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ESTIMATION OF THE MULTIFRACTIONAL FUNCTION AND THE STABILITY INDEX OF LINEAR MULTIFRACTIONAL STABLE PROCESSES

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Abstract. In this paper we are interested in multifractional stable processes where the self-similarity index H becomes time-dependent, while the stability index α remains constant. Using β- negative power variations ($-1/2 < \beta < 0$), we propose estimators for the value at a fixed time of the multifractional function H which satisfies an η-Hölder condition and for α in two cases: multifractional Brownian motion ($\alpha = 2$) and linear multifractional stable motion ($0 < \alpha < 2$). We get the consistency of our estimates for the underlying processes together with the rate of convergence.

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1. Introduction

Multifractional processes have been introduced in order to overcome limitations for some applications of the fractional Brownian motion, due to the constancy in time of its self-similarity index H. In these processes, the path regularity can now vary with the time variable t. The well-known example is mutilfractional Brownian motion which was introduced by Benassi *et al.* in [10] and independently by Peltier and Lévy Véhel [24], where the self-similarity index H of fractional Brownian motion is replaced by a multifractional function $H(t)$, allowing the Hurst index to change in a prescribed manner. This flexible stochastic model allows to separate the properties of local regularity and of long range dependence and to produce sample paths that are both highly correlated and irregular. In the last twenty years, many multifractional processes have been introduced and investigated, see *e.g.*, [5, 6, 9, 12, 13, 16–18, 21, 23, 25–28].

Therefore, the statistical estimation of the multifractional function H at a variable time-value t for multifractional processes, has interested many authors since about two decades. In the statistical literature on this topic, the value of $H(\cdot)$ at a fixed time t_0, is built via [1, 2, 8, 9, 12, 13, 20, 21]. One can mention the work of Peltier and Lévy Véhel (see [24]) for the estimation of the multifractional function of a multifractional Brownian motion, based on the average variation of the sampled process. In the case of multifractional Brownian motion, strongly consistent estimators of $H(t_0)$ has been presented in [9], using generalized quadratic variations of this process. For a more general Gaussian setting than that of the latter process, the increment ratio method is used to get the estimation of $H(t_0)$, see *e.g.*, [6]. Recently, the corresponding estimation problem of the stability function and the multifractional function for a class of multistable processes was considered in the Le Guével's paper, see

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[21], based on some conditions that involve the consistency of the estimators. For linear multifractional stable motions, in [2], the authors presented strongly consistent estimators of the multifractional function $H(\cdot)$ and the stability index α using wavelet coefficients when $\alpha \in (1,2)$ and $H(\cdot)$ is a Hölder function smooth enough, with values in a compact subinterval $[\underline{H}, \overline{H}]$ of $(1/\alpha, 1)$. One can refer to [3], in the setting of the symmetric α-stable non-anticipative moving average linear multifractional stable motion, for an almost surely and $L^p(\Omega), p \in (0,4]$, consistent estimator of the multifractional function $H(\cdot)$ when $\alpha \in (1,2)$.

The aim of this work is to construct consistent estimators for the value at an arbitrary fixed time t_0 of the multifractional function $H(\cdot)$ which satisfies an η-Hölder condition and for the stability index α, using β-negative power variations ($-1/2 < \beta < 0$) for multifractional Brownian motions ($\alpha = 2$) and linear multifractional stable motions ($0 < \alpha < 2$). This framework has been introduced recently in a paper by Dang and Istas (see [15]) to estimate the Hurst and the stability indices of a H-self-similar stable process, in the case H and α constant, based on the fact that β-negative power variations have expectations and covariances for $-1/2 < \beta < 0$. The authors showed that using these variations, one can obtain the estimate of H without a priori knowledge on α and vice versa, the estimator of α can be ascertained without assumptions on H. This type of variations have been used in [7, 22] to obtain the estimators of parameters for linear fractional stable motions. When $H(\cdot)$ becomes a function, moreover with only Hölder regularity, although there are some results on stable random processes presented in [15] that can be applied to this context, the methods of [15] do not go through. We then need some new techniques inspired in part by [25] and the paper of Falconer and Lévy Véhel (see [17]) to deal with the problem. In order to estimate the value of $H(\cdot)$ at t_0, using this new framework, requires no a priori knowledge on α, but only a weak a priori condition on the supremum of the function $H(t)$. In other words, these variations provide estimators of $H(\cdot)$ at a fixed time t_0 and α separately, without the assumption on the existence moment of the underlying processes. We also get the consistent estimator for the stability index α for the underlying processes. Moreover, the rate of convergence of our estimates is given.

This paper is organized as follows: in the next section, we present the setting and main results needed to construct the estimators for $H(\cdot)$ at a fixed time t_0 and for α in the two aforementioned cases: multifractional Brownian motion ($\alpha = 2$) and linear multifractional stable motion ($0 < \alpha < 2$). In Section 3, we gather all the proofs of the main results presented in Section 2. These proofs make use of several lemmas which are introduced and proved in Section 3.1.

2. Settings and main results

Definition 2.1. *Linear multifractional stable motion and multifractional Brownian motion.*
Let $0 < \alpha \leq 2$ and $H : U \to (0,1)$ be a function on a closed interval $U \subset \mathbb{R}$, H satisfies an η-Hölder condition: $|H(v) - H(u)| \leq C|v-u|^\eta, \eta \in (0,1]$ for all $u, v \in U$. Let

$$X(t) = \int_{\mathbb{R}} (|t-s|^{H(t)-1/\alpha} - |s|^{H(t)-1/\alpha}) M_\alpha(\mathrm{d}s) \tag{2.1}$$

where $t \in U$ and M_α is a symmetric α-stable random measure on $\mathbb{R}$ whose control measure ds is the Lebesgue measure.

When $0 < \alpha < 2$, the process $X(t)$ is called *a linear multifractional stable motion* (see, *e.g.*, [17]). The Gaussian case ($\alpha = 2$) occurs when $M(\mathrm{d}u)$ is the standard Gaussian measure $W(\mathrm{d}u)$ on $\mathbb{R}$ and the process is then called *a multifractional Brownian motion* (see, *e.g.*, [27]).

Let $X(t)$ be the process defined by (2.1) where the exponent η of the Hölder condition of $H(\cdot)$ satisfies $0 < \sup_{t \in U} H(t) < \eta \leq 1$ and t_0 be a fixed point in $\mathring{U}$, the interior U. We now construct estimators of $H(t_0)$ and α.

Let $L \geq 1, K \geq 1$ be fixed integers, $a = (a_0, \ldots, a_K)$ be a finite sequence with exactly $(L+1)$ vanishing first moments, that is for all $q \in \{0, \ldots, L\}$, one has

$$\sum_{k=0}^{K} k^q a_k = 0, \sum_{k=0}^{K} k^{L+1} a_k \neq 0 \tag{2.2}$$

with the convention that $0^0 = 1$. The aim is to estimate $H(t_0)$ and α from a discrete sample of X based on the filter a. The choice of vanishing moments is to be compared with the analogue in wavelet theory: it allows to unbiase the estimator up to exponent L. But finding the "best" sequence a is an open problem (see *e.g.* [4, 11, 14]). Classical sequences a are given by the discrete derivatives: $a = (1, -1)$, when $L = 0$, $a = (1, -2, 1)$ when $L = 1$, and, for $L > 1$, we can choose $K = L + 1$ and

$$a_k = (-1)^{L+1-k} \frac{(L+1)!}{k!(L+1-k)!}. \tag{2.3}$$

Let γ be fixed such that

$$0 < \sup_{t \in U} H(t) < \gamma < \eta \leq 1. \tag{2.4}$$

Define a set $\nu_{\gamma,n}(t_0)$ and its cardinal by

$$\nu_{\gamma,n}(t_0) = \left\{ k \in \mathbb{Z} : \forall p = 0, \ldots, K, |\frac{k+p}{n} - t_0| \leq \frac{1}{n^{\gamma/\eta}} \right\}, \tag{2.5}$$

$$\upsilon_{\gamma,n}(t_0) = \#\nu_{\gamma,n}(t_0). \tag{2.6}$$

We can choose $n \in \mathbb{N}$ large enough so that

$$\left\{ \frac{k+p}{n}, k \in \nu_{\gamma,n}(t_0), p = 0, \ldots, K \right\} \subset U.$$

Then for $k \in \nu_{\gamma,n}(t_0)$, $\frac{k+p}{n} \in U$, the discrete variations $\triangle_{k,n} X$ with respect to the sequence a are defined by

$$\triangle_{k,n} X = \sum_{p=0}^{K} a_p X \left(\frac{k+p}{n} \right). \tag{2.7}$$

Note that $\upsilon_{\gamma,n}(t_0) = [2n^{1-\gamma/\eta} - K]$ or $[2n^{1-\gamma/\eta} - K] + 1$ depending on the parity of $[2n^{1-\gamma/\eta} - K]$, where $[x]$ denotes the integer part of a real number x.

