ON $\mathbb{R}^d$-VALUED PEACOCKS

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Abstract. In this paper, we consider $\mathbb{R}^d$-valued integrable processes which are increasing in the convex order, i.e. $\mathbb{R}^d$-valued peacocks in our terminology. After the presentation of some examples, we show that an $\mathbb{R}^d$-valued process is a peacock if and only if it has the same one-dimensional marginals as an $\mathbb{R}^d$-valued martingale. This extends former results, obtained notably by Strassen [Ann. Math. Stat. 36 (1965) 423–439], Doob [J. Funct. Anal. 2 (1968) 207–225] and Kellerer [Math. Ann. 198 (1972) 99–122].

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1. Introduction

1.1. Terminology

First we fix the terminology. In the sequel, $d$ denotes a fixed integer and $\mathbb{R}^d$ is equipped with a norm which is denoted by $| \cdot |$.

We say that two $\mathbb{R}^d$-valued processes: $(X_t, \ t \geq 0)$ and $(Y_t, \ t \geq 0)$ are associated, if they have the same one-dimensional marginals, i.e. if:

$$\forall t \geq 0, \ X_t \overset{\text{law}}{=} Y_t.$$ 

A process which is associated with a martingale is called a 1-martingale.

An $\mathbb{R}^d$-valued process $(X_t, \ t \geq 0)$ will be called a peacock if:

(i) it is integrable, that is:

$$\forall t \geq 0, \ E[|X_t|] < \infty;$$

(ii) it increases in the convex order, meaning that, for every convex function $\psi : \mathbb{R}^d \to \mathbb{R}$, the map:

$$t \geq 0 \mapsto E[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

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This terminology was introduced in [5]. We refer the reader to this monograph for an explanation of the origin of the term: “peacock”, as well as for a comprehensive study of this notion in the case $d = 1$.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t, t \geq 0)$ of its one-dimensional marginals is involved. This makes it natural, in the following, to also call a peacock, a family $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}^d$ such that:

(i) $\forall t \geq 0$, $\int |x| \mu_t(dx) < \infty$;

(ii) for every convex function $\psi : \mathbb{R}^d \to \mathbb{R}$, the map:

$$t \geq 0 \to \int \psi(x) \mu_t(dx) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}^d$ and an $\mathbb{R}^d$-valued process $(Y_t, t \geq 0)$ will be said to be associated if, for every $t \geq 0$, the law of $Y_t$ is $\mu_t$, i.e. if $(\mu_t, t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t, t \geq 0)$.

Obviously, the above notions also are meaningful if one considers processes and families of measures indexed by a subset of $\mathbb{R}_+$ (for example $\mathbb{N}$) instead of $\mathbb{R}_+$.

It is an easy consequence of Jensen’s inequality that an $\mathbb{R}^d$-valued process which is a 1-martingale, is a peacock. So, a natural question is whether the converse holds.

1.2. Case $d = 1$

A remarkable result due to Kellerer [6] states that, actually, any $\mathbb{R}$-valued process which is a peacock, is a 1-martingale. More precisely, Kellerer’s result states that any $\mathbb{R}$-valued peacock admits an associated martingale which is Markovian.

Two more recent results now complete Kellerer’s theorem.

(i) Lowther [7] states that if $(\mu_t, t \geq 0)$ is an $\mathbb{R}$-valued peacock such that the map: $t \to \mu_t$ is weakly continuous (i.e. for any $\mathbb{R}$-valued, bounded and continuous function $f$ on $\mathbb{R}$, the map: $t \to \int f(x) \mu_t(dx)$ is continuous), then $(\mu_t, t \geq 0)$ is associated with a strongly Markovian martingale which moreover is “almost-continuous” (see [7] for the definition);

(ii) in a previous paper [4], we presented a new proof of the above mentioned theorem of Kellerer. Our method, which is inspired from the “Fokker–Planck Equation Method” ([5], Sect. 6.2, p. 229), then appears as a new application of M. Pierre’s uniqueness theorem for a Fokker–Planck equation ([5], Thm. 6.1, p. 223). Thus, we show that a martingale which is associated to an $\mathbb{R}$-valued peacock, may be obtained as a limit of solutions of stochastic differential equations. However, we do not obtain that such a martingale is Markovian.

1.3. Case $d \geq 1$

Concerning the case $\mathbb{R}^d$ with $d \geq 1$, and even much more general spaces, we would like to mention the following three important papers.

