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# RESCALED WEIGHTED DETERMINANTAL RANDOM BALLS

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**Abstract.** We consider a collection of weighted Euclidian random balls in $\mathbb{R}^d$ distributed according a determinantal point process. We perform a zoom-out procedure by shrinking the radii while increasing the number of balls. We observe that the repulsion between the balls is erased and three different regimes are obtained, the same as in the weighted Poissonian case.

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## 1. Introduction

In this work, we give a generalization of the existing results concerning the asymptotics study of random balls model, that are used to represent a variety of situations. In dimension one, a random balls model can represent the traffic in a communication network. In this case, the (half-)balls are intervals $[x, x+r]$ and represent sessions of connection to the network, $x$ being the date of connection and $r$ the duration of connection. Such a model is investigated in [12] in a Poissonian setting, see also [11]. In dimension two, the model can represent a wireless network with $x$ being the location of a base station emitting a signal with a range $r$ so that $B(x, r)$ represents the covering area of the station $x$ and the collection of the random balls gives the overall covering of the network, cf. [15]. The two-dimensional model is used also in imagery to represent Black and White pictures. In dimension three, such models are again used to represent porous media, for instance bones can be modeled in this way and an analysis of the model allows in this case to investigate anomalies such as osteoporosis, see [1].

The first results are obtained in 2007 by Kaj *et al.* in [10]. In their model, the balls are generated by an homogeneous Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ (see [5] for a general reference on point processes). In 2009, Breton and Dombry generalize this model adding in [3] a mark $m$ on the balls of the previous model and they obtain limit theorems on the so-called rescaled weighted random balls model. In 2010, Biermé *et al.* obtain in [2] results performing for the first time a zoom-in scaling. In 2014, Gobard in his paper [8] extends the results of [3] considering inhomogeneous weighted random balls, and adding a dependence between the centers and the radii. The next step is to consider repulsion between the balls. In [4], Breton *et al.* give results on determinantal random balls model, but no weight are considered in their model. In this note, we consider weighted random balls generated by a non-stationary determinantal point process. To that purpose, we give an extension of the Laplace transform of determinantal processes allowing to compute Laplace transform with not necessarily compactly supported function, but with instead a condition of integrability. The main contributions of this note

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thus are a simplication of the proof of [4] and the introduction of weights in the non-stationary determinantal random balls model.

The introduction of weights allowed the modeling of more general situations. If we take again the examples of the beginning of the introduction, in dimension one, the weights are modelling the bandwidth used per connection. Without the weights, we have to suppose that each connection to the network uses the same bandwidth, which is not realistic. In the same idea, in dimension two, the weights are simulating the fact that each network antenna will have a different transmission power.

This paper is organized as follows. In Sections 2 and 3, we give a description of the model and state our main results under the three different regimes. In Section 4, we give the Laplace transform of a determinantal point process for not compactly supported test functions and prove our results. Finally some technical results are gathered in the Appendix.

## 2. Model

We consider a model of random balls in $\mathbb{R}^d$ constructed in the following way. The centers of the balls are generated by a determinantal point process (DPP) $\phi$ on $\mathbb{R}^d$ characterized by its kernel $K$ with respect to the Lebesgue measure. The motivation for considering such processes is that it introduces repulsion between the centers in agreement with various real model of balls. We assume that the map $\mathbf{K}$ given for all $f \in L^2(\mathbb{R}^d, \mathrm{d}x)$ and $x \in \mathbb{R}^d$ by

$$\mathbf{K}f(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\ \mathrm{d}y \tag{2.1}$$

satisfies the following hypothesis:

**Hypothesis 2.1.** The map $\mathbf{K}$ given in (2.1) is a bounded symmetric integral operator $\mathbf{K}$ from $L^2(\mathbb{R}^d, \mathrm{d}x)$ into $L^2(\mathbb{R}^d, \mathrm{d}x)$, with a continuous kernel $K$ with spectrum included in $[0, 1[$. Moreover, $\mathbf{K}$ is locally trace-class, i.e. for all compact $\Lambda \subset \mathbb{R}^d$, the restriction $\mathbf{K}_\Lambda$ of $\mathbf{K}$ on $L^2(\Lambda, \lambda)$ is of trace-class.

Moreover, we also assume

$$x \longmapsto K(x,x) \in L^\infty(\mathbb{R}^d). \tag{2.2}$$

These assumptions imply that $K(x,x) \geq 0$.

**Example 2.2.** A typical example of DPP is given by Ginibre point processes. In our real framework, the Ginibre-type point process $\phi^G$ is a DPP with kernel

$$K^G(x,y) = \exp\Big(-\frac{1}{2}\|x-y\|^2\Big), \quad x, y \in \mathbb{R}^d,$$

with respect to the Lebesgue measure. Such processes have been used recently to model wireless networks of communication, see [7], [13].

At each center $x \in \mathbb{R}^d$, we attach two positive marks $r$ and $m$ independently. The first mark is interpreted as the radius and the second mark is the weight of the ball $B(x,r)$. The radii (resp. the weight) are independently and identically distributed according to $F$ (resp. according to $G$), assumed to admit a probability density $f$ (resp. a probability density $g$). We have a new point process $\Phi$ on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ and according to Proposition A.7 in [4], $\Phi$ is a DPP on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with kernel

$$\widehat{K}\big((x,r,m),(y,s,m')\big) = \sqrt{g(m)}\sqrt{f(r)}K(x,y)\sqrt{f(s)}\sqrt{g(m')},$$

with respect to the Lebesgue measure.

Moreover, we suppose that the probability measure $G$ belongs to the normal domain of attraction of the $\alpha$-stable distribution $S_\alpha(\sigma, b, \tau)$ with $\alpha \in (1, 2]$. Because $\alpha > 1$, we can note that

$$\int_{\mathbb{R}_+} mG(\mathrm{d}m) = \int_{\mathbb{R}_+} mg(m)\mathrm{d}m < +\infty. \tag{2.3}$$

In the sequel, we shall use the notation $\Phi$ both for the marked DPP (*i.e.* the random locally finite collection of points $(X_i, R_i, M_i)$) and for the associated random measure $\sum_{(X,R,M)\in\Phi} \delta_{(X,R,M)}$. We consider the contribution of the model in any suitable measure $\mu$ on $\mathbb{R}^d$ given by the following measure-indexed random field:

$$M(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+} m\mu\big(B(x, r)\big)\ \Phi(\mathrm{d}x, \mathrm{d}r, \mathrm{d}m). \tag{2.4}$$

