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ON $\mathbb{R}^d$-VALUED PEACOCKS

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Abstract. In this paper, we consider $\mathbb{R}^d$-valued integrable processes which are increasing in the convex order, *i.e.* $\mathbb{R}^d$-valued peacocks in our terminology. After the presentation of some examples, we show that an $\mathbb{R}^d$-valued process is a peacock if and only if it has the same one-dimensional marginals as an $\mathbb{R}^d$-valued martingale. This extends former results, obtained notably by Strassen [*Ann. Math. Stat.* **36** (1965) 423–439], Doob [*J. Funct. Anal.* **2** (1968) 207–225] and Kellerer [*Math. Ann.* **198** (1972) 99–122].

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1. Introduction

1.1. Terminology

First we fix the terminology. In the sequel, d denotes a fixed integer and $\mathbb{R}^d$ is equipped with a norm which is denoted by $|\cdot|$.

We say that two $\mathbb{R}^d$-valued processes: $(X_t,\ t \geq 0)$ and $(Y_t,\ t \geq 0)$ are *associated*, if they have the same one-dimensional marginals, *i.e.* if:

$$\forall t \geq 0, \quad X_t \overset{\text{(law)}}{=} Y_t.$$

A process which is associated with a martingale is called a 1-*martingale*.

An $\mathbb{R}^d$-valued process $(X_t,\ t \geq 0)$ will be called a *peacock* if:

(i) it is *integrable*, that is:

$$\forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty;$$

(ii) it *increases in the convex order*, meaning that, for every convex function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

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This terminology was introduced in [5]. We refer the reader to this monograph for an explanation of the origin of the term: "peacock", as well as for a comprehensive study of this notion in the case $d = 1$.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t,\ t \geq 0)$ of its one-dimensional marginals is involved. This makes it natural, in the following, to also call a *peacock*, a family $(\mu_t,\ t \geq 0)$ of probability measures on $\mathbb{R}^d$ such that:

(i) $\forall t \geq 0, \quad \int |x|\ \mu_t(\mathrm{d}x) < \infty;$

(ii) for every convex function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x)\ \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t,\ t \geq 0)$ of probability measures on $\mathbb{R}^d$ and an $\mathbb{R}^d$-valued process $(Y_t,\ t \geq 0)$ will be said to be *associated* if, for every $t \geq 0$, the law of Y_t is μ_t, *i.e.* if $(\mu_t,\ t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t,\ t \geq 0)$.

Obviously, the above notions also are meaningful if one considers processes and families of measures indexed by a subset of $\mathbb{R}_+$ (for example $\mathbb{N}$) instead of $\mathbb{R}_+$.

It is an easy consequence of Jensen's inequality that an $\mathbb{R}^d$-valued process which is a 1-martingale, is a peacock. So, a natural question is whether the converse holds.

1.2. Case $d = 1$

A remarkable result due to Kellerer [6] states that, actually, any $\mathbb{R}$-valued process which is a peacock, is a 1-martingale. More precisely, Kellerer's result states that any $\mathbb{R}$-valued peacock admits an associated martingale which is *Markovian*.

Two more recent results now complete Kellerer's theorem.

(i) Lowther [7] states that if $(\mu_t,\ t \geq 0)$ is an $\mathbb{R}$-valued peacock such that the map: $t \longrightarrow \mu_t$ is weakly continuous (*i.e.* for any $\mathbb{R}$-valued, bounded and continuous function f on $\mathbb{R}$, the map: $t \longrightarrow \int f(x)\ \mu_t(\mathrm{d}x)$ is continuous), then $(\mu_t,\ t \geq 0)$ is associated with a strongly Markovian martingale which moreover is "almost-continuous" (see [7] for the definition);

(ii) in a previous paper [4], we presented a new proof of the above mentioned theorem of Kellerer. Our method, which is inspired from the "Fokker−Planck Equation Method" ([5], Sect. 6.2, p. 229), then appears as a new application of M. Pierre's uniqueness theorem for a Fokker−Planck equation ([5], Thm. 6.1, p. 223). Thus, we show that a martingale which is associated to an $\mathbb{R}$-valued peacock, may be obtained as a limit of solutions of stochastic differential equations. However, we do not obtain that such a martingale is Markovian.