(i) in [1], Cartier et al. study the case of two probability measures $(\mu_1, \mu_2)$ on a metrizable convex compact $K$ of a locally convex space. They prove, using the Hahn–Banach theorem, that, if $(\mu_1, \mu_2)$ is a $K$-valued peacock (indexed by $\{1, 2\}$), then there exists a Markovian kernel $P$ on $K$ such that: $\theta(dx_1, dx_2) := \mu_1(dx_1) P(x_1, dx_2)$ is the law of a $K$-valued martingale $(Y_1, Y_2)$ associated to $(\mu_1, \mu_2)$;

(ii) in [8], Strassen extends the Cartier–Fell–Meyer result to $\mathbb{R}^d$-valued peacocks without making the assumption of compact support. Then he proves that, if $(\mu_n, n \geq 0)$ is an $\mathbb{R}^d$-valued peacock (indexed by $\mathbb{N}$), there exists an associated martingale which is obtained as a Markov chain.
(iii) in [3], Doob studies, in a very general extended framework, peacocks indexed by $\mathbb{R}_+$ and taking their values in a fixed compact set. In particular, he proves that they admit associated martingales. Note that in [3], the Markovian character of the associated martingales is not considered.

1.4. Organization

The remainder of this paper is organised as follows:
– in Section 2, we present some basic facts concerning the $\mathbb{R}^d$-valued peacocks and we describe some examples, thus extending results of [5];
– in Section 3, starting from Strassen’s theorem, we prove that a family $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}^d$, is associated to a right-continuous martingale, if and only if, $(\mu_t, t \geq 0)$ is a peacock such that the map: $t \rightarrow \mu_t$ is weakly right-continuous on $\mathbb{R}_+$;
– in Section 4, by approximation from the previous result, we extend this result to the case of general $\mathbb{R}^d$-valued peacocks.

2. Generalities, examples

2.1. Notation

In the sequel, $d$ denotes a fixed integer, $\mathbb{R}^d$ is equipped with a norm which is denoted by $\cdot$, and we adopt the terminology of Section 1.1.

We also denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}^d$, equipped with the topology of weak convergence (with respect to the space $C_b(\mathbb{R}^d)$ of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}^d$). We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x| \mu(dx) < \infty$. $\mathcal{M}_f$ is also equipped with the topology of weak convergence.

$C_c(\mathbb{R}^d)$ denotes the space of $\mathbb{R}$-valued continuous functions on $\mathbb{R}^d$ with compact support, and $C_c^+(\mathbb{R}^d)$ is the subspace consisting of all the nonnegative functions in $C_c(\mathbb{R}^d)$.

2.2. Basic facts

Proposition 2.1. Let $(X_t, t \geq 0)$ be an $\mathbb{R}^d$-valued integrable process. Then $(X_t, t \geq 0)$ is a peacock if (and only if) the map: $t \rightarrow \mathbb{E}[\psi(X_t)]$ is increasing, for every function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ which is convex, of $C^\infty$ class and such that the derivative $\psi'$ is bounded on $\mathbb{R}^d$.

Proof. Let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function. For every $a \in \mathbb{R}^d$, there exists an affine function $h_a$ such that:

$$\forall x \in \mathbb{R}^d, \quad \psi(x) \geq h_a(x) \quad \text{and} \quad \psi(a) = h_a(a).$$

Let $\{a_n; n \geq 1\}$ be a countable dense subset of $\mathbb{R}^d$. We set:

$$\forall n \geq 1, \quad \psi_n(x) = \sup_{1 \leq j \leq n} h_{a_j}(x).$$

Then:

$$\forall x \in \mathbb{R}^d, \quad \lim_{n \to \infty} \psi_n(x) = \psi(x).$$

The functions $\psi_n$ are convex and Lipschitz continuous.

Let $\phi$ be a nonnegative function, of $C^\infty$ class, with compact support and such that $\int \phi(x) \, dx = 1$. We set, for $n, p \geq 1$,

$$\forall x \in \mathbb{R}^d, \quad \psi_{n,p}(x) = \int \psi_n \left( x - \frac{1}{p} y \right) \phi(y) \, dy.$$ 

Clearly, $\psi_{n,p}$ is convex, of $C^\infty$ class and Lipschitz continuous. Consequently, its derivative is bounded on $\mathbb{R}^d$. Moreover, $\lim_{p \to \infty} \psi_{n,p} = \psi_n$ uniformly on $\mathbb{R}^d$.

