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ON BERNSTEIN–KANTOROVICH INVARIANCE PRINCIPLE IN HÖLDER SPACES AND WEIGHTED SCAN STATISTICS*

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Abstract. Let ξ_n be the polygonal line partial sums process built on i.i.d. centered random variables X_i, $i \geq 1$. The Bernstein-Kantorovich theorem states the equivalence between the finiteness of $\mathrm{E}\,|X_1|^{\max(2,r)}$ and the joint weak convergence in $C[0,1]$ of $n^{-1/2}\xi_n$ to a Brownian motion W with the moments convergence of $\mathrm{E}\,\|n^{-1/2}\xi_n\|_\infty^r$ to $\mathrm{E}\,\|W\|_\infty^r$. For $0 < \alpha < 1/2$ and $p(\alpha) = (1/2-\alpha)^{-1}$, we prove that the joint convergence in the separable Hölder space $\mathcal{H}_\alpha^o$ of $n^{-1/2}\xi_n$ to W jointly with the one of $\mathrm{E}\,\|n^{-1/2}\xi_n\|_\alpha^r$ to $\mathrm{E}\,\|W\|_\alpha^r$ holds if and only if $P(|X_1| > t) = o(t^{-p(\alpha)})$ when $r < p(\alpha)$ or $\mathrm{E}\,|X_1|^r < \infty$ when $r \geq p(\alpha)$. As an application we show that for every $\alpha < 1/2$, all the α-Hölderian moments of the polygonal uniform quantile process converge to the corresponding ones of a Brownian bridge. We also obtain the asymptotic behavior of the rth moments of some α-Hölderian weighted scan statistics where the natural border for α is $1/2 - 1/p$ when $\mathrm{E}\,|X_1|^p < \infty$. In the case where the X_i's are p regularly varying, we can complete these results for $\alpha > 1/2 - 1/p$ with an appropriate normalization.

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1. Introduction

Let $(Z_n)_{n\geq 1}$ be a sequence of random elements in some separable metric space $\mathbb{S}$ endowed with its Borel σ-field $\mathscr{S}$. Let Z be a random element in $\mathbb{S}$. Assume for notational simplicity that Z and the Z_n's are all defined on the same probability space $(\Omega, \mathscr{F}, \mathrm{P})$. Then Z_n converges in distribution to Z, denoted by

$$Z_n \xrightarrow[n\to\infty]{\mathrm{d}} Z,$$

if its distribution $\mu_n = \mathrm{P} \circ Z_n^{-1}$ converges weakly to $\mu = \mathrm{P} \circ Z^{-1}$. This means that for every continuous bounded function $f : \mathbb{S} \to \mathbb{R}$,

$$\int_{\mathbb{S}} f \,\mathrm{d}\mu_n = \mathrm{E}\, f(Z_n) \xrightarrow[n\to\infty]{} \mathrm{E}\, f(Z) = \int_{\mathbb{S}} f \,\mathrm{d}\mu. \tag{1.1}$$

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Relaxing the boundedness assumption on f in (1.1) leads to the classical question of convergence of moments. When $\mathbb{S}$ is a separable Banach space with norm $\| \ \|$, one is interested in extending the convergence in (1.1) to the case of functions satisfying for some positive constants c_1, c_2, r,

$$|f(x)| \leq c_1 \|x\|^r + c_2, \quad x \in \mathbb{S}. \tag{1.2}$$

It is well known that this extension is valid if and only if $(\|Z_n\|^r)_{n\geq 1}$ is uniformly integrable (see [3], Thm. 5.4) that is

$$\lim_{a\to\infty} \sup_{n\geq 1} \mathrm{E} \|Z_n\|^r \mathbf{1}_{\{\|Z_n\|>a\}} = 0. \tag{1.3}$$

Let us note that if $(\|Z_n\|^r)_{n\geq 1}$ is uniformly integrable, necessarily

$$\sup_{n\geq 1} \mathrm{E} \|Z_n\|^r < \infty. \tag{1.4}$$

In this paper we focus on the convergence of moments in the functional central limit theorem. Let $(X_i)_{i\geq 1}$ be an i.i.d. sequence of real valued random variables with null expectation and variance one if they exist, $S_n := X_1 + \cdots + X_n$ and ξ_n the random polygonal line with vertices $(k/n, S_k)$, $k = 0, 1, \ldots, n$:

$$\xi_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0, 1]. \tag{1.5}$$

From Bernstein theorem [2] it is known that for $r > 0$ the joint convergence

$$n^{-1/2} S_n \xrightarrow[n\to\infty]{\mathrm{d}} G \quad \text{and} \quad \lim_{n\to\infty} \mathrm{E} \left| n^{-1/2} S_n \right|^r = \mathrm{E} |G|^r,$$

where G is a Gaussian $\mathfrak{N}(0, 1)$ random variable, is equivalent to the finiteness of $\mathrm{E} |X_1|^{\max\{2,r\}}$. Note also that in the case where $r = 2$ the convergence of the corresponding moment is trivial and that for $0 < r < 2$ the convergence of $\mathrm{E} \left|n^{-1/2} S_n\right|^r$ follows immediately from $\mathrm{E} X_1^2 < \infty$ by uniform integrability of $\left(n^{-1} S_n^2, n \geq 1\right)$.

Let us denote by W a standard Brownian motion viewed as a random element in the space $\mathrm{C}[0, 1]$ of continuous functions $x : [0, 1] \to \mathbb{R}$ endowed with the uniform norm $\|x\|_\infty = \sup\{|x(t)|, t \in [0, 1]\}$. The classical Donsker-Prokhorov theorem provides the equivalence :

$$n^{-1/2} \xi_n \xrightarrow[n\to\infty]{\mathrm{d}} W \text{ in } \mathrm{C}[0, 1] \quad \text{if and only if} \quad \mathrm{E} X_1^2 < \infty.$$

For $r > 0$, the Bernstein–Kantorovich functional central limit theorem (see [11], Thm. 11.2.1, p. 219) provides the equivalence between $\mathrm{E} |X_1|^{\max\{2,r\}} < \infty$ and the joint convergence

$$n^{-1/2} \xi_n \xrightarrow[n\to\infty]{\mathrm{d}} W \text{ in } \mathrm{C}[0, 1] \quad \text{with} \quad \mathrm{E} \|n^{-1/2} \xi_n\|_\infty^r \xrightarrow[n\to\infty]{} \mathrm{E} \|W\|_\infty^r. \tag{1.6}$$

It turns out that the condition $\mathrm{E} |X_1|^r < \infty$ for some $r > 2$ provides also the convergence in distribution of $n^{-1/2} \xi_n$ to W in a stronger topology than the $\mathrm{C}[0, 1]$'s one. Define for $0 \leq \alpha < 1$ the Hölder space $\mathcal{H}_\alpha^o[0, 1]$ as the set of functions $x : [0, 1] \to \mathbb{R}$ such that

$$\omega_\alpha(x, \delta) := \sup_{\substack{0<t-s\leq\delta \\ s,t\in[0,1]}} \frac{|x(t) - x(s)|}{(t - s)^\alpha} \xrightarrow[\delta\to 0]{} 0, \tag{1.7}$$

endowed with the norm

$$\|x\|_\alpha = |x(0)| + \omega_\alpha(x, 1), \tag{1.8}$$

which makes it a separable Banach space (isomorphic to C$[0, 1]$ in the special case $\alpha = 0$).

Let $\alpha \in (0, 1/2)$ and $p(\alpha) = (1/2 - \alpha)^{-1}$. By the necessary and sufficient condition for Lamperti's Hölderian invariance principle [12, 13], we know that $n^{-1/2}\xi_n$ converges in distribution in the space $\mathcal{H}^o_\alpha[0, 1]$ to the standard Brownian motion if and only if $\mathrm{P}(|X_1| > t) = o(t^{-p(\alpha)})$ when t tends to infinity. When $\mathrm{E}\,|X_1|^{p(\alpha)} < \infty$, this condition is satisfied. Our first result extends the Bernstein–Kantorovich functional central limit theorem to the spaces $\mathcal{H}^o_\alpha[0, 1]$.

Theorem 1.1. *Let $\alpha \in (0, 1/2)$ and $p(\alpha) = (1/2 - \alpha)^{-1}$. Let $r > 0$. Then the joint convergence*

$$n^{-1/2}\xi_n \xrightarrow[n\to\infty]{\mathrm{d}} W \text{ in } \mathcal{H}^o_\alpha[0, 1] \quad \text{with} \quad \mathrm{E}\,\|n^{-1/2}\xi_n\|^r_\alpha \xrightarrow[n\to\infty]{} \mathrm{E}\,\|W\|^r_\alpha \tag{1.9}$$

holds if and only if

$$\lim_{t\to\infty} t^{p(\alpha)}\,\mathrm{P}(|X_1| > t) = 0, \quad \text{when } r < p(\alpha), \tag{1.10}$$

$$\mathrm{E}\,|X_1|^r < \infty, \quad \text{when } r \geq p(\alpha). \tag{1.11}$$

It is worth noticing here that (1.9) is equivalent to the convergence of the distribution of $n^{-1/2}\xi_n$ to the one of W with respect to the Wasserstein distance of order r associated to the norm $\|.\|_\alpha$, *i.e.* with the mass transportation cost function $c(x, y) = \|x - y\|^r_\alpha$, see Section 3.2 for details.