Let $\beta \in (-1/2, 0)$ be fixed and

$$V_{t_0,n}(\beta) = \frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{k \in \nu_{\gamma,n}(t_0)} |\triangle_{k,n} X|^{\beta} \tag{2.8}$$

$$W_{t_0,n}(\beta) = n^{\beta H(t_0)} V_{t_0,n}(\beta). \tag{2.9}$$

Let $\widehat{H}_n(t_0)$ be defined by

$$\widehat{H}_n(t_0) = \frac{1}{\beta} \log_2 \frac{V_{t_0,n/2}(\beta)}{V_{t_0,n}(\beta)}. \tag{2.10}$$

We will prove later that $\widehat{H}_n(t_0)$ is a consistent estimator of $H(t_0)$ of the multifractional stable process defined by (2.1) at a fixed time t_0 where $t_0 \in \mathring{U}$.

We now present a consistent estimator of α.

We first define auxiliary functions $\psi_{u,v}, h_{u,v}, \varphi_{u,v}$, where $u > v > 0$.

Let $\psi_{u,v}$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\psi_{u,v}(x,y) = -v \ln x + u \ln y + C(u,v), \tag{2.11}$$

where

$$\begin{aligned} C(u,v) =& \frac{u-v}{2} \ln \pi + u \ln \left(\Gamma(1+\frac{v}{2})\right) + v \ln \left(\Gamma(\frac{1-u}{2})\right) \\ & - v \ln \left(\Gamma(1+\frac{u}{2})\right) - u \ln \left(\Gamma(\frac{1-v}{2})\right). \end{aligned}$$

Let $h_{u,v} : (0, +\infty) \to (-\infty, 0)$ be the function defined by

$$h_{u,v}(x) = u \ln \left(\Gamma(1+\frac{v}{x})\right) - v \ln \left(\Gamma(1+\frac{u}{x})\right). \tag{2.12}$$

Let $\varphi_{u,v} : \mathbb{R} \to [0, +\infty)$ be the function defined by

$$\varphi_{u,v}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ h_{u,v}^{-1}(x) & \text{if } x < 0 \end{cases} \tag{2.13}$$

where $h_{u,v}$ has been defined in (2.12). One refers to [15] for the elementary results on functions $\psi_{u,v}, h_{u,v}, \varphi_{u,v}$. Let β_1, β_2 be in $\mathbb{R}$ such that $-1/2 < \beta_1 < \beta_2 < 0$. The estimator of α is defined by

$$\widehat{\alpha}_n = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_{t_0,n}(\beta_1), V_{t_0,n}(\beta_2))\right), \tag{2.14}$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined as in (2.11) and (2.13), respectively.

The main results in this paper are presented in Theorems 2.2, 2.3 and 2.4. A consistent estimator of the value of the multifractional function $H(\cdot)$ at a fixed time $t_0 \in \mathring{U}$ is given in Theorem 2.2 for the multifractional Brownian motion and in Theorem 2.3 for the linear multifractional stable motion. Theorem 2.4 is devoted to giving a consistent estimator of the stability index α for those two cases.

2.1. Estimation of the multifractional function *H*

In this section, we state theorems which precise the statement that (2.10) gives a consistent estimator in the sense that it gives the rate of convergence, for the value of the multifractional function $H(\cdot)$ at a fixed time $t_0 \in \mathring{U}$ for linear multifractional stable motion ($0 < \alpha < 2$) and multifractional Brownian motion ($\alpha = 2$). We start with some definitions.

For $n \in \mathbb{N}, k \in \mathbb{Z}, s \in \mathbb{R}$, let

$$f_{k,n}(s) = \sum_{p=0}^{K} a_p \left|\frac{k+p}{n} - s\right|^{H(\frac{k+p}{n})-1/\alpha}, \tag{2.15}$$

$$g_{k,n}(s) = \sum_{p=0}^{K} a_p \left|\frac{k+p}{n} - s\right|^{H(t_0)-1/\alpha}. \tag{2.16}$$

Since $\sum_{k=0}^{K} a_k = 0$ it follows that

$$\triangle_{k,n} X = \int_{\mathbb{R}} f_{k,n}(s) M_\alpha(\mathrm{d}s). \tag{2.17}$$

Let $\beta \in (-1/2, 0)$, and $t_0 \in \mathring{U}$. We set

$$M_{t_0} = \left(\int_{\mathbb{R}} \left| \sum_{p=0}^{K} a_p |p-s|^{H(t_0)-1/\alpha} \right|^\alpha \mathrm{d}s \right)^{1/\alpha}. \tag{2.18}$$

$$M_{t_0,\beta} = \frac{M_{t_0}^\beta C_\beta}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+\beta}} \mathrm{d}y, \tag{2.19}$$

where C_β is defined by

$$C_\beta = \frac{2^{\beta+1}\Gamma(\frac{\beta+1}{2})}{\Gamma(-\frac{\beta}{2})}. \tag{2.20}$$

2.1.1. Multifractional Brownian motion ($\alpha = 2$)

In this case which corresponds to $\alpha = 2$ in the multifractional stable process defined by (2.1), for a fixed $t_0 \in \mathring{U}$, we get a consistent estimator for $H(t_0)$ with rate of convergence d_n defined by

$$d_n = \max(n^{H(t_0)-\gamma}, n^{\frac{\gamma/\eta - 1}{2}}) \tag{2.21}$$

(we recall that $H(t_0) < \gamma < 1$ and $\gamma/\eta < 1$).

The precise statement is

Theorem 2.2. *Let X be a multifractional Brownian motion defined by (2.1) with $\alpha = 2$ and $M_\alpha(\mathrm{d}s)$ is the standard Gaussian measure on $\mathbb{R}$. For a fixed $t_0 \in \mathring{U}$, one has*

1. $$W_{t_0,n}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_n) \tag{2.22}$$

   where $W_{t_0,n}, M_{t_0,\beta}, d_n$ are defined by (2.9), (2.19), (2.21), respectively and $O_{\mathbb{P}}$ is defined by:
   •$X_n = O_{\mathbb{P}}(1)$ iff for all $\epsilon > 0$, there exists $M > 0$ such that $\sup_n \mathbb{P}(|X_n| > M) < \epsilon$.
   •$Y_n = O_{\mathbb{P}}(a_n)$ means $Y_n = a_n X_n$ with $X_n = O_{\mathbb{P}}(1)$.
2. $$\widehat{H}_n(t_0) - H(t_0) = O_{\mathbb{P}}(d_n) \tag{2.23}$$

   where $\widehat{H}_n(t_0)$ is defined by (2.10).

The proof of this theorem is given in Section 3.2.