The desired result now follows directly. \qed

The next result will be useful in the sequel.
Proposition 2.2. Let \((X_t, t \geq 0)\) be an \(\mathbb{R}^d\)-valued peacock. Then:

1. the map: \(t \mapsto \mathbb{E}[X_t]\) is constant;
2. the map: \(t \mapsto \mathbb{E}[|X_t|]\) is increasing, and therefore, for every \(T \geq 0\),
   \[
   \sup_{0 \leq t \leq T} \mathbb{E}[|X_t|] = \mathbb{E}[|X_T|] < \infty;
   \]
3. for every \(T \geq 0\), the random variables \((X_t; 0 \leq t \leq T)\) are uniformly integrable.

Proof. Properties 1 and 2 are obvious.

If \(c \geq 0\),
\[
|x| 1_{\{|x| \geq c\}} \leq (2|x| - c)^+.
\]
As the function \(x \mapsto (2|x| - c)^+\) is convex,
\[
\sup_{t \in [0,T]} \mathbb{E}[|X_t| 1_{\{|X_t| \geq c\}}] \leq \mathbb{E}[(2|X_T| - c)^+].
\]
Now, by dominated convergence,
\[
\lim_{c \to +\infty} \mathbb{E}[(2|X_T| - c)^+] = 0.
\]
Hence, property 3 holds.

\[\square\]

2.3. Examples

The following examples are given in [5] for \(d = 1\). The proofs given below are essentially the same as in [5].

Proposition 2.3. Let \(X\) be a centered \(\mathbb{R}^d\)-valued random variable. Then \((tX, t \geq 0)\) is a peacock.

Proof. Let \(\psi : \mathbb{R}^d \to \mathbb{R}\) be a convex function, and \(0 \leq s < t\). Then,
\[
\psi(sX) \leq \left(1 - \frac{s}{t}\right) \psi(0) + \frac{s}{t} \psi(tX).
\]
Since \(X\) is centered, by Jensen’s inequality:
\[
\psi(0) = \psi(\mathbb{E}[tX]) \leq \mathbb{E}[\psi(tX)].
\]
Hence,
\[
\mathbb{E}[\psi(sX)] \leq \left(1 - \frac{s}{t}\right) \mathbb{E}[\psi(tX)] + \frac{s}{t} \mathbb{E}[\psi(tX)] = \mathbb{E}[\psi(tX)].
\]
\[\square\]

Proposition 2.4. Let \((X_t, t \geq 0)\) be a family of centered, \(\mathbb{R}^d\)-valued, Gaussian variables. We denote by \(C(t) = (c_{i,j}(t))_{1 \leq i,j \leq d}\) the covariance matrix of \(X_t\). Then, \((X_t, t \geq 0)\) is a peacock if and only if the map: \(t \mapsto C(t)\) is increasing in the sense of quadratic forms, i.e.:
\[
\forall a = (a_1, \ldots, a_d) \in \mathbb{R}^d, \quad t \mapsto \sum_{1 \leq i,j \leq d} c_{i,j}(t) a_i a_j \quad \text{is increasing}.
\]

Proof.

(1) For every \(a \in \mathbb{R}^d\), the function:
\[
x \mapsto \sum_{1 \leq i,j \leq d} a_i a_j x_i x_j = \left(\sum_{i=1}^d a_i x_i\right)^2
\]
is convex. This entails that, if \((X_t, t \geq 0)\) is a peacock, then the map: \(t \mapsto C(t)\) is increasing in the sense of quadratic forms.
(2) Conversely, suppose that the map: \( t \mapsto C(t) \) is increasing in the sense of quadratic forms. By the proof of [5], Theorem 2.16, page 132, there exists a centered \( \mathbb{R}^d \)-valued Gaussian process: \((I_t = (\Gamma_{1,t}, \ldots, \Gamma_{d,t}), \ t \geq 0)\), such that:

\[ \forall s, t \geq 0, \ \forall 1 \leq i, j \leq d, \quad \mathbb{E}[\Gamma_{i,s}\Gamma_{j,t}] = c_{i,j}(s \wedge t). \]

Therefrom we deduce that \((I_t, \ t \geq 0)\) is a martingale which is associated to \((X_t, \ t \geq 0)\), and consequently, \((X_t, \ t \geq 0)\) is a peacock. \(\blacksquare\)

**Corollary 2.5.** Let \( A \) be a \( d \times d \) matrix. We consider the \( \mathbb{R}^d \)-valued Ornstein–Uhlenbeck process \((U_t, \ t \geq 0)\), defined as (the unique) solution, started from 0, of the SDE:

\[ dU_t = dB_t + AU_t \ dt \]

where \((B_t, \ t \geq 0)\) denotes a \( d \)-dimensional Brownian motion. Then, \((U_t, \ t \geq 0)\) is a peacock.