1.3. Case $d \geq 1$

Concerning the case $\mathbb{R}^d$ with $d \geq 1$, and even much more general spaces, we would like to mention the following three important papers.

(i) in [1], Cartier *et al.* study the case of two probability measures (μ_1, μ_2) on a metrizable convex compact K of a locally convex space. They prove, using the Hahn−Banach theorem, that, if (μ_1, μ_2) is a K-valued peacock (indexed by $\{1, 2\}$), then there exists a Markovian kernel P on K such that: $\theta(\mathrm{d}x_1, \mathrm{d}x_2) := \mu_1(\mathrm{d}x_1)\, P(x_1, \mathrm{d}x_2)$ is the law of a K-valued martingale (Y_1, Y_2) associated to (μ_1, μ_2);

(ii) in [8], Strassen extends the Cartier−Fell−Meyer result to $\mathbb{R}^d$-valued peacocks without making the assumption of compact support. Then he proves that, if $(\mu_n,\ n \geq 0)$ is an $\mathbb{R}^d$-valued peacock (indexed by $\mathbb{N}$), there exists an associated martingale which is obtained as a Markov chain;

(iii) in [3], Doob studies, in a very general extended framework, peacocks indexed by $\mathbb{R}_+$ and taking their values in a fixed compact set. In particular, he proves that they admit associated martingales. Note that in [3], the Markovian character of the associated martingales is not considered.

1.4. Organization

The remainder of this paper is organised as follows:

– in Section 2, we present some basic facts concerning the $\mathbb{R}^d$-valued peacocks and we describe some examples, thus extending results of [5];
– in Section 3, starting from Strassen's theorem, we prove that a family $(\mu_t,\ t \geq 0)$ of probability measures on $\mathbb{R}^d$, is associated to a *right-continuous* martingale, if and only if, $(\mu_t,\ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is *weakly right-continuous* on $\mathbb{R}_+$;
– in Section 4, by approximation from the previous result, we extend this result to the case of general $\mathbb{R}^d$-valued peacocks.

2. Generalities, examples

2.1. Notation

In the sequel, d denotes a fixed integer, $\mathbb{R}^d$ is equipped with a norm which is denoted by $|\cdot|$, and we adopt the terminology of Section 1.1.

We also denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}^d$, equipped with the topology of weak convergence (with respect to the space $C_b(\mathbb{R}^d)$ of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}^d$). We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x|\ \mu(\mathrm{d}x) < \infty$. $\mathcal{M}_f$ is also equipped with the topology of weak convergence.

$C_c(\mathbb{R}^d)$ denotes the space of $\mathbb{R}$-valued continuous functions on $\mathbb{R}^d$ with compact support, and $C_c^+(\mathbb{R}^d)$ is the subspace consisting of all the nonnegative functions in $C_c(\mathbb{R}^d)$.

2.2. Basic facts

Proposition 2.1. *Let $(X_t,\ t \geq 0)$ be an $\mathbb{R}^d$-valued integrable process. Then $(X_t,\ t \geq 0)$ is a peacock if (and only if) the map: $t \longrightarrow \mathbb{E}[\psi(X_t)]$ is increasing, for every function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ which is convex, of C^∞ class and such that the derivative ψ' is bounded on $\mathbb{R}^d$.*

Proof. Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function. For every $a \in \mathbb{R}^d$, there exists an affine function h_a such that:

$$\forall x \in \mathbb{R}^d, \quad \psi(x) \geq h_a(x) \quad \text{and} \quad \psi(a) = h_a(a).$$

Let $\{a_n;\ n \geq 1\}$ be a countable dense subset of $\mathbb{R}^d$. We set:

$$\forall n \geq 1, \quad \psi_n(x) = \sup_{1 \leq j \leq n} h_{a_j}(x).$$

Then:

$$\forall x \in \mathbb{R}^d, \quad \lim_{n \uparrow \infty} \uparrow \psi_n(x) = \psi(x).$$

The functions ψ_n are convex and Lipschitz continuous.