2.1.2. Linear multifractional stable motion $(0<\alpha<2)$

In this case again for a given point $t_0 \in \mathring{U}$, we define

$$d_n = \begin{cases} \max\left(n^{\frac{\gamma/\eta-1}{2}}, n^{\frac{\alpha(H(t_0)-\gamma)}{4}}\right) & \text{if } \frac{\alpha H(t_0)-\alpha(L+1)}{2} < -1 \\ \max\left(n^{\frac{\alpha(H(t_0)-\gamma)}{4}}, n^{\frac{(1-\gamma/\eta)(\alpha H(t_0)-\alpha(L+1))}{4}}\right) & \text{if } -1 < \frac{\alpha H(t_0)-\alpha(L+1)}{2} < 0 \\ \max\left(n^{\frac{\gamma/\eta-1}{2}} \ln n, n^{\frac{\alpha(H(t_0)-\gamma)}{4}}\right) & \text{if } \frac{\alpha H(t_0)-\alpha(L+1)}{2} = -1 \end{cases} \tag{2.24}$$

The precise theorem in this case is

Theorem 2.3. *Let X be a linear multifractional stable motion defined by (2.1) with $0<\alpha<2$. For a fixed $t_0 \in \mathring{U}$, we have*

1.

$$W_{t_0,n}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_n) \tag{2.25}$$

where $W_{t_0,n}, M_{t_0,\beta}, d_n$ are defined by (2.9), (2.19) and (2.24), respectively.

2.

$$\widehat{H}_n(t_0) - H(t_0) = O_{\mathbb{P}}(d_n) \tag{2.26}$$

where $\widehat{H}_n(t_0)$ is defined by (2.10).

The proof of this theorem will be given in Subsection 3.3.

2.2. Estimation of the stable index α

For the multifractional stable process defined by (2.1), we consider the stable index $\alpha \in (0,2]$. Recall that $\widehat{\alpha}_n$ was defined in (2.14) by:

$$\widehat{\alpha}_n = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_{t_0,n}(\beta_1), V_{t_0,n}(\beta_2))\right).$$

Theorem 2.4. *Let X be a multifractional stable process defined by (2.1). If $t_0 \in \mathring{U}$, then $\widehat{\alpha}_n$ is a consistent estimator of α, moreover*

$$\widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(d_n),$$

where $\widehat{\alpha}_n$ is defined by (2.14), d_n is defined by (2.21) in the case of multifractional Brownian motion $(\alpha=2)$ and d_n is defined by (2.24) in the case of linear multifractional stable motion $(0<\alpha<2)$.

See Section 3.4 for the proof of this theorem.

Remark 2.5. We remark that in this framework, one needs an a priori upper bound γ for the function $H(\cdot)$ and the rate of convergence depending on the value of γ. Therefore, the obtained results are not as good as the ones presented in [15] in the case of fractional Brownian motion and well-balanced linear fractional stable motion where H and α are constants.

3. Proofs

3.1. Auxiliary results

We present here some results related to discrete variations of linear multifractional stable motion and multifractional Brownian motion. These results will be used in the proof of the main results.

Lemma 3.1. *Let X be a multifractional stable process defined by (2.1). For $0 < \alpha \leq 2$, $t_0 \in \mathring{U}$ and $k \in \nu_{\gamma,n}(t_0)$, let*

$$\sigma_{k,n}(t_0) = ||\frac{\triangle_{k,n}X}{n^{-H(t_0)}}||_\alpha, \tag{3.1}$$

with the notation $||X||_\alpha = (\int_{\mathbb{R}} |f(s)|^\alpha \mu(\mathrm{d}s))^{1/\alpha}$, where $X = \int_{\mathbb{R}} f(s) M_\alpha(\mathrm{d}s), f \in L^\alpha(\mathbb{R}, \mu)$. Then

$$|\sigma_{k,n}(t_0) - M_{t_0}| = O\left(n^{-(\alpha\wedge 1)(\gamma - H(t_0))}\right), \tag{3.2}$$

where M_{t_0} is defined by (2.18) and $x \wedge y = min(x, y)$.

Remark 3.2. From Lemma 3.1 and since $H(t_0) < \gamma$, it follows that

$$\lim_{n\to+\infty} \sigma_{k,n}(t_0) = M_{t_0}.$$

Therefore, there exist $n_0 \in \mathbb{N}$ and constants $0 < M_1 < M_{t_0} < M_2$ such that for all $n \geq n_0$ and $k \in v_{\gamma,n}(t_0)$, then $M_1 < \sigma_{k,n}(t_0) < M_2$.

Proof. We denote by C a running constant which may change from an occurrence to another occurrence. We have

$$\sigma_{k,n}(t_0) = \left(\int_{\mathbb{R}} \left|\sum_{p=0}^{K} a_p n^{H(t_0)} \left(|\frac{k+p}{n} - s|^{H(\frac{k+p}{n})-1/\alpha} - |s|^{H(\frac{k+p}{n})-1/\alpha}\right)\right|^\alpha \mathrm{d}s\right)^{1/\alpha}.$$

Using the change of variable $s = s_1/n$, and $s_2 = s_1 - k$, combining with the fact that $\sum_{p=0}^{K} a_p = 0$, one gets

$$\begin{aligned} ||g_{k,n}(\cdot)||_\alpha^\alpha &= n^{-\alpha H(t_0)} \int_{\mathbb{R}} |\sum_{p=0}^{K} a_p |k+p-s_1|^{H(t_0)-1/\alpha}|^\alpha \mathrm{d}s_1 \\ &= n^{-\alpha H(t_0)} \int_{\mathbb{R}} |\sum_{p=0}^{K} a_p |p-s_2|^{H(t_0)-1/\alpha}|^\alpha \mathrm{d}s_2 \\ &= M_{t_0}^\alpha n^{-\alpha H(t_0)} \end{aligned} \tag{3.3}$$

where $g_{k,n}(\cdot)$ is defined by (2.16). Then following Lemma 4.7.2 in [25], one has

$$
\begin{aligned}
|\sigma^\alpha_{k,n}(t_0) - M^\alpha_{t_0}| &= \left| \int_{\mathbb{R}} n^{\alpha H(t_0)} \left(|f_{k,n}(s)|^\alpha - |g_{k,n}(s)|^\alpha \right) \mathrm{d}s \right| \\
&\le \int_{\mathbb{R}} \left| |f^1_{k,n}(s)|^\alpha - |g^1_{k,n}(s)|^\alpha \right| \mathrm{d}s \\
&\le C_*(\alpha) \left(\int_{\mathbb{R}} \left| f^1_{k,n}(s) - g^1_{k,n}(s) \right|^\alpha \mathrm{d}s \right)^{1 \wedge 1/\alpha}, \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
C_*(\alpha) &= \begin{cases} 1 & \text{if } 0 < \alpha \le 1, \\ 2^{1/\alpha} \alpha (||f^1_{k,n}(\cdot)||^{\alpha-1}_\alpha + ||g^1_{k,n}(\cdot)||^{\alpha-1}_\alpha) & \text{if } 1 < \alpha \le 2, \end{cases} \\
f^1_{k,n}(s) &= \sum_{p=0}^{K} a_p n^{H(t_0)} \left(|\frac{k+p}{n} - s|^{H(\frac{k+p}{n}) - 1/\alpha} - |s|^{H(\frac{k+p}{n}) - 1/\alpha} \right) = n^{H(t_0)} f_{k,n}(s), \\
g^1_{k,n}(s) &= \sum_{p=0}^{K} a_p n^{H(t_0)} \left(|\frac{k+p}{n} - s|^{H(t_0) - 1/\alpha} - |s|^{H(t_0) - 1/\alpha} \right) = n^{H(t_0)} g_{k,n}(s).
\end{aligned}
$$

We now turn to prove that $C_*(\alpha)$ is bounded independently of α.