An immediate consequence of Theorem 1.1 is that (1.11) implies the convergence of $\mathrm{E}\,f(n^{-1/2}\xi_n)$ to $\mathrm{E}\,f(W)$ for any continuous functional $f : \mathcal{H}^o_\alpha[0, 1] \to \mathbb{R}$ satisfying

$$|f(x)| \leq c_1\|x\|^r_\alpha + c_2. \tag{1.12}$$

Among the various functionals $f(n^{-1/2}\xi_n)$ where f satisfies a condition like (1.12) are the (powers of the) following weighted scan type statistics :

$$M_{n,\alpha} = \max_{1\leq j\leq n} j^{-\alpha} \max_{0\leq k\leq n-j} |S_{k+j} - S_k|.$$

We refer to [1, 10] for valuable information about scan statistics and their applications. The following result is a corollary of more general results obtained in this paper (see Thms. 4.1 and 4.2).

Theorem 1.2. *Let $p > 2$.*

(a) If $0 \leq \alpha \leq 1/2 - 1/p$, and $\mathrm{E}\,|X_1|^p < \infty$ then

$$\lim_{n\to\infty} n^{-1}\,\mathrm{E}\,M^p_{n,\alpha} = \mathrm{E}\,\omega^p_\alpha(W, 1).$$

(b) If $1/2 - 1/p < \alpha < 1$ and X_1 is regularly varying with exponent p then for any $0 \leq r < p$,

$$\lim_{n\to\infty} b^{-r}_n\,\mathrm{E}\,M^r_{n,\alpha} = \mathrm{E}\,Y^r_p,$$

where

$$b_n = \inf\{t > 0 : \mathrm{P}(|X_1| \le t) \ge 1 - 1/n\} \tag{1.13}$$

and Y_p has Fréchet distribution with exponent p.

The paper is organized as follows. Section 2 is devoted to preliminaries where uniform integrability, regularly varying random variables are discussed and necessary tools on Hölder spaces are presented. In Section 3, one proves Theorem 1.1 and some comments concerning the upper bound for admissible Hölder index in the Bernstein–Kantorovich invariance principle are presented. Convergence of moments of weighted scan statistics is considered in Section 4. The paper ends with an appendix devoted to some facts from Karamata theory.

2. Preliminaries

2.1. Uniform integrability

Lemma 2.1. *Let $(Z_n)_{n\ge1}$ be a sequence of random elements in the Banach space $(\mathbb{S}, \|\ \|)$. For $r > 0$, $(\|Z_n\|^r)_{n\ge1}$ is uniformly integrable if and only if*

$$\lim_{a\to\infty} \sup_{n\ge1} \int_a^\infty t^{r-1}\, \mathrm{P}(\|Z_n\| > t)\, \mathrm{d}t = 0. \tag{2.1}$$

The proof is elementary and will be omitted.

2.2. Hölderian tools

Let D_j denotes the set of dyadic numbers of level j in $[0,1]$, that is $D_0 := \{0,1\}$ and for $j \ge 1$, $D_j := \big\{(2l-1)2^{-j};\ 1 \le l \le 2^{j-1}\big\}$. For $d \in D_j$ set $d^- := d - 2^{-j}$, $d^+ := d + 2^{-j}$, $j \ge 0$. For $x : [0,1] \to \mathbb{R}$ and $d \in D_j$ let us define

$$\lambda_d(x) := \begin{cases} x(d) - \frac{x(d^+)+x(d^-)}{2} & \text{if } j \ge 1,\\ x(d) & \text{if } j = 0. \end{cases}$$

The following sequential norm defined on $\mathcal{H}^o_\alpha[0,1]$ by

$$\|x\|^{\mathrm{seq}}_\alpha := \sup_{j\ge0} 2^{\alpha j} \max_{d\in D_j} |\lambda_d(x)|,$$

is equivalent to the natural norm $\|x\|_\alpha$, see [5]. Let us define also $\mathbb{D}_j := \{k2^{-j},\ 0 \le k < 2^j\}$, so that $\mathbb{D}_j = \{0\} \cup \bigcup_{1\le i\le j} D_i$.

The Hölder norm of a polygonal line function is very easy to compute according to the following lemma for which we refer *e.g.* to [8] Lemma 3, where it is proved in a more general setting.

Lemma 2.2. *Let $t_0 = 0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ and x be a real-valued polygonal line function on $[0,1]$ with vertices at the t_i's, i.e. x is continuous on $[0,1]$ and its restriction to each interval $[t_i, t_{i+1}]$ is an affine function. Then for any $0 \le \alpha < 1$,*

$$\sup_{0\le s<t\le1} \frac{|x(t)-x(s)|}{(t-s)^\alpha} = \max_{0\le i<j\le n} \frac{|x(t_j)-x(t_i)|}{(t_j-t_i)^\alpha}.$$

2.3. Regularly varying random variables

Throughout this paper we implicitly assume that all the random variables considered are defined on the same probability space $(\Omega, \mathscr{F}, \mathrm{P})$ and we use the following notion of regularly varying random variable.

Definition 2.3. The random variable X is regularly varying with index $p > 0$ (denoted $X \in \mathrm{RV}_p$) if there exists a slowly varying function L such that the distribution function $F(t) = \mathrm{P}(X \leq t)$ satisfies the tail balance condition

$$F(-x) \sim bL(x)x^{-p} \quad \text{and} \quad 1 - F(x) \sim aL(x)x^{-p}, \quad \text{as} \quad x \to \infty, \tag{2.2}$$

where $a, b \in (0, 1)$ and $a + b = 1$.

We refer to [4] for an encyclopaedic treatment of regular variation.

Writing $\mathcal{L}_p$ or $\mathcal{L}^o_{p,\infty}$ for the sets of random variables X verifying respectively $\mathrm{E}\,|X|^p < \infty$ or $\lim_{t\to\infty} t^p\,\mathrm{P}(|X| > t) = 0$, we note that

$$\begin{aligned}
&\mathrm{RV}_p \subset \mathcal{L}_r, \quad \text{for} \quad 0 \leq r < p,\\
&\mathrm{RV}_p \cap \mathcal{L}_p \neq \emptyset, \quad \mathrm{RV}_p \cap \mathcal{L}^o_{p,\infty} \neq \emptyset,\\
&\mathrm{RV}_p \cap \mathcal{L}_{p'} = \emptyset \quad \text{for} \quad p' > p.
\end{aligned}$$

The next lemma plays a key role in our results on the scan statistics built on RV_p random variables. Its proof is detailed in the Appendix.

Lemma 2.4. *Let $X \in \mathrm{RV}_p$ and b_n be its $(1 - 1/n)$ quantile defined as in (1.13).*

i) For any $0 < s < p$,

$$\mathrm{E}\,|X|^s \mathbf{1}_{\{|X|>yb_n\}} \leq \frac{2p}{p-s} b_n^s \frac{1}{n} y^{s-p}, \tag{2.3}$$

for n large enough, uniformly in $y \in [1, \infty)$.

ii) For any $s > p$,

$$\mathrm{E}\,|X|^s \mathbf{1}_{\{|X|\leq yb_n\}} \leq \frac{2p}{s-p} b_n^s \frac{1}{n} y^{s-p}, \tag{2.4}$$

for n large enough, uniformly in $y \in [1, \infty)$.

iii) Let $\widetilde{X} = X\mathbf{1}_{\{|X|\leq yb_n\}}$, $X' = \widetilde{X} - \mathrm{E}\,\widetilde{X}$. Then for any $0 < s \neq p$,

$$\mathrm{E}\,|X'|^s \leq K(s,p) b_n^s \frac{1}{n} y^{s-p}, \tag{2.5}$$

for n large enough, uniformly in $y \in [1, \infty)$, where

$$K(s,p) = \max(1, 2^{s-1}) \frac{4p}{|p-s|}.$$

3. Bernstein–Kantorovich theorem in $\mathcal{H}^o_\alpha[0,1]$

In this section first we prove Theorem 1.1 and then discuss some aspects of Bernstein–Kantorovich theorem in Hölder framevork.

3.1. Proof of Theorem 1.1

The necessity of the X_1's integrability conditions for the joint convergence (1.9) is easily seen. Indeed when $r < p(\alpha)$, (1.10) follows from the first convergence in (1.9), see [12]. When $r \geq p(\alpha)$, we note that $|X_1| = |\xi_n(1/n) - \xi_n(0)| \leq \|\xi_n\|_\alpha n^{-\alpha}$. From the second convergence in (1.9), $\mathrm{E}\,\|n^{-1/2}\xi_n\|_\alpha^r$ is finite, at least for n large enough, whence $\mathrm{E}\,|X_1|^r \leq n^{(1/2-\alpha)r}\,\mathrm{E}\,\|n^{-1/2}\xi_n\|_\alpha^r < \infty$, giving the necessity of (1.11).