**Proof.** One has:

\[ U_t = t \int_0^t \exp((t - s) \ A) \ dB_s. \]

Hence, for every \( t \geq 0 \), \( U_t \) is a centered, \( \mathbb{R}^d \)-valued Gaussian variable whose covariance matrix is:

\[ C(t) = t \int_0^t \exp(s \ A) \exp(s \ A^*) \ ds \]

where \( A^* \) denotes the transpose matrix of \( A \). Therefrom it is clear that the map: \( t \mapsto C(t) \) is increasing in the sense of quadratic forms, and Proposition 2.4 applies. \(\blacksquare\)

**Proposition 2.6.** Let \((M_t, \ t \geq 0)\) be an \( \mathbb{R}^d \)-valued, right-continuous martingale such that:

\[ \forall T > 0, \ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t| \right] < \infty. \]

Then,

1. \( \left( X_t := \frac{1}{t} \int_0^t M_s \ ds; \ t \geq 0 \right) \) is a peacock;
2. \( \left( \tilde{X}_t := \int_0^t (M_s - M_0) \ ds; \ t \geq 0 \right) \) is a peacock.

**Proof.** Using Proposition 2.1, we may use the proof of [5], Theorem 1.4, page 26. For the convenience of the reader, we reproduce this proof below.

(1) Let \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a convex function, of \( C^\infty \) class and such that the derivative \( \psi' \) is bounded on \( \mathbb{R}^d \). Setting:

\[ \tilde{M}_t = \int_0^t s \ dM_s, \]

one has, by integration by parts:

\[ X_t = M_t - t^{-1} \tilde{M}_t \quad \text{and} \quad dX_t = t^{-2} \tilde{M}_t \ dt. \]

Denoting by \( \mathcal{F}_s \) the \( \sigma \)-algebra generated by \( \{M_u; \ 0 \leq u \leq s\} \), one gets, for \( 0 \leq s \leq t \),

\[ \mathbb{E}[X_t | \mathcal{F}_s] = X_s + (s^{-1} - t^{-1}) \tilde{M}_s. \]
Consequently, by Jensen’s inequality,
\[ \mathbb{E}[\psi(X_s)] \geq \mathbb{E}[\psi(X_s + (s^{-1} - t^{-1}) \hat{M}_s)]. \]

Using again the fact that \( \psi \) is convex, one obtains:
\[ \mathbb{E}[\psi(X_s)] \geq \mathbb{E}[\psi(X_s)] + (s^{-1} - t^{-1}) \mathbb{E}[\psi'(X_s) \cdot \hat{M}_s]. \]

Now,
\[ \psi'(X_s) \cdot \hat{M}_s = \int_0^s u^{-2} \psi''(X_u)(\hat{M}_u, \hat{M}_u) \, du + \int_0^s u \psi'(X_u) \cdot dM_u \]
and therefore
\[ \mathbb{E}[\psi(X_s)] - \mathbb{E}[\psi(X_s)] \geq (s^{-1} - t^{-1}) \mathbb{E}[\psi'(X_s) \cdot \hat{M}_s] \geq 0, \]
which, by Proposition 2.1, yields the desired result.

(2) Let \( \psi \) be as above. One may suppose that \( M_0 = 0 \). One has, for \( 0 \leq s \leq t, \)
\[ \mathbb{E}[\tilde{X}_t | \mathcal{F}_s] = \tilde{X}_s + (t - s) M_s. \]

Consequently, by Jensen’s inequality,
\[ \mathbb{E}[\psi(\tilde{X}_t)] \geq \mathbb{E}[\psi(\tilde{X}_s + (t - s) M_s)]. \]

Using again the fact that \( \psi \) is convex, one obtains:
\[ \mathbb{E}[\psi(\tilde{X}_t)] \geq \mathbb{E}[\psi(\tilde{X}_s)] + (t - s) \mathbb{E}[\psi'(\tilde{X}_s) \cdot M_s]. \]

Now,
\[ \psi'(\tilde{X}_s) \cdot M_s = \int_0^s \psi''(\tilde{X}_u)(M_u, M_u) \, du + \int_0^s \psi'(\tilde{X}_u) \cdot dM_u \]
and therefore
\[ \mathbb{E}[\psi(\tilde{X}_t)] - \mathbb{E}[\psi(\tilde{X}_s)] \geq (t - s) \mathbb{E}[\psi'(\tilde{X}_s) \cdot M_s] \geq 0, \]
which, by Proposition 2.1, yields the desired result. \( \square \)

3. Right-continuous peacocks

In this section, we shall show that any right continuous peacock admits an associated right-continuous martingale. For this, we start from Strassen’s theorem, which we now recall.