From (3.1) and (3.3), one has

$$
||f^1_{k,n}(\cdot)||_\alpha = \sigma_{k,n}(t_0), \ ||g^1_{k,n}(\cdot)||_\alpha = M_{t_0}. \qquad (3.5)
$$

Applying Lemma 2.7.13 in [25], one obtains

$$
\begin{aligned}
||f^1_{k,n}(\cdot)||^\alpha_\alpha &= n^{\alpha H(t_0)} \int_{\mathbb{R}} |f_{k,n}(s)|^\alpha \mathrm{d}s \\
&\le 2^{0 \wedge (\alpha-1)} n^{\alpha H(t_0)} \int_{\mathbb{R}} \left(|f_{k,n}(s) - g_{k,n}(s)|^\alpha + |g_{k,n}(s)|^\alpha \right) \mathrm{d}s \qquad (3.6)
\end{aligned}
$$

We come now to estimate $\int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^\alpha \mathrm{d}s$. Applying again K times Lemma 2.7.13 in [25], for $0 < \alpha \le 2$, one gets

$$
\begin{aligned}
&|f_{k,n}(s) - g_{k,n}(s)|^\alpha \\
&\le C \sum_{p=0}^{K} \left| a_p \left(|\frac{k+p}{n} - s|^{H(\frac{k+p}{n}) - 1/\alpha} - |s|^{H(\frac{k+p}{n}) - 1/\alpha} - |\frac{k+p}{n} - s|^{H(t_0) - 1/\alpha} - |s|^{H(t_0) - 1/\alpha} \right) \right|^\alpha \\
&= C \sum_{p=0}^{K} \left| a_p \right|^\alpha \left| h\left(\frac{k+p}{n}, \frac{k+p}{n}, s \right) - h\left(\frac{k+p}{n}, t_0, s \right) \right|^\alpha \qquad (3.7)
\end{aligned}
$$

where

$$h(t,v,s) = |t-s|^{H(v)-1/\alpha} - |s|^{H(v)-1/\alpha}.$$

As in the proof of Theorem 7.4 in [17], let h_-, h_+ be such that $0 < h_- < H(t) < h_+ < 1$ for all $t \in U$. With $s \neq 0, s \neq \frac{k+p}{n}$, applying the mean value theorem, since $k \in \nu_{\gamma,n}(t_0)$, one obtains

$$\begin{aligned}
&\left| h\left(\frac{k+p}{n}, \frac{k+p}{n}, s\right) - h\left(\frac{k+p}{n}, t_0, s\right)\right| \\
&= \left| H\left(\frac{k+p}{n}\right) - H(t_0)\right| \left| |\frac{k+p}{n} - s|^{H(\cdot)-1/\alpha} \ln|\frac{k+p}{n} - s| - |s|^{H(\cdot)-1/\alpha}\ln|s|\right| \\
&\le \frac{C}{n^\gamma}\left| |\frac{k+p}{n} - s|^{H(\cdot)-1/\alpha} \ln|\frac{k+p}{n} - s| - |s|^{H(\cdot)-1/\alpha}\ln|s|\right| \\
&\le \frac{K_1(\frac{k+p}{n}, s)}{n^\gamma},
\end{aligned}$$

where $H(\cdot)$ is on a line segment connecting $H(\frac{k+p}{n})$ and $H(t_0)$ and

$$K_1(t,s) = \begin{cases} c_1 \max\{1, |t-s|^{h_- - 1/\alpha} + |s|^{h_- - 1/\alpha}\} & \text{if } |s| \le 1 + 2\max_{t\in U}|t|, \\ c_2 |s|^{h_+ - 1/\alpha - 1} & \text{if } |s| > 1 + 2\max_{t\in U}|t| \end{cases}$$

where c_1 and c_2 are appropriate constants, see [17]. Then $\int_{\mathbb{R}} K_1(t,s)^\alpha \mathrm{d}s < +\infty$ and uniformly bounded for $t \in U$. Combining with (3.7), it follows that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^\alpha \mathrm{d}s \le \frac{C}{n^{\alpha\gamma}}. \tag{3.8}$$

From (3.3), (3.6), (3.8) and since $H(t_0) < \gamma$, there exists a running constant $C > 0$ such that

$$||f^1_{k,n}(\cdot)||_\alpha \le C. \tag{3.9}$$

From (3.5) and (3.9), it follows that C_* is bounded as a function of α.

We now consider $\int_{\mathbb{R}} \left| f^1_{k,n}(s) - g^1_{k,n}(s)\right|^\alpha \mathrm{d}s$. From (3.8), one gets

$$\begin{aligned}
\int_{\mathbb{R}} \left|f^1_{k,n}(s) - g^1_{k,n}(s)\right|^\alpha \mathrm{d}s &= n^{\alpha H(t_0)} \int_{\mathbb{R}} \left|f_{k,n}(s) - g_{k,n}(s)\right|^\alpha \mathrm{d}s \\
&\le C n^{\alpha(H(t_0)-\gamma)}.
\end{aligned}$$

It follows that

$$|\sigma^\alpha_{k,n}(t_0) - M^\alpha_{t_0}| = O\left(n^{-(\alpha\wedge 1)(\gamma - H(t_0))}\right). \tag{3.10}$$

Thus, there exist $n_0 \in \mathbb{N}$ and constants $0 < M_1 < M_{t_0} < M_2$ such that for all $n \ge n_0$, then $0 < M_1 < \sigma_{k,n}(t_0) < M_2$.

For $0<\alpha\le 2, \alpha\neq 1$, the mean value theorem gives

$$|\sigma^{\alpha}_{k,n}(t_0)-M^{\alpha}_{t_0}|=\alpha|\sigma_{k,n}(t_0)-M_{t_0}|x_0^{\alpha-1}$$

where $x_0\in(M_1,M_2)$. One gets $|\sigma_{k,n}(t_0)-M_{t_0}|=O\left(n^{-(\alpha\wedge 1)(\gamma-H(t_0))}\right)$. □

Lemma 3.3. *Let X be a multifractional stable process defined by (2.1). For $0<\alpha\le 2$ and $k\in\nu_{\gamma,n}(t_0)$,*

$$\left|\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta}-M_{t_0,\beta}\right|=O\left(n^{-(\alpha\wedge 1)(\gamma-H(t_0))}\right)$$

and

$$|\mathbb{E}W_{t_0,n}(\beta)-M_{t_0,\beta}|=O\left(n^{-(\alpha\wedge 1)(\gamma-H(t_0))}\right)$$

where $M_{t_0,\beta}$ is defined by (2.19)

Proof. Since $\frac{\triangle_{k,n}X}{n^{-H(t_0)}}$ is a $S\alpha S$- stable random variable, combining with (2.17) and (3.1), one gets

$$\mathbb{E}e^{iy\frac{\triangle_{k,n}X}{n^{-H(t_0)}}}=e^{-|y|^{\alpha}\sigma_{k,n}(t_0)^{\alpha}}.$$

Following Theorem 4.1 in [15] and using the change of variable $y_1=y\sigma_{k,n}$, one may write

$$\begin{aligned}\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta}&=\frac{C_\beta}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\mathbb{E}e^{iy\frac{\triangle_{k,n}X}{n^{-H(t_0)}}}}{|y|^{1+\beta}}\mathrm{d}y=\frac{C_\beta}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-|y|^{\alpha}\sigma_{k,n}(t_0)^{\alpha}}}{|y|^{1+\beta}}\mathrm{d}y\\&=\frac{\sigma^{\beta}_{k,n}(t_0)C_\beta}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-|y_1|^{\alpha}}}{|y_1|^{1+\beta}}\mathrm{d}y_1\end{aligned}$$

where C_β is defined by (2.20). It follows that

$$\begin{aligned}\left|\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta}-M_{t_0,\beta}\right|&=|\sigma^{\beta}_{k,n}(t_0)-M^{\beta}_{t_0}|\frac{C_\beta}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-|y|^{\alpha}}}{|y|^{1+\beta}}\mathrm{d}y\\&=C|\sigma^{\beta}_{k,n}(t_0)-M^{\beta}_{t_0}|.\end{aligned}$$

