|Y| \le M$ and $\forall t \ge 0, \mathbb{E}\big[\int_0^{t\wedge\tau}\|Z_s\|^2\mathrm{d}s\big] < \infty$.*

Proof. We only sketch the proof. We define a process $\phi(t) := \int_0^t (u^2(s) + l(s) + 1)\mathrm{d}s$ with which we associate time-change. Obviously $\phi^{-1}(t) \le t$. Let $\widetilde{f}$ denote the driver of time-changed BSDE.

Then it is easy to see that $\widetilde{f}$ is uniformly Lipschitz in z with Lipschitz coefficient 1 and monotone decreasing in y. It also satisfies general growth condition with coefficient 1. We can easily check that $\widetilde{f}(\omega, t, 0, 0) = 0$.

So there exists a solution (y_t, z_t) to the time-changed BSDE such that $|y| < M$ and for any t, $\int_0^{t\wedge\widetilde{\tau}} \|z_s\|^2 \mathrm{d}s < \infty$ from [41], Theorem 3.1. Noting that $Y_t = y_{\phi(t)}$ and $\int_0^{t\wedge\tau} \|Z_s\|^2 \mathrm{d}s \le \int_0^{\phi^{-1}(t)\wedge\tau} \|Z_s\|^2 \mathrm{d}s = \int_0^{\phi^{-1}(t)\wedge\tau} \|Z_{\phi(s)}\|^2 \phi'(s)\mathrm{d}s = \int_0^{t\wedge\widetilde{\tau}} \|z_s\|^2 \mathrm{d}s < \infty$, we can complete the proof. □

Remark 5.5. We note that the uniqueness and comparison can be stated under the further conditions using Theorems 3.6 and 3.7 in [41]. We can also study the stability of solutions by referring to [43].

Remark 5.6. In Theorem 5.4, the exponential integrability condition on terminal value and the driver are not made and the same conditions as the case of uniformly Lipschitz were used for the study of the BSDE with stochastic one. This is because the monotone coefficient which makes discounting rate is equal to zero.

5.2. Markov chain BSDEs with stochastic Lipschitz coefficients

As a direct corollary of Theorem 4.1, we can get the results of Markov chain BSDEs with stochastic Lipschitz conditions. But we shall study this class of BSDEs in other direction. In the preceding discussions, we used the results obtained under the monotonicity assumption. So, the stronger integrability conditions on the driver and the solutions were still required as in the previous works. In this subsection, we study the BSDEs driven by Markov chains without using the monotonicity condition to avoid the stronger integrability assumptions. BSDEs on Markov chains were first introduced in [13] and have developed in several papers, for example, the comparison theorem in [14] or the case of random terminal time in [12]. We present some preliminaries of the Markov chain BSDEs below.

Consider a continuous time, countable state Markov chain X on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F}$ is the natural filtration generated by X. Without loss of generality, we assume that X takes values from the unit vector e_i in $\mathbb{R}^N, (N \in \mathbb{N} \cup \{\infty\})$, where N is the number of states of the chain. We denote by Π the state space. If A_t denotes the rate matrix of the chain at time t, then $(A_t)_{ij} \ge 0,\ i \ne j$ and $\forall j, \Sigma_i (A_t)_{ij} = 0$. For the simplicity, we shall assume that A is uniformly bounded. The Markov chain X has the following Doob-Meyer decomposition (see [22], Appendix B).

$$X_t = X_0 + \int_0^t A_u X_{u-} \mathrm{d}u + M_t, \tag{5.4}$$

where M is a pure discontinuous martingale with finite variation. In this section, we further assume the Markov chain has the strong Markov property. Let us consider the following BSDE to stopping time on Markov chain.

$$Y_t = \xi + \int_t^\tau f(\omega, u, Y_{u-}, Z_u)\mathrm{d}u - \int_t^\tau Z_u \mathrm{d}M_u, \quad 0 \le t \le \tau, \tag{5.5}$$

where $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\int_0^t Z_u \mathrm{d}M_u = \sum_{i=1}^N \int_0^t Z_u^i \mathrm{d}M_u^i$. Note that (5.5) is contained in the class of BSDEs defined by (1.3) due to [13], Lemma 3.1 where it was shown that the sequence $(M^i), i = 1, 2, ...$ has the martingale representation.

Definition 5.7. We define $\psi_t := diag(A_t X_{t-}) - A_t diag(X_{t-}) - diag(X_{t-}) A_t^\intercal$. Then the matrix ψ_t is symmetric and positive (semi-)definite and $\mathrm{d}\langle M\rangle_t = \psi_t \mathrm{d}t$ (see [13]).

Due to (1.6), we can set the stochastic semi-norm $\|\cdot\|_{M_t}$ as follows.

$$\|Z\|_{M_t}^2 := Z_u^\intercal \psi_u Z_u, \quad Z \in \mathbb{R}^{1\times N}. \tag{5.6}$$

We give further definitions from [12].