However, in order to ensure that $M(\mu)$ in (2.4) is well defined, we restrain to measures $\mu$ with finite total variation (see Prop. 2.3). In the sequel, $\mathcal{Z}(\mathbb{R}^d)$ stands the set of signed (Borelian) measures $\mu$ on $\mathbb{R}^d$ with finite total variation $\|\mu\|_{var}(\mathbb{R}^d) < +\infty$. Moreover as in [10], we assume the following assumption on the radius behaviour, for $d < \beta < 2d$,

$$f(r) \underset{r\to+\infty}{\sim} \frac{C_\beta}{r^{\beta+1}}, \qquad r^{\beta+1} f(r) \le C_0. \tag{2.5}$$

Since $\beta > d$, condition (2.5) implies that the mean volume of the random ball is finite:

$$v_d \int_0^{+\infty} r^d f(r)\ \mathrm{d}r < +\infty, \tag{2.6}$$

where $v_d$ is the Lebesgue measure of the unit ball of $\mathbb{R}^d$. On the contrary, $\beta < 2d$ implies that $F$ does not admit a moment of order $2d$ and the volume of the balls has an infinite variance. The asymptotics condition in (2.5) is of constant use in the following.

**Proposition 2.3.** *Assume (2.5) is in force. For all $\mu \in \mathcal{Z}(\mathbb{R}^d)$, $\mathbb{E}\,[M(|\mu|)] < +\infty$. As a consequence, $M(\mu)$ in (2.4) is almost surely well defined for all $\mu \in \mathcal{Z}(\mathbb{R}^d)$.*

*Proof.* The proof follows the same lines as that of Proposition 1.1 in [4], replacing $K(0)$ by $K(x, x)$ and controlling it thanks to Hypothesis (2.2). □

## 3. Asymptotics and main results

The zooming-out procedure acts accordingly both on the centers and on the radii. First, a scaling $S_\rho : r \mapsto \rho r$ of rate $\rho \in (0, 1]$ changes balls $B(x, r)$ into $B(x, \rho r)$; this scaling changes the distribution $F$ of the radius into $F_\rho = F \circ S_\rho^{-1}$. Second, the intensity of the centers is simultaneously adapted; to do this, we introduce actually a family of new kernels $K_\rho$, $\rho \in ]0, 1]$, that we shall refer to as scaled kernels, and we denote by $\phi_\rho$ the DPP with kernel $K_\rho$ (with respect to the Lebesgue measure).The zoom-out procedure consists now in introducing the family of DPPs $\phi_\rho$, $\rho \in ]0, 1]$, with kernels $K_\rho$ with respect to the Lebesgue measure satisfying

$$K_\rho(x, x) \underset{\rho\to 0}{\sim} \lambda(\rho) K(x, x), \tag{3.1}$$

with $\lim_{\rho\to 0} \lambda(\rho) = +\infty$. We also suppose

$$\sup_{x\in\mathbb{R}^d} K_\rho(x, x) \le \lambda(\rho) \sup_{x\in\mathbb{R}^d} K(x, x), \tag{3.2}$$

and observe that with (2.2) and (3.2), Proposition A.6 in [4] gives the following uniform bound

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|K_\rho(x,y)\right|^2\,\mathrm{d}y \underset{\rho\to 0}{=} \mathcal{O}\big(\lambda(\rho)\big). \tag{3.3}$$

**Example 3.1.** If we come back to the example of the **Ginibre** process, we consider the family of Ginibre point processes $\phi_\rho^G$, $\rho\in]0,1]$, with kernels:

$$K_\rho^G(x,y)=\lambda(\rho)\exp\Big(-\frac{\lambda(\rho)}{2}\|x-y\|^2\Big),\quad x,y\in\mathbb{R}^d, \tag{3.4}$$

with respect to the Lebesgue measure, where $\lambda:\mathbb{R}_+\to\mathbb{R}_+$ is a decreasing function with $\lim_{\rho\to 0}\lambda(\rho)=+\infty$, so that (3.3) is satisfied.

The zoom-out procedure consists in considering a new marked DPP $\Phi_\rho$ on $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+$ with kernel:

$$\widehat{K}_\rho\big((x,r,m),(y,s,m')\big)=\sqrt{g(m)}\sqrt{\frac{f(r/\rho)}{\rho}}K_\rho(x,y)\sqrt{\frac{f(s/\rho)}{\rho}}\sqrt{g(m')},$$

with respect to the Lebesgue measure. The so-called scaled version of $M(\mu)$ is then the field

$$M_\rho(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+} m\mu\big(B(x,r)\big)\ \Phi_\rho(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m).$$

In the sequel, we are interested in the fluctuations of $M_\rho(\mu)$ with respect to its expectation

$$\mathbb{E}\big[M_\rho(\mu)\big]=\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+} m\mu\big(B(x,r)\big)\ K_\rho(x,x)\frac{f(r/\rho)}{\rho}g(m)\mathrm{d}x\mathrm{d}r\mathrm{d}m$$

and we introduce

$$\widetilde{M}_\rho(\mu)=M_\rho(\mu)-\mathbb{E}\big[M_\rho(\mu)\big]=\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+} m\mu\big(B(x,r)\big)\ \widetilde{\Phi}_\rho(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m), \tag{3.5}$$

where $\widetilde{\Phi}_\rho$ stands for the compensated random measure associated to $\Phi_\rho$.

We introduce a subspace $\mathcal{M}_{\alpha,\beta}\subset\mathcal{Z}$ on which we will investigate the convergence of the random field $M_\rho(\mu)$. The next definition comes from [3].

**Definition 3.2.** For $1<\alpha\le 2$ and $\beta>0$, we denote by $\mathcal{M}_{\alpha,\beta}$ the subset of measures $\mu\in\mathcal{Z}(\mathbb{R}^d)$ satisfying for some finite constant $C_\mu$ and some $0<p<\beta<q$:

$$\int_{\mathbb{R}^d}|\mu(B(x,r))|^\alpha\,\mathrm{d}x\le C_\mu(r^p\wedge r^q)$$

where $a\wedge b=\min(a,b)$.

We denote by $\mathcal{M}^+_{\alpha,\beta}$ the space of positive measures $\mu\in\mathcal{M}_{\alpha,\beta}$. Now, we can state the main result of this note. The proof consists in a combination of the arguments of [4, 8]. It is given in Section 4 where for some required technical points, it is referred to [3, 4].