Now let us prove that the integrability conditions (1.10) or (1.11) are sufficient for the joint convergence (1.9). By the Hölderian invariance principle [12], (1.10) or *a fortiori* (1.11) implies that $n^{-1/2}\xi_n$ converges in distribution to W in $\mathcal{H}_\alpha^o[0,1]$ and by continuous mapping that $\|n^{-1/2}\xi_n\|_\alpha$ converges in distribution to $\|W\|_\alpha$. It remains to check the uniform integrability of the sequence $\bigl(\|n^{-1/2}\xi_n\|_\alpha^r\bigr)_{n\geq 1}$. It is enough to consider the case where $r \leq p(\alpha)$ only. Indeed, if $r > p(\alpha)$, then we choose $\beta = 1/2 - 1/r$ (so that $r = p(\beta)$) and notice that uniform integrability of the sequence $\bigl(\|n^{-1/2}\xi_n\|_\beta^{p(\beta)}\bigr)_{n\geq 1}$ yields that of the sequence $\bigl(\|n^{-1/2}\xi_n\|_\alpha^{p(\beta)}\bigr)_{n\geq 1}$ since $\beta > \alpha$.

Now to prove the uniform integrability of the sequence $(\|n^{-1/2}\xi_n\|_\alpha^r)_{n\geq 1}$ we can obviously replace $\|\ \|_\alpha$ by an equivalent norm. The choice of $\|\ \|_\alpha^{\mathrm{seq}}$ seems more convenient here. So we have to prove that for $r \leq p(\alpha)$,

$$\lim_{a\to\infty} \sup_{n\geq 1} \int_a^\infty r t^{r-1}\, \mathrm{P}\left(\|n^{-1/2}\xi_n\|_\alpha^{\mathrm{seq}} > t\right) \mathrm{d}t = 0. \tag{3.1}$$

The following proof of (3.1) is essentially common to the cases $r < p(\alpha)$ and $r = p(\alpha)$ except for some nuance in the exploitation of the integrability of X_1. From now on, we write p for $p(\alpha)$.

The first task in establishing (3.1) is to obtain a good estimate for

$$P(n,t) := \mathrm{P}\left(\|n^{-1/2}\xi_n\|_\alpha^{\mathrm{seq}} > t\right). \tag{3.2}$$

Write for simplicity $t_{k,j} = k2^{-j}$, $k = 0, 1, \dots, 2^j$, $j = 1, 2, \dots$ and $t_k = t_{k,j}$ whenever the context dispels any doubt on the value of j. It is easily seen that for any $x \in \mathcal{H}_\alpha$ such that $x(0) = 0$,

$$\|x\|_\alpha^{\mathrm{seq}} \leq \sup_{j\geq 0} 2^{j\alpha} \max_{0\leq k<2^j} |x(t_{k+1,j}) - x(t_{k,j})|.$$

From this we deduce that

$$P(n,t) \leq P_1(n,t) + P_2(n,t), \tag{3.3}$$

with

$$P_1(n,t) = \mathrm{P}\left(\max_{0\leq j\leq \log n} 2^{j\alpha} \max_{0\leq k<2^j} |\xi_n(t_{k+1,j}) - \xi_n(t_{k,j})| > n^{1/2}t\right), \tag{3.4}$$

$$P_2(n,t) = \mathrm{P}\left(\sup_{j>\log n} 2^{j\alpha} \max_{0\leq k<2^j} |\xi_n(t_{k+1,j}) - \xi_n(t_{k,j})| > n^{1/2}t\right), \tag{3.5}$$

where log denotes the logarithm *with basis* 2 ($\log 2 = 1$).

Estimation of $P_2(n,t)$. If $j > \log n$, then $t_{k+1} - t_k = 2^{-j} < 1/n$ and therefore with $t_k \in [i/n, (i+1)/n)$, either t_{k+1} is in $(i/n, (i+1)/n]$ or belongs to $\bigl((i+1)/n, (i+2)/n\bigr]$, where $1 \leq i \leq n-2$ depends on k and j.

In the first case we have

$$|\xi_n(t_{k+1}) - \xi_n(t_k)| = |X_{i+1}| 2^{-j} n \leq 2^{-j} n \max_{1\leq i\leq n} |X_i|.$$

If t_k and t_{k+1} are in consecutive intervals, noticing that the slope of each of the two involved segments of the polygonal line is bounded in absolute value by $n\max_{1\le i\le n}|X_i|$, we get

$$\begin{aligned}|\xi_n(t_{k+1})-\xi_n(t_k)| &\le |\xi_n(t_k)-\xi_n((i+1)/n)|+|\xi_n((i+1)/n)-\xi_n(t_{k+1})|\\ &\le 2^{-j}n\max_{1\le i\le n}|X_i|.\end{aligned}$$

With both cases taken into account we obtain

$$P_2(n,t)\le \mathrm{P}\left(\sup_{j>\log n}2^{j\alpha}n^{-1/2}2^{-j}n\max_{1\le i\le n}|X_i|>t\right).$$

Noting that for $j>\log n$, $2^{j(-1+\alpha)}n^{1/2}<n^{-1/2+\alpha}=n^{-1/p}$, this leads to

$$P_2(n,t)\le n\,\mathrm{P}\left(|X_1|>tn^{1/p}\right). \tag{3.6}$$

To control the contribution of $P_2(n,t)$ when estimating the integral in (3.1), we note that for every $n\ge 1$,

$$\int_a^\infty rt^{r-1}n\,\mathrm{P}\left(|X_1|>tn^{1/p}\right)\mathrm{d}t=n^{1-r/p}\int_{an^{1/p}}^\infty rs^{r-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s. \tag{3.7}$$

In the case where $r=p$, as $\mathrm{E}\,|X_1|^p=\int_0^\infty ps^{p-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s$ is supposed finite, we can bound the right hand side of (3.7) by $\int_a^\infty ps^{p-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s$ uniformly in $n\ge 1$. In the case where $r<p$, the hypothese (1.10) implies that

$$\mathrm{P}(|X_1|>t)\le Kt^{-p},\quad t>0, \tag{3.8}$$

for some constant K depending only on the distribution of X_1. Hence

$$n^{1-r/p}\int_{an^{1/p}}^\infty rs^{r-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s\le\frac{Kr}{p-r}a^{r-p},\quad n\ge 1. \tag{3.9}$$

Gathering both cases we obtain that for $r\le p$,

$$\lim_{a\to\infty}\sup_{n\ge 1}\int_a^\infty rt^{r-1}n\,\mathrm{P}\left(|X_1|>tn^{1/p}\right)\mathrm{d}t=0 \tag{3.10}$$

and consequently,

$$\lim_{a\to\infty}\sup_{n\ge 1}\int_a^\infty rt^{r-1}P_2(n,t)\,\mathrm{d}t=0. \tag{3.11}$$

Estimation of $P_1(n,t)$. Let $u_k=[nt_k]$. Then $u_k\le nt_k\le 1+u_k$ and $1+u_k\le u_{k+1}\le nt_{k+1}\le 1+u_{k+1}$. Therefore

$$|\xi_n(t_{k+1})-\xi_n(t_k)|\le|\xi_n(t_{k+1})-S_{u_{k+1}}|+|S_{u_{k+1}}-S_{u_k}|+|S_{u_k}-\xi_n(t_k)|.$$

Since $|S_{u_k} - \xi_n(t_k)| \le |X_{1+u_k}|$ and $|\xi_n(t_{k+1}) - S_{u_{k+1}}| \le |X_{1+u_{k+1}}|$ we obtain

$$P_1(n,t) \le P_{1,1}(n,t) + 2P_{1,2}(n,t), \tag{3.12}$$

where

$$P_{1,1}(n,t) := \mathrm{P}\left(\sup_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{0\le k<2^j} |S_{u_{k+1}} - S_{u_k}| > \frac{t}{2}\right)$$
$$P_{1,2}(n,t) := \mathrm{P}\left(\max_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{1\le i\le n} |X_i| > \frac{t}{4}\right).$$

In $P_{1,2}(n,t)$, $\max_{0\le j\le \log n} 2^{j\alpha} \le n^\alpha$, so

$$P_{1,2}(n,t) \le \mathrm{P}\left(n^{-1/2+\alpha} \max_{1\le i\le n} |X_i| > \frac{t}{4}\right) \le n\,\mathrm{P}\left(|X_1| > \frac{tn^{1/p}}{4}\right).$$

Using (3.10) we obtain that for $r \le p$,

$$\lim_{a\to\infty} \sup_{n\ge 1} \int_a^\infty rt^{r-1} P_{1,2}(n,t)\,\mathrm{d}t = 0. \tag{3.13}$$

To estimate $P_{1,1}(n,t)$, we use a truncation method. Define for $t > 0$ and $0 < \delta \le 1$,

$$\widetilde{X}_i := X_i \mathbf{1}_{\{|X_i|\le \delta t n^{1/p}\}}, \qquad X_i' := \widetilde{X}_i - \mathrm{E}\,\widetilde{X}_i.$$

Let $\widetilde{S}_{u_k}$ and S'_{u_k} be the random variables obtained by replacing X_i with respectively $\widetilde{X}_i$ in S_{u_k} or with X_i' in S_{u_k}. We introduce also

$$\widetilde{P}_{1,1}(n,t,\delta) := \mathrm{P}\left(\sup_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{0\le k<2^j} |\widetilde{S}_{u_{k+1}} - \widetilde{S}_{u_k}| > \frac{t}{2}\right), \tag{3.14}$$

$$P'_{1,1}(n,t,\delta) := \mathrm{P}\left(\sup_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{0\le k<2^j} |S'_{u_{k+1}} - S'_{u_k}| > \frac{t}{4}\right). \tag{3.15}$$

First, since on the event $\{\max_{1\le i\le n} |X_i| \le \delta t n^{1/p}\}$, $\widetilde{S}_{u_k} = S_{u_k}$ for every k, we note that

$$\begin{aligned} P_{1,1}(n,t) &\le \widetilde{P}_{1,1}(n,t,\delta) + \mathrm{P}\left(\max_{1\le i\le n} |X_i| > \delta t n^{1/p}\right) \\ &\le \widetilde{P}_{1,1}(n,t,\delta) + n\,\mathrm{P}\big(|X_1| > \delta t n^{1/p}\big). \end{aligned} \tag{3.16}$$

Invoking again (3.10) reduces the control of the contribution of $P_{1,1}(n,t)$ when bounding the integral in (3.1) to the one of $\widetilde{P}_{1,1}(n,t,\delta)$.