Definition 5.8. We say that the driver f is $\gamma-$balanced if there exists a random field $\eta : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, with $\eta(\cdot, \cdot, z, z')$ predictable and $\eta(\omega, t, 0, 0)$ Borel-measurable, such that

- $f(\omega, t, y, z) - f(\omega, t, y, z') = (z - z')^\top(\eta(\omega, t, z, z') - AX_{t-})$
- for each $e^i \in \Pi$, $(e_i^\top \eta(\omega, t, z, z'))/(e_i^\top AX_{t-}) \in [\gamma, \gamma^{-1}]$ for some $\gamma > 0$, where $0/0 := 1$
- $\mathbf{1}^\top \eta(\omega, t, z, z') = 0$ for $\mathbf{1} \in \mathbb{R}^N$ the vector with all entries 1
- $\eta(\omega, t, z + \alpha \mathbf{1}, z') = \eta(\omega, t, z, z')$ for all $\alpha \in \mathbb{R}$

Remark 5.9. Note that the above definition of $\gamma-$balanced driver is closely connected to the notion of balanced driver in Section 2 (see [12], Lem. 3).

Definition 5.10. Let $\mathcal{Q}_\gamma$ denote the family of all measures Q where X has the compensator $\eta(t, \omega)$, for η a predictable process with $\mathbf{1}^\top \eta(t, \omega) = 0$ and $\frac{e_i^\top \eta(t,\omega)}{e_i^\top AX_{t-}} \in [\gamma, \gamma^{-1}]$ for all $e_i \in \Pi$, where $0/0 := 1$. That is, $X_t = X_0 + \int_0^t \eta_t \mathrm{d}t + Q$-martingale, $Q \in \mathcal{Q}^\gamma$.

We give the key result of [12] (see Thm. 3, Rem. 4 therein).

Lemma 5.11. *Suppose that the following conditions are verified for Markov chain BSDE* (4.2)*.*

1. ξ *is* $\mathcal{F}_\tau-$*measurable.*
2. *There exist non-decreasing functions* $K_1, K_2 : \mathbb{R}^+ \to [1, \infty)$ *and some constants* $\beta, \widetilde{\beta} > 0$ *such that*

$$\mathbb{E}^Q[\xi|\mathcal{F}_t] \leq K_1(t), \mathbb{E}^Q[(1+\tau)^{1+\beta}|\mathcal{F}_t] \leq K_1(t), \mathbb{E}^Q[K_1(\tau)^{1+\widetilde{\beta}}|\mathcal{F}_t] \leq K_2(t),$$

   for all $\mathbb{P} - a.s.$ *all* $Q \in \mathcal{Q}_\gamma$ *and all* t*.*
3. $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is* $\gamma-$*balanced.*
4. *The discounting terms are uniformly bounded above, that is, there exists a constant* $C_1 \in \mathbb{R}$ *such that for any* y, y', z *and* $s < t$*,*

$$\int_s^t r(\omega, u, y, y', z)\mathrm{d}u < C_1, \quad r(\omega, u, y, y', z) := \frac{f(\omega, t, y, z) - f(\omega, t, y', z)}{y - y'}.$$

5. *There exists* $C_2 \in \mathbb{R}$, $\hat{\beta} \in [0, \beta]$ *such that* $|f(\omega, t, 0, 0)| \leq C_2(1 + t^{\hat{\beta}})$*.*
6. $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is uniformly Lipschitz in* y*. That is, there exists a constant* C *such that* $|f(\omega, t, y, z) - f(\omega, t, y', z)| \leq C|y - y'|$ *for all* y, y', z*.*

Then the BSDE (5.5) *has a unique solution such that* $|Y_t| \leq (1 + C_2)\exp(C_1)|K_1(t)|$*.*

In Lemma 5.11, the Lipschitz condition in y is restrictive. We give a simple illustration below with the motion of a particle on graph. Consider a model for transmission of messages from a node to another node over a network. Let the chain X describe the motion of a message. Then the probability that the message reaches its target is given as the solution of the following BSDE (see [12], Sect. 4).

$$I_{\{X_\tau = x_1\}} = Y_t - \int_t^\tau -r_{X_{u-}} Y_{u-}\mathrm{d}u + \int_t^\tau Z_u \mathrm{d}M_u, \quad 0 \leq t \leq \tau, \tag{5.7}$$

where r_x is the rate by which the node x loses a message. To suppose that the losing rate at each node is bounded is an assumption rarely satisfied. It depends on the time variable in general and it should be written as $r(t, X_{t-})$ which may be unbounded. So, we need to relax the uniform Lipschitz assumption for Markov chain BSDE. In addition to that, we also aim to relax the growth assumption on driver by using the time-change, more effectively.

Usually, if one wants to relax the Lipschitz continuity as the stochastic one, it has to be considered that the stronger integrability conditions on terminal value, driver and the solution are required, instead. However, this is not true for undiscounted BSDE. It is because the terminal value and driver of this BSDEs are not needed to be discounted at some rate and one can consider the direct conditions on them respectively.

The main result of this subsection is as follows (we shall give the proof later).

Theorem 5.12. *Suppose that the conditions* (1)−(4) *in Lemma 5.11 are satisfied for BSDE* (5.5)*. Let the driver f be stochastic Lipschitz in y, that is, there exists a non-negative predictable process $C(t)$ such that for any t, y, z, z',*

$$|f(\omega, t, y, z) - f(\omega, t, y', z)| \leq C(t)|y - y'|. \tag{5.8}$$

We further assume that f is integrable on every finite interval in $\mathbb{R}^+$. Then BSDE (5.5) *has a unique solution such that $|Y_t| \leq \exp(C_1)|K_1(t)|$.*

Example 5.13. Let $f(\omega, t, y, z) = r(t)y,\ r(t) := \frac{2}{\tau\sqrt{\pi}}\sqrt{\ln \frac{\tau}{\tau - t\mathbf{1}_{t<\tau}}}$. Then $r(t)$ is unbounded, but $\int_{\mathbb{R}^+} r(t)\mathrm{d}t = \int_0^\tau r(t)\mathrm{d}t = 1$. So f is stochastic Lipschitz and all the discounting terms are uniformly bounded.