**Theorem 3.3.** *Assume (2.5) and $\phi_\rho$ is a DPP with kernel satisfying (2.2), (3.1), (3.2) and Hypothesis 2.1 for its associated operator $\mathbf{K}_\rho$ in (2.1).*

*(i) Large-balls scaling: Assume $\lambda(\rho)\rho^\beta \to +\infty$ and set $n(\rho) = \big(\lambda(\rho)\rho^\beta\big)^{1/\alpha}$. Then, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}^+_{\alpha,\beta}$ to $W_\alpha(\cdot)$ where*

$$W_\alpha(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x,r)\big)\ M_\alpha(\mathrm{d}x,\mathrm{d}r)$$

*is a stable integral with respect to the $\alpha$-stable random measure $M_\alpha$ with control measure $\sigma^\alpha K(x,x)C_\beta r^{-\beta-1}\,\mathrm{d}x\mathrm{d}r$ and constant skewness function $b$ given in the domain of attraction of $G$.*

*(ii) Intermediate scaling: Assume $\lambda(\rho)\rho^\beta \to a^{d-\beta} \in ]0,+\infty[$ and set $n(\rho) = 1$. Then, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}^+_{\alpha,\beta}$ to $\widetilde{P}\circ D_a$ where*

$$\widetilde{P}(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+} m\mu\big(B(x,r)\big)\ \widetilde{\Pi}(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)$$

*with $\widetilde{\Pi}$ a (compensated) PPP with compensator measure $K(x,x)C_\beta r^{-\beta-1}\,\mathrm{d}x\mathrm{d}rG(\mathrm{d}m)$ and $D_a$ is the dilatation defined by $(D_a\mu)(B) = \mu(a^{-1}B)$.*

*(iii) Small-balls scaling: Suppose $\lambda(\rho)\rho^\beta \to 0$ when $\rho\to 0$ for $d<\beta<\alpha d$ and set $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\gamma}$ with $\gamma = \beta/d \in ]1,\alpha[$. Then, the field $n(\rho)^{-1}\widetilde{M}_\rho(\cdot)$ converges when $\rho\to 0$ in the finite-dimensional distributions sense in $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)\cap\{\mu\geq 0\}$ to $Z_\gamma(\cdot)$ where*

$$Z_\gamma(\mu) = \int_{\mathbb{R}^d}\phi(x)\ M_\gamma(\mathrm{d}x) \quad \text{for} \quad \mu(\mathrm{d}x) = \phi(x)\mathrm{d}x \text{ with } \phi\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),\ \phi\geq 0,$$

*is a stable integral with respect to the $\gamma$-stable random measure $M_\gamma$ with control measure $\sigma_\gamma K(x,x)\mathrm{d}x$ where*

$$\sigma_\gamma^\gamma = \frac{C_\beta v_d^\gamma}{d}\int_0^{+\infty}\frac{1-\cos(r)}{r^{1+\gamma}}\mathrm{d}r\int_0^{+\infty} m^\gamma G(\mathrm{d}m),$$

*and constant unit skewness.*

Here, and in the sequel, we follow the notations of the standard reference [14] for stable random variables and integrals.

## 4. Major auxiliary result

To investigate the behaviour of $\widetilde{M}_\rho(\mu)$ in the determinantal case, we use the Laplace transform of determinantal measures. An explicit expression is well known when the test functions are compactly supported, see Theorem A.4 in [4]. However, in our situation, the test functions $(x,r,m)\longmapsto m\mu(B(x,r))$ are not compactly supported on $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+$ for $\mu\in\mathcal{M}^+_{\alpha,\beta}$. In order to overpass this issue we use Proposition 4.1 below for the Laplace transform of determinantal measures with non-compactly supported test functions, but with a condition of integration with respect to the kernel of the determinantal process (see (4.1)).

In addition to generalizing the model studied in [4] by adding a weight, the following proposition has the further consequence of simplifying the proofs of the results in [4], since there is no more need to study the truncated model and obtain uniform convergence to exchange the limit in $R$, the truncation parameter and the limit in $\rho$, the scaling parameter.

Independently of the random balls model studied in this paper, this result opens numerous computational perspectives for determinantal point processes, which were limited to local behavior studies.

**Proposition 4.1.** *Let $\Phi$ a determinantal point process on a locally compact Polish space $E$ with a continuous kernel $K$ such that the associated operator $\boldsymbol{K}$ satisfies Hypothesis 2.1. Let $h$ be a nonnegative function such that the kernel $K\left[1-e^{-h}\right] \in L^2(E\times E)$ and also satisfying the integrability condition*

$$\int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x < +\infty. \tag{4.1}$$

*Then $\mathbf{K}\left[1-e^{-h}\right]$ is a trace-class operator with*

$$\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right) = \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x$$

*and we have*

$$\mathbb{E}\left[\exp\left(-\int_E h(x)\Phi(\mathrm{d}x)\right)\right] = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big)\right), \tag{4.2}$$

*where $\mathbf{K}\left[1-e^{-h}\right]$ is the operator with kernel*

$$K\left[1-e^{-h}\right](x,y) = \sqrt{1-e^{-h(x)}}K(x,y)\sqrt{1-e^{-h(y)}}.$$

*Proof.* Expression (4.2) is known to be true when $h$ has a compact support (see [4], Thm. A.4 and Eq. (37)), but it is not the case here. Let $(h_q)_{q\in\mathbb{N}}$ a non-decreasing sequel of positive functions with compact support defined by

$$h_q(x) = h(x)\mathbf{1}_{B(0,q)}(x).$$

Thanks to Theorem A.4 in [4], we have for all $q\in\mathbb{N}$:

$$\mathbb{E}\left[\exp\left(-\int_E h_q(x)\Phi(\mathrm{d}x)\right)\right] = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right). \tag{4.3}$$

We want to take the limit when $q\to+\infty$ in equation (4.3). The proof is structured in four steps, described below in the order in which they will be proved:

- **Step 1**: We prove that $\lim\limits_{q\to+\infty}\mathbb{E}\left[\exp\left(-\int_E h_q(x)\Phi(\mathrm{d}x)\right)\right] = \mathbb{E}\left[\exp\left(-\int_E h(x)\Phi(\mathrm{d}x)\right)\right]$.

To take the limit in the right term of equation (4.3), we need three steps:

- **Step 2**: We prove that $\mathbf{K}\left[1-e^{-h}\right]$ is a trace-class operator.
- **Step 3**: We prove that $\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right) = \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x$.
- **Step 4**: We compute the limit in the right term of equation (4.3) and prove that

$$\lim_{q\to+\infty}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big)\right).$$

Combination of **Step 1** and **Step 4** gives

$$\mathbb{E}\left[\exp\left(-\int_E h(x)\Phi(\mathrm{d}x)\right)\right] = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big)\right),$$

which is the result of Proposition 4.1.