Applying the mean value theorem and Remark 3.2, we get

$$|\sigma^{\beta}_{k,n}(t_0)-M^{\beta}_{t_0}|=|\beta||\sigma_{k,n}(t_0)-M_{t_0}|\theta^{\beta-1}$$

where $\theta\in(M_1,M_2)$. Combining with Lemma 3.1, for $n\ge n_0$, it follows that

$$\left|\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta}-M_{t_0,\beta}\right|=O\left(n^{-(\alpha\wedge 1)(\gamma-H(t_0))}\right).$$

One also gets

$$
\begin{aligned}
|\mathbb{E}W_{t_0,n}(\beta) - M_{t_0,\beta}| &= \left| \frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{k\in\nu_{\gamma,n}(t_0)} \mathbb{E}\left(\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta - M_{t_0,\beta}\right)\right| \\
&\leq \frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{k\in\nu_{\gamma,n}(t_0)} \left| \mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta - M_{t_0,\beta}\right|.
\end{aligned}
$$

Since $\upsilon_{\gamma,n} = \#\nu_{\gamma,n}$, one obtains

$$
|\mathbb{E}W_{t_0,n}(\beta) - M_{t_0,\beta}| = O\left(n^{-(\alpha\wedge 1)(\gamma - H(t_0))}\right).
$$

□

Lemma 3.4. *Let X be a linear multifractional stable process defined by (2.1). Then there exists a constant $C>0$ such that for $k,k'\in\nu_{\gamma,n}(t_0)$, one has*

$$
\left| cov\left(\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta, \left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^\beta \right)\right| \leq C.
$$

Proof. Since $-1/2<\beta<0$, from Theorem 4.1 in [15] and Lemma 3.1, for $n\geq n_0$, we can write

$$
\begin{aligned}
\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{2\beta} &= \frac{C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}e^{iy\frac{\triangle_{k,n}X}{n^{-H(t_0)}}}}{|y|^{1+2\beta}}\mathrm{d}y = \frac{C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha\sigma_{k,n}^\alpha}}{|y|^{1+2\beta}}\mathrm{d}y \\
&= \frac{\sigma_{k,n}^{2\beta}C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+2\beta}}\mathrm{d}y \leq \frac{M_1^{2\beta}C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+2\beta}}\mathrm{d}y
\end{aligned}
$$

where $C_{2\beta}$ is defined by (2.20) and M_1 is defined as in Remark 3.2. Applying Cauchy-Schwarz's inequality, one gets

$$
\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta \left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^\beta \leq \left(\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{2\beta} \mathbb{E}\left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^{2\beta}\right)^{1/2} \leq \frac{M_1^{2\beta}C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+2\beta}}\mathrm{d}y.
$$

Moreover

$$
\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta \leq \left(\mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{2\beta}\right)^{1/2} \leq \left(\frac{M_1^{2\beta}C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+\beta}}\mathrm{d}y\right)^{1/2}.
$$

We deduce:

$$
\left| cov\left(\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta, \left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^\beta \right)\right| \leq \mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta \left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^\beta + \mathbb{E}\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^\beta \mathbb{E}\left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^\beta
$$

$$\leq \frac{2M_1^{2\beta}C_{2\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+\beta}} \mathrm{d}y = C.$$

□

Lemma 3.5. *Let X be a a process defined by (2.1) and $t_0 \in \mathring{U}$ be in the interior of U. Then*

1. *if $\alpha = 2$, there exist $n_1, k_0 \in \mathbb{N}$ and a constant $C > 0$ such that for all $n \geq n_1, k, k' \in \nu_{\gamma,n}(t_0)$, and $|k - k'| > k_0$, we have*

$$\left|cov(|\triangle_{k,n}X|^\beta, |\triangle_{k',n}X|^\beta)\right| \leq Cn^{-2\beta H(t_0)} \left(n^{2(H(t_0)-\gamma)} + |k - k'|^{2H(t_0)-2(L+1)}\right). \tag{3.11}$$

2. *if $0 < \alpha < 2$, there exist $n_1, k_0 \in \mathbb{N}$ and a constant $C^* > 0$ such that for all $n \geq n_1, k, k' \in \nu_{\gamma,n}(t_0)$, $|k - k'| > k_0$, we have*

$$\left|cov(|\triangle_{k,n}X|^\beta, |\triangle_{k',n}X|^\beta)\right| \leq C^* n^{-2\beta H(t_0)} \left(n^{\frac{\alpha(H(t_0)-\gamma)}{2}} + |k - k'|^{\frac{\alpha H(t_0)-\alpha(L+1)}{2}}\right). \tag{3.12}$$

Proof. Let

$$I = \int_{\mathbb{R}} |f_{k,n}(s) f_{k',n}(s)|^{\frac{\alpha}{2}} \, \mathrm{d}s,$$

where $f_{k,n}(s)$ is defined by (2.15). Since $0 < \alpha/2 < 1$, following Lemma 2.7.13 in [25], one gets

$$|f_{k,n}(s)|^{\alpha/2} \leq |f_{k,n}(s) - g_{k,n}(s)|^{\alpha/2} + |g_{k,n}(s)|^{\alpha/2},$$
$$|f_{k',n}(s)|^{\alpha/2} \leq |f_{k',n}(s) - g_{k',n}(s)|^{\alpha/2} + |g_{k',n}(s)|^{\alpha/2}$$

where $g_{k,n}(s)$ is defined by (2.16). It follows that

$$\begin{aligned} I &\leq \int_{\mathbb{R}} \left(|f_{k,n}(s) - g_{k,n}(s)|^{\alpha/2} + |g_{k,n}(s)|^{\alpha/2}\right) \left(|f_{k',n}(s) - g_{k',n}(s)|^{\alpha/2} + |g_{k',n}(s)|^{\alpha/2}\right) \mathrm{d}s \\ &:= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.13}$$

where

$$I_1 = \int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^{\alpha/2} |f_{k',n}(s) - g_{k',n}(s)|^{\alpha/2} \mathrm{d}s$$

$$I_2 = \int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^{\alpha/2} |g_{k',n}(s)|^{\alpha/2} \mathrm{d}s$$

$$I_3 = \int_{\mathbb{R}} |f_{k',n}(s) - g_{k',n}(s)|^{\alpha/2} |g_{k,n}(s)|^{\alpha/2} \mathrm{d}s$$

$$I_4 = \int_{\mathbb{R}} |g_{k,n}(s)|^{\alpha/2} |g_{k',n}(s)|^{\alpha/2} \mathrm{d}s.$$

We estimate I_1, I_2, I_3, I_4 separately: For I_4, we have

$$I_4 = \int_{\mathbb{R}} |g_{k,n}(s)|^{\alpha/2} |g_{k',n}(s)|^{\alpha/2} \mathrm{d}s$$
$$= \int_{\mathbb{R}} \left| \sum_{p=0}^{K} a_p \left|\frac{k+p}{n} - s\right|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \left| \sum_{p'=0}^{K} a_{p'} \left|\frac{k'+p'}{n} - s\right|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \mathrm{d}s.$$

Making the change of variable $s = s_1/n$, it induces

$$I_4 = n^{-\alpha H(t_0)} \int_{\mathbb{R}} \left| \sum_{p=0}^{K} a_p |k+p-s_1|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \left| \sum_{p'=0}^{K} a_{p'} |k'+p'-s_1|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \mathrm{d}s_1.$$

Let $s_2 = s_1 - k$, then

$$I_4 = n^{-\alpha H(t_0)} \int_{\mathbb{R}} \left| \sum_{p=0}^{K} a_p |p-s_2|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \left| \sum_{p'=0}^{K} a_{p'} |p'-(s_2+k-k'|^{H(t_0)-1/\alpha} \right|^{\alpha/2} \mathrm{d}s_2.$$

Following Lemma 3.6 in [19], then there exists $k^* > 0$ such that for $|k-k'| \geq k^*$, one has

$$I_4 \leq C n^{-\alpha H(t_0)} |k-k'|^{\frac{\alpha H(t_0) - \alpha(L+1)}{2}}. \tag{3.14}$$

To find a bound for I_2, one can apply Cauchy-Schwarz's inequality and obtains

$$I_2 \leq \left(\int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^{\alpha} \mathrm{d}s \int_{\mathbb{R}} |g_{k',n}(s)|^{\alpha} \mathrm{d}s| \right)^{1/2}.$$

Moreover, from (3.3) and (3.8), one gets

$$\int_{\mathbb{R}} |g_{k',n}(s)|^{\alpha} \mathrm{d}s = M_{t_0}^{\alpha} n^{-\alpha H(t_0)}$$
$$\int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^{\alpha} \mathrm{d}s \leq C n^{-\alpha\gamma}$$

where M_{t_0} is defined by (2.18), C is a constant.