In Theorem 5.12, the conditions on stopping time seem to be unfamiliar and it is required to afford an example when they are satisfied. In this context, S. N. Cohen [12] showed that the direct conditions on stopping time are satisfied when the stopping time is a hitting time of a subset of Π under the uniform ergodicity of the chain by the way of examining the exponential ergodicity of the chain under the perturbations of rate matrix. One can observe that the above hitting time only depends on the character of the chain. Our driver is stochastic Lipschitz only in y and $\gamma-$balanced condition related to z was still assumed as in the Lipschitz setting. On the other hand, Theorem 5.12 implies that the conditions on stopping time and terminal value for the wellposedness of BSDE (5.5) coincide in two uniform and stochastic Lipschitz settings. These lead to the following result (see [12], Lem. 6).

Proposition 5.14. *Suppose that rate matrix is time-homogeneous under the measure $\mathbb{P}$ and the chain is uniformly ergodic. Let τ be the first hitting time of a set $\Xi \subseteq \Pi$ and ξ be a random variable of a Markovian form $\xi = g(\tau, X_\tau)$ for some function $g(t, x) \leq k(1 + t^\beta)$ for some $k, \beta > 0$. Then there exist functions K_1, K_2 satisfying the requirements of Theorem 5.12.*

In Markovian setting, the BSDE (5.5) is closely connected to the ODE system with boundary condition (see [12], Thms. 6 or 7). If the terminal time and terminal value have the forms like in Proposition 5.14 and the driver is Markovian, that is, $f(\omega, t, y, z) = \bar{f}(X_{t-}, t, y, z)$ for some $\bar{f}$, we can give the ODE system with boundary condition which describes the solution of BSDE in the context of stochastic Lipschitz assumption in the same way as in [12]. Now we seem to prove Theorem 5.12 by means of time-change described in Sect. 2.

Proof of Theorem 5.12 For any $m \in N$, let us define the following process ϕ as follows (this is based on the same idea as in Sect. 2).

$$\phi(t) := \int_0^t \alpha^2(s)\mathrm{d}s, \quad \alpha^2(s) := C(s) + m|f_0(s)| + 1,\ f_0(s) := f(\omega, s, 0, 0). \tag{5.9}$$

It follows from $\phi(t) \geq \int_0^t 1\mathrm{d}s = t$ that $t \geq \phi^{-1}(t)$. We set $\widetilde{\mathcal{F}}_t := \mathcal{F}_{\phi^{-1}(t)}, \widetilde{\mathbb{F}} := \{\widetilde{\mathcal{F}}_t\}_{t\geq 0}$ as in Section 2. As X is a strong Markov chain, $\widetilde{X} := X_{\phi^{-1}(t)}$ is also a strong Markov chain with respect to $\widetilde{\mathbb{F}}$ (see *e.g.* [4], Chap. 22, Sect. 3).

Using the expression (5.4),

$$\widetilde{X}_t = X_0 + \int_0^{\phi^{-1}(t)} A_u X_u \mathrm{d}u + M_{\phi^{-1}(t)} = \widetilde{X}_0 + \int_0^t \widetilde{A}_u \widetilde{X}_u \mathrm{d}u + \widetilde{M}_t, \tag{5.10}$$

where $\widetilde{A}_u := A_{\phi^{-1}(u)} \cdot (\phi^{-1})'(u)$ and $\widetilde{M}_t := M_{\phi^{-1}}(t)$. We recall that $\widetilde{M} = (\widetilde{M}^i), i = 1, 2, ..., N$ is a sequence of orthogonal martingales which has martingale representation on $(\Omega, \mathcal{F}, \mathbb{P}, \widetilde{\mathbb{F}})$ (see Lem. 2.8). Therefore, (5.10) is the (unique) Doob-Meyer decomposition of $\widetilde{X}$. If we denote by R_u the rate matrix of $\widetilde{X}$, then

$$R_u \widetilde{X}_u = \widetilde{A}_u \widetilde{X}_u, \quad \mathrm{d}t \times \mathrm{d}\mathbb{P} - a.s. \,. \tag{5.11}$$

It follows from $(\phi^{-1})'(t) = \widetilde{\alpha}^{-2}(t) \leq 1$ that R_u is uniformly bounded, so the $\widetilde{\mathcal{F}}-$chain $\widetilde{X}$ is also regular. We can consider that the random rate matrix $\widetilde{A}_t$ plays the role of transition rate matrix of $\widetilde{X}$. Next, we shall show that $\widetilde{f}(\omega, s, y, z) := f(\omega, \phi^{-1}(s), y, z) \cdot (\phi^{-1})'(s)$ is $\gamma-$balanced with respect to $\widetilde{\mathbb{F}}$. We define $\widetilde{\eta}(\omega, t, z, z') := \eta(\omega, \phi^{-1}(t), z, z') \cdot (\phi^{-1})'(t)$. Then by the definition and (5.11), we have $\mathrm{d}t \times \mathrm{d}\mathbb{P}-$a.s.,