Let ϕ be a nonnegative function, of C^∞ class, with compact support and such that $\int \phi(x)\ \mathrm{d}x = 1$. We set, for $n, p \geq 1$,

$$\forall x \in \mathbb{R}^d, \quad \psi_{n,p}(x) = \int \psi_n \left(x - \frac{1}{p}\, y \right) \phi(y)\ \mathrm{d}y.$$

Clearly, $\psi_{n,p}$ is convex, of C^∞ class and Lipschitz continuous. Consequently, its derivative is bounded on $\mathbb{R}^d$. Moreover, $\lim_{p \to \infty} \psi_{n,p} = \psi_n$ uniformly on $\mathbb{R}^d$.

The desired result now follows directly. $\square$

The next result will be useful in the sequel.

Proposition 2.2. *Let $(X_t,\ t \geq 0)$ be an $\mathbb{R}^d$-valued peacock. Then:*

1. *the map: $t \longrightarrow \mathbb{E}[X_t]$ is constant;*
2. *the map: $t \longrightarrow \mathbb{E}[|X_t|]$ is increasing, and therefore, for every $T \geq 0$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t|] = \mathbb{E}[|X_T|] < \infty;$$

3. *for every $T \geq 0$, the random variables $(X_t;\ 0 \leq t \leq T)$ are uniformly integrable.*

Proof. Properties 1 and 2 are obvious.
If $c \geq 0$,

$$|x|\, 1_{\{|x| \geq c\}} \leq (2\,|x| - c)^+.$$

As the function $x \longrightarrow (2\,|x| - c)^+$ is convex,

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t|\, 1_{\{|X_t| \geq c\}}\right] \leq \mathbb{E}[(2\,|X_T| - c)^+].$$

Now, by dominated convergence,

$$\lim_{c \to +\infty} \mathbb{E}[(2\,|X_T| - c)^+] = 0.$$

Hence, property 3 holds. □

2.3. Examples

The following examples are given in [5] for $d = 1$. The proofs given below are essentially the same as in [5].

Proposition 2.3. *Let X be a centered $\mathbb{R}^d$-valued random variable. Then $(t\,X,\ t \geq 0)$ is a peacock.*

Proof. Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function, and $0 \leq s < t$. Then,

$$\psi(s\,X) \leq \left(1 - \frac{s}{t}\right)\, \psi(0) + \frac{s}{t}\, \psi(t\,X).$$

Since X is centered, by Jensen's inequality:

$$\psi(0) = \psi\left(\mathbb{E}[t\,X]\right) \leq \mathbb{E}[\psi(t\,X)].$$

Hence,

$$\mathbb{E}[\psi(s\,X)] \leq \left(1 - \frac{s}{t}\right)\, \mathbb{E}[\psi(t\,X)] + \frac{s}{t}\, \mathbb{E}[\psi(t\,X)] = \mathbb{E}[\psi(t\,X)].$$ □

Proposition 2.4. *Let $(X_t,\ t \geq 0)$ be a family of centered, $\mathbb{R}^d$-valued, Gaussian variables. We denote by $C(t) = (c_{i,j}(t))_{1 \leq i,j \leq d}$ the covariance matrix of X_t. Then, $(X_t,\ t \geq 0)$ is a peacock if and only if the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms,* i.e.*:*

$$\forall a = (a_1, \ldots, a_d) \in \mathbb{R}^d, \quad t \longrightarrow \sum_{1 \leq i,j \leq d} c_{i,j}(t)\, a_i a_j \quad \text{is increasing.}$$

Proof.