Thus

$$I_2 \leq C n^{-\frac{\alpha(H(t_0)+\gamma)}{2}} \tag{3.15}$$

Similarly, one also gets

$$I_3 \leq C n^{-\frac{\alpha(H(t_0)+\gamma)}{2}}. \tag{3.16}$$

For I_1, applying Cauchy-Schwarz's inequality, then

$$I_1 \leq \left(\int_{\mathbb{R}} |f_{k,n}(s) - g_{k,n}(s)|^\alpha \mathrm{d}s \int_{\mathbb{R}} |f_{k',n}(s) - g_{k',n}(s)|^\alpha \mathrm{d}s| \right)^{1/2} \leq Cn^{-\gamma\alpha}. \tag{3.17}$$

Combining with (3.14), (3.15), (3.16), one gets

$$I \leq C \left(n^{-\alpha\gamma} + n^{-\frac{\alpha(H(t_0)+\gamma)}{2}} + n^{-\alpha H(t_0)} |k-k'|^{\frac{\alpha H(t_0) - \alpha(L+1)}{2}} \right). \tag{3.18}$$

Let

$$\eta_{k,k'} = \left[\frac{\triangle_{k,n}X}{||\triangle_{k,n}X||_\alpha}, \frac{\triangle_{k',n}X}{||\triangle_{k',n}X||_\alpha} \right]_2, \tag{3.19}$$

where

$$\left[\int_{\mathbb{R}} f(s) M_\alpha(\mathrm{d}s), \int_{\mathbb{R}} g(s) M_\alpha(\mathrm{d}s) \right]_2 = \int_{\mathbb{R}} |f(s)g(s)|^{\alpha/2} \mathrm{d}s.$$

One has

$$\eta_{k,k'} = \frac{\int_{\mathbb{R}} |f_{k,n}(s) f_{k',n}(s)|^{\alpha/2} \mathrm{d}s}{\left(\int_{\mathbb{R}} |f_{k,n}(s)|^\alpha \mathrm{d}s \int_{\mathbb{R}} |f_{k',n}(s)|^\alpha \right)^{1/2} \mathrm{d}s}.$$

For $n \geq n_0$, from Lemma 3.1 and Remark 3.2, for $n \geq n_0, |k-k'| \geq k*$ and $M_1 < \sigma_{k,n}(t_0) < M_2$, using (3.18) and the fact that $\int_{\mathbb{R}} |f_{k,n}(s)|^\alpha \mathrm{d}s = n^{-\alpha H(t_0)} \sigma^\alpha_{k,n}(t_0)$, it follows that

$$\begin{aligned} \eta_{k,k'} &\leq C \frac{n^{-\alpha\gamma} + n^{-\frac{\alpha(H(t_0)+\gamma)}{2}} + n^{-\alpha H(t_0)} |k-k'|^{\frac{\alpha H(t_0) - \alpha(L+1)}{2}}}{n^{-\alpha H(t_0)}} \\ &= C \left(n^{\alpha(H(t_0)-\gamma)} + n^{\frac{\alpha(H(t_0)-\gamma)}{2}} + |k-k'|^{\frac{\alpha H(t_0) - \alpha(L+1)}{2}} \right). \end{aligned}$$

Since $H(t_0) < \gamma < 1$ and $\frac{\alpha H(t_0) - \alpha(L+1)}{2} < 0$ then there exist $n_1, k_0 \in \mathbb{N}, n_1 \geq n_0$ and $0 < \eta_0 < 1$ such that for $n \geq n_1, |k-k'| > k_0$, we have $0 < \eta_{k,k'} \leq \eta_0 < 1$.
To get the bound for $\left| cov(|\triangle_{k,n}X|^\beta, |\triangle_{k',n}X|^\beta) \right|$, one can follow Lemma A.1 in [15] for Gaussian case ($\alpha = 2$) and Theorem 4.2 in [15] for non-Gaussian case ($0 < \alpha < 2$). We get of course better bounds in the Gaussian case that leads different bounds for two cases.

We can now consider each case in details.

1. In the case of multifractional Brownian motion ($\alpha = 2$), since $X(t)$ is a centered Gaussian variable (see *e.g.* [14])) and following Lemma A.1 in [15], there exist a constant C such that

$$\begin{aligned} \left| cov \left(\left| \frac{\triangle_{k,n}X}{||\triangle_{k,n}X||_2} \right|^\beta, \left| \frac{\triangle_{k',n}X}{||\triangle_{k',n}X||_2} \right|^\beta \right) \right| &\leq C\eta^2_{k,k'} \\ &\leq C \left(\int_{\mathbb{R}} \frac{|f_{k,n}(s) f_{k',n}(s)|}{||\triangle_{k,n}X||_2 ||\triangle_{k',n}X||_2} \mathrm{d}s \right)^2. \end{aligned}$$

Thus, one gets

$$\left|cov(|\triangle_{k,n}X|^{\beta},|\triangle_{k',n}X|^{\beta})\right| \leq C\left(\int_{\mathbb{R}}|f_{k,n}(s)|^2\mathrm{d}s\right)^{\frac{\beta}{2}-1}\left(\int_{\mathbb{R}}|f_{k',n}(s)|^2\mathrm{d}s\right)^{\frac{\beta}{2}-1}$$
$$\times\left(\int_{\mathbb{R}}|f_{k,n}(s)f_{k',n}(s)|\,\mathrm{d}s\right)^{2}.$$

Moreover, from Remark 3.2, there exists $M_1>0$ such that $M_1<\sigma_{k,n}(t_0), M_1<\sigma_{k',n}(t_0)$. Then using the fact that $\frac{\beta}{2}-1<0$ and

$$\int_{\mathbb{R}}|f_{k,n}(s)|^2\mathrm{d}s = n^{-2H(t_0)}\sigma_{k,n}(t_0)^2 > M_1^2 n^{-2H(t_0)},$$
$$\int_{\mathbb{R}}|f_{k',n}(s)|^2\mathrm{d}s = n^{-2H(t_0)}\sigma_{k',n}(t_0)^2 > M_1^2 n^{-2H(t_0)},$$

combining with (3.18) it follows that

$$\begin{aligned}
&\left|cov(|\triangle_{k,n}X|^{\beta},|\triangle_{k',n}X|^{\beta})\right| \\
&\leq C\left(M_1^2 n^{-2H(t_0)}\right)^{2\left(\frac{\beta}{2}-1\right)}\left(\int_{\mathbb{R}}|f_{k,n}(s)f_{k',n}(s)|\,\mathrm{d}s\right)^{2} \\
&\leq Cn^{-2\beta H(t_0)+4H(t_0)}\left(n^{-4\gamma}+n^{-2(H(t_0)+\gamma)}+n^{-4H(t_0)}|k-k'|^{2H(t_0)-2(L+1)}\right) \\
&\leq Cn^{-2\beta H(t_0)}\left(n^{2(H(t_0)-\gamma)}+|k-k'|^{2H(t_0)-2(L+1)}\right),
\end{aligned}$$

from which the conclusion follows.