$$\begin{aligned}
\widetilde{f}(\omega, t, y, z) - \widetilde{f}(\omega, t, y, z') &= (f(\omega, \phi^{-1}(t), y, z) - f(\omega, \phi^{-1}(t), y, z'))(\phi^{-1})'(t) \\
&= (z - z')^\top (\eta(\omega, \phi^{-1}(t), z, z') - A_{\phi^{-1}(t)} X_{\phi^{-1}(t)})(\phi^{-1})'(t) \\
&= (z - z')^\top (\widetilde{\eta}(\omega, t, z, z') - \widetilde{A}_t \widetilde{X}_t) = (z - z')^\top (\widetilde{\eta}(\omega, t, z, z') - R_t \widetilde{X}_t), \\
(e_i^\top \widetilde{\eta}(\omega, t, z, z'))/(e_i^\top R_t \widetilde{X}_t) &= (e_i^\top \widetilde{\eta}(\omega, t, z, z'))/(e_i^\top \widetilde{A}_t \widetilde{X}_t) \\
&= (e_i^\top \eta(\omega, \phi^{-1}(t), z, z'))/(e_i^\top A_{\phi^{-1}(t)} X_{\phi^{-1}(t)}) \in [\gamma, \gamma^{-1}], \\
\mathbf{1}^\top \widetilde{\eta}(\omega, t, z, z') &= \mathbf{1}^\top \eta(\omega, \phi^{-1}(t), z, z')(\phi^{-1})'(t) = 0, \\
\widetilde{\eta}(\omega, t, z + \alpha \mathbf{1}, z') &= \eta(\omega, \phi^{-1}(t), z + \alpha \mathbf{1}, z') \cdot (\phi^{-1})'(t) \\
&= \eta(\omega, \phi^{-1}(t), z, z') \cdot (\phi^{-1})'(t) = \widetilde{\eta}(\omega, t, z, z').
\end{aligned}$$

So $\widetilde{f}$ is $\gamma-$balanced. We note that $\widetilde{f}$ is uniformly Lipschitz in z under norm $\|\cdot\|_{\widetilde{M}_t}$ because it is $\gamma-$balanced (see [12], Lem. 1). From the expressions (5.10) and (5.11), it is trivial that the family of probability measures, where $\widetilde{X}$ has the $\widetilde{\mathbb{F}}-$predictable compensator $\widetilde{\eta}(t, \omega)$ such that $\mathbf{1}^\top \widetilde{\eta} = 0$ and $\forall 1 \leq i \leq N; (e_i^\top \widetilde{\eta}(t, \omega))/(e_i^\top R_t \widetilde{X}_{t-}) \in [\gamma, \gamma^{-1}]$, is also $\mathcal{Q}_\gamma$.

Finally, we show that the following time-changed BSDE has a unique solution ($\widetilde{\tau} := \phi(\tau)$).

$$y_t = \xi + \int_t^{\widetilde{\tau}} \widetilde{f}(\omega, s, y_s, z_s) \mathrm{d}s - \int_t^{\widetilde{\tau}} z_s \mathrm{d}\widetilde{M}_s. \tag{5.12}$$

We have already seen that $\widetilde{f}$ is $\gamma-$balanced.

Let us define the non-decreasing functions $\widetilde{K}_1(t) := K_1(\phi^{-1}(t))$ and $\widetilde{K}_2(t) := K_2(\phi^{-1}(t))$. Then $\forall Q \in \mathcal{Q}_\gamma; \mathbb{E}^Q[\xi|\widetilde{\mathcal{F}}_t] \leq K_1(\phi^{-1}(t)) = \widetilde{K}_1(t)$ and $\mathbb{E}^Q[\widetilde{K}_1(\widetilde{\tau})^{1+\widetilde{\beta}}|\widetilde{\mathcal{F}}_t] = \mathbb{E}^Q[K_1(\tau)^{1+\widetilde{\beta}}|\mathcal{F}_{\phi^{-1}(t)}] \leq K_2(\phi^{-1}(t)) = \widetilde{K}_2(t)$.

Using the assumptions on f, we can get the following expressions on $\widetilde{f}$.

$$|\widetilde{f}(\omega, t, 0, 0)| = \widetilde{\alpha}^{-2}(t)|\widetilde{f}_0(t)| = \frac{|\widetilde{f}_0(t)|}{\widetilde{C}(t) + m|\widetilde{f}_0(t)| + 1} \leq \frac{1}{m},$$

$$\int_s^t \frac{\widetilde{f}(\omega, u, y, z) - \widetilde{f}(\omega, u, y', z)}{y - y'} \mathrm{d}u = \int_s^t r(\omega, \phi^{-1}(u), y, y', z) \mathrm{d}\phi^{-1}(u) = \int_{\phi^{-1}(s)}^{\phi^{-1}(t)} r(\omega, u, y, y', z) \mathrm{d}u \leq C_1,$$

$$|\widetilde{f}(\omega, t, y, z) - \widetilde{f}(\omega, t, y', z)| \leq \frac{\widetilde{C}(t)}{\widetilde{C}(t) + m|\widetilde{f}_0(t)| + 1}|y - y'| \leq |y - y'|.$$

