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MODERATE DEVIATIONS FOR A CURIE–WEISS MODEL WITH DYNAMICAL EXTERNAL FIELD *

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Abstract. In the present paper we prove moderate deviations for a Curie–Weiss model with external magnetic field generated by a dynamical system, as introduced by Dombry and Guillotin-Plantard in [C. Dombry and N. Guillotin-Plantard, *Markov Process. Related Fields* **15** (2009) 1–30]. The results extend those already obtained for the Curie–Weiss model without external field by Eichelsbacher and Löwe in [P. Eichelsbacher and M. Löwe, *Markov Process. Related Fields* **10** (2004) 345–366]. The Curie–Weiss model with dynamical external field is related to the so called dynamic $\mathbb{Z}$-random walks (see [N. Guillotin-Plantard and R. Schott, *Theory and applications*, Elsevier B. V., Amsterdam (2006).]). We also prove a moderate deviation result for the dynamic $\mathbb{Z}$-random walk, completing the list of limit theorems for this object.

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1. Introduction

Mean field models from statistical mechanics have attained much interest in the recent decades among probabilists. This is due to the fact that even in simple models as the Curie–Weiss model of ferromagnetism interesting critical phenomena can be observed on the level of probabilistic limit theorems, which might have role model character for more intricate lattice based models. In recent years there is increasing interest in the rigorous treatment of probabilistic models from spin glass theory, *i.e.* models which have an additional random parameter in the Gibbs measure, also called the disorder parameter. A natural extension of models from classical equilibrium statistical mechanics towards some additonal random mechanism is to consider an additional *random external field* in the hamiltonian. In this paper, we are interested in the extension of certain limit theorems already known for Curie–Weiss models without external field to a situation, where we have additional randomness in terms of an external field driven by a dynamical system [5]. We note here that mean field models with random external field also exhibit interesting behaviour on the level of so called "metastates", see [11] for recent results.

We consider the following physical model: For a fixed positive integer d and a finite subset $\Lambda \subset \mathbb{Z}^d$ a ferromagnetic crystal is described by a *configuration space* $\Omega_\Lambda = \Omega^\Lambda$, Ω called *spin space*, and random variables

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$\Sigma_i^\Lambda : \Omega_\Lambda \longrightarrow \Omega$, $\Sigma_i^\Lambda(\sigma) = \sigma_i$. Σ_i^Λ is called the *spin* at site i. We restrict ourselves to the classical Curie–Weiss model, where the spins take values in $\Omega = \{+1, -1\}$. The crystal is exposed to an external magnetic field, described by a dynamical system $S = (E, \mathcal{A}, \mu, T)$, *i.e.* a probability space $(E, \mathcal{A}, \mu)$, a measure-preserving transformation $T : E \to E$ and a measurable function $f : E \to [0, 1]$. We denote by $\beta > 0$ the *inverse temperature* und by $J > 0$ a *coupling constant*. For a spin configuration, *i.e.* a realization $(\Sigma_i^\Lambda)_{i\in\Lambda} = (\sigma_i)_{i\in\Lambda}$ and $x \in E$ we define the *Hamiltonian* (see [5]), which specifies the energy of a given configuration $\sigma = (\sigma_i)_{i\in\Lambda}$:

$$H_{\Lambda,x}(\sigma) = -\frac{\beta J}{2|\Lambda|}\left(\sum_{i\in\Lambda}\sigma_i\right)^2 - \frac{1}{2}\sum_{i\in\Lambda}\log\left(\frac{f(T^i x)}{1 - f(T^i x)}\right)\sigma_i.$$

The energy is due to the interaction of the spins and the force of the external magnetic field. The probability of observing the system in state $\sigma = (\sigma_i)_{i\in\Lambda}$ is specified by the Gibbs measure

$$P_{\Lambda,x}(\sigma) \equiv P_{\Lambda,x,\beta}(\sigma) = \frac{1}{Z_{\Lambda,x}}\exp\left(-\left[H_{\Lambda,x}(\sigma)\right]\right)$$ [2]

The normalizing factor $Z_{\Lambda,x}$ is called *partition function*. Equivalently one can say that the distribution of σ shall have the density $P_{\Lambda,x}(\sigma)$ with respect to the n-fold product measure $P_n(\sigma) = \prod_{i=1}^n \rho(\sigma_i)$ on Ω^n, with $\rho(1) = \rho(-1) = \frac{1}{2}$. The single site measure ρ is called the *a priori* measure. So the Gibbs measure turns the independent situation in a dependent one and the parameter β somehow determines the strength of this dependence. For each configuration $\sigma = (\sigma_i)_{i\in\Lambda}$ we define the *total magnetization* $M_\Lambda = \sum_{i\in\Lambda}\Sigma_i^\Lambda$. Without loss of generality we set $d = 1$, $\Lambda = \{1, \ldots, n\}$ in the sequel and we write n instead of Λ, as well as $\Sigma_i^{(n)}$, $P_{n,x}$ and M_n. So we consider a spin model on the complete graph with n edges. This model belongs to the class of *mean field* models, *i.e.* the spatial interaction is the same for every pair of spins (J constant). For $x \in E$ the sequence $\left\{\log\left(\frac{f(T^i x)}{1-f(T^i x)}\right)\right\}_{i\geq 1}$ specifies a magnetic field, which is inhomogeneous in space. The special case $f \equiv \frac{1}{2}$ corresponds to the Curie–Weiss model with zero external field. Furthermore, any external field $\left\{g(T^i x)\right\}_{i\geq 1}$ can be considered choosing the function $f = \frac{e^g}{1-e^g}$.

Another way of looking at this model is to put the summand of the Hamiltonian depending on the external field into the *a priori* measure. By a suitable normalization one gets a mean field model with random and site dependent *a priori* measure, which motivates the definition of a "dynamic $\mathbb{Z}$-random walk" in the following way (for further details see [5] or [12]): Consider a dynamical system $S = (E, \mathcal{A}, \mu, T)$, where $(E, \mathcal{A}, \mu)$ is a probability space and T is a measure preserving transformation defined on E. Let $f : E \to [0, 1]$ be a measurable function. For each $x \in E$ denote by $\mathbb{P}_x$ the distribution of the time inhomogeneous random walk

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i \quad \text{for } n \geq 1,$$

where $(X_i)_{i\in\Lambda}$ are independent random variables with step distribution

$$\mathbb{P}_x(X_i = z) = \begin{cases} f(T^i x), & \text{if } z = 1 \\ 1 - f(T^i x), & \text{if } z = -1 \\ 0, & \text{otherwise.} \end{cases}$$

In the case $\beta = 0$ (infinite temperature) $P_{n,x}$ is equal to the product measure

$$\prod_{i=1}^n \left(f(T^i x)\delta_1 + (1 - f(T^i x))\delta_{-1}\right).$$

[2]In the case that β is fixed we keep quiet about this dependency in the notation of the measure.