**Step 1**: Thanks to Lemma A.3, we have

$$\lim_{q\to+\infty}\mathbb{E}\left[\exp\left(-\int_E h_q(x)\Phi(\mathrm{d}x)\right)\right] = \mathbb{E}\left[\exp\left(-\int_E h(x)\Phi(\mathrm{d}x)\right)\right].$$

This point is essentially proved by an application of the dominated convergence theorem.

Now we work on the right term of equation (4.3).

**Step 2**: We prove that $\mathbf{K}\left[1-e^{-h}\right]$ is a trace-class operator.

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(E)$.

We recall that $\mathbf{K}\left[1-e^{-h}\right]$ is trace-class if $\sum_{n=0}^{+\infty}\left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle < +\infty$ and in this case we have $\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right) = \sum_{n=0}^{+\infty}\left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle$.

The strategy is to prove the following inequality:

$$\sum_{n=0}^{+\infty}\left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle \le \int_E\left(1-e^{-h(x)}\right)K(x,x)\mathrm{d}x < +\infty.$$

⋆ Thanks to Lemma A.4, we have the following approximation:

$$\lim_{q\to+\infty}\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle = \left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle. \tag{4.4}$$

Thus we have:

$$\begin{aligned}\sum_{n=0}^{+\infty}\left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle &= \sum_{n=0}^{+\infty}\liminf_{q\to+\infty}\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle\\ &\underset{(a)}{\le}\liminf_{q\to+\infty}\sum_{n=0}^{+\infty}\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle\\ &= \liminf_{q\to+\infty}\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right)\end{aligned}$$

where $(a)$ is allowed by the Fatou lemma because $\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle \ge 0$.

Thanks to Lemma A.5, we have

$$\liminf_{q\to+\infty}\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right) = \int_E\left(1-e^{-h(x)}\right)K(x,x)\mathrm{d}x.$$

Finally

$$\sum_{n=0}^{+\infty} \left\langle \mathbf{K}\left[1-e^{-h}\right](e_n), e_n \right\rangle \leq \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x < +\infty$$

which proves that $\mathbf{K}\left[1-e^{-h}\right]$ is trace-class.

**Step 3**: We prove that $\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right) = \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x$.

Because $\mathbf{K}\left[1-e^{-h}\right]$ is trace-class and $K\left[1-e^{-h}\right] \in L^2(E\times E)$, Lemma 4.2.2 in [9] ensures that

$$\sqrt{1-e^{-h(x)}} K(x,y) \sqrt{1-e^{-h(y)}} = \sum_{i=1}^{+\infty} \lambda_i \varphi_i(x)\varphi_i(y) \tag{4.5}$$

where $(\varphi_i)$ is an orthonormal basis of $L^2(E)$. Thus, thanks to (4.5) we have:

$$\begin{aligned} \left\langle \mathbf{K}\left[1-e^{-h}\right](e_n), e_n \right\rangle &= \int_{E\times E} \sqrt{1-e^{-h(x)}} K(x,y) \sqrt{1-e^{-h(y)}} e_n(x) e_n(y) \mathrm{d}x\mathrm{d}y \\ &= \int_{E\times E} \sum_{i=1}^{+\infty} \lambda_i \varphi_i(x)\varphi_i(y) e_n(x) e_n(y)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{4.6}$$

We suppose at first that we can invert the sum and the integral (we will justify it immediately after). After inverting the integral and the sum in (4.6) we obtain

$$\begin{aligned} \left\langle \mathbf{K}\left[1-e^{-h}\right](e_n), e_n \right\rangle &= \sum_{i=1}^{+\infty} \int_{E\times E} \lambda_i \varphi_i(x)\varphi_i(y) e_n(x) e_n(y)\mathrm{d}x\mathrm{d}y \\ &= \sum_{i=1}^{+\infty} \lambda_i \left(\int_E \varphi_i(x) e_n(x)\mathrm{d}x\right)^2 = \sum_{i=1}^{+\infty} \lambda_i \left\langle \varphi_i, e_n \right\rangle^2 \end{aligned}$$

and finally

$$\begin{aligned} \sum_{n=0}^{+\infty} \left\langle \mathbf{K}\left[1-e^{-h}\right](e_n), e_n \right\rangle &= \sum_{n=0}^{+\infty}\sum_{i=1}^{+\infty} \lambda_i \left\langle \varphi_i, e_n \right\rangle^2 \underset{(b)}{=} \sum_{i=1}^{+\infty} \lambda_i \sum_{n=0}^{+\infty} \left\langle \varphi_i, e_n \right\rangle^2 \\ &= \sum_{i=1}^{+\infty} \lambda_i \left\|\varphi_i\right\|_2^2 \\ &= \sum_{i=1}^{+\infty} \lambda_i \int_E \left|\varphi_i(x)\right|^2 \mathrm{d}x \\ &\underset{(b)}{=} \int_E \sum_{i=1}^{+\infty} \lambda_i \left|\varphi_i(x)\right|^2 \mathrm{d}x \\ &= \int_E (1-e^{-h(x)}) K(x,x)\mathrm{d}x < +\infty, \end{aligned}$$

where the two equalities $(b)$ are allowed because the terms are non-negative.

Then, $\mathbf{K}\left[1-e^{-h}\right]$ is a trace-class operator with

$$\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right)=\int_E\left(1-e^{-h(x)}\right)K(x,x)\mathrm{d}x.$$

The computations above gives in particular

$$\lim_{q\to+\infty}\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right)=\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right).$$

Now we prove that the inversion of the sum and the integral in (4.6) is allowed:

We can invert the sum and the integral if

$$\sum_{i=1}^{+\infty}\int_{E\times E}\left|\lambda_i\varphi_i(x)\varphi_i(y)e_n(x)e_n(y)\right|\mathrm{d}x\mathrm{d}y<+\infty.$$

$$\begin{aligned}\sum_{i=1}^{+\infty}\int_{E\times E}\left|\lambda_i\varphi_i(x)\varphi_i(y)e_n(x)e_n(y)\right|\mathrm{d}x\mathrm{d}y&=\sum_{i=1}^{+\infty}\int_{E\times E}\lambda_i\left|\varphi_i(x)\right|\left|\varphi_i(y)\right|\left|e_n(x)\right|\left|e_n(y)\right|\mathrm{d}x\mathrm{d}y\\&=\sum_{i=1}^{+\infty}\lambda_i\left(\int_E\left|\varphi_i(x)\right|\left|e_n(x)\right|\mathrm{d}x\right)^2\\&\underset{(c)}{\leq}\sum_{i=1}^{+\infty}\lambda_i\left(\int_E\left|\varphi_i(x)\right|^2\mathrm{d}x\right)\left(\int_E\left|e_n(x)\right|^2\mathrm{d}x\right)\\&\underset{(d)}{=}\sum_{i=1}^{+\infty}\lambda_i\int_E\left|\varphi_i(x)\right|^2\mathrm{d}x\\&\underset{(e)}{=}\int_E\left(\sum_{i=1}^{+\infty}\lambda_i\left|\varphi_i(x)\right|^2\right)\mathrm{d}x\\&\underset{(f)}{=}\int_E(1-e^{-h(x)})K(x,x)\mathrm{d}x<+\infty.\end{aligned}$$