(1) For every $a \in \mathbb{R}^d$, the function:

$$x \in \mathbb{R}^d \longrightarrow \sum_{1 \leq i,j \leq d} a_i\, a_j\, x_i\, x_j = \left(\sum_{i=1}^{d} a_i\, x_i\right)^2$$

is convex. This entails that, if $(X_t,\ t \geq 0)$ is a peacock, then the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms.

(2) Conversely, suppose that the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms. By the proof of [5], Theorem 2.16, page 132, there exists a centered $\mathbb{R}^d$-valued Gaussian process: $(\Gamma_t = (\Gamma_{1,t}, \dots, \Gamma_{d,t}),\ t \geq 0)$, such that:

$$\forall s, t \geq 0,\ \ \forall 1 \leq i, j \leq d, \quad \mathbb{E}[\Gamma_{i,s}\, \Gamma_{j,t}] = c_{i,j}(s \wedge t).$$

Therefrom we deduce that $(\Gamma_t,\ t \geq 0)$ is a martingale which is associated to $(X_t,\ t \geq 0)$, and consequently, $(X_t,\ t \geq 0)$ is a peacock. □

Corollary 2.5. *Let A be a $d \times d$ matrix. We consider the $\mathbb{R}^d$-valued Ornstein– Uhlenbeck process $(U_t,\ t \geq 0)$, defined as (the unique) solution, started from 0, of the SDE:*

$$\mathrm{d}U_t = \mathrm{d}B_t + A\, U_t\, \mathrm{d}t$$

where $(B_t,\ t \geq 0)$ denotes a d-dimensional Brownian motion. Then, $(U_t,\ t \geq 0)$ is a peacock.

Proof. One has:

$$U_t = \int_0^t \exp((t-s)\, A)\, \mathrm{d}B_s.$$

Hence, for every $t \geq 0$, U_t is a centered, $\mathbb{R}^d$-valued Gaussian variable whose covariance matrix is:

$$C(t) = \int_0^t \exp(s\, A)\, \exp(s\, A^*)\, \mathrm{d}s$$

where A^* denotes the transpose matrix of A. Therefrom it is clear that the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms, and Proposition 2.4 applies. □

Proposition 2.6. *Let $(M_t,\ t \geq 0)$ be an $\mathbb{R}^d$-valued, right-continuous martingale such that:*

$$\forall T > 0, \quad \mathbb{E}\left[\sup_{0 \leq t \leq T} |M_t|\right] < \infty.$$

Then,

1. $\left(X_t := \dfrac{1}{t} \displaystyle\int_0^t M_s\, \mathrm{d}s;\ t \geq 0\right)$ *is a peacock;*
2. $\left(\widetilde{X}_t := \displaystyle\int_0^t (M_s - M_0)\, \mathrm{d}s;\ t \geq 0\right)$ *is a peacock.*

Proof. Using Proposition 2.1, we may use the proof of [5], Theorem 1.4, page 26. For the convenience of the reader, we reproduce this proof below.

(1) Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function, of C^∞ class and such that the derivative ψ' is bounded on $\mathbb{R}^d$. Setting:

$$\widehat{M_t} = \int_0^t s\, \mathrm{d}M_s,$$

one has, by integration by parts:

$$X_t = M_t - t^{-1} \widehat{M_t} \quad \text{ and } \quad \mathrm{d}X_t = t^{-2} \widehat{M_t}\, \mathrm{d}t.$$

Denoting by $\mathcal{F}_s$ the σ-algebra generated by $\{M_u;\ 0 \leq u \leq s\}$, one gets, for $0 \leq s \leq t$,

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s + (s^{-1} - t^{-1})\, \widehat{M_s}.$$

Consequently, by Jensen's inequality,

$$\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s + (s^{-1} - t^{-1})\, \widehat{M}_s)].$$

Using again the fact that ψ is convex, one obtains:

$$\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s)] + (s^{-1} - t^{-1})\, \mathbb{E}[\psi'(X_s) \cdot \widehat{M}_s].$$