2. In the case of linear multifractional stable motion ($0<\alpha<2$), applying Theorem 4.2 in [15], one has

$$\left|cov\left(\left|\frac{\triangle_{k,n}X}{||\triangle_{k,n}X||_{\alpha}}\right|^{\beta},\left|\frac{\triangle_{k',n}X}{||\triangle_{k',n}X||_{\alpha}}\right|^{\beta}\right)\right| \leq C\int_{\mathbb{R}}\left|\frac{f_{k,n}(s)f_{k',n}(s)}{||\triangle_{k,n}X||_{\alpha}||\triangle_{k',n}X||_{\alpha}}\right|^{\frac{\alpha}{2}}\mathrm{d}s,$$

where C is a constant. Thus, one obtains

$$\left|cov(|\triangle_{k,n}X|^{\beta},|\triangle_{k',n}X|^{\beta})\right| \leq C\left(\int_{\mathbb{R}}|f_{k,n}s)|^{\alpha}\mathrm{d}s\right)^{\frac{\beta}{\alpha}-\frac{1}{2}}\left(\int_{\mathbb{R}}|f_{k',n}(s)|^{\alpha}\mathrm{d}s\right)^{\frac{\beta}{\alpha}-\frac{1}{2}}$$
$$\times\int_{\mathbb{R}}|f_{k,n}(s)f_{k',n}(s)|^{\frac{\alpha}{2}}\,\mathrm{d}s.$$

Moreover,

$$\int_{\mathbb{R}}|f_{k,n}(s)|^{\alpha}\mathrm{d}s = n^{-\alpha H(t_0)}\sigma_{k,n}^{\alpha}(t_0) > M_1^{\alpha}n^{-\alpha H(t_0)},$$
$$\int_{\mathbb{R}}|f_{k',n}(s)|^{\alpha}\mathrm{d}s = n^{-\alpha H(t_0)}\sigma_{k',n}^{\alpha}(t_0) > M_1^{\alpha}n^{-\alpha H(t_0)}.$$

From (3.18) and using the fact that $\frac{\beta}{\alpha}-\frac{1}{2}<0$, it follows that

$$\begin{aligned}\left|cov(|\triangle_{k,n}X|^{\beta},|\triangle_{k',n}X|^{\beta})\right| &\leq C\left(M_1^{\alpha}n^{-\alpha H(t_0)}\right)^{2\left(\frac{\beta}{\alpha}-\frac{1}{2}\right)}\int_{\mathbb{R}}|f_{k,n}(s)f_{k',n}(s)|^{\frac{\alpha}{2}}\,\mathrm{d}s\\ &\leq C^{*}n^{-2\beta H(t_0)}\left(n^{\frac{\alpha(H(t_0)-\gamma)}{2}}+|k-k'|^{\frac{\alpha H(t_0)-\alpha(L+1)}{2}}\right)\end{aligned}$$

where C^{*} is a constant. □

3.2. Proof of Theorem 2.2

Proof.

1. One gets

$$\mathbb{E}(W_{t_0,n}(\beta)-M_{t_0,\beta})^2=\mathbb{E}W_{t_0,n}^2(\beta)-\left(\mathbb{E}W_{t_0,n}(\beta)\right)^2+\left(\mathbb{E}W_{t_0,n}(\beta)-M_{t_0,\beta}\right)^2.$$

Applying Lemma 3.4 and Lemma 3.5, for $n\geq n_1$, one has

$$\begin{aligned}&\mathbb{E}W_{t_0,n}^2(\beta)-\left(\mathbb{E}W_{t_0,n}(\beta)\right)^2\\ &=\frac{1}{\upsilon_{\gamma,n}(t_0)^2}\sum_{k,k'\in\nu_{\gamma,n}(t_0)}cov\left(\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta},\left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^{\beta}\right)\\ &\leq\frac{1}{\upsilon_{\gamma,n}(t_0)^2}\sum_{k,k'\in\nu_{\gamma,n}(t_0),|k-k'|\leq k_0}\left|cov\left(\left|\frac{\triangle_{k,n}X}{n^{-H(t_0)}}\right|^{\beta},\left|\frac{\triangle_{k',n}X}{n^{-H(t_0)}}\right|^{\beta}\right)\right|\\ &\quad+\frac{n^{2\beta H(t_0)}}{\upsilon_{\gamma,n}(t_0)^2}\sum_{k,k'\in\nu_{\gamma,n}(t_0),|k-k'|>k_0}\left|cov\left(|\triangle_{k,n}X|^{\beta},|\triangle_{k',n}X|^{\beta}\right)\right|\\ &\leq\frac{1}{\upsilon_{\gamma,n}(t_0)^2}\sum_{|p|\leq k_0,p\in\mathbb{Z}}(\upsilon_{\gamma,n}(t_0)-|p|)C\\ &\quad+\frac{n^{2\beta H(t_0)}}{\upsilon_{\gamma,n}(t_0)^2}\times\sum_{k_0<|p|\leq\upsilon_{\gamma,n}(t_0),p\in\mathbb{Z}}(\upsilon_{\gamma,n}(t_0)-|p|)Cn^{-2\beta H(t_0)}\left(n^{2(H(t_0)-\gamma)}+|p|^{2H(t_0)-2(L+1)}\right)\\ &\leq\frac{1}{\upsilon_{\gamma,n}(t_0)^2}\sum_{|p|\leq k_0,p\in\mathbb{Z}}(\upsilon_{\gamma,n}(t_0)-|p|)C+\frac{C}{\upsilon_{\gamma,n}(t_0)}\sum_{k_0<|p|\leq\upsilon_{\gamma,n}(t_0),p\in\mathbb{Z}}\left(n^{2(H(t_0)-\gamma)}+|p|^{2H(t_0)-2(L+1)}\right).\end{aligned}$$

Combining with Lemma 3.3 and using the fact that $\upsilon_{\gamma,n}(t_0)=[2n^{1-\gamma/\eta}-K]$ or $[2n^{1-\gamma/\eta}-K]+1$ depending on the parity of $[2n^{1-\gamma/\eta}-K]$, it follows that

$$\begin{aligned}\mathbb{E}(W_{t_0,n}(\beta)-M_{t_0,\beta})^2&\leq C\left(n^{2(H(t_0)-\gamma)}+n^{\gamma/\eta-1}\right)\\ &\quad+\frac{C}{\upsilon_{\gamma,n}(t_0)}\sum_{|p|\leq\upsilon_{\gamma,n}(t_0)}|p|^{2H(t_0)-2(L+1)}.\end{aligned}$$

Since $2H(t_0) - 2(L+1) < -1$, from Lemma A.4 in [15], one gets

$$\frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{|p| \le \upsilon_{\gamma,n}(t_0) p \in \mathbb{Z}} |p|^{2H(t_0)-2(L+1)} = O(\upsilon_{\gamma,n}(t_0)^{-1}) = O(n^{\gamma/\eta - 1}).$$

Finally we can write

$$\begin{aligned} \mathbb{E}(W_{t_0,n}(\beta) - M_{t_0,\beta})^2 &\le C\left(n^{\gamma/\eta-1} + n^{2(H(t_0)-\gamma)}\right) \\ &\le C d_n^2 \end{aligned} \tag{3.20}$$

where d_n is defined by (2.21).
For $\epsilon > 0$ fixed, using Markov's inequality, one gets

$$\mathbb{P}(|W_{t_0,n}(\beta) - M_{t_0,\beta}| > \epsilon) \le \frac{\mathbb{E}(W_{t_0,n}(\beta) - M_{t_0,\beta})^2}{\epsilon^2} \le \frac{C d_n^2}{\epsilon}. \tag{3.21}$$