Then M_n is a sum of independent $Ber(f(T^i x))-$distributed random variables σ_i and it defines a *dynamic* $\mathbb{Z}$*-random walk* ([12]).

For a general introduction to the theory of large deviations and its applications to statistical mechanics we refer to the book [8]. Let us recall the definition of a large deviation principle:

Let E be a metric space, endowed with the Borel sigma-field $\mathcal{B}(E)$ and $(\gamma_n)_n$ be a sequence of positive reals with $\gamma_n \to \infty$ as $n \to \infty$. A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ on E is said to satisfy a large deviation principle (LDP for short) with speed γ_n and rate function $I : E \to [0, +\infty]$ if the following properties are satisfied:

- I has compact level sets $\Phi(s) = \{x \in E : I(x) \le s\}$, $s \in E$.
- For every open set $G \subset E$ it holds

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(G) \ge - \inf_{x\in G} I(x).$$

- For every closed set $A \subset E$ it holds

$$\limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu_n(A) \le - \inf_{x\in A} I(x).$$

Similarly we say that a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ with values in E obeys a large deviation principle with speed γ_n and rate function $I : E \to [0, +\infty]$ if the sequence of their distributions does. We will speak about a moderate deviation principle (MDP), whenever the scaling of the corresponding random variable is between that of an ordinary law of large numbers and that of a central limit theorem.

In [12] the authors proved a LDP for the mean magnetization M_n/n in the above defined Curie–Weiss model with dynamical external field. We briefly recall these statements and outline the main ideas of the proofs.

Using the Gärtner–Ellis Theorem (Birkhoff's theorem implies the needed convergence of the logarithmic moment generating function) the authors proved the following result in [12]:

For μ-almost every $x \in E$, the sequence $(S_n/n)_{n\in\mathbb{N}}$ satisfies a LDP with speed n and rate function

$$\Lambda_x^*(y) = \sup_{\lambda\in\mathbb{R}} \{\langle \lambda, y\rangle - \Lambda_x(\lambda)\},$$

where

$$\Lambda_x(\lambda) = \mathbb{E}\left(\log\left(f \mathrm{e}^{\lambda} + (1-f)\mathrm{e}^{-\lambda}\right) \mid \mathcal{T}\right)(x),$$

$\mathcal{T}$ being the σ-field generated by the fixed points of the transformation T.

Under further assumptions on the dynamical system one can apply a stronger version of Birkhoff's theorem (see [14]), which states pointwise convergence against a constant instead of μ-almost sure convergence. The result reads as follows:

Suppose that the above defined dynamical system $S = (E, \mathcal{A}, \mu, T)$ is uniquely ergodic, with compact metric space E, continuous transformation T and continuous function f. Then the above LDP holds for every point $x \in E$ with deterministic rate function

$$\Lambda(\lambda) = \int_E \log\left(f(y)\mathrm{e}^{\lambda} + (1-f)\mathrm{e}^{-\lambda}\right)\, \mathrm{d}\mu(y).$$

The authors in [12] also prove a functional central limit theorem for the dynamic random walk.

The first result in the present paper is a MDP for the dynamic random walk. To this end, we consider the centered random variables $\hat{X}_i = X_i - (2f(T^i x) - 1)$ and first define precisely what a MDP is in our case of partial sums of independent random variables. We say that $\frac{1}{a_n}\hat{S_n} = \frac{1}{a_n}\sum_{i=1}^n \hat{X}_i$ obeys a MDP with rate function I and speed $\frac{a_n^2}{n} \to \infty$, under the quenched measure $\mathbb{P}_x$, if $(a_n)_n$ is an increasing sequence of reals such that $\frac{a_n}{\sqrt{n}} \nearrow \infty$, $\frac{a_n}{n} \searrow 0$ and for all $A \in \mathcal{B}(\mathbb{R})$

$$-\inf_{t\in A^\circ} I(t) \leq \liminf_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}_x\left(\frac{1}{a_n}\hat{S}_n \in A\right) \leq \limsup_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}_x\left(\frac{1}{a_n}\hat{S}_n \in A\right) \leq -\inf_{t\in \overline{A}} I(t).$$

Our deviation results will depend on the speed of convergence in Birkhoff's Ergodic Theorem. One defines for every $\alpha \in [0,1]$ the class of μ-integrable functions

$$\mathcal{C}_\alpha(S) := \left\{ h : E \to \mathbb{R} : \left| \sum_{k=1}^n \left(h(T^k x) - \int_E h \, \mathrm{d}\mu \right) \right| = o(n^\alpha) \quad \forall x \in E \right\}.$$

We get the following theorem.

Theorem 1.1 (Moderate deviations for the dynamic random walk under the quenched measure)**.** *Suppose that* $f(1-f) \in \mathcal{C}_1(S)$ *and* $a := \int_E 4f(1-f)\mathrm{d}\mu > 0$. *Then for all* $x \in E$, $\frac{1}{a_n}\hat{S}_n$ *obeys a MDP with speed* $\frac{a_n^2}{n}$ *and rate function* $I(t) = \frac{t^2}{2a}\cdot$

Corollary 1.2. *If the dynamical system S is uniquely ergodic with compact metric space E, f is continuous and* $a := \int_E 4f(1-f)\mathrm{d}\mu > 0$, *then the assertion of Theorem 1.1 holds.*

Remark 1.3.

(i) In the case $f \equiv \frac{1}{2}$ Theorem 1.1 reproduces the classical MDP for i.i.d. random variables X_i with $P(X_1 = 1) = \frac{1}{2} = P(X_1 = -1)$.

(ii) Note that a MDP under the annealed measure, *i.e.* under the measure $\mathbb{P}(\mathrm{d}y) = \int_E \mathbb{P}_x(dy)\, \mathrm{d}\mu(x)$, can be obtained combining Theorems 2.1 and 2.2 in [4] with Theorem 1.1 above.

Now the LDP for the dynamic random walk on the integers yields a LDP for the mean magnetization M_n/n *via* Varadhan's lemma, since its distribution is absolutely continuous with respect to the distribution of the dynamic random walk and equal to

$$\frac{\mathrm{d}P_{n,x}^{M_n}}{\mathrm{d}\mathbb{P}_x^{S_n}}(y) = \frac{1}{\widehat{Z}_{n,x}} \exp\left[\frac{\beta J}{2n} y^2\right],$$

where $\widehat{Z}_{n,x} = \mathbb{E}_x\left\{\exp\left[\frac{\beta J}{2n}(S_n)^2\right]\right\}$ is a normalizing constant, *i.e.* the integrand is a continuous and bounded function on $[-1.1]$. So M_n/n under $P_{n,x}$ obeys a LDP with speed n and rate function

$$I_{\beta,x}(s) = \Lambda^*(s) - \frac{\beta J}{2}s^2 - \inf_{z\in\mathbb{R}}\left\{\Lambda^*(z) - \frac{\beta J}{2}z^2\right\}, \tag{1.1}$$

where Λ^* denotes the Fenchel–Legendre transform from above.