Now,

$$\psi'(X_s) \cdot \widehat{M}_s = \int_0^s u^{-2} \psi''(X_u)(\widehat{M}_u, \widehat{M}_u)\, \mathrm{d}u + \int_0^s u\, \psi'(X_u) \cdot \mathrm{d}M_u$$

and therefore

$$\mathbb{E}[\psi(X_t)] - \mathbb{E}[\psi(X_s)] \geq (s^{-1} - t^{-1})\, \mathbb{E}[\psi'(X_s) \cdot \widehat{M}_s] \geq 0,$$

which, by Proposition 2.1, yields the desired result.

(2) Let ψ be as above. One may suppose that $M_0 = 0$. One has, for $0 \leq s \leq t$,

$$\mathbb{E}[\widetilde{X}_t \mid \mathcal{F}_s] = \widetilde{X}_s + (t-s)\, M_s.$$

Consequently, by Jensen's inequality,

$$\mathbb{E}[\psi(\widetilde{X}_t)] \geq \mathbb{E}[\psi(\widetilde{X}_s + (t-s)\, M_s)].$$

Using again the fact that ψ is convex, one obtains:

$$\mathbb{E}[\psi(\widetilde{X}_t)] \geq \mathbb{E}[\psi(\widetilde{X}_s)] + (t-s)\, \mathbb{E}[\psi'(\widetilde{X}_s) \cdot M_s].$$

Now,

$$\psi'(\widetilde{X}_s) \cdot M_s = \int_0^s \psi''(\widetilde{X}_u)(M_u, M_u)\, \mathrm{d}u + \int_0^s \psi'(\widetilde{X}_u) \cdot \mathrm{d}M_u$$

and therefore

$$\mathbb{E}[\psi(\widetilde{X}_t)] - \mathbb{E}[\psi(\widetilde{X}_s)] \geq (t-s)\, \mathbb{E}[\psi'(\widetilde{X}_s) \cdot M_s] \geq 0,$$

which, by Proposition 2.1, yields the desired result. □

3. Right-continuous peacoks

In this section, we shall show that any right continuous peacock admits an associated right-continuous martingale. For this, we start from Strassen's theorem, which we now recall.

Theorem 3.1 (Strassen [8], Thm. 8)**.** *Let $(\mu_n,\ n \in \mathbb{N})$ be a sequence in $\mathcal{M}$. Then $(\mu_n,\ n \in \mathbb{N})$ is a peacock if and only if there exists a martingale $(M_n,\ n \in \mathbb{N})$ which is associated to $(\mu_n,\ n \in \mathbb{N})$.*

We shall extend this theorem to right-continuous peacocks indexed by $\mathbb{R}_+$. In the case $d = 1$, the following theorem is proven in [4], by a quite different method. In particular, in [4], we do not use Strassen's theorem, nor the Hahn−Banach theorem, but an explicit approximation by solutions of SDE's.

Theorem 3.2. *Let $(\mu_t,\ t \geq 0)$ be a family in $\mathcal{M}$. Then the following properties are equivalent:*

(i) there exists a right-continuous martingale associated to $(\mu_t,\ t \geq 0)$;

(ii) $(\mu_t,\ t \geq 0)$ is a peacock and the map:

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous.

Proof.

(1) We first assume that property (i) is satisfied. Then, the fact that $(\mu_t,\ t \geq 0)$ is a peacock follows classically from Jensen's inequality. Let $(M_t,\ t \geq 0)$ be a right-continuous martingale associated to $(\mu_t,\ t \geq 0)$. Then, if $f \in C_b(\mathbb{R}^d)$, dominated convergence yields that, for any $t \geq 0$,

$$\lim_{s\to t, s>t} \int f(x)\ \mu_s(\mathrm{d}x) = \lim_{s\to t, s>t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x)\ \mu_t(\mathrm{d}x).$$

Therefore, the map:

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous, and property (ii) is satisfied.