So BSDE (5.12) has a unique solution satisfying $|y_t| \leq (1+1/m)\exp(C_1)|\widetilde{K}_1(t)|$ by Lemma 5.11. Taking $m \to \infty$, we get $|y_t| \leq \exp(C_1)|\widetilde{K}_1(t)|$. Set $(Y_t, Z_t) := (y_{\phi(t)}, z_{\phi(t)})$. Theorem 2.9 shows that (Y, Z) is a solution of BSDE (5.5). Since the solution y_t of (5.12) is unique up to indistinguishability, Y_t is also unique up to indistinguishability. And $|Y_t| = |y_{\phi(t)}| \leq \exp(C_1)|\widetilde{K}_1(\phi(t))| = \exp(C_1)|K_1(t)|$. □

Remark 5.15. We note that the comparison theorem for BSDE (5.5) holds under the stochastic Lipschitz condition from the corresponding comparison theorem for BSDE (5.12) (see [12], Thm. 5).

6. Conclusion

In this paper, we suggested a technique for dealing with the BSDEs with stochastic Lipschitz coefficients by random time-change. The technique says that we can study the BSDEs with stochastic Lipschitz coefficients in a manner that replaces the randomness of the Lipschitz coefficients by the randomness of the terminal time. We mention that our approach is applicable to other class of BSDEs. For example, we can study reflected BSDE in the context of stochastic Lipschitz (or monotonicity) condition by referring to [1]. We can use the results in [37] to study the BSDEs with jumps under stochastic Lipschitz (or monotonicity) condition. To study the backward doubly SDEs(BDSDEs) with stochastic Lipschitz coefficients, we can refer to [29], which is concerning with the BDSDEs with random terminal time. By referring to [27], where the results of second-order BSDEs (2BSDEs) with random terminal time are established, we can study 2BSDEs with stochastic Lipschitz condition.

Appendix A.

Lemma A.1. *Suppose g and g' satisfy the stochastic Lipschitz condition with the same coefficients $a(t)$ and $b(t)$. Then for any $\kappa \geq 0, \delta > 0$,*

$$2(y-y')\big[g(t,y,z) - g'(t,y',z')\big] \leq [b^2(t)(1+\kappa) + \delta + 2a(t)] \cdot |y-y'|^2 + \frac{\|z-z'\|^2_{M_t}}{1+\kappa} + \frac{|g(t,y,z) - g'(t,y,z)|^2}{\delta}.$$

Moreover, if $g = g'$ then

$$2(y-y')\big[g(t,y,z) - g(t,y',z')\big] \leq [b^2(t)(1+\kappa) + 2a(t)] \cdot |y-y'|^2 + \frac{\|z-z'\|^2_{M_t}}{1+\kappa}.$$

Lemma A.2. *Let $\lambda(t)$ be a bounded process such that $e^{\lambda_\tau}\xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P})$, where $\lambda_t := \int_0^t \lambda(s)\mathrm{d}s$. By the Theorem 1.2, a square-integrable martingale $\mathbb{E}[e^{\lambda_\tau}\xi|\mathcal{F}_t]$ has a unique representation:*

$$\zeta(t) := \mathbb{E}[e^{\lambda_\tau}\xi|\mathcal{F}_t] = \mathbb{E}[e^{\lambda_\tau}\xi] + \sum_i \int_0^{t\wedge\tau} \eta^i(s)\mathrm{d}M^i_s. \tag{A.1}$$

Then, for any bounded process $\theta(t) \geq 0$ such that $e^{(\theta_\tau/2+\lambda_\tau)}\xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P})$ (where $\theta_t := \int_0^t \theta(s)\mathrm{d}s$), the processes ζ and η satisfy

$$\mathbb{E}\bigg[\int_0^\tau e^{\theta_s}\bigg(|\zeta(s)|^2\mathrm{d}s + \theta(s)\sum_i \|\eta^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg] = \mathbb{E}\big[e^{(\theta_\tau+2\lambda_\tau)}|\xi|^2\big] - \big|\mathbb{E}\big[e^{\lambda_\tau}\xi\big]\big|^2.$$

Proof. We can adopt the same strategy such as in [19], Lemma 4.1, so we omit the proof. □

Proof of Proposition 3.1

Existence. Set $\lambda(t) := \gamma(t)/2$. Recall that λ_t means the integral $\int_0^t \lambda(s)\mathrm{d}s$. For each $n \geq 1$, let us consider the following BSDE with finite interval (we omit ω for the simplicity).

$$\hat{Y}_n(t) = \mathbb{E}[e^{\lambda_\tau}|\mathcal{F}_n] + \int_{t\wedge\tau}^{n\wedge\tau} [e^{\lambda_s} g(s, e^{-\lambda_s}\hat{Y}_n(s), e^{-\lambda_s}\hat{Z}_n(s)) - \lambda(s)\hat{Y}_n(s)]\mathrm{d}s - \sum_{i=1}^{\infty} \hat{Z}_n^i(s)\mathrm{d}M_s^i, \quad 0 \leq t \leq n. \quad \text{(A.2)}$$