The authors also prove central limit theorems for the associated magnetization. Analogously to the treatment in [9] and [10], the asymptotic behaviour of M_n depends on the extremal points of a function G, which is a transformation of the rate function of the above LDP for the mean magnetization and defined by

$$G(s) = \frac{\beta J}{2}s^2 - \int_E L(f(y), \beta J s)\, \mathrm{d}\mu(y).$$

Furthermore, one defines for every $n \geq 1$ the function

$$G_n(s) = \frac{\beta J}{2}s^2 - \frac{1}{n}\log \mathbb{E}_x(\exp(\beta J s S_n)) \tag{1.2}$$

$$= \frac{\beta J}{2}s^2 - \frac{1}{n}\sum_{i=1}^n L(f(T^i x), \beta J s), \tag{1.3}$$

where

$$L(\phi, s) := \begin{cases} [0,1] \times \mathbb{R} \to \mathbb{R} \\ (\phi, s) \mapsto \log(\phi \mathrm{e}^s + (1-\phi)\mathrm{e}^{-s}). \end{cases} \tag{1.4}$$

The function G is real analytic, and the set where G achieves its minimum is non-empty and finite (see Thm. 3.1 in [5]). So we denote by $g = \min\{G(s) : s \in \mathbb{R}\}$ the value of the global minimum (which is nonpositive since $G(0) = 0$) and by $m_1, \ldots, m_r$ the minimizers of G. Furthermore, one defines the *type* $2k_i$ and the strength $\lambda_i > 0$ of the minimum m_i by

$$\begin{aligned} 2k_i &= \min\{j \geq 0 \mid G^{(j)}(m_i) \neq 0\} \\ \lambda_i &= G^{(2k_i)}(m_i). \end{aligned}$$

Usually, multiple minima occur for values of β larger than some critical value β_c and this phenomenon is called a "phase transition". For an explicit class of dynamical systems, the authors in [5] can compute a critical temperature β_c for the model. But the situation for $\beta > \beta_c$, *i.e.* multiple minima of G, seems not to be well understood as we try to outline in the following: In [10] Ellis and Newman proved a law of large numbers (for the Curie–Weiss model with constant external field), *i.e.* they showed that the mean magnetization converges weakly to a linear combination of the minima of maximal type of G, whose weights can be written explicitly in terms of the types and strenghts of the corresponding minima. On the other hand, an LDP for the mean magnetization of this model also yields weak convergence to the zeros of the respective rate function. Interestingly enough, Ellis et al. recently proved in [2] (Thm. A.1) by means of convex analysis, that the set of global minimizers of G coincides with the set of zeros of the LDP rate function. This general theorem can also be applied to the Curie–Weiss model with dynamical external field and yields coincidence of the set of zeros of the above LDP rate function (1.1) and the set of minimizers of G. Astonishingly, in the treatment of the Curie–Weiss model with dynamical external field in [5], the authors claimed that the mean magnetization did not converge in distribution in the case of multiple minima of G. Nevertheless, they proved exponential equivalence of M_n/n to a linear combination of the minimizers of G, *i.e.* for every continuous bounded function h, the expectation of $h(M_n/n)$ under $P_{n,x}$ is equivalent, as $n \to \infty$, to

$$\frac{\sum_{i=1}^r b_{i,n} h(m_i)}{\sum_{i=1}^r b_{i,n}}.$$

For details on the n-dependent weights $b_{i,n}$ see Theorem 3.2 in [5].

For the case of a unique minimum m of G, the following limit theorem for the fluctuations of M_n/n around m has been proved in [5] (Thm. 3.3): Assume that the unique minimum m of G is of type $2k$ and strenght λ and that for every $j \in \{1, \ldots, 2k\}$, the function $\frac{\partial^j}{\partial s^j} L(f(.), \beta J m)$ belongs to the set $\mathcal{C}_{j/2k}(S)$. Then, the following convergence of measures holds:

$$\frac{M_n - nm}{n^{1-1/2k}} \Rightarrow Z(2k, \tilde{\lambda}),$$

where $Z(2k, \tilde{\lambda})$ is a probability measure with density function

$$C \exp\left(-\tilde{\lambda} s^{2k}/(2k)!\right),$$

C being a normalizing constant and

$$\tilde{\lambda} = \begin{cases} \left(\frac{1}{\lambda} - \frac{1}{\beta J}\right)^{-1}, & \text{if } k = 1 \\ \lambda, & \text{if } k \geq 2. \end{cases}$$

The purpose of the present paper is to analyze the asymptotic behaviour of M_n on a moderate deviation scale. Our first result considers a smoothed version of the mean magnetization, called the Hubbard–Stratonovich transform.

Theorem 1.4 (Moderate deviations for the Hubbard–Stratonovich transform, conditioned version)**.** *Let m be a (local or global) minimum of G of type $2k$ and strength λ and assume that for every $j \in \{1, \ldots, 2k\}$ the function $y \mapsto \frac{\partial^j}{\partial s^j} L(f(y), \beta J m)$ belongs to the class $\mathcal{C}_{\frac{j}{2k}}(S)$. Let W be a $\mathcal{N}(0, \frac{1}{\beta J})$-distributed random variable, defined on some probability space $(\Omega, \mathcal{F}, Q)$ and independent of M_n for every $n \geq 1$. Write $Q_{n,x} := P_{n,x} \otimes Q$. Then there exists an $A = A(m) > 0$ such that for all $0 < a < A$ and for all $1 - \frac{1}{2k} < \alpha < 1$ the sequence of measures*

$$\left\{ Q_{n,x} \left(\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha - \frac{1}{2}}} \in \bullet \,\middle|\, \frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha - \frac{1}{2}}} \in [-an^{1-\alpha}, an^{1-\alpha}] \right) \right\}_{n \in \mathbb{N}}$$

satisfies a MDP with speed $n^{1-2k+2k\alpha}$ and rate function

$$I(z) \equiv I_{k,\lambda,\beta,J}(z) := \lambda \frac{z^{2k}}{(2k)!}.$$

In the case of a unique minimum of G we get a MDP for the mean magnetization out of Theorem 1.4 using some transfer principle, Proposition A.1 in [7]. In the case of multiple minima a similar, general transfer principle for conditional measures is still an open question. In the recent preprint [13] such a transfer principle was proved under additional conditions on the inverse temperature for the random field Curie–Weiss model (Lem. 5.2 (ii)).