**Theorem 3.1** (Strassen [8], Thm. 8). Let \((\mu_n, n \in \mathbb{N})\) be a sequence in \( \mathcal{M} \). Then \((\mu_n, n \in \mathbb{N})\) is a peacock if and only if there exists a martingale \((M_n, n \in \mathbb{N})\) which is associated to \((\mu_n, n \in \mathbb{N})\).

We shall extend this theorem to right-continuous peacocks indexed by \( \mathbb{R}_+ \). In the case \( d = 1 \), the following theorem is proven in [4], by a quite different method. In particular, in [4], we do not use Strassen’s theorem, nor the Hahn–Banach theorem, but an explicit approximation by solutions of SDE’s.

**Theorem 3.2.** Let \((\mu_t, t \geq 0)\) be a family in \( \mathcal{M} \). Then the following properties are equivalent:

(i) there exists a right-continuous martingale associated to \((\mu_t, t \geq 0)\);
(ii) \((\mu_t, t \geq 0)\) is a peacock and the map:
\[ t \geq 0 \rightarrow \mu_t \in \mathcal{M} \]
is right-continuous.
Proof.

1. We first assume that property (i) is satisfied. Then, the fact that \((\mu_t, t \geq 0)\) is a peacock follows classically from Jensen’s inequality. Let \((M_t, t \geq 0)\) be a right-continuous martingale associated to \((\mu_t, t \geq 0)\). Then, if \(f \in C_b(\mathbb{R}^d)\), dominated convergence yields that, for any \(t \geq 0\),

\[
\lim_{s \to t, s > t} \int f(x) \mu_s(dx) = \lim_{s \to t, s > t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x) \mu_t(dx).
\]

Therefore, the map:

\[ t \geq 0 \mapsto \mu_t \in \mathcal{M} \]

is right-continuous, and property (ii) is satisfied.

2. Conversely, we now assume that property (ii) is satisfied. For every \(n \in \mathbb{N}\), we set:

\[ \mu_k^{(n)} = \mu_{k2^{-n}}, \quad k \in \mathbb{N}. \]

By Strassen’s theorem (Thm. 3.1), there exists a martingale \((M_k^{(n)}, k \in \mathbb{N})\) which is associated to \((\mu_k^{(n)}, k \in \mathbb{N})\). We set:

\[ X_t^{(n)} = M_k^{(n)} \text{ if } t = k2^{-n} \quad \text{and} \quad X_t^{(n)} = 0 \text{ otherwise.} \]

Consequently, the law of \(X_t^{(n)}\) is \(\mu_t\) if \(t \in \{k2^{-n}; k \in \mathbb{N}\}\), and is \(\delta\) (the Dirac measure at 0) if \(t \notin \{k2^{-n}; k \in \mathbb{N}\}\).

Note that, due to the lack of uniqueness in Strassen’s theorem, the law of \((X_{k2^{-n}}^{(n)}, k \in \mathbb{N})\) may not be the same as the law of \((X_{(k+1)2^{-n}}^{(n)}, k \in \mathbb{N})\).

Only the one-dimensional marginals are identical.

3. Let \(D = \{k2^{-n}; k, n \in \mathbb{N}\}\) the set of dyadic numbers. For every \(n \in \mathbb{N}\), for every \(r \geq 1\) and \(\tau_r = (t_1, t_2, \ldots, t_r) \in D^r\), we denote by \(\Pi_{\tau_r}^{(r,n)}\) the law of \((X_{t_1}^{(n)}, \ldots, X_{t_r}^{(n)})\), a probability on \((\mathbb{R}^d)^r\).

Lemma 3.3. For every \(\tau_r \in D^r\), the set of probability measures: \(\{\Pi_{\tau_r}^{(r,n)}; n \in \mathbb{N}\}\) is tight.