(2) Conversely, we now assume that property (ii) is satisfied. For every $n \in \mathbb{N}$, we set:

$$\mu_k^{(n)} = \mu_{k2^{-n}}, \quad k \in \mathbb{N}.$$

By Strassen's theorem (Thm. 3.1), there exists a martingale $(M_k^{(n)},\ k \in \mathbb{N})$ which is associated to $(\mu_k^{(n)},\ k \in \mathbb{N})$. We set:

$$X_t^{(n)} = M_k^{(n)} \ \text{ if } \ t = k\,2^{-n} \quad \text{and} \quad X_t^{(n)} = 0 \ \text{ otherwise.}$$

Consequently, the law of $X_t^{(n)}$ is μ_t if $t \in \{k\,2^{-n};\ k \in \mathbb{N}\}$, and is δ (the Dirac measure at 0) if $t \notin \{k\,2^{-n};\ k \in \mathbb{N}\}$.

Note that, due to the lack of uniqueness in Strassen's theorem, the law of $(X_{k2^{-n}}^{(n)},\ k \in \mathbb{N})$ may not be the same as the law of $(X_{k2^{-n}}^{(n+1)},\ k \in \mathbb{N})$.

Only the one-dimensional marginals are identical.

(3) Let $D = \{k\,2^{-n};\ k, n \in \mathbb{N}\}$ the set of dyadic numbers. For every $n \in \mathbb{N}$, for every $r \geq 1$ and $\tau_r = (t_1, t_2, \ldots, t_r) \in D^r$, we denote by $\Pi_{\tau_r}^{(r,n)}$ the law of $(X_{t_1}^{(n)}, \ldots, X_{t_r}^{(n)})$, a probability on $(\mathbb{R}^d)^r$.

Lemma 3.3. *For every $\tau_r \in D^r$, the set of probability measures: $\{\Pi_{\tau_r}^{(r,n)};\ n \in \mathbb{N}\}$ is tight.*

Proof. We set, for $x = (x^1, \ldots, x^r) \in (\mathbb{R}^d)^r$, $|x|_r = \sum_{j=1}^r |x^j|$. Then, for $p > 0$,

$$\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p}\, \Pi_{\tau_r}^{(r,n)}(|x|_r) = \frac{1}{p} \sum_{j=1}^r \mathbb{E}[|X_{t_j}^{(n)}|] \leq \frac{1}{p} \sum_{j=1}^r \mu_{t_j}(|x|)$$

since, by point (2), the law of $X_{t_j}^{(n)}$ is either μ_{t_j} or δ. Hence,

$$\lim_{p\to\infty} \sup_{n\geq 0} \Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) = 0. \qquad \square$$

(4) As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a subsequence $(n_l)_{l\geq 0}$ such that, for every $\tau_r \in D^r$, the sequence of probabilities on $(\mathbb{R}^d)^r$: $(\Pi_{\tau_r}^{(r,n_l)},\ l \geq 0)$, weakly converges to a probability which we denote by $\Pi_{\tau_r}^{(r)}$. We remark that, for l large enough, the law of $X_{t_j}^{(n_l)}$ is μ_{t_j}. Then, there exists an $\mathbb{R}^d$-valued process $(X_t,\ t \in D)$ such that, for every $r \geq 1$ and every $\tau_r = (t_1, \ldots, t_r) \in D^r$, the law of $(X_{t_1}, \ldots, X_{t_r})$ is $\Pi_{\tau_r}^{(r)}$, and $\Pi_t^{(1)} = \mu_t$ for every $t \in D$.