Theorem 1.5 (Moderate deviations for the Curie–Weiss model with dynamical external field, unconditioned version)**.** *Assume that G has a unique global minimum m of type $2k$ and strength $\lambda \neq \beta J$ and that for all $j \in \{1, \ldots, 2k\}$ the function $y \mapsto \frac{\partial^j}{\partial s^j} L(f(y), \beta J m)$ belongs to the class $\mathcal{C}_{\frac{j}{2k}}(S)$. Then for all $1 - \frac{1}{2k} < \alpha < 1$ the sequence of measures*

$$\left\{ P_{n,x} \left(\frac{M_n - nm}{n^\alpha} \in \bullet \right) \right\}_{n \in \mathbb{N}}$$

satisfies a MDP with speed $n^{1-2k+2k\alpha}$ and rate function

$$I(z) \equiv I_{k,\lambda,\beta,J}(z) := \begin{cases} \frac{z^2}{2\sigma^2}, & k = 1 \\ \lambda \frac{z^{2k}}{(2k)!}, & k \geq 2, \end{cases}$$

where $\sigma^2 = \frac{1}{\lambda} - \frac{1}{\beta J}$.

Remark 1.6.

(i) Theorem 1.5 can be seen as a refinement of the scaling limits in [5] (Thm. 3.3).

(ii) Comparing Theorem 1.1 with Theorem 1.5 we see that the MDP for the rescaled dynamic $\mathbb{Z}$-random walk has the same kind of rate function, *i.e.* the Gaussian rate, as the MDP for the rescaled total magnetization under the Curie–Weiss measure with dynamical external field in the case $k = 1$. This is due to the fact that the case $k = 1$ corresponds to the high temperature phase (small β) where one has only weak correlation among the individual spins. The case $k \geq 2$ corresponds to the high temperature regime and one observes a phase transition even on the level of the MDP rate function.

2. Auxiliary results

In this section we state several lemmas that we will need in the proofs of our main theorems. The first lemma contains some important information about the sequence of functions G_n and the function G, as defined in the introduction of the present paper. For the proof we refer to Theorem 3.1, Lemma 3.2 and Lemma 3.4 in [5] respectively.

Lemma 2.1.

(*i*) *The function G is real analytic and the set where G attains its global minimum is non-empty and finite.*

(*ii*) *The sequence of functions $(G_n)_{n\geq 1}$ converges uniformly to G on compacta of $\mathbb{R}$ as $n \to \infty$. Furthermore, the sequence of derivative functions $(G_n^{(k)})_{n\geq 1}$ converges uniformly to $G^{(k)}$ for every $k \geq 1$ on compacta of $\mathbb{R}$ as $n \to \infty$.*

(*iii*) *Let $A \subset \mathbb{R}$ be a closed subset containing no global minima of G. Then there exists $\epsilon > 0$ such that*

$$\mathrm{e}^{ng} \int_A \mathrm{e}^{-nG_n(s)} \, \mathrm{d}s = \mathcal{O}(\mathrm{e}^{-n\epsilon}),$$

where g is the value of the global minimum of G.

The following Lemma is a key ingredient for the proof of our MDPs. It is based on the Taylor expansion of G and a slight generalization of Lemma 3.3 in [5]. We skip the proof since it differs only minorly from the one given in [5].

Lemma 2.2. *Let m be a (local or global) minimum of G of type $2k$ and strenght λ. Suppose that for every $j \in \{1, \ldots, 2k\}$ the function $y \mapsto \frac{\partial^j}{\partial s^j} L(f(y), \beta J m)$ belongs to the set $\mathcal{C}_{\frac{j}{2k}}(S)$. Let $1 - \frac{1}{2k} < \alpha < 1$. Then the following assertions hold:*

(*i*) *For every $s \in \mathbb{R}$*

$$\lim_{n\to\infty} n^{2k(1-\alpha)}(G_n(m + sn^{-(1-\alpha)}) - G_n(m)) = \lambda \frac{s^{2k}}{(2k)!}. \tag{2.1}$$

The convergence is uniform on compact intervals of the form $[-M, M]$.

(*ii*) *There exist $r > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and $s \in \left[-rn^{(1-\alpha)}, rn^{(1-\alpha)}\right]$ the following lower bound is valid:*

$$n^{2k(1-\alpha)}(G_n(m + sn^{-(1-\alpha)}) - G_n(m)) \geq \frac{\lambda}{2} \frac{s^{2k}}{(2k)!} - \sum_{j=1}^{2k-1} |s|^j \,. \tag{2.2}$$

The following lemma concerns a well known transformation of our mean field measure, sometimes called the Hubbard–Stratonovich transform in the literature. For the proof we refer to Lemma 3.1 in [5].

Lemma 2.3. *Let W be a $\mathcal{N}(0, \frac{1}{\beta J})$-distributed random variable, defined on some probability space $(\Omega, \mathcal{F}, Q)$ and independent of M_n for every $n \geq 1$, and let m and α be some real numbers. Then the random variable*

$$\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha - 1/2}}$$

under the measure $Q_{n,x} := P_{n,x} \otimes Q$ has a density with respect to the Lebesgue measure given by

$$\frac{\exp(-nG_n(m + sn^{-(1-\alpha)}))}{\int_{\mathbb{R}} \exp(-nG_n(m + sn^{-(1-\alpha)}))\mathrm{d}s}. \tag{2.3}$$

The usefulness of the previous lemma lies in the fact that one can often prove MDPs for the convolution, using the Taylor series expansion of G. Clearly, the type $2k$ and strength λ of the global minimum m of G will therefore play an important role. We next state two lemmas which ensure that it does not matter whether we consider the sequence of measures $P_{n,x} \circ \left(\frac{M_n - nm}{n^\alpha}\right)^{-1}$ or the sequence $P_{n,x} \otimes Q \circ \left(\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}}\right)^{-1}$ as long as $k \geq 2$.