It follows from [16] that the BSDE (A.2) has a unique solution $(\hat{Y}_n, \hat{Z}_n)$ in $U_0^2(0,n) \times L_0^2(0,n,M)$. Now, we extend this solution to the whole axis by setting

$$\hat{Y}_n(t) := \zeta(t) = \mathbb{E}[e^{\lambda_\tau}\xi|\mathcal{F}_t], \;\; \hat{Z}_n(t) := \eta(t),$$

for $t > n$. Define, for all $t \geq 0$, $Y_n(t) := e^{-\lambda_t}\hat{Y}_n(t), Z_n(t) := e^{-\lambda_t}\hat{Z}_n(t)$. Then, by Itô's formula, (Y_n, Z_n) satisfies

$$Y_n(t) = \xi + \int_{t\wedge\tau}^{\tau} g_n(s, Y_n(s), Z_n(s))\mathrm{d}s - \sum_{i=1}^{\infty} Z_n^i(s)\mathrm{d}M_s^i. \quad \text{(A.3)}$$

where $g_n(s,y,z) := \mathbf{1}_{\{s\leq n\}} g(s,y,z) + \lambda(s)\mathbf{1}_{\{s>n\}} y$. Fix $m > n$ and we put $\Delta Y := Y_m - Y_n, \Delta Z := Z_m - Z_n$. We observe that $\Delta Y = \Delta Z = 0$ for $t > m$. Since $\Delta Y(m\wedge\tau) = \Delta Z(m\wedge\tau)$, we get

$$\Delta Y(t\wedge\tau) = \int_{t\wedge\tau}^{m\wedge\tau} \Delta g_{m,n}(s)\mathrm{d}s - \sum_{i=1}^{\infty} \Delta Z^i(s)\mathrm{d}M_s^i. \quad \text{(A.4)}$$

where $\Delta g_{m,n}(s) := g_m(s, Y_m(s), Z_m(s)) - g_n(s, Y_n(s), Z_n(s))$. By integration by parts, we deduce

$$e^{\rho_{t\wedge\tau}/2}\Delta Y(t\wedge\tau) = \int_{t\wedge\tau}^{m\wedge\tau} e^{\rho_s/2}(\Delta g_{m,n}(s) - \rho(s)/2 \cdot \Delta Y(s))\mathrm{d}s - \sum_i \int_{t\wedge\tau}^{m\wedge\tau} e^{\rho_s/2}\Delta Z^i(s)\mathrm{d}M_s^i. \quad \text{(A.5)}$$

Using Itô's formula, we obtain

$$\begin{aligned} &e^{\rho_{t\wedge\tau}}|\Delta Y(t\wedge\tau)|^2 + \int_{t\wedge\tau}^{m\wedge\tau} \rho(s)e^{\rho_s}|\Delta Y(s)|^2\mathrm{d}s \\ &= 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho_s}\Delta Y(s)\Delta g_{m,n}(s)\mathrm{d}s - 2\int_{t\wedge\tau}^{m\wedge\tau} \left\{ \sum_i e^{\rho_s}\Delta Y(s-)\Delta Z^i(s)\mathrm{d}M_s^i + \sum_{i,j} e^{\rho_s}\Delta Z^i(s)\Delta Z^j(s)\mathrm{d}[M^i, M^j]_s \right\}. \end{aligned} \quad \text{(A.6)}$$

Since $(\Delta Y, \Delta Z) \in U_0^2(0,m) \times L_0^2(0,m,M)$, we can use BDG-inequality to show that $\int_0^{t\wedge\tau}\sum_i e^{\rho_s}\Delta Y(s-)\Delta Z^i(s)\mathrm{d}M_s^i$ is a uniformly integrable martingale. It follows from the orthogonality of the martingales M^i and $\Delta Z \in L_0^2(0,m,M)$ that

$$\int_0^{t\wedge\tau} \left\{ \sum_{i,j} e^{\rho_s}\Delta Z^i(s)\Delta Z^j(s)\mathrm{d}[M^i, M^j]_s - \sum_i e^{\rho_s}\|\Delta Z^i(s)\|^2\mathrm{d}\langle M^i\rangle_s \right\}$$

is a martingale. Thus, taking expectations through (A.6) then gives

$$\mathbb{E}\big[e^{\rho_{t\wedge\tau}}|\Delta Y(t\wedge\tau)|^2\big]+\mathbb{E}\bigg[\int_{t\wedge\tau}^{m\wedge\tau}\rho(s)e^{\rho_s}|\Delta Y(s)|^2\mathrm{d}s+\sum_i e^{\rho s}\|\Delta Z^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg]$$
$$=2\mathbb{E}\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho_s}\Delta Y(s)\Delta g_{m,n}(s)\mathrm{d}s. \tag{A.7}$$

Set $\vartheta(s):=\rho(s)-(b^2(s)+2a(s)+b^2(s)\kappa+\delta)$. Let d be a constant such that $b^2(s)\le d$ for any $s\ge 0$. We take $\delta=\frac{\varepsilon}{3},\kappa=\frac{\varepsilon}{3d}$, so that $\vartheta(s)>\varepsilon/3$ for all s. By the Lemma A.1, we obtain