Proof. We set, for \(x = (x^1, \ldots, x^r) \in (\mathbb{R}^d)^r\), \(|x_r| = \sum_{j=1}^r |x^j|\). Then, for \(p > 0\),

\[ \Pi_{\tau_r}^{(r,n)}(|x_r| \geq p) \leq \frac{1}{p} \Pi_{\tau_r}^{(r,n)}(|x_r|) = \frac{1}{p} \sum_{j=1}^r \mathbb{E}[|X_{t_j}^{(n)}|] \leq \frac{1}{p} \sum_{j=1}^r \mu_{t_j}(|x|) \]

since, by point (2), the law of \(X_{t_j}^{(n)}\) is either \(\mu_{t_j}\) or \(\delta\). Hence,

\[ \lim_{p \to \infty} \sup_{n \geq 0} \Pi_{\tau_r}^{(r,n)}(|x_r| \geq p) = 0. \]

(4) As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a subsequence \((n_l)_{l \geq 0}\) such that, for every \(\tau_r \in D^r\), the sequence of probabilities on \((\mathbb{R}^d)^r\): \(\Pi_{\tau_r}^{(r,n_l)}\), \(l \geq 0\), weakly converges to a probability which we denote by \(\Pi_{\tau_r}^{(r)}\). We remark that, for \(l\) large enough, the law of \(X_{t_j}^{(n_l)}\) is \(\mu_{t_j}\). Then, there exists an \(\mathbb{R}^d\)-valued process \((X_t, t \in D)\) such that, for every \(r \geq 1\) and every \(\tau_r = (t_1, \ldots, t_r) \in D^r\), the law of \((X_{t_1}, \ldots, X_{t_r})\) is \(\Pi_{\tau_r}^{(r)}\), and \(\Pi_{t}^{(1)} = \mu_t\) for every \(t \in D\).
Theorem 4.1. Let $0 \leq \langle X, t \in D \rangle$ be a martingale associated to $(\mu_t, t \in D)$. We set:

$$\forall p > 0, \forall x \in \mathbb{R}^d, \varphi_p(x) = \left(1 + \frac{|x|}{p}\right)^{-1} x.$$ 

Then,

$$\varphi_p \in C_b(\mathbb{R}^d; \mathbb{R}^d) \quad \text{and} \quad \varphi_p(x) = x \quad \text{for} \quad |x| \leq p.$$ 

Let $0 \leq s_1 < s_2 < \ldots < s_r \leq s \leq t$ be elements of $D$, and let $f \in C_b((\mathbb{R}^d)^r)$. Then, for $l$ large enough,

$$E[f(X_{s_1}^{(n_l)}, \ldots, X_{s_r}^{(n_l)}) X_t^{(n_l)}] = E[f(X_{s_1}^{(n_l)}, \ldots, X_{s_r}^{(n_l)}) X_s^{(n_l)}].$$

On the other hand,

$$|E[f(X_{s_1}, \ldots, X_{s_r}) \varphi_p(X_t)] - E[f(X_{s_1}, \ldots, X_{s_r}) X_t]| \leq \|f\|_{\infty} \mu_t(|x| 1_{\{|x| \geq p\}}), \quad \text{for every} \quad p > 0,$$

and likewise, replacing $t$ by $s$. Moreover,

$$\lim_{l \to \infty} E[f(X_{s_1}^{(n_l)}, \ldots, X_{s_r}^{(n_l)}) \varphi_p(X_t^{(n_l)})] = E[f(X_{s_1}, \ldots, X_{s_r}) \varphi_p(X_t)],$$

and likewise, replacing $t$ by $s$. Finally, we obtain, for $p > 0$,

$$|E[f(X_{s_1}, \ldots, X_{s_r}) X_t] - E[f(X_{s_1}, \ldots, X_{s_r}) X_s]| \leq 2 \|f\|_{\infty} [\mu_t(|x| 1_{\{|x| \geq p\}}) + \mu_s(|x| 1_{\{|x| \geq p\}})],$$

and the desired result follows, letting $p$ go to $\infty$. \hfill \Box

(5) By the classical theory of martingales (see, for example, [2]), almost surely, for every $t \geq 0$,

$$M_t = \lim_{s \to t, s \in D, s > t} X_s$$

is well defined, and $(M_t, t \geq 0)$ is a right-continuous martingale. Besides, since, by hypothesis, the map:

$$t \geq 0 \quad \mapsto \quad \mu_t \in \mathcal{M}$$

is right-continuous, we deduce from Lemma 3.4 that this martingale $(M_t, t \geq 0)$ is associated to $(\mu_t, t \geq 0)$.