Now to work with centered random variables, we prove that $\widetilde{P}_{1,1}(n,t,\delta) \le P'_{1,1}(n,t,\delta)$ for t large enough, uniformly in $n \ge 1$. Comparing (3.14) and (3.15), we see that $\widetilde{P}_{1,1}(n,t,\delta) \le P'_{1,1}(n,t,\delta)$ will be satisfied provided that

$$\max_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{0\le k<2^j} \sum_{i=1+u_k}^{u_{k+1}} |\operatorname{E}\widetilde{X}_i| < \frac{t}{4}. \tag{3.17}$$

As $j \le \log n$, $1 \le u_{k+1}-u_k \le 2n2^{-j}$, (3.17) reduces to $2n^{1/2}|\operatorname{E}\widetilde{X}_1| < t/4$. Since $\operatorname{E} X_1 = 0$, $\operatorname{E}\widetilde{X}_1 = -\operatorname{E} X_1 \mathbf{1}_{\{|X_1|>\delta t n^{1/p}\}}$, so we just have to check that

$$\operatorname{E}|X_1|\mathbf{1}_{\{|X_1|>\delta t n^{1/p}\}} < \frac{t}{8} n^{-1/2}, \tag{3.18}$$

for t large enough, uniformly in $n \ge 1$. By a Fubini argument,

$$\begin{aligned}
\operatorname{E}|X_1|\mathbf{1}_{\{|X_1|>\delta t n^{1/p}\}} &= \int_0^\infty \operatorname{P}\left(|X_1| > \max(\delta t n^{1/p}, s)\right) \mathrm{d}s \\
&= \int_0^{\delta t n^{1/p}} \operatorname{P}(|X_1| > \delta t n^{1/p})\, \mathrm{d}s + \int_{\delta t n^{1/p}}^\infty \operatorname{P}(|X_1| > s)\, \mathrm{d}s \\
&= \delta t n^{1/p} \operatorname{P}(|X_1| > \delta t n^{1/p}) + \int_{\delta t n^{1/p}}^\infty \operatorname{P}(|X_1| > s)\, \mathrm{d}s.
\end{aligned}$$

Now, in the case where $r < p$, using (3.8) we obtain

$$\operatorname{E}|X_1|\mathbf{1}_{\{|X_1|>\delta t n^{1/p}\}} \le \delta^{1-p}\frac{K}{t^p}\left(1+\frac{1}{p-1}\right) t n^{1/p-1} = \delta^{1-p}\frac{pK}{(p-1)t^p} t n^{-1/2-\alpha}.$$

Therefore (3.18) is satisfied for every $t > t_0$ not depending on n, since we can choose

$$t_0 = \delta^{1/p-1}\left(\frac{8pK}{(p-1)}\right)^{1/p}. \tag{3.19}$$

The same holds in the case where $r = p$, replacing (3.8) by Markov's inequality and K by $\operatorname{E}|X_1|^p$.

Now it only remains to deal with $\sup_{n\ge 1}\int_a^\infty r t^{r-1} P'_{1,1}(n,t,\delta)\, \mathrm{d}t$. For any $q > p$, we have

$$\begin{aligned}
P'_{1,1}(n,t,\delta) &= \operatorname{P}\left(\max_{0\le j\le \log n} 2^{j\alpha} n^{-1/2} \max_{0\le k<2^j} |S'_{u_{k+1}} - S'_{u_k}| > \frac{t}{4}\right) \\
&\le \sum_{j=0}^{\log n} \sum_{0\le k<2^j} \operatorname{P}\left(|S'_{u_{k+1}} - S'_{u_k}| > \frac{t n^{1/2}}{2^{j\alpha+2}}\right) \\
&\le \frac{4^q}{t^q n^{q/2}} \sum_{j=0}^{\log n} 2^{j\alpha q} \sum_{0\le k<2^j} \operatorname{E}|S'_{u_{k+1}} - S'_{u_k}|^q.
\end{aligned} \tag{3.20}$$

Next we bound up $\operatorname{E}|S'_{u_{k+1}} - S'_{u_k}|^q$ by using the Rosenthal inequality:

$$\operatorname{E}|S'_{u_{k+1}} - S'_{u_k}|^q \le C_q\left(\operatorname{Var}(S'_{u_{k+1}} - S'_{u_k})^{q/2} + \sum_{i=1+u_k}^{u_{k+1}} \operatorname{E}|X'_i|^q\right),$$

were C_q is a universal constant, *i.e.* not depending on the distribution of the X_i's. As the X_i's are i.i.d. and $u_{k+1} - u_k \leq 2n2^{-j}$, this gives

$$\mathrm{E}\,|S'_{u_{k+1}} - S'_{u_k}|^q \leq C_q \left((2n2^{-j})^{q/2} (\mathrm{E}\, {X'_1}^2)^{q/2} + 2n2^{-j}\, \mathrm{E}\,|X'_1|^q \right).$$

Going back to (3.20) with this bound, we obtain

$$P'_{1,1}(n,t,\delta) \leq \frac{4^q C_q}{t^q n^{q/2}} \left((2n)^{q/2} (\mathrm{E}\, {X'_1}^2)^{q/2} \sum_{j=0}^{\log n} 2^{j(\alpha q - q/2 + 1)} + 2n\, \mathrm{E}\,|X'_1|^q \sum_{j=0}^{\log n} 2^{j\alpha q} \right)$$

As $q > p = (1/2 - \alpha)^{-1}$, $1 - q/2 + q\alpha$ is negative, whence for every n,

$$\sum_{j=0}^{\log n} 2^{j(\alpha q - q/2 + 1)} < \frac{1}{1 - 2^{1 - q/2 + q\alpha}} = \frac{1}{1 - 2^{1 - q/p}}.$$

Next, for every $n \geq 1$,

$$\sum_{j=0}^{\log n} 2^{j\alpha q} < \frac{2^{q\alpha}}{2^{q\alpha} - 1} n^{q\alpha}.$$

Moreover

$$\mathrm{E}\, {X'_1}^2 = \mathrm{Var}\, \widetilde{X}_1 \leq \mathrm{E}\, \widetilde{X_1}^2 \leq \mathrm{E}\, X_1^2$$

and

$$\mathrm{E}\,|X'_1|^q = \mathrm{E}\,|\widetilde{X_1} - \mathrm{E}\,\widetilde{X_1}|^q \leq 2^q\, \mathrm{E}\,|\widetilde{X}_1|^q.$$

With all these partial estimates, the upper bound obtained for $P'_{1,1}(n,t,\delta)$ becomes

$$P'_{1,1}(n,t,\delta) \leq \frac{C_1(p,q)(\mathrm{E}\, X_1^2)^{q/2}}{t^q} + \frac{C_2(p,q) n^{1-q/p}\, \mathrm{E}\,|\widetilde{X}_1|^q}{t^q},$$

with

$$C_1(p,q) = \frac{2^{5q/2} C_q}{1 - 2^{1-q/p}} \quad \text{and} \quad C_2(p,q) = \frac{2^{3q + q\alpha + 1} C_q}{2^{q\alpha} - 1}.$$

Now we have for $r \leq p$,

$$\int_a^\infty r t^{r-1} P'_{1,1}(n,t,\delta)\, \mathrm{d}t \leq \frac{C_1(p,q)(\mathrm{E}\, X_1^2)^{q/2} r}{q - r} a^{r-q} + C_2(p,q) I_{r,q}(a,n), \tag{3.21}$$

where

$$I_{r,q}(a,n) = n^{1-q/p} \int_a^\infty r t^{r-q-1}\, \mathrm{E}\,|\widetilde{X}_1|^q\, \mathrm{d}t.$$