$$\begin{aligned}\mathbb{E}\bigg[e^{\rho_{t\wedge\tau}}|\Delta Y(t\wedge\tau)|^2+\int_{t\wedge\tau}^{m\wedge\tau}\bigg(\vartheta(s)e^{\rho_s}|\Delta Y(s)|^2\mathrm{d}s+\sum_i e^{\rho_s}\cdot\frac{\kappa}{1+\kappa}\|\Delta Z^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg]\\
\le\delta^{-1}\mathbb{E}\bigg[\int_{t\wedge\tau}^{m\wedge\tau}e^{\rho_s}|g_m(s,Y_m(s),Z_m(s))-g_n(s,Y_n(s),Z_n(s))|^2\mathrm{d}s\bigg]\\
\le\delta^{-1}\mathbb{E}\bigg[\int_{n\wedge\tau}^{m\wedge\tau}e^{\rho_s}|g(s,Y_n(s),Z_n(s))-\lambda(s)Y_n(s)|^2\mathrm{d}s\bigg]\\
\le c\mathbb{E}\bigg[\int_{n\wedge\tau}^{m\wedge\tau}e^{\rho_s}\bigg(\big(|Y_n(s)|^2+|g(s,0,0)|^2\big)\mathrm{d}s+\sum_i\|Z_n^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg]\\
=c\mathbb{E}\bigg[\int_{n\wedge\tau}^{m\wedge\tau}e^{\rho_s-2\lambda_s}\bigg(\big(|\zeta(s)|^2+|g(s,0,0)|^2\big)\mathrm{d}s+\sum_i\|\eta^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg].\end{aligned} \tag{A.8}$$

for some constant $c\ge 0$.

In the above expression, we used that $\mathbb{E}\big[\int_A\|Z(t)\|^2_{M_t}\mathrm{d}t\big]\le\mathbb{E}\big[\sum_i\int_A\|Z^i(s)\|^2\mathrm{d}\langle M^i\rangle_t\big]$ for any $A\subset\mathcal{B}(\Omega\times\mathbb{R}^+)$. As $g(s,0,0)\in L^2_{\rho(\cdot)}(0,\tau)$, we can use dominated convergence theorem to see that

$$\lim_{n,m\to\infty}\mathbb{E}\bigg[\int_{n\wedge\tau}^{m\wedge\tau}e^{\rho_s}|g(s,0,0)|^2\mathrm{d}s\bigg]=0. \tag{A.9}$$

Set $\theta(t):=\rho(t)-2\lambda(t)>\varepsilon$. By the Lemma A.2,

$$\mathbb{E}\bigg[\int_0^\tau e^{\theta_s}\bigg(|\zeta(s)|^2\mathrm{d}s+\theta(s)\sum_i\|\eta^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg]=\mathbb{E}\big[e^{\rho_\tau/2}|\xi|^2\big]-\big|\mathbb{E}\big[e^{\lambda_\tau}\xi\big]\big|^2<\infty. \tag{A.10}$$

Therefore, dominated convergence theorem ensures that

$$\lim_{n,m\to\infty}\mathbb{E}\bigg[\int_{n\wedge\tau}^{m\wedge\tau}e^{\theta_s}\bigg(|\zeta(s)|^2\mathrm{d}s+\sum_i\|\eta^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg]=0 \tag{A.11}$$

From (A.8), (A.9), (A.11), $\{(Y_n,Z_n)\}$ is a Caushy sequence in $L^2_{\rho(\cdot)}(0,\tau)\times L^2_{\rho(\cdot)}(0,\tau,M)$ and it has a limit (Y,Z). The pair (Y,Z) satisfies the desired equation. Now, it only remains to prove $Y\in U^2_{\rho(\cdot)}(0,\tau)$. But this is seen as usual by BDG-inequality (see *e.g.* [19], Prop. 4.3 or [42], Lem. 5.5).

Uniqueness. Let (Y,Z) and (Y',Z') be two solutions of (3.1). Define $\Delta Y:=Y-Y',\Delta Z:=Z-Z'$. In the similar way as the existence argument , we first apply Itô's formula to $e^{\rho_{t\wedge\tau}}|\Delta Y(t\wedge\tau)|^2$ and then use

BDG-inequality in order that martingale terms vanish. Then, we obtain for any $\kappa \geq 0$,

$$\mathbb{E}\bigg[e^{\rho_{t\wedge\tau}}|\Delta Y(t\wedge\tau)|^2 + \int_{t\wedge\tau}^{\tau}\bigg(e^{\rho_s}(\rho(s)-\gamma(s)-b^2(s)\kappa)|\Delta Y(s)|^2\mathrm{d}s + \sum_i e^{\rho_s}\cdot\frac{\kappa}{1+\kappa}\|\Delta Z^i(s)\|^2\mathrm{d}\langle M^i\rangle_s\bigg)\bigg] \leq 0.$$

which implies $(\Delta Y, \Delta Z) = (0,0)$ in $L^2_{\rho(\cdot)}(0,\tau)\times L^2_{\rho(\cdot)}(0,\tau,M)$. □

Proof of Proposition 4.2 First, we note that $\mathcal{F}_\infty = \widetilde{\mathcal{F}}_\infty$. In fact, since $\{\phi^{-1}(n)\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of $\mathbb{F}-$stopping times and $\lim_{n\to\infty}\phi^{-1}(n)=\infty$, we have

$$\mathcal{F}_\infty = \sigma\bigg(\bigcup_n \mathcal{F}_{\phi^{-1}(n)-}\bigg) \subset \sigma\bigg(\bigcup_{t\geq 0}\mathcal{F}_{\phi^{-1}(t)-}\bigg) \subset \sigma\bigg(\bigcup_{t\geq 0}\mathcal{F}_{\phi^{-1}(t)}\bigg) = \widetilde{\mathcal{F}}_\infty \subset \mathcal{F}_\infty.$$