4. THE GENERAL CASE

Theorem 3.2 shall now be extended, by approximation, to the general case.

Theorem 4.1. Let $(\mu_t, t \geq 0)$ be a family in $\mathcal{M}$. Then the following properties are equivalent:

(i) there exists a martingale associated to $(\mu_t, t \geq 0)$;
(ii) $(\mu_t, t \geq 0)$ is a peacock.
Proof. Let \((\mu_t, t \geq 0)\) be a peacock.

Lemma 4.2. There exists a countable set \(\Delta \subset \mathbb{R}_+\) such that the map:

\[
t \mapsto \mu_t \in \mathcal{M}
\]

is continuous at any \(s \not\in \Delta\).

Proof. Let \(\chi : \mathbb{R}^d \to \mathbb{R}_+\) be defined by:

\[
\chi(x) = (1 - |x|)^+ = (1 \vee |x|) - |x|.
\]

Then \(\chi \in C^+_c(\mathbb{R}^d)\) and \(\chi\) is the difference of two convex functions. We set: \(\chi_m(x) = m^d \chi(mx)\), and we define the countable set \(\mathcal{H}\) by:

\[
\mathcal{H} = \left\{ \sum_{j=0}^{r} a_j \chi_m(x - q_j); \ r \in \mathbb{N}, \ m \in \mathbb{N}, \ a_j \in \mathbb{Q}_+, \ q_j \in \mathbb{Q}^d \right\}.
\]

For \(h \in \mathcal{H}\), the function: \(t \to \mu_t(h)\) is the difference of two increasing functions, and hence admits a countable set \(\Delta_h\) of discontinuities. We set \(\Delta = \bigcup_{h \in \mathcal{H}} \Delta_h\). Then \(\Delta\) is a countable subset of \(\mathbb{R}_+\), and \(t \to \mu_t(h)\) is continuous at any \(s \not\in \Delta\), for every \(h \in \mathcal{H}\). Now, it is easy to see that \(\mathcal{H}\) is dense in \(C^+_c(\mathbb{R}^d)\) in the following sense: for every \(\varphi \in C^+_c(\mathbb{R}^d)\), there exist a compact set \(K \subset \mathbb{R}^d\) and a sequence \((h_n)_{n \geq 0} \subset \mathcal{H}\) such that:

\[
\forall n, \ \text{Supp} \ h_n \subset K \quad \text{and} \quad \lim_{n \to \infty} h_n = \varphi \ 	ext{uniformly.}
\]

Consequently, \(t \to \mu_t\) is vaguely continuous at any \(s \not\in \Delta\), and, since measures \(\mu_t\) are probabilities, \(t \to \mu_t\) is also weakly continuous at any \(s \not\in \Delta\). \(\square\)

We may write \(\Delta = \{d_j; \ j \in \mathbb{N}\}\). For \(n \in \mathbb{N}\), we denote by \((k_l^{(n)}, \ l \geq 0)\) the increasing rearrangement of the set:

\[
\{k2^{-n}; \ k \in \mathbb{N}\} \cup \{d_j; \ 0 \leq j \leq n\}.
\]

We define \((\mu_t^{(n)}, \ t \geq 0)\) by:

\[
\mu_t^{(n)} = \frac{k_{l+1}^{(n)} - t}{k_{l+1}^{(n)} - k_l^{(n)}} \mu_k^{(n)} + \frac{t - k_l^{(n)}}{k_{l+1}^{(n)} - k_l^{(n)}} \mu_{k+l+1}^{(n)} \quad \text{if} \quad t \in [k_l^{(n)}, k_{l+1}^{(n)}].
\]

Lemma 4.3. The following properties hold:

1. for every \(n \geq 0\), \((\mu_t^{(n)}, \ t \geq 0)\) is a peacock and the map: \(t \to \mu_t^{(n)} \in \mathcal{M}\) is continuous;
2. for any \(t \geq 0\), \(\sup \{\mu_t^{(n)}(|x|); \ n \in \mathbb{N}\} < \infty;\)
3. for any \(t \geq 0\), the set \(\{\mu_t^{(n)}; \ n \in \mathbb{N}\}\) is uniformly integrable;
4. for \(t \geq 0\), \(\lim_{n \to \infty} \mu_t^{(n)} = \mu_t\) in \(\mathcal{M}\).