Lemma 2.4. *If the sequence of random variables $\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}}$ satisfies a MDP with respect to $Q_{n,x} = P_{n,x} \otimes Q$ with speed n^γ, $\gamma < 2\alpha - 1$ and rate function I, then so does $\frac{M_n - nm}{n^\alpha}$ with respect to $P_{n,x}$ and the speeds and rate functions agree.*

Proof. The proof is based on exponential equivalence and can be found in [7]. Nevertheless we give the proof in order to allude to the problems that arise in the case $k = 1$. One shows that the two sequences $\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}}$ and $\frac{M_n-nm}{n^\alpha}$ are exponentially equivalent and therefore have the same moderate deviation behaviour (see Thm. 4.2.13 in [3]). For all $\epsilon > 0$ the following estimate holds:

$$P_{n,x} \otimes Q\left(\left|\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}} - \frac{M_n - nm}{n^\alpha}\right| > \epsilon\right) = P_{n,x} \otimes Q\left(\left|\frac{W}{n^{\alpha-1/2}}\right| > \epsilon\right)$$
$$= Q\left(|W| > \epsilon n^{\alpha - \frac{1}{2}}\right) \leq \sqrt{\frac{2\beta J}{\pi}} \frac{1}{\epsilon n^{\alpha-\frac{1}{2}}} \exp\left(-\frac{\beta J}{2}\epsilon^2 n^{2\alpha-1}\right).$$

This implies

$$\limsup_{n\to\infty} \frac{1}{n^\gamma} \log P_{n,x} \otimes Q\left(\left|\frac{M_n - nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}} - \frac{M_n - nm}{n^\alpha}\right| > \epsilon\right)$$
$$\leq \limsup_{n\to\infty}\left(\frac{-\frac{\beta J}{2}\epsilon^2 n^{2\alpha-1} - \log\left(\sqrt{\frac{\pi}{2\beta J}}\epsilon n^{\alpha-\frac{1}{2}}\right)}{n^\gamma}\right) = \limsup_{n\to\infty}\left(-\frac{\beta J}{2}\epsilon^2 n^{(2\alpha-1)-\gamma} - \frac{\log\left(\sqrt{\frac{\pi}{2\beta J}} n^{\alpha-\frac{1}{2}}\right)}{n^\gamma}\right) = -\infty,$$

since $\gamma < 2\alpha - 1$ by assumption. □

In the case $k = 1$ the speed of the MDP in Theorems 1.4 and 1.5 is of the same order as the variance of the respective Gaussian random variable. Therefore the above argument for exponential equivalence fails. In [7] the authors proved a "transfer principle" for LDP which can be applied in this special case (see Prop. A.1 in [7]). The application of this Proposition reads as follows (for the proof see Lem. 3.6 in [7]):

Lemma 2.5. *Suppose that* $\frac{M_n-nm}{n^\alpha} + \frac{W}{n^{\alpha-1/2}}$ *satisfies a MDP with respect to* $Q_{n,x} = P_{n,x} \otimes Q$ *with speed* $n^{2\alpha-1}$ *and rate function* $\lambda\frac{z^2}{2}$ *for* $\lambda \neq \beta J$. *Then so does* $\frac{M_n-nm}{n^\alpha}$ *with respect to* $P_{n,x}$ *with the same speed and rate function* $\frac{y^2}{2\sigma^2}$, *where* $\sigma^2 = \frac{1}{\lambda} - \frac{1}{\beta J}$.

Our last lemma can be considered as a starting point of the *Laplace method* in the theory of large deviations. It will be used in the proof of our main theorems.

Lemma 2.6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function,* $M > 0$ *a real number and* $\gamma_n \to \infty$ *a sequence of positive integers. Then*

$$\lim_{n\to\infty} \frac{1}{\gamma_n} \log \int_{\{|x|\leq M\}} \exp\left[\gamma_n f(x)\right] \mathrm{d}x = \max_{\{|x|\leq M\}} f(x).$$

3. Proofs

In this section we first give the proof of our moderate deviation result 1.1 for the dynamic random walk. Thereafter we prove our main Theorems 1.4 and 1.5, *i.e.* MDPs for the fluctuations of the mean magnetization around the minimizer of G. Our proofs use *Laplace method*, an equivalent formulation of a LDP which is based on the asymptotic analysis of the scaled logarithms of certain expectations. We refer to the book [6] for a detailed introduction to this approach to large deviation theory. The Laplace method has been successfully applied in the context of the Blume–Emery–Griffiths model in [1], where the authors prove MDP's for the case of size dependent temperatures, *i.e.* the limit results are obtained as the pair (β, J) converges along appropriate sequences (β_n, J_n) to points belonging to various subsets of the phase diagram. Our proof has been inspired by this approach since we have a similar n-dependence in the Hubbard–Stratonovich transform *via* the functions G_n.

Proof of Theorem 1.1. In order to apply Gärtner–Ellis–Theorem [3] we consider the Laplace transform of our random variable of interest. Let us fix $x \in E$ and write $\lambda_i \equiv \lambda_i(x) = f(T^i x)$. We then get for each $t \in \mathbb{R}$

$$\begin{aligned}\log \mathbb{E}_x\left[\mathrm{e}^{t\frac{a_n^2}{n}(\hat{S}_n/a_n)}\right] &= \sum_{i=1}^n \log \mathbb{E}_x\left[ta_n(X_i - 2\lambda_i + 1)/n\right] \\ &= \sum_{i=1}^n \log\left(\lambda_i \mathrm{e}^{ta_n(1-2\lambda_i+1)/n} + (1-\lambda_i)\mathrm{e}^{ta_n(-2\lambda_i)/n}\right) \\ &= \frac{-ta_n\, 2\sum_{i=1}^n \lambda_i}{n} + \sum_{i=1}^n \log\left(1 + \lambda_i(\mathrm{e}^{2ta_n/n} - 1)\right).\end{aligned}$$

For n large enough we have $\lambda_i(\mathrm{e}^{2ta_n/n} - 1) \in [0,1]$ so that we can use Taylor expansion for the logarithm.

$$\begin{aligned}\log\left(1 + \lambda_i\left(\mathrm{e}^{2ta_n/n} - 1\right)\right) &= \lambda_i(\mathrm{e}^{2ta_n/n} - 1) - \frac{\lambda_i^2}{2}(\mathrm{e}^{2ta_n/n} - 1)^2 + \frac{\lambda_i^3}{3}(\mathrm{e}^{2ta_n/n} - 1)^3 + o\left(\lambda_i^4(\mathrm{e}^{2ta_n/n} - 1)^4\right) \\ &= \lambda_i \frac{2ta_n}{n} + \lambda_i \frac{(2t)^2 a_n^2}{2n^2} + \lambda_i \frac{(2t)^3 a_n^3}{3!n^3} + o\left(\frac{\lambda_i a_n^4}{n^4}\right) - \lambda_i^2 \frac{(2t)^2 a_n^2}{2n^2} \\ &\quad - \lambda_i^2 \frac{(2t)^3 a_n^3}{2n^3} + o\left(\frac{\lambda_i^2 a_n^4}{n^4}\right) + \lambda_i^3 \frac{(2t)^3 a_n^3}{3n^3} + o\left(\frac{\lambda_i^3 a_n^4}{n^4}\right) \qquad (3.1) \\ &= \lambda_i \frac{2ta_n}{n} + \lambda_i \frac{(2t)^2 a_n^2}{2n^2} - \lambda_i^2 \frac{(2t)^2 a_n^2}{2n^2} + o\left(\lambda_i(1-\lambda_i)\frac{a_n^3}{n^3}\right) \qquad (3.2) \\ &= \lambda_i \frac{2ta_n}{n} + \lambda_i(1-\lambda_i)\frac{(2t)^2 a_n^2}{2n^2} + o\left(\frac{\lambda_i(1-\lambda_i)}{n^3}\right).\end{aligned}$$