The inequality $(c)$ is the Cauchy-Schwarz inequality, the equality $(d)$ is because

$$\int_E\left|e_n(x)\right|^2\mathrm{d}x=\|e_n\|_2^2=1,$$

the inversion of the sum and the integral in $(e)$ is allowed because all the terms are non-negative and the equality $(f)$ stands because $K$ is continuous. Thus, the inversion in (4.6) is allowed.

**Step 4**: Now, we have to take the limit in the right term of equation (4.3).

$\star$ First, we prove that we can exchange the limit and the infinite sum. To do that, we show that the sum normally converges. For all $n,q\geq 0$:

$$\begin{aligned}\left|\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]^n\right)\right|&\leq\left\|\mathbf{K}\left[1-e^{-h_q}\right]\right\|^{n-1}\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right)\\&\leq\left\|\mathbf{K}\left[1-e^{-h}\right]\right\|^{n-1}\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h}\right]\right),\end{aligned}$$

Thanks to Lemma A.1 in Appendix A we have $\left\|\mathbf{K}\left[1-e^{-h_q}\right]\right\| \leq \left\|\mathbf{K}\left[1-e^{-h}\right]\right\|$ and because $K(x,x) \geq 0$,

$$\begin{aligned} \operatorname{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]\Big) &= \int_E \left(1-e^{-h_q(x)}\right) K(x,x)\mathrm{d}x \\ &\leq \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x \\ &= \operatorname{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]\Big), \end{aligned} \tag{4.7}$$

which is finite as proved above.

So we obtain the following domination: for all $n, q \geq 0$:

$$\left|\operatorname{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right| \leq \left\|\mathbf{K}\left[1-e^{-h}\right]\right\|^{n-1} \operatorname{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]\Big),$$

this domination being independant of the parameter $q \in \mathbb{N}$ and summable on $n \in \mathbb{N}$ because $\left\|\mathbf{K}\left[1-e^{-h}\right]\right\| < 1$. In fact, we have:

$$\begin{aligned} \left\|\mathbf{K}\left[1-e^{-h}\right]\right\| &= \sup_{\|g\|_2=1} \left\langle \mathbf{K}\left[1-e^{-h}\right](g), g\right\rangle \\ &= \sup_{\|g\|_2=1} \lim_{q\to+\infty} \left\langle \mathbf{K}\left[1-e^{-h_q}\right](g), g\right\rangle \\ &\leq \lim_{q\to+\infty} \sup_{\|g\|_2=1} \left\langle \mathbf{K}\left[1-e^{-h_q}\right](g), g\right\rangle \\ &= \lim_{q\to+\infty} \sup_{\|g\|_2=1} \left\langle \mathbf{K}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle \end{aligned}$$

Because $\sqrt{1-e^{-h_q}}g$ has a compact support,

$$\left\langle \mathbf{K}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle = \left\langle \mathbf{K}_{|B(0,q)}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle,$$

where $\mathbf{K}_{|B(0,q)}$ is the restriction of $\mathbf{K}$ on $L^2(B(0,p))$. Since $\mathbf{K}_{|B(0,q)}$ is trace-class, we have

$$\begin{aligned} \left\langle \mathbf{K}_{|B(0,q)}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle &\leq \lambda_q^{\max} \left\|\sqrt{1-e^{-h_q}}g\right\|^2 \\ &\leq \lambda_q^{\max} \\ &\leq \lambda^{\max}, \end{aligned}$$

where $\lambda_q^{\max}$ (resp. $\lambda^{\max}$) is the greatest eigenvalue of $\mathbf{K}_{|B(0,q)}$ (resp $\mathbf{K}$). Then

$$\sup_{\|g\|_2=1} \left\langle \mathbf{K}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle \leq \lambda^{\max}$$

and

$$\lim_{q\to+\infty} \sup_{\|g\|_2=1} \left\langle \mathbf{K}\left(\sqrt{1-e^{-h_q}}g\right), \sqrt{1-e^{-h_q}}g\right\rangle \leq \lambda^{\max}.$$

Thus,

$$\left\|\mathbf{K}\left[1-e^{-h}\right]\right\| \leq \lambda^{\max} < 1.$$

So we have an upper bound of $\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)$ independent of $q$ which is summable so we can exchange the limit and the sum.

$\star$ Secondly, we prove that $\lim\limits_{q\to+\infty}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big) = \mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big)$.

For all $n \geq 0$:

$$\begin{aligned}
\Big|\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big) - \mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\Big| &= \Big|\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n - \mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\Big| \\
&= \left|\mathrm{Tr}\left(\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\sum_{k=0}^{n-1}\mathbf{K}\left[1-e^{-h}\right]^k\mathbf{K}\left[1-e^{-h_q}\right]^{n-1-k}\right)\right| \\
&\leq \sum_{k=0}^{n-1}\mathrm{Tr}\Big(\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\mathbf{K}\left[1-e^{-h}\right]^k\mathbf{K}\left[1-e^{-h_q}\right]^{n-1-k}\Big).
\end{aligned}$$

Moreover we have the following inequalities

$$\begin{aligned}
&\mathrm{Tr}\left(\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\mathbf{K}\left[1-e^{-h}\right]^k\mathbf{K}\left[1-e^{-h_q}\right]^{n-1-k}\right) \\
&\qquad \leq \mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\left\|\mathbf{K}\left[1-e^{-h}\right]^k\mathbf{K}\left[1-e^{-h_q}\right]^{n-1-k}\right\| \\
&\qquad \leq \mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\left\|\mathbf{K}\left[1-e^{-h}\right]\right\|^k\left\|\mathbf{K}\left[1-e^{-h_q}\right]\right\|^{n-1-k} \\
&\qquad \leq \mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)\left\|\mathbf{K}\left[1-e^{-h}\right]\right\|^{n-1},
\end{aligned}$$

the last inequality taking place according to Lemma A.1. We finally have the following inequality

$$\begin{aligned}
&\Big|\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big) - \mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\Big| \\
&\qquad\qquad \leq \mathrm{Tr}\left(\mathbf{K}\left[1-e^{-h}\right] - \mathbf{K}\left[1-e^{-h_q}\right]\right)n\left\|\mathbf{K}\left[1-e^{-h}\right]\right\|^{n-1} \\
&\qquad\qquad \underset{q\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