It is worth recalling here that $\mathrm{E}\,|\widetilde{X}_1|^q$ depends on n, δ and t. As the first term in the upper bound (3.21) neither depends on n or on δ and goes to 0 as a tends to infinity, it remains only to investigate the asymptotic behavior of $\sup_{n\geq 1} I_{r,q}(a,n)$ when a tends to infinity. To transform $I_{r,q}(a,n)$, we use the fact that if Y is a positive random variable and f a C^1 non decreasing function on $[0,\infty)$ with $f(0)=0$, then by the Fubini-Tonelli theorem, for any positive constant c,

$$\mathrm{E}\left(f(Y)\mathbf{1}_{\{Y\leq c\}}\right) = \int_0^c f'(s)\,\mathrm{P}(s<Y\leq c)\,\mathrm{d}s \leq \int_0^c f'(s)\,\mathrm{P}(Y>s)\,\mathrm{d}s.$$

Applying this to $\mathrm{E}\,|\widetilde{X}_1|^q = \mathrm{E}\,|X_1|^q\mathbf{1}_{\{|X_1|\leq \delta t n^{1/p}\}}$, we obtain

$$I_{r,q}(a,n) \leq n^{1-q/p}\int_a^\infty rt^{r-q-1}\int_0^{\delta t n^{1/p}} qs^{q-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s\,\mathrm{d}t =: J_{r,q}(a,n).$$

Exchanging the order of integrations in $J_{r,q}(a,n)$ gives

$$\begin{aligned} J_{r,q}(a,n) &= n^{1-q/p}\int_0^\infty \left\{\int_{\max(a,s\delta^{-1}n^{-1/p})}^\infty rt^{r-q-1}\,\mathrm{d}t\right\} qs^{q-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s \\ &= \frac{qr}{q-r}n^{1-q/p}\int_0^\infty \max(a,s\delta^{-1}n^{-1/p})^{r-q}s^{q-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s \\ &= \frac{qr}{q-r}\left(n^{1-q/p}a^{r-q}J' + n^{1-r/p}\delta^{q-r}J''\right), \end{aligned} \tag{3.22}$$

where

$$J' = \int_0^{\delta a n^{1/p}} s^{q-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s, \quad J'' = \int_{\delta a n^{1/p}}^\infty s^{r-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s.$$

We bound J' for $r\leq p$, using (3.8), agreeing for simplicity that $K=\mathrm{E}\,|X_1|^p$ when $r=p$. This gives

$$J' \leq K\int_0^{\delta a n^{1/p}} s^{q-p-1}\,\mathrm{d}s = \frac{K}{q-p}a^{q-p}\delta^{q-p}n^{q/p-1}. \tag{3.23}$$

For J'', the same method would lead to a divergent integral in the special case $r=p$, so we restrict the use of (3.8) to the case where $r<p$. This gives

$$J'' \leq \begin{cases} \frac{K}{p-r}a^{r-p}\delta^{r-p}n^{r/p-1} & \text{when } r<p, \\ \int_{\delta a n^{1/p}}^\infty s^{p-1}\,\mathrm{P}(|X_1|>s)\,\mathrm{d}s & \text{when } r=p. \end{cases} \tag{3.24}$$

Going back to $I_{r,q}(a,n)$ and accounting (3.22)–(3.24) we obtain (recalling that $\delta\leq 1$)

$$I_{r,q}(a,n) \leq \frac{Kqr}{(q-p)(p-r)}a^{r-p}\delta^{r-p} \leq \frac{Kqr}{(q-p)(p-r)}a^{r-p}, \quad (r<p). \tag{3.25}$$

In the case $r=p$, bounding the integral in (3.24) by $\frac{1}{p}\,\mathrm{E}\,|X_1|^p = K/p$, we obtain

$$I_{p,q}(a,n) \leq \frac{q^2}{(q-p)^2}K\delta^{q-p}. \tag{3.26}$$

Recapitulating all the estimate proposed throughout the proof, we see that for every $a > t_0(\delta)$ defined by (3.19) and every $n \geq 1$,

$$\begin{aligned}\int_a^\infty rt^{r-1}P(n,t)\,\mathrm{d}t \leq \int_a^\infty rt^{r-1}P_2(n,t)\,\mathrm{d}t + 2\int_a^\infty rt^{r-1}P_{1,2}(n,t)\,\mathrm{d}t \\ + \int_a^\infty rt^{r-1}n\,\mathrm{P}\left(|X_1| > \delta t n^{1/p}\right)\mathrm{d}t \\ + \frac{pC_1(p,q)(\mathrm{E}\,X_1^2)^{q/2}}{q-p}a^{p-q} + C_2(p,q)I_{r,q}(a,n).\end{aligned} \tag{3.27}$$

In the case where $r < p$, recalling (3.11), (3.13), (3.10) and (3.25), it follows

$$\lim_{a\to\infty}\sup_{n\geq 1}\int_a^\infty rt^{r-1}P(n,t)\,\mathrm{d}t = 0.$$

We note in passing that in this case we did not really need the freedom to tune the value of δ, the simple choice $\delta = 1$ would have done the job as well.

In the special case where $r = p$, we have only to modify the treatment of the last term in the bound (3.27). As $q > p$, for any $\varepsilon > 0$, we can fix a $\delta > 0$ such that $C_2(p,q)(1-p/q)^{-2}K\delta^{q-p} < \varepsilon$. Accounting (3.11), (3.13), (3.10) and (3.26), there is some a_1 depending on ε, p, q and on the distribution of X_1, such that for every $a \geq a_1$ and every $n \geq 1$,

$$\int_a^\infty pt^{p-1}P(n,t)\,\mathrm{d}t < 6\varepsilon.$$

As ε was arbitrary, the uniform convergence (3.1) is established and the proof is complete.

3.2. Comments

If we fix $p > 2$ and consider $X_1 \in \mathcal{L}_p$, then the best possible Hölderian index corresponding to the p'th moments convergence is $\alpha = \alpha(p) := 1/2 - 1/p$ as shows the following result.

Theorem 3.1. *Let $p > 2$. If for any $X_1 \in \mathcal{L}_p$,*

$$\sup_{n\geq 1}\mathrm{E}\,\|n^{-1/2}\xi_n\|_\beta^p < \infty \tag{3.28}$$

then $\beta \leq \alpha(p) = 1/2 - 1/p$.

Proof. Put $r = (1/2-\beta)^{-1}$. By looking at the increments of $n^{-1/2}\xi_n$ between k/n and $(k+1)/n$, $0 \leq k < n$, we see that

$$\|n^{-1/2}\xi_n\|_\beta^p \geq \left(n^{-1/2+\beta}\max_{1\leq i\leq n}|X_i|\right)^p,$$

so (3.28) implies

$$\sup_{n\geq 1}\int_0^\infty t^{p-1}\,\mathrm{P}\left(\max_{1\leq i\leq n}|X_i| > n^{1/r}t\right)\mathrm{d}t < \infty,$$

which can be recast as

$$\sup_{n\geq 1} n^{-p/r}\int_0^\infty s^{p-1}\,\mathrm{P}\left(\max_{1\leq i\leq n}|X_i|>s\right)\mathrm{d}s<\infty. \tag{3.29}$$

It is well known that when the X_i's are i.i.d.,

$$n\,\mathrm{P}(|X_1|>s)\leq 1 \;\Rightarrow\; \mathrm{P}\left(\max_{1\leq i\leq n}|X_i|>s\right)\geq (1-\mathrm{e}^{-1})n\,\mathrm{P}(|X_1|>s). \tag{3.30}$$

Now choose for $|X_1|$ the distribution given by

$$\mathrm{P}(|X_1|>s)=\frac{c}{s^p(\ln s)^2}\mathbf{1}_{[2,\infty)}(s).$$

Then $\mathrm{P}(|X_1|>n^{1/p})=cp^2n^{-1}(\ln n)^{-2}\leq n^{-1}$ for $n\geq n_0$, so we deduce from (3.29) and (3.30) that

$$\sup_{n\geq n_0} n^{1-p/r}\int_{n^{1/p}}^\infty \frac{1}{s(\ln s)^2}\,\mathrm{d}s=\sup_{n\geq n_0}\frac{pn^{1-p/r}}{\ln n}<\infty,$$

which holds if and only if $p/r\geq 1$ or equivalently $\beta\leq\alpha(p)$. □

Let us turn now on some probability distances considerations. Recall the Kantorovich functionals which are involved in the Monge-Kantorovich minimal cost of mass transportation problem :

$$\mathcal{A}_c(P_1,P_2)=\inf\left\{\int_{\mathbb{S}\times\mathbb{S}}c(x,y)P(\mathrm{d}x,\mathrm{d}y),\ P\in\mathcal{P}^{(P_1,P_2)}\right\}, \tag{3.31}$$

where $(\mathbb{S},d)$ is a separable metric space, $\mathcal{P}^{(P_1,P_2)}$ denotes the set of all probabilities on the Borel σ-field of $\mathbb{S}\times\mathbb{S}$ with given marginals P_1, P_2 and $c(x,y)=H(d(x,y))$ where $H(0)=0$, H is non decreasing on $[0,\infty)$ and satisfies the Orlicz condition $\sup_{t>0}H(2t)/H(t)<\infty$. It is known, see Theorem 11.1.1 in [11] that if for some $a\in\mathbb{S}$, $\int_{\mathbb{S}}c(x,a)P_n(\,\mathrm{d}x)<\infty$ for every $n\geq 1$, then

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{A}_c(P_n,P_0)=0\quad\text{if and only if}\\ &\qquad P_n\xrightarrow[n\to\infty]{\text{weakly}}P_0\text{ and }\lim_{n\to\infty}\int_{\mathbb{S}}c(x,b)(P_n-P_0)(\mathrm{d}x)=0,\end{aligned} \tag{3.32}$$

for some (and therefore for any) $b\in\mathbb{S}$.