It follows that

$$W_{t_0,n}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_n).$$

2. Let $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\phi(x,y) = \frac{1}{\beta}\log_2 \frac{x}{y}. \tag{3.22}$$

Then

$$\begin{aligned} \phi(W_{t_0,n/2}(\beta), W_{t_0,n}(\beta)) &= \frac{1}{\beta}\log_2 \frac{W_{t_0,n/2}(\beta)}{W_{t_0,n}(\beta)} \\ &= \frac{1}{\beta}\log_2 \frac{V_{t_0,n/2}(\beta)}{V_{t_0,n}(\beta)} - H(t_0) \\ &= \widehat{H}_n(t_0) - H(t_0). \end{aligned}$$

From the proof of the latter part, one has

$$W_{t_0,n}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_n), W_{t_0,n/2}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_{n/2}) = O_{\mathbb{P}}(d_n).$$

Moreover, ϕ is differentiable at $(M_{t_0,\beta}, M_{t_0,\beta})$, applying Lemma 4.10 in [15], one deduces that $\widehat{H}_n - H(t_0) = O_{\mathbb{P}}(d_n)$. □

3.3. Proof of Theorem 2.3

Proof.

1. Following similar changes as in the proof of Theorem 2.2 in Section 3.2 and combining with Lemma 3.3, Lemma 3.4 and Lemma 3.5, it follows that

$$\begin{aligned} \mathbb{E}(W_{t_0,n}(\beta) - M_{t_0,\beta})^2 \le C &\left(n^{-2(\alpha\wedge 1)(\gamma - H(t_0))} + n^{\gamma/\eta-1} + n^{\frac{\alpha(H(t_0)-\gamma)}{2}}\right) \\ &+ \frac{C}{\upsilon_{\gamma,n}(t_0)} \sum_{|p|\le \upsilon_{\gamma,n}(t_0)} |p|^{\frac{\alpha H(t_0) - \alpha(L+1)}{2}}. \end{aligned}$$

From Lemma A.4 in [15], one gets

$$\frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{|p|\leq \upsilon_{\gamma,n}(t_0), p\in\mathbb{Z}} |p|^{\frac{\alpha H(t_0)-\alpha(L+1)}{2}}$$
$$= \begin{cases} O\left(\upsilon_{\gamma,n}(t_0)^{-1}\right) = O\left(n^{\gamma/\eta-1}\right) & \text{if } \frac{\alpha H(t_0)-\alpha(L+1)}{2} < -1, \\ O\left(\upsilon_{\gamma,n}(t_0)^{\frac{\alpha H(t_0)-\alpha(L+1)}{2}}\right) = O\left(n^{\frac{(1-\gamma/\eta)(\alpha H(t_0)-\alpha(L+1))}{2}}\right) & \text{if } -1 < \frac{\alpha H(t_0)-\alpha(L+1)}{2} < 0, \\ O\left(\frac{\ln(\upsilon_{\gamma,n}(t_0))}{\upsilon_{\gamma,n}(t_0)}\right) = O\left(n^{\gamma/\eta-1}\ln n\right) & \text{if } \frac{\alpha H(t_0)-\alpha(L+1)}{2} = -1. \end{cases}$$

Combining with the fact that $-2(\alpha\wedge 1)(\gamma - H(t_0)) \leq \frac{\alpha(H(t_0)-\gamma)}{2} < 0$, one can write

$$\begin{aligned} \mathbb{E}(W_{t_0,n}(\beta) - M_{t_0,\beta})^2 &\leq C\left(n^{\gamma/\eta-1} + n^{\frac{\alpha(H(t_0)-\gamma)}{2}} + \frac{1}{\upsilon_{\gamma,n}(t_0)} \sum_{|p|\leq \upsilon_{\gamma,n}(t_0), p\in\mathbb{Z}} |p|^{\frac{\alpha H(t_0)-\alpha(L+1)}{2}}\right) \\ &\leq Cd_n^2 \end{aligned} \tag{3.23}$$

where d_n defined by (2.24). For $\epsilon > 0$ fixed, using Markov's inequality, one gets

$$\mathbb{P}(|W_{t_0,n}(\beta) - M_{t_0,\beta}| > \epsilon) \leq \frac{\mathbb{E}(W_{t_0,n}(\beta) - M_{t_0,\beta})^2}{\epsilon^2} \leq \frac{Cd_n^2}{\epsilon^2}.$$

It follows that

$$W_{t_0,n}(\beta) - M_{t_0,\beta} = O_{\mathbb{P}}(d_n).$$

2. One can use the same function ϕ as in (3.22) and mimic the proof of Theorem 2.2 to get the conclusion.

□

3.4. Proof of Theorem 2.4

Proof. For a fixed $t_0 \in \mathring{U}$, we mimic the proof of Theorem 2.1 in [15] and get

$$h_{-\beta_1,-\beta_2}(\alpha) = \psi_{-\beta_1,-\beta_2}(M_{t_0,\beta_1}, M_{t_0,\beta_2}),$$

where the function $\psi_{u,v}, h_{u,v}$ and $M_{t_0,\beta}$ are defined by (2.11), (2.12) and (2.19) respectively.
From Lemma 4.11 in [15], it follows that $h_{u,v}$ is a strictly increasing function on $(0, +\infty)$ and

$$\lim_{x\to+\infty} h_{u,v}(x) = 0, \lim_{x\to 0} h_{u,v}(x) = -\infty.$$

Furthermore, there exists an inverse function

$$h_{u,v}^{-1} : (-\infty, 0) \to (0, +\infty)$$

which is continuous and differentiable on $(-\infty, 0)$.
In addition,

$$\psi_{-\beta_1,-\beta_2}(W_{u,n}(\beta_1), W_{u,n}(\beta_2)) = \psi_{-\beta_1,-\beta_2}(V_{u,n}(\beta_1), V_{u,n}(\beta_2)).$$

and since $\alpha \in (0,2]$, one gets

$$h_{-\beta_1,-\beta_2}(\alpha) = \psi_{-\beta_1,-\beta_2}(M_{t_0,\beta_1}, M_{t_0,\beta_2}) < 0.$$

Then

$$\begin{aligned}\widehat{\alpha}_n - \alpha &= \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_{t_0,n}(\beta_1), V_{t_0,n}(\beta_2))\right) - h^{-1}_{-\beta_1,-\beta_2}\left(h_{-\beta_1,-\beta_2}(\alpha)\right) \\ &= \varphi_{-\beta_1,-\beta_2}\left(\psi_{\beta_1,\beta_2}(W_{t_0,n}(\beta_1), W_{t_0,n}(\beta_2))\right) - \varphi_{-\beta_1,-\beta_2}\left(h_{-\beta_1,-\beta_2}(\alpha)\right) \\ &= \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(W_{t_0,n}(\beta_1), W_{t_0,n}(\beta_2))\right) \\ &\quad - \varphi_{-\beta_1,-\beta_2}(\psi_{-\beta_1,-\beta_2}(M_{t_0,\beta_1}, M_{t_0,\beta_2})). \end{aligned} \tag{3.24}$$

Moreover, from Theorem 2.2 and Theorem 2.3, one has

$$W_{t_0,n}(\beta_1) - M_{t_0,\beta_1} = O_{\mathbb{P}(d_n)}, W_{t_0,n}(\beta_2) - M_{t_0,\beta_2} = O_{\mathbb{P}(d_n)}$$

where d_n is defined by (2.21) for multifractional Brownian motion and by (2.24) for linear multifractional stable motion. Combining with (3.24) and the fact that $\varphi_{-\beta_1,-\beta_2} \circ \psi_{-\beta_1,-\beta_2}$ is differentiable at $(M_{t_0,\beta_1}, M_{t_0,\beta_2})$, we apply Lemma 4.10 in [15] and get the conclusion. □

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