$\star$ Thus,

$$\begin{aligned}
\lim_{q\to+\infty}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right) &= \exp\left(-\lim_{q\to+\infty}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right) \\
&= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\lim_{q\to+\infty}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h_q}\right]^n\Big)\right) \\
&= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\Big(\mathbf{K}\left[1-e^{-h}\right]^n\Big)\right)
\end{aligned}$$

and the **Step 4** is proved, which ends the proof of Proposition 4.1. □

## 5. Proof of Theorem 3.3

We give a short proof of Theorem 3.3. In this non-stationary case, the proof follows the same general strategy as in [4].

In order to prove the convergence in distribution of $n(\rho)^{-1}\widetilde{M_\rho}(\mu)$, for $\mu \in \mathcal{M}^+_{\alpha,\beta}$, we study the convergence of its Laplace transform : for $\theta \geq 0$,

$$\begin{aligned}\mathbb{E}\Big[\exp\big(-\theta n(\rho)^{-1}\widetilde{M_\rho}(\mu)\big)\Big] &= \exp\big(\theta\mathbb{E}[n(\rho)^{-1}M_\rho(\mu)]\big)\ \mathbb{E}\Big[\exp\big(-\theta n(\rho)^{-1}M_\rho(\mu)\big)\Big]\\ &= \exp\big(\theta\mathbb{E}[n(\rho)^{-1}M_\rho(\mu)]\big)\ \mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}\theta n(\rho)^{-1}m\mu(B(x,r))\Phi_\rho(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)\right)\right].\end{aligned}$$

It is here that the result of this paper, Proposition 4.1, makes perfect sense. Indeed, without this result, can not directly compute the last term because the integrand is not compactly supported. That why in [4] we use another parameter $R$ to make the integrand artificially compactly supported, but when we want to take the limit when $R \to +\infty$, we have to prove that the convergence in $R$ is uniform we respect to the parameter $\rho$, so that we can exchange $\lim_{\rho\to 0}$ and $\lim_{R\to+\infty}$.

To apply Proposition 4.1, we check its hypothesis. The hypothesis (4.1) in this proposition is satisfied because in our context of weighted balls model, we have $h(x,r,m) = m\mu(B(x,r))$ and therefore:

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}(1-e^{-m\mu(B(x,r))})K_\rho(x,x)f(r/\rho)g(m)\mathrm{d}x\frac{\mathrm{d}r}{\rho}\mathrm{d}m\\ &\qquad\leq \lambda(\rho)\sup_{x\in\mathbb{R}^d}K(x,x)\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}m\mu(B(x,r))f(r/\rho)g(m)\mathrm{d}x\frac{\mathrm{d}r}{\rho}\mathrm{d}m\\ &\qquad\leq \lambda(\rho)\rho^d v_d\mu(\mathbb{R}^d)\sup_{x\in\mathbb{R}^d}K(x,x)\left(\int_{\mathbb{R}_+}mg(m)\mathrm{d}m\right)\left(\int_{\mathbb{R}_+}r^d f(r)\mathrm{d}r\right) < +\infty.\end{aligned}$$

$\star$ Moreover, we have to check that $\widehat{K}_\rho\left[1-e^{-h}\right] \in L^2(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+)$, if we denote by $h$ the function given by $h(x,r,m) = m\mu(B(x,r))$ defined on $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+$.

$$\begin{aligned}&\int_{(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+)^2}\widehat{K}_\rho\left[1-e^{-h}\right]^2((x,r,m),(y,s,m'))\mathrm{d}x\mathrm{d}r\mathrm{d}m\mathrm{d}y\mathrm{d}s\mathrm{d}m'\\ &= \int_{(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+)^2}\left(1-e^{-m\mu(B(x,r))}\right)g(m)\frac{f(r/\rho)}{\rho}K^2_\rho(x,y)\\ &\quad\times\frac{f(s/\rho)}{\rho}g(m')\left(1-e^{-m'\mu(B(y,s))}\right)\mathrm{d}x\mathrm{d}r\mathrm{d}m\mathrm{d}y\mathrm{d}s\mathrm{d}m'\\ &\underset{(g)}{\leq}\lambda(\rho)^2\left(\sup_{x\in\mathbb{R}^d}K(x,x)\right)^2\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}\left(1-e^{-m\mu(B(x,r))}\right)g(m)\frac{f(r/\rho)}{\rho}\mathrm{d}x\mathrm{d}r\mathrm{d}m\right)^2\\ &\leq \lambda(\rho)^2\left(\sup_{x\in\mathbb{R}^d}K(x,x)\right)^2\left(\rho^d v_d\mu(\mathbb{R}^d)\left(\int_{\mathbb{R}_+}mg(m)\mathrm{d}m\right)\left(\int_{\mathbb{R}_+}r^d f(r)\mathrm{d}r\right)\right)^2 < +\infty,\end{aligned}$$

where inequality $(g)$ stands because $K^2_\rho(x,y) \leq K_\rho(x,x)K_\rho(y,y) \leq \lambda(\rho)^2\left(\sup_{x\in\mathbb{R}^d}K(x,x)\right)^2$ thanks to (3.2).

We can now apply Proposition 4.1 to obtain:

$$\mathbb{E}\left[\exp\left(-\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}\theta n(\rho)^{-1}m\mu(B(x,r))\Phi_\rho(\mathrm{d}x,\mathrm{d}r,\mathrm{d}m)\right)\right] = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left(\mathbf{K}\left[1-e^{-\theta n(\rho)^{-1}h}\right]^n\right)\right). \tag{5.1}$$

The Laplace transform of $n(\rho)^{-1}\widetilde{M}_\rho(\mu)$ thus rewrites

$$\mathbb{E}\Big[e^{-\theta n(\rho)^{-1}\widetilde{M}_\rho(\mu)}\Big] = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+}\psi\big(\theta n(\rho)^{-1}m\mu(B(x,r))\big)\ \lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}g(m)\mathrm{d}x\mathrm{d}r\mathrm{d}m\right) \times\exp\left(-\sum_{n\geq 2}\frac{1}{n}\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]^n\Big)\right) \tag{5.2}$$

with $\psi(u)=e^{-u}-1+u$.