Let us denote by $\mathcal{A}_r^\alpha$ the Kantorovich functional obtained by choosing $H(t)=t^r$ and $\mathbb{S}=\mathcal{H}_\alpha^o[0,1]$ with $d(x,y)=\|x-y\|_\alpha$ (recall that $\mathcal{H}_0^o[0,1]$ is isomorphic to $\mathrm{C}[0,1]$). We observe that $(\mathcal{A}_r^\alpha)^{1/r}$ is the Wasserstein distance W_r associated to the space $\mathcal{H}_\alpha^o[0,1]$. Write P_n for the distribution of $n^{-1/2}\xi_n$ and P_0 for the Wiener measure. Then (3.32) can be rewritten as

$$\begin{aligned}\lim_{n\to\infty}\mathcal{A}_r^\alpha(P_n,P_0)=0\quad\text{if and only if}\quad & n^{-1/2}\xi_n\xrightarrow[n\to\infty]{\mathrm{d}}W\text{ in }\mathcal{H}_\alpha^o[0,1]\text{ and}\\ &\mathrm{E}\,\|n^{-1/2}\xi_n\|_\alpha^r\xrightarrow[n\to\infty]{}\mathrm{E}\,\|W\|_\alpha^r.\end{aligned} \tag{3.33}$$

From this point of view, Theorem 1.1 means that the convergence of $\mathcal{A}_r^\alpha(P_n,P_0)$ to 0 is equivalent to the moment condition (1.10) or (1.11) according to $r<p(\alpha)$ or $r\geq p(\alpha)$. Similarly, from the Bernstein Kantorovich

invariance principle in $\mathrm{C}[0,1]$, the convergence of $\mathcal{A}_r^0(P_n, P_0)$ to 0 is equivalent to $\mathrm{E}\,|X_1|^{\max(r,2)} < \infty$. As already hinted in the introduction, we see that starting from the classical Donsker-Prokhorov invariance principle in $\mathrm{C}[0,1]$ ($\mathcal{A}_2^0(P_n, P_0) \to 0$ iff $\mathrm{E}\,X_1^2 < \infty$) and looking for a stronger convergence in the framework of $\mathrm{C}[0,1]$ ($\mathcal{A}_p^0(P_n, P_0) \to 0$) at the price of a stronger moment assumption $\mathrm{E}\,|X_1|^p < \infty$ ($p > 2$), we obtain a similar convergence ($\mathcal{A}_p^\alpha(P_n, P_0) \to 0$) with a stronger topological path's space.

3.3. An application to uniform quantile processes

As a corollary of Theorem 1.1, we look now at the convergence of moments for the uniform quantile process. For the weak-Hölderian convergence of the uniform quantile process we refer to [7].

Let $U_1, \dots, U_n$ be a sample of i.i.d. random variables uniformly distributed on $[0,1]$. We denote by $U_{n:i}$ the order statistics of the sample:

$$0 = U_{n:0} \le U_{n:1} \le \dots \le U_{n:n} \le U_{n:n+1} = 1,$$

which are distinct with probability one. For notational convenience, put

$$u_{n:i} = \mathrm{E}\,U_{n:i} = \frac{i}{n+1}, \quad i = 0, 1, \dots, n+1.$$

The polygonal uniform quantile process χ_n^{pg} is the random polygonal line on $[0,1]$ which is affine on each $[u_{n:i-1}, u_{n:i}]$, $i = 1, \dots, n+1$ and satisfies

$$\chi_n^{\mathrm{pg}}(u_{n:i}) = \sqrt{n}(U_{n:i} - u_{n:i}), \quad i = 0, 1, \dots, n+1. \tag{3.34}$$

As a corollary of Theorem 10 in [7], for any $0 < \alpha < 1/2$, χ_n^{pg} converges weakly in $\mathcal{H}_\alpha^o[0,1]$ to the Brownian bridge B. Theorem 1.1 enables us to complete this convergence by the following convergence of moments.

Corollary 3.2. *Let χ_n^{pg} be the polygonal uniform quantile process defined above. Then for every $0 \le \alpha < 1/2$, and every $r > 0$,*

$$\lim_{n\to\infty} \mathrm{E}\,\|\chi_n^{\mathrm{pg}}\|_\alpha^r = \mathrm{E}\,\|B\|_\alpha^r, \tag{3.35}$$

where B is the Brownian bridge on $[0,1]$.

Proof. We recall the distributional equality (see *e.g.* [15])

$$(U_{n:1}, \dots, U_{n:n}) \stackrel{d}{=} \Big(\frac{S_1}{S_{n+1}}, \dots, \frac{S_n}{S_{n+1}}\Big), \tag{3.36}$$

where $S_k = X_1 + \dots + X_k$ and the X_k's are i.i.d 1-exponential random variables. Following [7], introduce the polygonal process ζ_n which is affine on each interval $[u_{n:i-1}, u_{n:i}]$, $i = 1, \dots, n+1$ and such that

$$\zeta_n(u_{n:i}) = \sqrt{n}\Big(\frac{S_i}{S_{n+1}} - u_{n:i}\Big), \quad i = 0, 1, \dots, n+1. \tag{3.37}$$

It easily follows from the distributional equalities (3.36) that χ_n^{pg} and ζ_n have the same distribution as random elements in any $\mathcal{H}_\alpha^o[0,1]$ ($0 \le \alpha < 1$). Hence it is enough to prove the convergence of moments (3.35) with χ_n^{pg} replaced by ζ_n.

Putting $X_i' = X_i - \mathrm{E}\, X_i$ (note that $\mathrm{E}\, X_1'^2 = 1$) and $S_k' = S_k - \mathrm{E}\, S_k$, we consider also the normalized partial sums polygonal process Ξ_n built on the S_k''s, *i.e.* the random polygonal line with vertices $(k/n, n^{-1/2}S_k')$, $k = 0, 1, \dots, n$. As shown in the proof of Theorem 10 in [7],

$$\zeta_n(t) = \frac{\sqrt{n(n+1)}}{S_{n+1}}\bigl(\Xi_{n+1}(t) - t\,\Xi_{n+1}(1)\bigr), \quad t \in [0,1]. \tag{3.38}$$

Since we already know that ζ_n, as well as χ_n^{pg}, converges weakly to B in any $\mathcal{H}_\alpha^o[0,1]$ for $0 \le \alpha < 1/2$, it remains just to check the uniform integrability of the sequence $(\|\zeta_n\|_\alpha^r)_{n\ge 1}$, for which it suffices to prove that for some $s > r$,

$$\sup_{n\ge 1} \mathrm{E}\,\|\zeta_n\|_\alpha^s < \infty. \tag{3.39}$$

Define for each $x \in \mathrm{C}[0,1]$, the function x^{br}: $[0,1] \to \mathbb{R}$, $t \mapsto x(t) - tx(1)$. It is easily seen that the bridge linear operator $x \mapsto x^{\mathrm{br}}$ maps $\mathcal{H}_\alpha^o[0,1]$ into itself and that for every $x \in \mathcal{H}_\alpha^o[0,1]$, $\|x^{\mathrm{br}}\|_\alpha \le 2\|x\|_\alpha$. Now

$$\mathrm{E}\,\|\zeta_n\|_\alpha^s \le \mathrm{E}\left(\left|\frac{n+1}{S_{n+1}}\right| \|\Xi_{n+1}^{\mathrm{br}}\|_\alpha\right)^s \le 2^s\, \mathrm{E}^{1/2} \left|\frac{n+1}{S_{n+1}}\right|^{2s} \mathrm{E}^{1/2}\,\|\Xi_{n+1}\|_\alpha^{2s}.$$

To obtain (3.39) with any $s > r$, we just note the following facts. First, by elementary computation,

$$\mathrm{E}\left|\frac{n+1}{S_{n+1}}\right|^{2s} = \frac{(n+1)^{2s}}{n!}\Gamma(n+1-2s) = \frac{(n+1)^{2s}(n-2s)!}{n!} \xrightarrow[n\to\infty]{} 1.$$

Next, since X_1' has finite moments of every order, $\mathrm{E}\,\|\Xi_{n+1}\|_\alpha^{2s}$ converges to $\mathrm{E}\,\|W\|_\alpha^{2s}$ by Theorem 1.1. □

4. Weighted scan statistics

In this section we consider several weighted scan type statistics. For $\alpha \ge 0$, define

$$M_{n,\alpha} = \max_{1\le \ell\le n} \frac{1}{\ell^\alpha} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k|,$$

and

$$T_{n,\alpha} = \max_{1\le \ell\le n} \frac{1}{[\ell(1-\ell/n)]^\alpha} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k - (\ell/n)S_n|.$$

Theorem 4.1. *Let $p > 2$. If $\mathrm{E}\,|X_1|^p < \infty$, then for any $0 \le \alpha \le 1/2 - 1/p$ and any $0 \le r \le p$,*

$$\lim_{n\to\infty} n^{-r(1/2-\alpha)}\, \mathrm{E}\, M_{n,\alpha}^r = \mathrm{E}\,\omega_\alpha^r(W,1); \tag{4.1}$$

and

$$\lim_{n\to\infty} n^{-r(1/2-\alpha)}\, \mathrm{E}\, T_{n,\alpha}^r = \mathrm{E}\, T_\alpha^r(B), \tag{4.2}$$

where

$$T_\alpha(B) = \sup_{0<h<1} \frac{1}{[h(1-h)]^\alpha} \sup_{0\le t<1-h} |B(t+h) - B(t)|$$

and $B = (B_t = W_t - tW_1, 0 \le t \le 1)$ *denotes the Brownian bridge on* $[0,1]$.