Lemma 3.4. *The process $(X_t,\ t \in D)$ is a martingale associated to $(\mu_t,\ t \in D)$.*

Proof. As we have already seen, the process $(X_t,\ t \in D)$ is associated to $(\mu_t,\ t \in D)$. We now prove that it is a martingale. We set:

$$\forall p > 0,\ \ \forall x \in \mathbb{R}^d, \quad \varphi_p(x) = \left(1 \vee \frac{|x|}{p}\right)^{-1} x.$$

Then,

$$\varphi_p \in C_b(\mathbb{R}^d; \mathbb{R}^d) \quad \text{ and } \quad \varphi_p(x) = x \ \text{ for } \ |x| \leq p.$$

Let $0 \leq s_1 < s_2 < \ldots < s_r \leq s \leq t$ be elements of D, and let $f \in C_b((\mathbb{R}^d)^r)$. We set: $\| f \|_\infty = \sup\{|f(x)|;\ x \in (\mathbb{R}^d)^r\}$. Then, for l large enough,

$$\mathbb{E}[f(X^{(n_l)}_{s_1}, \ldots, X^{(n_l)}_{s_r})\, X^{(n_l)}_t] = \mathbb{E}[f(X^{(n_l)}_{s_1}, \ldots, X^{(n_l)}_{s_r})\, X^{(n_l)}_s].$$

On the other hand,

$$|\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, \varphi_p(X_t)] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, X_t]| \leq \| f \|_\infty\, \mu_t\left(|x|\, 1_{\{|x| \geq p\}}\right), \ \text{for every } p > 0,$$

$$\left|\mathbb{E}[f(X^{(n_l)}_{s_1}, \ldots, X^{(n_l)}_{s_r})\, \varphi_p(X^{(n_l)}_t)] - \mathbb{E}[f(X^{(n_l)}_{s_1}, \ldots, X^{(n_l)}_{s_r})\, X^{(n_l)}_t]\right| \leq \| f \|_\infty\, \mu_t\left(|x|\, 1_{\{|x| \geq p\}}\right), \quad \text{for every } l \text{ and every } p > 0,$$

and likewise, replacing t by s. Moreover,

$$\lim_{l \to \infty} \mathbb{E}[f(X^{(n_l)}_{s_1}, \ldots, X^{(n_l)}_{s_r})\, \varphi_p(X^{(n_l)}_t)] = \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, \varphi_p(X_t)],$$

and likewise, replacing t by s. Finally, we obtain, for $p > 0$,

$$|\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, X_t] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, X_s]| \leq 2\, \| f \|_\infty\, \left[\mu_t\left(|x|\, 1_{\{|x| \geq p\}}\right) + \mu_s\left(|x|\, 1_{\{|x| \geq p\}}\right)\right],$$

and the desired result follows, letting p go to ∞. □

(5) By the classical theory of martingales (see, for example, [2]), almost surely, for every $t \geq 0$,

$$M_t = \lim_{s \to t, s \in D, s > t} X_s$$

is well defined, and $(M_t,\ t \geq 0)$ is a right-continuous martingale. Besides, since, by hypothesis, the map: $t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$ is right-continuous, we deduce from Lemma 3.4 that this martingale $(M_t,\ t \geq 0)$ is associated to $(\mu_t,\ t \geq 0)$. □

4. The general case

Theorem 3.2 shall now be extended, by approximation, to the general case.

Theorem 4.1. *Let $(\mu_t,\ t \geq 0)$ be a family in $\mathcal{M}$. Then the following properties are equivalent:*

(*i*) *there exists a martingale associated to $(\mu_t,\ t \geq 0)$;*
(*ii*) *$(\mu_t,\ t \geq 0)$ is a peacock.*

Proof. Let $(\mu_t,\ t \geq 0)$ be a peacock.

Lemma 4.2. *There exists a countable set $\Delta \subset \mathbb{R}_+$ such that the map:*

$$t \longrightarrow \mu_t \in \mathcal{M}$$

is continuous at any $s \notin \Delta$.