Now, the rest of the proof consists in the combination of articles [4] and [8]. Indeed, the term $n=1$ in equation (5.2) is the Laplace transform of our random balls model but with a non-stationary Poisson point process. Note that, in this non-stationary setting, the Poissonian limits for $n=1$ come from Theorem 1, Theorem 2 and Theorem 3 in [8] taking $f(x,r)=K(x,x)f(r)$ in our situation.

For the terms $n\geq 2$ in (5.2), we follow the same strategy than in [4]. The convergence derives from the Lemmas A.6 and A.7 given in Appendix A.

As a consequence of Lemma A.6, we have

$$\left|-\sum_{n\geq 2}\frac{1}{n}\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\Big)\right| \leq \sum_{n\geq 1}\frac{1}{n}\left(\sqrt{\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\Big)}\right)^n = -\ln\left(1-\sqrt{\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\Big)}\right).$$

It is now enough to show now that

$$\lim_{\rho\to 0}\ \mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]^2\Big)=0$$

in the three regimes.

As a consequence of Lemma A.7, it remains to show that $\frac{\lambda(\rho)\rho^q}{n(\rho)^2}\underset{\rho\to 0}{\longrightarrow}0$.

(i) *Large-balls scaling.* Since $\lim_{\rho\to 0}\lambda(\rho)\rho^\beta=+\infty$, for $\rho$ small enough we have $\lambda(\rho)\rho^\beta\geq 1$ and so $\big(\lambda(\rho)\rho^\beta\big)^{1/\alpha}\geq\big(\lambda(\rho)\rho^\beta\big)^{1/2}$ with $\alpha\in]1,2]$. Thus since $q>\beta$ we have :

$$0\leq\frac{\lambda(\rho)\rho^q}{n(\rho)^2}\leq\frac{\lambda(\rho)\rho^q}{\lambda(\rho)\rho^\beta}=\rho^{q-\beta}\underset{\rho\to 0}{\longrightarrow}0.$$

(ii) *Intermediate scaling.* In this case, $n(\rho)=1$ and since $q>\beta$ we have :

$$0\leq\frac{\lambda(\rho)\rho^q}{n(\rho)^2}=\lambda(\rho)\rho^q=\lambda(\rho)\rho^\beta\rho^{q-\beta}\underset{\rho\to 0}{\longrightarrow}0.$$

(iii) *Small-balls scaling.* Since we consider $\mu \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, Proposition 2.5-(ii) in [4] (which remains correct for $\mu \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$) ensures that we can take $q = 2d$ and then with $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\gamma}$ since $\beta < \alpha d \leq 2d$ we have :

$$0 \leq \frac{\lambda(\rho)\rho^{2d}}{n(\rho)^2} = \lambda(\rho)^{(\beta-2d)/\beta} \underset{\rho\to 0}{\longrightarrow} 0.$$

## Appendix A. Lemmas for the Laplace transform of DPP

In this appendix, we state and prove three different lemmas used in Section 4 to prove Proposition 4.1 and in Section 5 to prove Theorem 3.3.

**Lemma A.1.** *If $f, g$ are two real functions on $E$ such that $0 \leq f \leq g$, then we have*

$$\|\mathbf{K}[f]\| \leq \|\mathbf{K}[g]\|$$

*where $\mathbf{K}[f]$ is the operator with kernel $K[f](x,y) = \sqrt{f(x)}K(x,y)\sqrt{f(y)}$ for $x, y \in E$.*

*Proof.* Recall that:

$$\|\mathbf{K}[f]\| = \sup_{h\in L^2(E)} \frac{\langle \mathbf{K}[f](h), h\rangle}{\|h\|_2^2}.$$

Let $h \in L^2(E)$.

$$\begin{aligned}
\langle \mathbf{K}[f](h), h\rangle &= \int_E \mathbf{K}[f](h)(x)h(x)\mathrm{d}x \\
&= \int_E\int_E \sqrt{f(x)}K(x,y)\sqrt{f(y)}h(y)h(x)\mathrm{d}y\mathrm{d}x \\
&= \int_E\int_E \sqrt{\frac{f(x)}{g(x)}}\sqrt{g(x)}K(x,y)\sqrt{g(y)}\sqrt{\frac{f(y)}{g(y)}}h(y)h(x)\mathrm{d}y\mathrm{d}x \\
&= \langle \mathbf{K}[g](lh), lh\rangle
\end{aligned}$$

where $l = \sqrt{f/g} \leq 1$. So we have:

$$\langle \mathbf{K}[f](h), h\rangle \leq \|\mathbf{K}[g]\|\,\|lh\|_2^2 \leq \|\mathbf{K}[g]\|\,\|h\|_2^2$$

and the result follows. ∎

We recall the following result from Proposition 280 in [6]:

**Lemma A.2.** *A trace-class operator $\boldsymbol{K}$ on a Hilbert space $\mathcal{H}$ is a compact operator.*

The following lemma is useful to approximate the Laplace transform of a determinantal point process for function which are not compactly supported. We use this result for the proof of Proposition 4.1.

**Lemma A.3.** *Let $\Phi$ a point process on a locally compact Polish space $E$. Let $h$ be a nonnegative function. Let $(h_q)_{q\in\mathbb{N}}$ a non-decreasing sequel of positive functions with compact support defined by*

$$h_q(x) = h(x)\boldsymbol{1}_{B(0,q)}(x).$$

*Then we have*

$$\lim_{q\to+\infty}\mathbb{E}\left[\exp\left(-\int_E h_q(x)\Phi(\mathrm{d}x)\right)\right]=\mathbb{E}\left[\exp\left(-\int_E h(x)\Phi(\mathrm{d}x)\right)\right].$$

*Proof.* If we denote by $M_p(E)$ the space of all point measures defined on $E$, we have for all $q\in\mathbb{N}$:

$$\mathbb{E}\left[\exp\left(-\int_E h_q(x)\Phi(\mathrm{d}x)\right)\right]=\int_{M_p(E)}\exp\left(-\int_E h_q(x)m(\mathrm{d}x)\right)\mathbb{P}_\Phi(\mathrm{d}m).$$

Because $q\longmapsto h_q$ is increasing, by monotone convergence we have

$$\lim_{q\to+\infty}\int_E h_q(x)m(\mathrm{d}x)=\int_E h(x)m(\mathrm{d}x).$$

We apply the dominated convergence theorem:

- $\lim_{q\to+\infty}\exp\left(-\int_E h_q(x)m(\mathrm{d}x)\right)=\exp\left(-\int_E h(x)m(\mathrm{d}x)\right)$,
- $\left|\exp\left(-\int_E h_q(x)m(\mathrm{d}x)\right)\right|\le 1$ because $h_q\ge 0$, and 1 is integrable on $M_p(E)$ with respect to $\mathbb{P}_\Phi$.