Proof. Properties 1 and 4 are clear by construction. Property 2 (resp. property 3) follows directly from property 2 (resp. property 3) in Proposition 2.2. \(\square\)

By Theorem 3.2, there exists, for each \(n\), a right-continuous martingale \((M_t^{(n)}, \ t \geq 0)\) which is associated to \((\mu_t^{(n)}, \ t \geq 0)\). For any \(r \in \mathbb{N}\) and \(\tau_r = (t_1, \ldots, t_r) \in \mathbb{R}_+^r\), we denote by \(\Pi_{\tau_r}^{(r,n)}\) the law of \((M_{t_1}^{(n)}, \ldots, M_{t_r}^{(n)})\), a probability measure on \((\mathbb{R}^d)^r\).
**Lemma 4.4.** For every \( \tau_r \in \mathbb{R}_+^r \), the set of probability measures \( \{ \Pi^{(r,n)}_{\tau_r}; n \in \mathbb{N} \} \) is tight.

**Proof.** As in Lemma 3.3, for \( p > 0 \),

\[
\Pi^{(r,n)}_{\tau_r}(|x|_r \geq p) \leq \frac{1}{p} \sum_{j=1}^{r} \mu_{\tau_j}^{(n)}(|x|),
\]

and by property 2 in Lemma 4.3,

\[
\lim_{p \to \infty} \sup_{n \geq 0} \Pi^{(r,n)}_{\tau_r}(|x|_r \geq p) = 0.
\]

Let now \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \), which refines Fréchet’s filter. As a consequence of the previous lemma, for every \( r \in \mathbb{N} \) and every \( \tau_r \in \mathbb{R}_+^r \), \( \lim_{\mathcal{U}} \Pi^{(r,n)}_{\tau_r} \) exists for the weak convergence and we denote this limit by \( \Pi^{(r)}_{\tau_r} \).

By property 4 in Lemma 4.3, \( \Pi^{(1)}_{\tau} = \mu_t \). There exists a process \( (M_t, t \geq 0) \) such that, for every \( r \in \mathbb{N} \) and every \( \tau_r = (t_1, \ldots, t_r) \in \mathbb{R}_+^r \), the law of \( (M_{t_1}, \ldots, M_{t_r}) \) is \( \Pi^{(r)}_{\tau_r} \). In particular, this process \( (M_t, t \geq 0) \) is associated to \( (\mu_t, t \geq 0) \).

**Lemma 4.5.** The process \( (M_t, t \geq 0) \) is a martingale.

**Proof.** The proof is quite similar to that of Lemma 3.4, but we give the details for the sake of completeness. We recall the notation:

\[
\forall p > 0, \ \forall x \in \mathbb{R}^d, \ \varphi_p(x) = \left( 1 \vee \frac{|x|}{p} \right)^{-1} x.
\]

Let \( 0 \leq s_1 < s_2 < \ldots < s_r \leq s \leq t \) be elements of \( \mathbb{R}_+ \), and let \( f \in C_b((\mathbb{R}^d)^r) \). We set: \( \| f \|_\infty = \sup \{|f(x)|; x \in (\mathbb{R}^d)^r \} \). Then, for every \( n \),

\[
\mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) M_t^{(n)}] = \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) M_s^{(n)}].
\]

On the other hand,

\[
\left| \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r}) \varphi_p(M_t)] - \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r}) M_t] \right| \leq \| f \|_\infty \mu_t \left( |x| 1_{\{|x| \geq p\}} \right), \text{ for every } p > 0,
\]

\[
\left| \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) \varphi_p(M_t^{(n)})] - \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) M_t^{(n)}] \right| \leq \| f \|_\infty \mu_t^{(n)} \left( |x| 1_{\{|x| \geq p\}} \right),
\]

for every \( n \) and every \( p > 0 \),

and likewise, replacing \( t \) by \( s \). Moreover,

\[
\lim_{\mathcal{U}} \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) \varphi_p(M_t^{(n)})] = \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r}) \varphi_p(M_t)],
\]

and likewise, replacing \( t \) by \( s \). Finally, we obtain, for \( p > 0 \),

\[
\left| \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_t] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_s] \right| \leq 2 \| f \|_\infty \sup_{n \geq 0} \left[ \mu_t^{(n)} \left( |x| 1_{\{|x| \geq p\}} \right) + \mu_s^{(n)} \left( |x| 1_{\{|x| \geq p\}} \right) \right],
\]

and, by property 3 in Lemma 4.3, the desired result follows, letting \( p \) go to \( \infty \).

This lemma completes the proof of Theorem 4.1.

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REFERENCES