Proof. By Lemma 2.2 we have

$$n^{-1/2+\alpha} M_{n,\alpha} = \omega_\alpha(n^{-1/2}\xi_n, 1)$$

and the convergence (4.1) follows from Theorem 1.1.

To prove (4.2), we use the representation

$$n^{-1/2+\alpha} T_{n,\alpha} = g_n(n^{-1/2}\xi_n) = g(n^{-1/2}\xi_n) + o_P(1), \tag{4.3}$$

which is explained in details in [14]. Here the functional g is defined by

$$g : \mathcal{H}^o_\alpha[0,1] \to \mathbb{R}_+, \quad x \mapsto \sup_{0<h<1} \frac{1}{[h(1-h)]^\alpha} \sup_{0\le t<1-h} |x(t+h) - x(t) - hx(1)|$$

and $(g_n)_{n\ge1}$ is an equicontinuous sequence of non negative, subadditive, homogeneous ($g_n(cx) = |c|g_n(x)$) functionals $\mathcal{H}^o_\alpha[0,1] \to \mathbb{R}_+$ converging pointwise to g on $\mathcal{H}^o_\alpha$ and verifying for some positive constant C

$$0 \le g_n(x) \le g(x) \le C\|x\|_\alpha. \tag{4.4}$$

By the Hölderian invariance principle [12], continuous mapping and Slutsky lemma, (4.3) provides the convergence in distribution of $g_n(n^{-1/2}\xi_n)$ to $g(W)$. Then in view of (4.4), Theorem 1.1 gives (4.2) since $g(W) = T_\alpha(B)$. □

Theorem 4.2. *Let* $p > 0$. *Assume that* $X_1 \in \mathrm{RV}_p$ *and* $\mathrm{E}\,X_1 = 0$ *when* $p > 1$. *Then for* $\alpha > \max\{0, 1/2 - 1/p\}$, *and any* $0 \le r < p$ *it holds*

$$\lim_{n\to\infty} b_n^{-r}\,\mathrm{E}\,M^r_{n,\alpha} = \mathrm{E}\,Y^r_p \tag{4.5}$$

and

$$\lim_{n\to\infty} b_n^{-r}\,\mathrm{E}\,T^r_{n,\alpha} = \mathrm{E}\,Y^r_p, \tag{4.6}$$

where b_n *is defined by* (1.13) *and* Y_p *has the Fréchet distribution with exponent* p.

Proof. From [9] we know, that

$$b_n^{-1} M_{n,\alpha} \xrightarrow[n\to\infty]{\mathrm{d}} Y_p.$$

Hence, in order to prove convergence of moments we need to check uniform integrability of $(b_n^{-r} M^r_{n,\alpha})$ for each $0 < r < p$. Actually it is enough to prove that *for each* $0 < r < p$,

$$\sup_{n\ge1} \mathrm{E}[b_n^{-1} M_{n,\alpha}]^r < \infty. \tag{4.7}$$

And to establish (4.7) it is clearly sufficient to prove that for some positive constant c and some integer n_0 (possibly depending on r),

$$\sup_{n\ge n_0} \int_c^\infty y^{r-1}\,\mathrm{P}(M_{n,\alpha} > b_n y)\,\mathrm{d}y < \infty. \tag{4.8}$$

To this aim we shall use the following lemma (see Lem. 3.3 in [9]).

Lemma 4.3. *Assume that $(Y_i, i \geq 1)$ are i.i.d. random variables. Then for all $y > 0$ and $n \geq 1$,*

$$\mathrm{P}\left(\max_{1\leq \ell\leq n} \ell^{-\alpha} \max_{1\leq k\leq n}\left|\sum_{i=k+1}^{k+\ell} Y_i\right| > y\right) \leq 2\sum_{j=1}^{\log n} 2^j\, \mathrm{P}\left(\max_{1\leq k\leq n2^{-j}}\left|\sum_{i=1}^{k} Y_i\right| > y(n2^{-j})^{\alpha}\right).$$

We shall estimate for $y > 0$ the probability

$$P(y) = \mathrm{P}(b_n^{-1} M_{n,\alpha} > y).$$

To this aim, we introduce the truncated random variables

$$\widetilde{X}_k = X_k \mathbf{1}_{\{|X_k|\leq y b_n\}}, \quad X'_k = \widetilde{X}_k - \mathrm{E}\,\widetilde{X}_k, \quad k = 1, \ldots, n.$$

Then

$$P(y) \leq \mathrm{P}\left(\max_{1\leq k\leq n} |X_k| > y b_n\right) + \mathrm{P}(b_n^{-1}\widetilde{M}_{n,\alpha} > y),$$

where $\widetilde{M}_{n,\alpha}$ is defined as $M_{n,\alpha}$ substituting X_k by $\widetilde{X}_k$. Let $M'_{n,\alpha}$ be defined as $M_{n,\alpha}$ substituting X_k by X'_k. We have

$$\begin{aligned} b_n^{-1}\widetilde{M}_{n,\alpha} &= b_n^{-1} \max_{1\leq \ell\leq n} \ell^{-\alpha} \max_{0\leq k\leq n-\ell} |\widetilde{S}_{k+\ell} - \widetilde{S}_k| \\ &\leq b_n^{-1} M'_{n,\alpha} + b_n^{-1} \max_{1\leq \ell\leq n} \ell^{-\alpha} \max_{0\leq k\leq n-\ell} \left|\sum_{i=k+1}^{k+\ell} \mathrm{E}\,\widetilde{X}_k\right| \\ &= b_n^{-1} M'_{n,\alpha} + b_n^{-1} n^{1-\alpha} |\,\mathrm{E}\,\widetilde{X}_1|. \end{aligned}$$

Since X_1 is regularly varying with index p, Lemma 2.4 gives the estimates

$$\mathrm{E}\,|X_1|\mathbf{1}_{\{|X_1|\leq y b_n\}} \leq c_p n^{-1} b_n y^{1-p} \quad \text{if } p < 1, \tag{4.9}$$
$$\mathrm{E}\,|X_1|\mathbf{1}_{\{|X_1|> y b_n\}} \leq c_p n^{-1} b_n y^{1-p} \quad \text{if } p > 1. \tag{4.10}$$

When $0 < p < 1$, (4.9) gives

$$b_n^{-1} n^{1-\alpha} |\,\mathrm{E}\,\widetilde{X}_1| \leq c_p n^{-\alpha} y^{1-p} \leq \frac{1}{2} y, \quad \text{for } n \geq c_p^{1/\alpha} \text{ and } y \geq 2^{1/p}. \tag{4.11}$$

In the case where $p > 1$, X_1 is integrable and $\mathrm{E}\,X_1 = 0$, hence $\mathrm{E}\,X_1 \mathbf{1}_{\{|X_1|\leq y b_n\}} = -\,\mathrm{E}\,X_1 \mathbf{1}_{\{|X_1|> y b_n\}}$, so (4.10) leads also to (4.11). It follows that for $n \geq c_p^{1/\alpha}$ and $y \geq 2^{1/p}$,

$$\mathrm{P}(b_n^{-1}\widetilde{M}_{n,\alpha} > y) \leq \mathrm{P}(b_n^{-1} M'_{n,\alpha} > y/2). \tag{4.12}$$

Consequently,

$$P(y) \leq \mathrm{P}\left(\max_{1\leq k\leq n} |X_k| > y b_n\right) + \mathrm{P}(b_n^{-1} M'_{n,\alpha} > y/2).$$

By Lemma 4.3, Markov and Doob inequalities with $q > p$,

$$\begin{aligned} \mathrm{P}(b_n^{-1} M'_{n,\alpha} > y/2) &\le 2 \sum_{j=1}^{\log n} 2^j \,\mathrm{P}\left(\max_{1 \le k \le n2^{-j}} \left| \sum_{i=1}^{k} X'_i \right| > (n2^{-j})^{\alpha} y b_n/2 \right) \\ &\le 2 \left(\frac{2q}{q-1} \right)^q \sum_{j=1}^{\log n} 2^j (n2^{-j})^{-\alpha q} y^{-q} b_n^{-q} \,\mathrm{E} \left| \sum_{i=1}^{n2^{-j}} X'_i \right|^q . \end{aligned} \tag{4.13}$$