Let $\{\widetilde{\mathbb{Q}}_t\}_{t\geq 0}$ be a family of consistent probability measures with respect to $\widetilde{\mathbb{F}}$. For any $\widetilde{\mathbb{F}}-$stopping time $\widetilde{\tau}$, we define a measure on $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}$ in a following manner. For any $A\in\widetilde{\mathcal{F}}_{\widetilde{\tau}}$, we construct a sequence $\{\widetilde{\mathbb{Q}}_n(\{\widetilde{\tau}\leq n\}\cap A)\}_{n\in\mathbb{N}}$. Since the measures $\widetilde{\mathbb{Q}}_n$ are consistent, we get for any $m>n$,

$$\widetilde{\mathbb{Q}}_m(\{\widetilde{\tau}\leq m\}\cap A) \geq \widetilde{\mathbb{Q}}_m(\{\widetilde{\tau}\leq n\}\cap A) = \widetilde{\mathbb{Q}}_n(\{\widetilde{\tau}\leq n\}\cap A).$$

And $\widetilde{\mathbb{Q}}_n(\{\widetilde{\tau}\leq n\}\cap A)\leq 1$ for any $n\in\mathbb{N}$. Thus, $\{\widetilde{\mathbb{Q}}_n(\{\widetilde{\tau}\leq n\}\cap A)\}_{n\in\mathbb{N}}$ is a bounded, non-decreasing sequence and so it has a unique limit. We define $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(A) := \lim_{n\to\infty}\widetilde{\mathbb{Q}}_n(\{\widetilde{\tau}\leq n\}\cap A)$. It is easy to check that $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}$ is a measure on $\widetilde{\mathcal{F}}_{\widetilde{\tau}}$.

Particularly, we have $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\Omega) = \lim_{n\to\infty}\widetilde{\mathbb{Q}}_n(\widetilde{\tau}\leq n) = 1$. This is seen as follows. Suppose that $\lim_{n\to\infty}\widetilde{\mathbb{Q}}_n(\widetilde{\tau}\leq n) < 1$. Then, there exists $N\in\mathbb{N}$ such that for all $n>N$, $\widetilde{\mathbb{Q}}_n(\widetilde{\tau}\leq n)<1$. Observing that

$$\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\widetilde{\tau}\leq n) = \lim_{m\to\infty}\widetilde{\mathbb{Q}}_m(\widetilde{\tau}\leq n) = \lim_{m\to\infty}\widetilde{\mathbb{Q}}_n(\widetilde{\tau}\leq n) = \widetilde{\mathbb{Q}}_n(\widetilde{\tau}\leq n) < 1,$$

we get $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\widetilde{\tau}>n)>0$ for all $n>N$. Let a_n be a number such that $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\widetilde{\tau}>a_n)\leq 2^{-n}$. Since $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\widetilde{\tau}>n)>0$, for all $n>N$, we deduce that $a_n\to\infty$ as $n\to\infty$. And $\sum_{n=1}^\infty\widetilde{\mathbb{Q}}_{\widetilde{\tau}}(\widetilde{\tau}>a_n) < \sum_{n=1}^\infty 2^{-n}<\infty$. By Borel-Kantelli lemma, it follows that $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}\big(\limsup_{n\to\infty}\{\widetilde{\tau}>a_n\}\big)=0$. This is a contradiction.

So, $\widetilde{\mathbb{Q}}_{\widetilde{\tau}}$ is a probability measure on $\widetilde{\mathcal{F}}_{\widetilde{\tau}}$. Consequently, we can construct a family of probability measures $\{\mathbb{Q}_t\}_{t\geq 0}$ such that $\mathbb{Q}_t := \widetilde{\mathbb{Q}}_{\phi(t)}$ is a measure on $\mathcal{F}_t\subset\widetilde{\mathcal{F}}_{\phi(t)}$. Let us see that the measures $\mathbb{Q}_t$ are consistent. For any $0\leq t<s$ and $A\in\mathcal{F}_t$, we have

$$\mathbb{Q}_s(A) = \widetilde{\mathbb{Q}}_{\phi(s)}(A) = \lim_{n\to\infty}\widetilde{\mathbb{Q}}_n(\{\phi(s)\leq n\}\cap A) \leq \lim_{n\to\infty}\widetilde{\mathbb{Q}}_n(\{\phi(t)\leq n\}\cap A) = \widetilde{\mathbb{Q}}_{\phi(t)}(A) = \mathbb{Q}_t(A).$$

If we suppose that $\mathbb{Q}_s(A)<\mathbb{Q}_t(A)$, then we get $\widetilde{\mathbb{Q}}_n(\phi(s)>n)\geq\widetilde{\mathbb{Q}}_n(\{\phi(t)\leq n<\phi(s)\}\cap A)>0$ except for some finite number of n, which is a contradiction. This implies that $\mathbb{Q}_s(A)=\mathbb{Q}_t(A)$. By the assumption, there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty$ such that $\mathbb{Q}|_{\mathcal{F}_t}=\mathbb{Q}_t|_{\mathcal{F}_t}$. From the construction, it is obvious that $\mathbb{Q}|_{\widetilde{\mathcal{F}}_t}=\widetilde{\mathbb{Q}}_t|_{\widetilde{\mathcal{F}}_t}$ □

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