Here we used Taylor series for the exponential function at 0 in (3.1) and the following estimate in equation (3.2).

$$\begin{aligned}\left|\lambda_i \frac{(2t)^3 a_n^3}{3!n^3} - \lambda_i^2 \frac{(2t)^3 a_n^3}{3!n^3} + \lambda_i^3 \frac{(2t)^3 a_n^3}{3n^3}\right| &= \left|\lambda_i^2 \frac{(2t)^3 a_n^3}{3!n^3}\left(1 - 3\lambda_i + 2\lambda_i^2\right)\right| \\ &= \left|\frac{(2t)^3 a_n^3}{3!n^3}\lambda_i(1-\lambda_i)(1-2\lambda_i)\right| \\ &\le \left|\frac{(2t)^3 a_n^3}{3!n^3}\lambda_i(1-\lambda_i)\right|.\end{aligned}$$

We therefore finally get

$$\begin{aligned}\log\left(1 + \lambda_i\left(\mathrm{e}^{2ta_n/n} - 1\right)\right) &= \frac{(2t)^2 a_n^2 \sum_{i=1}^n \lambda_i(1-\lambda_i)}{2n^2} + o\left(\frac{a_n^3 \sum_{i=1}^n \lambda_i(1-\lambda_i)}{n^3}\right) \\ &= \frac{t^2 a_n^2}{2n} \cdot \frac{\sum_{i=1}^n 4f(T^i x)(1 - f(T^i x))}{n} + o\left(\frac{a_n^3}{n^2} \cdot \frac{\sum_{i=1}^n f(T^i x)(1 - f(T^i x))}{n}\right)\end{aligned}$$

for all $t \in \mathbb{R}$ and $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log\left(1 + \lambda_i\left(\mathrm{e}^{2ta_n/n} - 1\right)\right) = \frac{t^2 a}{2} \qquad (3.3)$$

since $f(1-f) \in \mathcal{C}_1(\mathcal{S})$ and $a_n/n \to 0$ as $n \to \infty$ by assumption. An application of the Gärtner–Ellis–Theorem (see Thm. 3.2.6 in [3]) now yields a MDP for $\frac{1}{a_n}\hat{S}_n$ with speed $\frac{a_n^2}{n}$ and rate function

$$I(t) = \sup_x \left\{xt - \frac{t^2 a}{2}\right\} = \frac{t^2}{2a}.$$

□

Proof of Theorem 1.4. We would like to prove a MDP for the sequence of measures

$$\left\{Q_{n,x}\left(\frac{M_n-nm}{n^\alpha}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in\bullet\,\middle|\,\frac{M_n-nm}{n^\alpha}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in[-an^{1-\alpha},an^{1-\alpha}]\right)\right\}_{n\in\mathbb{N}},$$

for some $A=A(m)$ and all $0<a<A$. Lemma 2.3 yields for every Borel set B

$$Q_{n,x}\left(\frac{M_n-nm}{n^\alpha}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in B\,\middle|\,\frac{M_n-nm}{n^\alpha}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in[-an^{1-\alpha},an^{1-\alpha}]\right)$$
$$=\frac{\int_{B\cap[-an^{1-\alpha},an^{1-\alpha}]}\exp(-nG_n(m+sn^{-(1-\alpha)}))\,\mathrm{d}s}{\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp(-nG_n(m+sn^{-(1-\alpha)}))\,\mathrm{d}s}.$$

We will prove a MDP for this sequence of measures *via Laplace principle*. Theorem 1.2.3 in [6] states that it satisfies a MDP with the respective speed and rate funtion if and only if it satisfies the *Laplace principle*. So let $\Psi\in\mathcal{C}_b(\mathbb{R})$ be a continuous and bounded function. To verify the Laplace principle we have to show that

$$\lim_{n\to\infty}\frac{1}{n^{1-2k+2k\alpha}}\log\int_{\mathbb{R}}\exp\left[n^{1-2k+2k\alpha}\Psi(s)\right]\;Q_{n,x}\left(Y_n\in\mathrm{d}s\,|\,Y_n\in[-an^{1-\alpha},an^{1-\alpha}]\right)$$
$$=\sup_{s\in\mathbb{R}}\left\{\Psi(s)-\lambda\frac{s^{2k}}{(2k)!}\right\},\tag{3.4}$$

where we used the abbreviation $Y_n:=\frac{M_n-nm}{n^\alpha}+\frac{W}{n^{\alpha-\frac{1}{2}}}$.

We substitute the density of $Q_{n,x}(Y_n\in\bullet\,|\,Y_n\in[-an^{1-\alpha},an^{1-\alpha}])$ on the left hand side of (3.4) and thus have to analyze the following object:

$$\frac{1}{n^{1-2k+2k\alpha}}\log\left\{\frac{\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[n^{1-2k+2k\alpha}\Psi(s)-nG_n(m+sn^{-(1-\alpha)})\right]\;\mathrm{d}s}{\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[-nG_n(m+sn^{-(1-\alpha)})\right]\;\mathrm{d}s}\right\},$$

or equivalently

$$\frac{1}{n^{1-2k+2k\alpha}}\log\left\{\frac{\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[n^{1-2k+2k\alpha}\Psi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\;\mathrm{d}s}{\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\;\mathrm{d}s}\right\}.\tag{3.5}$$

We consider the nominator and the denominator in (3.5) seperately.
Lemma 2.2 states that there exist $r>0$, $N\in\mathbb{N}$ and a polynomial $H(s)=\frac{\lambda}{2}\frac{s^{2k}}{(2k)!}-\sum_{j=1}^{2k-1}|s|^j$ such that for all $n\geq N$ and for all s with $|s|<rn^{1-\alpha}$ the following estimate holds:

$$n^{2k(1-\alpha)}(G_n(m+sn^{-(1-\alpha)})-G_n(m))\geq H(s).\tag{3.6}$$

We choose $A(m):=r$. Since the leading coefficient of H is positive, $H(s)\to\infty$ for $|s|\to\infty$. Since $H(s)\to\infty$ and $\lambda\frac{s^{2k}}{(2k)!}\to\infty$ for $|s|\to\infty$, there exists $M>0$ such that

$$\sup_{|s|>M}\{\Psi(s)-H(s)\}\leq-\left|\sup_{s\in\mathbb{R}}\{\Psi(s)-\lambda\frac{s^{2k}}{(2k)!}\}\right|-1$$

and the supremum of $\Psi - \lambda\frac{s^{2k}}{(2k)!}$ over $\mathbb{R}$ is attained on the interval $[-M, M]$. This and inequality (3.6) together imply that for all $0 < a < A$ and for all $n \geq N$ with $an^{(1-\alpha)} > M$ it holds

$$\begin{aligned}
&\sup_{\{M<|s|<an^{(1-\alpha)}\}} \left\{ n^{1-2k+2k\alpha}\Psi(s) - n(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right\} \\
&= \sup_{\{M<|s|<an^{(1-\alpha)}\}} \left\{ n^{1-2k+2k\alpha} \left[\Psi(s) - n^{2k(1-\alpha)}(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right] \right\} \\
&\leq -n^{1-2k+2k\alpha} \left(\left| \sup_{s\in\mathbb{R}} \{\Psi(s) - \lambda\frac{s^{2k}}{(2k)!}\} \right| + 1 \right). \qquad (3.7)
\end{aligned}$$