So we obtain:

$$\lim_{q\to+\infty}\int_{M_p(E)}\exp\left(-\int_E h_q(x)m(\mathrm{d}x)\right)\mathbb{P}_\Phi(\mathrm{d}m)=\int_{M_p(E)}\exp\left(-\int_E h(x)m(\mathrm{d}x)\right)\mathbb{P}_\Phi(\mathrm{d}m)$$

which is the result of the lemma.

□

**Lemma A.4.** *Let $\Phi$ a determinantal point process on a locally compact Polish space $E$ with a continuous kernel $K$ such that the associated operator $\boldsymbol{K}$ satisfies Hypothesis 2.1. Let $h$ be a nonnegative function such that the kernel $K\left[1-e^{-h}\right]\in L^2(E\times E)$ and also satisfying the integrability condition*

$$\int_E\left(1-e^{-h(x)}\right)K(x,x)\mathrm{d}x<+\infty. \tag{A.1}$$

*Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(E)$ and $(h_q)_{q\in\mathbb{N}}$ a non-decreasing sequel of positive functions with compact support defined by $h_q(x)=h(x)\boldsymbol{1}_{B(0,q)}(x)$. Then for all $n\in\mathbb{N}$,*

$$\lim_{q\to+\infty}\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle=\left\langle\mathbf{K}\left[1-e^{-h}\right](e_n),e_n\right\rangle. \tag{A.2}$$

*Proof.*

$$\left\langle\mathbf{K}\left[1-e^{-h_q}\right](e_n),e_n\right\rangle=\int_{E\times E}\sqrt{1-e^{-h_q(x)}}K(x,y)\sqrt{1-e^{-h_q(y)}}e_n(x)e_n(y)\mathrm{d}x\mathrm{d}y$$

We apply the dominated convergence theorem:

- Computation of the limit when $q \to +\infty$:

$$\lim_{q\to+\infty} \sqrt{1-e^{-h_q(x)}}K(x,y)\sqrt{1-e^{-h_q(y)}}e_n(x)e_n(y) \\ = \sqrt{1-e^{-h(x)}}K(x,y)\sqrt{1-e^{-h(y)}}e_n(x)e_n(y)$$

- Domination:

$$\left|\sqrt{1-e^{-h_q(x)}}K(x,y)\sqrt{1-e^{-h_q(y)}}e_n(x)e_n(y)\right| \\ \leq \sqrt{1-e^{-h(x)}}\,|K(x,y)|\,\sqrt{1-e^{-h(y)}}\,|e_n(x)|\,|e_n(y)|$$

  which is integrable on $E \times E$ because:

$$\int_{E\times E} \sqrt{1-e^{-h(x)}}\,|K(x,y)|\,\sqrt{1-e^{-h(y)}}\,|e_n(x)|\,|e_n(y)|\,\mathrm{d}x\mathrm{d}y \\ \leq \sqrt{\int_{E\times E}(1-e^{-h(x)})\,|K(x,y)|^2\,(1-e^{-h(y)})\mathrm{d}x\mathrm{d}y} \times \sqrt{\int_{E\times E}|e_n(x)|^2\,|e_n(y)|^2\,\mathrm{d}x\mathrm{d}y} \\ \leq \int_E (1-e^{-h(x)})K(x,x)\mathrm{d}x < +\infty$$

  because $|K(x,y)|^2 \leq K(x,x)K(y,y)$ and $\int_E |e_n(x)|^2\,\mathrm{d}x = 1$,

and equation (A.2) follows. □

**Lemma A.5.** *With the same hypothesis and notations that in Lemma A.4, we have*

$$\lim_{q\to+\infty} \operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right) = \int_E \left(1-e^{-h(x)}\right) K(x,x)\mathrm{d}x.$$

*Proof.* To compute $\lim_{q\to+\infty} \operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right)$, we apply the dominated convergence theorem. We have the following expression:

$$\operatorname{Tr}\left(\mathbf{K}\left[1-e^{-h_q}\right]\right) = \int_E \left(1-e^{-h_q(x)}\right) K(x,x)\mathrm{d}x.$$

The hypothesis of the dominated convergence theorem are satisfied because:

1. $\lim_{q\to+\infty} \left(1-e^{-h_q(x)}\right) K(x,x) = \left(1-e^{-h(x)}\right) K(x,x)$,
2. For all $q \geq 0$ we have:

$$\left|\left(1-e^{-h_q(x)}\right) K(x,x)\right| \leq \left(1-e^{-h(x)}\right) K(x,x)$$

   which is integrable on $E$ by hypothesis (A.1).

Thus

$$\lim_{q\to+\infty}\int_E \left(1-e^{-h_q(x)}\right)K(x,x)\mathrm{d}x=\int_E \left(1-e^{-h(x)}\right)K(x,x)\mathrm{d}x.$$

and the result follows. □

We finish by giving two lemmas needed in the proof of Theorem 3.3, with complete proofs that can be found in [4].

**Lemma A.6.** *For all $n\geq 2$, we have*

$$\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]^n\Big)\leq \mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]^2\Big)^{n/2}.$$

*Proof.* The key point is that $\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]$ is trace-class, and as a consequence a Hilbert-Schmidt operator thanks to Lemma A.2 so the proof of Lemma 2.9 in [4] applies in the same way. □

**Lemma A.7.** *Assume conditions (2.2), (2.6) and (3.2), and consider $\mu\in\mathcal{M}^+_{\alpha,\beta}$. Then there is a constant $M\in]0,+\infty[$ such that,*

$$\mathrm{Tr}\Big(\widehat{\mathbf{K}}_\rho\big[1-e^{-\theta n(\rho)^{-1}h}\big]^2\Big)\leq M\theta^2\frac{\lambda(\rho)\rho^q}{n(\rho)^2}.$$

*Proof.* The computations are analogous to that in [4] for the model without weight. It is important to observe that the key point, namely inequality (25) in [4], remains true because $\mu\in\mathcal{M}^+_{\alpha,\beta}$ so thanks Proposition 2.2 (iii) in [3], $\mu\in\mathcal{M}^+_{2,\beta}=\mathcal{M}^+_\beta$ using the notations of [4]. To be complete, the constant $M$ is equal to $C_K C_\mu C_f\left(\int_{\mathbb{R}_+} mg(m)\mathrm{d}m\right)^2$ with the notations of Lemma 2.10 in [4]. □

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