By Rosenthal's inequality,

$$\mathrm{E} \left| \sum_{i=1}^{n2^{-j}} X'_i \right|^q \le C_q \big((n2^{-j})^{q/2} (\mathrm{E}\, {X'_1}^2)^{q/2} + n2^{-j}\,\mathrm{E}\,|X'_1|^q \big).$$

By iii) in Lemma 2.4, assuming $q > 1$

$$\mathrm{E}\, {X'_1}^2 \le \frac{8p}{|p-2|} b_n^2 \frac{1}{n} y^{2-p}, \quad \mathrm{E}\,|X'_1|^q \le \frac{p2^{q+1}}{q-p} b_n^q \frac{1}{n} y^{q-p},$$

whence

$$\mathrm{E} \left| \sum_{i=1}^{n2^{-j}} X'_i \right|^q \le C_q \left(\frac{(8p)^{q/2}}{|p-2|^{q/2}} 2^{-jq/2} b_n^q y^{q-pq/2} + \frac{p2^{q+1}}{q-p} 2^{-j} b_n^q y^{q-p} \right).$$

It is worth noticing here that for $s > 0$,

$$\sum_{j=1}^{\log n} 2^{js} \le \frac{1}{1-2^{-s}} n^s,$$

recalling that $\log n$ denotes the dyadic logarithm: $2^{\log n} = n$. Moreover we can always choose q such that

$$q > \max(2, p) \quad \text{and} \quad 1 + (\alpha - 1/2) q > 0.$$

Indeed if $\alpha \ge 1/2$, the second inequality above is automatically satisfied for any $q > 0$ which leaves us free to choose $q > \max(2, p)$. If $1/2 > \alpha > \max(1/2 - 1/p, 0)$, one can find q so that

$$\alpha > 1/2 - 1/q > \max(1/2 - 1/p, 0).$$

The first inequality above gives $1 + (\alpha - 1/2)q > 0$ while the second gives $q > \max(2, p)$.

Now, going back to (4.13), we obtain for large n, uniformly in $y \in [2^{1/p}, \infty)$,

$$\mathrm{P}\left(b_n^{-1} M'_{n,\alpha} > y/2 \right) \le C y^{-p},$$

with

$$C := 2 \left(\frac{2q}{q-1} \right)^q \left(\left(\frac{8p}{|p-2|} \right)^{q/2} + \frac{p2^{q+1}}{q-p} \right) \frac{2^{q\alpha}}{2^{q\alpha} - 2^{-1+q/2}}.$$

Using (4.12) yields

$$P_n(y) \le \mathrm{P}\left(\max_{1\le k\le n} |X_k| > yb_n\right) + Cy^{-p}.$$

Integrating this inequality over the interval $[2^{1/p}, \infty)$ gives

$$\int_{2^{1/p}}^{\infty} y^{r-1} P_n(y)\,\mathrm{d}y \le \int_{2^{1/p}}^{\infty} y^{r-1} n\,\mathrm{P}\left(|X_1| > yb_n\right)\,\mathrm{d}y + \frac{2^{r/p-1}C}{p-r}$$

By Markov inequality and i) in Lemma 2.4,

$$\mathrm{P}\left(|X_1| > yb_n\right) \le b_n^{-r} y^{-r}\,\mathrm{E}\,|X_1|^r \mathbf{1}_{\{|X_1|>b_n y\}} \le \frac{2p}{p-r}\frac{1}{n} y^{-p},$$

so finally,

$$\int_{2^{1/p}}^{\infty} y^{r-1} n\,\mathrm{P}\left(|X_1| > yb_n\right)\,\mathrm{d}y \le \frac{p2^{r/p}}{(p-r)^2},$$

which completes the verification of (4.8) and the proof of the convergence (4.5) in Theorem 4.2.

To prove the convergence (4.6), we already know from [9] that

$$b_n^{-1} T_{n,\alpha} \xrightarrow[n\to\infty]{\mathrm{d}} Y_p.$$

Hence it remains only to check that

$$\sup_{n\ge 1} \mathrm{E}[b_n^{-1} T_{n,\alpha}]^r < \infty. \tag{4.14}$$

Using the same representation as in the proof of Theorem 4.1, we see from (4.3) and (4.4) that

$$n^\alpha T_{n,\alpha} = g_n(\xi_n) \le C\|\xi_n\|_\alpha.$$

As $\xi_n(0) = 0$, $\|\xi_n\|_\alpha = \omega_\alpha(\xi_n, 1) = n^\alpha M_{n,\alpha}$, whence

$$b_n^{-1} T_{n,\alpha} \le C b_n^{-1} M_{n,\alpha},$$

so (4.14) follows from (4.7) established above. □

Appendix A. Proof of Lemma 2.4

Before proving the Lemma 2.4, let us recall some basic facts from Karamata theory. A positive measurable function f defined on some neighbourhood $[a, \infty)$ $(a \ge 0)$ of infinity and satisfying for some real p and every $y > 0$,

$$\frac{f(xy)}{f(x)} \xrightarrow[x\to\infty]{} y^p,$$

is said regularly varying (at infinity) with exponent p. In the special case where $p = 0$, f is said slowly varying. It is easily seen that each regulary varying function f with exponent p can be writen as $f(x) = x^p \ell(x)$ where ℓ is slowly varying. So for $X \in \mathrm{RV}_p$, the tail function $t \mapsto \mathrm{P}(|X| > t)$ is regularly varying with exponent $-p$.

Assuming for notational simplicity that $a = 0$ and that f is locally bounded and regularly varying with exponent p, let us define for r real,

$$f_r(x) = \int_0^x y^r f(y)\,\mathrm{d}y, \quad f_r^*(x) = \int_x^\infty y^r f(y)\,\mathrm{d}y. \tag{A.1}$$

From Karamata's theorem (see *e.g.* [6], Thm. 1, p.281) we know that

i) If $f_r^*(x)$ is finite (for some and then for every positive x),

$$\frac{x^{r+1} f(x)}{f_r^*(x)} \xrightarrow[x\to\infty]{} \lambda = -(r+p+1) \geq 0. \tag{A.2}$$

ii) If $r \geq -p-1$, then

$$\frac{x^{r+1} f(x)}{f_r(x)} \xrightarrow[x\to\infty]{} \lambda = r+p+1. \tag{A.3}$$

Proof of Lemma 2.4. We take $f(x) = \mathrm{P}(|X| > x)$ and use the notations (A.1). Then f is regularly varying with exponent $-p$.

To prove i), we note first that by a Fubini argument,

$$\begin{aligned} \mathrm{E}\,|X|^s \mathbf{1}_{\{|X|>yb_n\}} &= (yb_n)^s\,\mathrm{P}(|X| > yb_n) + s\int_{yb_n}^\infty t^{s-1}\,\mathrm{P}(|X| > t)\,\mathrm{d}t \\ &= (yb_n)^s f(yb_n) + s f_{s-1}^*(yb_n). \end{aligned} \tag{A.4}$$

Then applying (A.2) with $\lambda = -(s-1-p+1) = p-s > 0$ and fixed y, we obtain

$$\frac{f_{s-1}^*(yb_n)}{(yb_n)^s f(yb_n)} \xrightarrow[n\to\infty]{} \frac{1}{\lambda} = \frac{1}{p-s}.$$

For $y \geq 1$, $b_n y \geq b_n \to \infty$, so the above convergence is obviously uniform in $y \in [1,\infty)$. Hence when n tends to infinity,

$$\mathrm{E}\,|X|^s \mathbf{1}_{\{|X|>yb_n\}} \sim \frac{p}{p-s}(yb_n)^s f(yb_n),$$

uniformly in $y \in [1,\infty)$. Now since f has regular variation with exponent $-p < 0$, $f(yb_n) \sim y^{-p} f(b_n)$, uniformly in $y \in [1,\infty)$, see *e.g.* Theorem 1.5.2 in [4]. Recalling now the definition (1.13) of b_n as the $1-1/n$ quantile of $|X|$ and using the right continuity of the distribution function of $|X|$ we see that $f(b_n) \leq 1/n$, which gives (2.3) and complete the proof of i).

The proof of ii) is completely similar and will be omitted.

To check iii), let us begin with the case where $s \geq 1$. Then since $x \mapsto x^s$ is convex on $\mathbb{R}$ for $s \geq 1$, $|X'|^s \leq 2^{s-1}\big(|\widetilde{X}|^s + \mathrm{E}(|\widetilde{X}|^s)\big)$, whence $\mathrm{E}\,|X'|^s \leq 2^s\,\mathrm{E}\,|\widetilde{X}|^s$. Using (2.3) or (2.4) according to $s < p$ or $s > p$, we obtain

$$\mathrm{E}\,|X'|^s \leq \frac{2^{s+1}p}{|p-s|} b_n^s \frac{1}{n} y^{s-p}$$

for n large enough, uniformly in $y \in [1, \infty)$,

For $s < 1$, we have directly $(|\widetilde{X}| + \mathrm{E}\,|\widetilde{X}|)^s \leq |\widetilde{X}|^s + (\mathrm{E}\,|\widetilde{X}|)^s$, which leads similarly to

$$\mathrm{E}\,|X'|^s \leq \frac{4p}{|p-s|} b_n^s \frac{1}{n} y^{s-p}.$$

$\square$

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