Now let $0 < a < A$, with $A = A(m)$ being the constant chosen above. Lemma 2.2 then implies that for all $\delta > 0$ and n large enough

$$\left| n^{2k(1-\alpha)}(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) - \lambda\frac{s^{2k}}{(2k)!} \right| < \delta,$$

for all $s \in [-M, M]$, where M is the constant chosen in (3.7). Thus

$$\begin{aligned}
&\exp\left(-n^{1-2k+2k\alpha}\cdot\delta\right) \int_{\{|s|\leq M\}} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right] \right) \,\mathrm{d}s \\
&\leq \int_{\{|s|\leq M\}} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - n^{2k(1-\alpha)}(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right] \right) \,\mathrm{d}s \\
&\leq \exp\left(n^{1-2k+2k\alpha}\cdot\delta\right) \int_{\{|s|\leq M\}} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right] \right) \,\mathrm{d}s. \qquad (3.8)
\end{aligned}$$

Estimate (3.7) implies for n large enough

$$\begin{aligned}
&\int_{\{M<|s|<an^{1-\alpha}\}} \exp\left(n^{1-2k+2k\alpha}\Psi(s) - n(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right) \,\mathrm{d}s \\
&\leq 2an^{1-\alpha} \exp\left(-n^{1-2k+2k\alpha} \left(\left| \sup_{s\in\mathbb{R}} \{\Psi(s) - \lambda\frac{s^{2k}}{(2k)!}\} \right| + 1 \right) \right). \qquad (3.9)
\end{aligned}$$

For sufficiently large n we get *via* the estimates (3.8) and (3.9)

$$\begin{aligned}
&\exp\left(-n^{1-2k+2k\alpha}\cdot\delta\right) \int_{\{|s|\leq M\}} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right] \right) \,\mathrm{d}s \\
&\leq \int_{-an^{1-\alpha}}^{an^{1-\alpha}} \exp\left[n^{1-2k+2k\alpha}\Psi(s) - n(G_n(m+sn^{(1-\alpha)}) - G_n(m)) \right] \,\mathrm{d}s \\
&\leq \exp\left(n^{1-2k+2k\alpha}\cdot\delta\right) \int_{|s|\leq M} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right] \right) \,\mathrm{d}s \\
&\quad + 2an^{1-\alpha} \exp\left(-n^{1-2k+2k\alpha} \left(\left| \sup_{s\in\mathbb{R}} \{\Psi(s) - \lambda\frac{s^{2k}}{(2k)!}\} \right| + 1 \right) \right).
\end{aligned}$$

Lemma 2.6 and the fact that the supremum of $\Psi - \lambda\frac{s^{2k}}{(2k)!}$ is attained on the interval $[-M, M]$ now yield

$$\lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\{|s|\leq M\}} \exp\left(n^{1-2k+2k\alpha} \left[\Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right] \right) \,\mathrm{d}s = \sup_{s\in\mathbb{R}} \left\{ \Psi(s) - \lambda\frac{s^{2k}}{(2k)!} \right\}.$$

Therefore

$$\sup_{s\in\mathbb{R}}\left\{\varPsi(s)-\lambda\frac{s^{2k}}{(2k)!}\right\}-\delta$$
$$\leq \liminf_{n\to\infty}\frac{1}{n^{1-2k+2k\alpha}}\log\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s$$
$$\leq \limsup_{n\to\infty}\frac{1}{n^{1-2k+2k\alpha}}\log\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s$$
$$\leq \sup_{s\in\mathbb{R}}\left\{\varPsi(s)-\lambda\frac{s^{2k}}{(2k)!}\right\}+\delta.$$

Since this equality is valid for all $\delta>0$ we finally get

$$\lim_{n\to\infty}\frac{1}{n^{1-2k+2k\alpha}}\log\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s$$
$$=\sup_{s\in\mathbb{R}}\left\{\varPsi(s)-\lambda\frac{s^{2k}}{(2k)!}\right\}.$$

Considering the special case $\varPsi\equiv 0$ yields

$$\lim_{n\to\infty}\frac{1}{n^{1-2k+2k\alpha}}\log\int_{-an^{1-\alpha}}^{an^{1-\alpha}}\exp\left[-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s=0.$$

Taking these two limits together we get (3.4), which proves the assertion. □

Proof of Theorem 1.5. The Lemmas 2.4 and 2.5 state that it suffices to prove a MDP for the sequence of measures

$$\left\{Q_{n,x}\left(\frac{M_n-nm}{n^{\alpha}}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in\bullet\right)\right\}_{n\in\mathbb{N}}.$$

By Theorem 1.4 we already know that for some $A(m)$ and all $0<a<A(m)$ the sequence

$$\left\{Q_{n,x}\left(\frac{M_n-nm}{n^{\alpha}}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in\bullet\;\middle|\;\frac{M_n-nm}{n^{\alpha}}+\frac{W}{n^{\alpha-\frac{1}{2}}}\in[-an^{1-\alpha},an^{1-\alpha}]\right)\right\}_{n\in\mathbb{N}}$$

obeys a MDP with the above speed and rate function. We now prove the MDP again *via* Laplace method, *i.e.* we consider for $\varPsi\in\mathcal{C}_b(\mathbb{R})$

$$\frac{1}{n^{1-2k+2k\alpha}}\log\int_{\mathbb{R}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)\right]\,\mathrm{d}Q_{n,x}(s)$$
$$=\frac{1}{n^{1-2k+2k\alpha}}\log\left\{\frac{\int_{\mathbb{R}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-nG_n(m+sn^{-(1-\alpha)})\right]\,\mathrm{d}s}{\int_{\mathbb{R}}\exp\left[-nG_n(m+sn^{-(1-\alpha)})\right]\,\mathrm{d}s}\right\}$$

or equivalently

$$\frac{1}{n^{1-2k+2k\alpha}}\log\left\{\frac{\int_{\mathbb{R}}\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s}{\int_{\mathbb{R}}\exp\left[-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right]\,\mathrm{d}s}\right\}$$

and we study the nominator and the denominator seperately. To this end we apply the conditioned version of the MDP as stated in Theorem 1.4 and the remaining work consists in controlling the missing integrals. For that purpose we make use of the assumed uniqueness of the global minimum of G. Define

$$h(s):=\exp\left[n^{1-2k+2k\alpha}\varPsi(s)-n(G_n(m+sn^{-(1-\alpha)})-G_n(m))\right].$$

We then get

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{[-an^{1-\alpha},an^{1-\alpha}]} h(s)\ \mathrm{d}s \\
&\leq \limsup_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \left\{ \int_{[-an^{1-\alpha},an^{1-\alpha}]} h(s)\ \mathrm{d}s + \int_{\{|s|>an^{1-\alpha}\}} h(s)\ \mathrm{d}s \right\} \\
&= \max\left\{ \limsup_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{[-an^{1-\alpha},an^{1-\alpha}]} h(s)\ \mathrm{d}s\ ,\ \limsup_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\{|s|>an^{1-\alpha}\}} h(s)\ \mathrm{d}s \right\}. \qquad (3.10)
\end{aligned}$$

Theorem 1.4 implies

$$\lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{[-an^{1-\alpha},an^{1-\alpha}]} h(s)\ \mathrm{d}s = \sup_{s\in\mathbb{R}} \left\{ \Psi(s) - \lambda \frac{s^{2k}}{(2k)!} \right\},$$

since this is the Laplace principle for the conditional Hubbard–Stratonovich transform. We now show that the second argument of the maximum in (3.10) converges to $-\infty$ as $n\to\infty$. Thereby we make use of the uniqueness of the global minimum m of G. Lemma 2.1 *(iii)* implies

$$\begin{aligned}
&\frac{1}{n^{1-2k+2k\alpha}} \log \int_{\{|s|>an^{1-\alpha}\}} h(s)\ \mathrm{d}s \\
&= \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\{|s|>an^{1-\alpha}\}} \exp\left[n^{1-2k+2k\alpha}\Psi(s) - n(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right]\ \mathrm{d}s \\
&= \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\{|t-m|>a\}} \exp\left[n^{1-2k+2k\alpha}\Psi((t-m)n^{1-\alpha}) - n(G_n(t) - G_n(m)) \right] n^{1-\alpha}\ \mathrm{d}t \\
&\leq \frac{1}{n^{1-2k+2k\alpha}} \log \left\{ \exp\left[n^{1-2k+2k\alpha} \|\Psi\|_\infty \right] \cdot n^{1-\alpha} \mathrm{e}^{nG_n(m)} \underbrace{\int_{\{|t-m|>a\}} \exp(-nG_n(t))\ \mathrm{d}t}_{=\mathcal{O}(\mathrm{e}^{-n\epsilon-ng})} \right\} \\
&= \|\Psi\|_\infty + \frac{1}{n^{1-2k+2k\alpha}} \log\left\{ n^{1-\alpha} \mathcal{O}\left(\exp\left[-n(\epsilon + (g - G_n(m))) \right] \right) \right\} \\
&= \|\Psi\|_\infty + \underbrace{\frac{\log(n^{1-\alpha})}{n^{1-2k+2k\alpha}}}_{\to 0} + \frac{1}{n^{1-2k+2k\alpha}} \log\left\{ \mathcal{O}\left(\exp\left[-n(\epsilon + \underbrace{(g - G_n(m))}_{\to 0}) \right] \right) \right\} \\
&\underset{n\to\infty}{\longrightarrow} -\infty, \qquad (3.11)
\end{aligned}$$

since the set $\{|t-m|>a\}$ does not contain a minimum of G. This together with inequality (3.10) yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\mathbb{R}} h(s)\ \mathrm{d}s &= \lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{[-an^{1-\alpha},an^{1-\alpha}]} h(s)\ \mathrm{d}s \\
&= \sup_{s\in\mathbb{R}} \left\{ \Psi(s) - \lambda \frac{s^{2k}}{(2k)!} \right\}.
\end{aligned}$$

Considering the special case $\Psi \equiv 0$ implies

$$\lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\mathbb{R}} \exp\left[-n(G_n(m+sn^{-(1-\alpha)}) - G_n(m)) \right]\ \mathrm{d}s = 0$$

These two limits together yield

$$\lim_{n\to\infty} \frac{1}{n^{1-2k+2k\alpha}} \log \int_{\mathbb{R}} \exp\left[n^{1-2k+2k\alpha}\Psi(s)\right] \, dQ_{n,x}(s) = \sup_{s\in\mathbb{R}} \left\{\Psi(s) - \lambda \frac{s^{2k}}{(2k)!}\right\}.$$

By Lemma 2.4 and Lemma 2.5 we get the assertion for the cases $k \geq 2$ and $k = 1$ respectively. □

For illustrational purposes we give a concrete example of a dynamical system and compute the rate function of the respective MDP explicitly. Note that this dynamical system has already been considered in [5]. We refer to this paper in order to check that our conditions imposed in Theorem 1.5 hold true for this concrete example.

4. Example: Irrational rotation on the torus

In [5] the authors consider the dynamical system $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \lambda_{\mathbb{T}}, T_\alpha)$. There $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1[$ denotes the one-dimensional torus, $\lambda_{\mathbb{T}}$ the restricted Lebesgue measure and T_α the irrational rotation with angle α of type η (see Def. 5.3 in [5]), *i.e.* $x \mapsto x + \alpha \mod 1$.

Let $f(x) = x$ be the identity on $\mathbb{T}$. In [5] it is proved that the Curie–Weiss model with dynamical external field according to this dynamical system exhibits a phase transition at the critical temperature $\beta_c = \frac{3}{2J}$.

Theorem 4.1 (MDP for the irrational rotation on the one-dimensional torus)**.** *For the above defined dynamical system the following assertions hold true.*

1. *For $\beta < \beta_c$, $\eta < 2$ and for all $\frac{1}{2} < \alpha < 1$ the sequence of measures*

$$\left\{P_{n,x}\left(\frac{M_n}{n^\alpha} \in \bullet\right)\right\}_{n\in\mathbb{N}}$$

satisfies a MDP with scale $n^{2\alpha-1}$ and rate function

$$I(z) = \frac{z^2}{2\sigma^2},$$

where $\sigma^2 = \frac{2}{3-2\beta J}$.

2. *For $\beta = \beta_c$, $\eta < 4/3$ and for all $\frac{3}{4} < \alpha < 1$ the sequence of measures*

$$\left\{P_{n,x}\left(\frac{M_n}{n^\alpha} \in \bullet\right)\right\}_{n\in\mathbb{N}}$$

satisfies a MDP with scale $n^{4\alpha-3}$ and rate function

$$I(z) = \frac{9}{80} z^4.$$

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