Proof. Let $\chi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be defined by:

$$\chi(x) = (1 - |x|)^+ = (1 \vee |x|) - |x|.$$

Then $\chi \in C_c^+(\mathbb{R}^d)$ and χ is the difference of two convex functions. We set: $\chi_m(x) = m^d\,\chi(m\,x)$, and we define the countable set $\mathcal{H}$ by:

$$\mathcal{H} = \left\{ \sum_{j=0}^{r} a_j\,\chi_m(x - q_j);\ r \in \mathbb{N},\ m \in \mathbb{N},\ a_j \in \mathbb{Q}_+,\ q_j \in \mathbb{Q}^d \right\}.$$

For $h \in \mathcal{H}$, the function: $t \longrightarrow \mu_t(h)$ is the difference of two increasing functions, and hence admits a countable set Δ_h of discontinuities. We set $\Delta = \bigcup_{h \in \mathcal{H}} \Delta_h$. Then Δ is a countable subset of $\mathbb{R}_+$, and $t \longrightarrow \mu_t(h)$ is continuous at any $s \notin \Delta$, for every $h \in \mathcal{H}$. Now, it is easy to see that $\mathcal{H}$ is dense in $C_c^+(\mathbb{R}^d)$ in the following sense: for every $\varphi \in C_c^+(\mathbb{R}^d)$, there exist a compact set $K \subset \mathbb{R}^d$ and a sequence $(h_n)_{n\geq 0} \subset \mathcal{H}$ such that:

$$\forall n,\ \ \operatorname{Supp} h_n \subset K \quad \text{and} \quad \lim_{n\to\infty} h_n = \varphi \ \text{ uniformly.}$$

Consequently, $t \longrightarrow \mu_t$ is vaguely continuous at any $s \notin \Delta$, and, since measures μ_t are probabilities, $t \longrightarrow \mu_t$ is also weakly continuous at any $s \notin \Delta$. □

We may write $\Delta = \{d_j;\ j \in \mathbb{N}\}$. For $n \in \mathbb{N}$, we denote by $(k_l^{(n)},\ l \geq 0)$ the increasing rearrangement of the set:

$$\{k\,2^{-n};\ k \in \mathbb{N}\} \cup \{d_j;\ 0 \leq j \leq n\}.$$

We define $(\mu_t^{(n)},\ t \geq 0)$ by:

$$\mu_t^{(n)} = \frac{k_{l+1}^{(n)} - t}{k_{l+1}^{(n)} - k_l^{(n)}}\,\mu_{k_l^{(n)}} + \frac{t - k_l^{(n)}}{k_{l+1}^{(n)} - k_l^{(n)}}\,\mu_{k_{l+1}^{(n)}} \qquad \text{if } t \in \left[k_l^{(n)}, k_{l+1}^{(n)}\right].$$

Lemma 4.3. *The following properties hold:*

1. *for every $n \geq 0$, $(\mu_t^{(n)},\ t \geq 0)$ is a peacock and the map: $t \longrightarrow \mu_t^{(n)} \in \mathcal{M}$ is continuous;*
2. *for any $t \geq 0$, $\sup\{\mu_t^{(n)}(|x|);\ n \in \mathbb{N}\} < \infty$;*
3. *for any $t \geq 0$, the set $\{\mu_t^{(n)};\ n \in \mathbb{N}\}$ is uniformly integrable;*
4. *for $t \geq 0$, $\lim_{n\to\infty} \mu_t^{(n)} = \mu_t$ in $\mathcal{M}$.*

Proof. Properties 1 and 4 are clear by construction. Property 2 (resp. property 3) follows directly from property 2 (resp. property 3) in Proposition 2.2. □

By Theorem 3.2, there exists, for each n, a right-continuous martingale $(M_t^{(n)},\ t \geq 0)$ which is associated to $(\mu_t^{(n)},\ t \geq 0)$. For any $r \in \mathbb{N}$ and $\tau_r = (t_1, \ldots, t_r) \in \mathbb{R}_+^r$, we denote by $\Pi_{\tau_r}^{(r,n)}$ the law of $(M_{t_1}^{(n)}, \ldots, M_{t_r}^{(n)})$, a probability measure on $(\mathbb{R}^d)^r$.

Lemma 4.4. *For every $\tau_r \in \mathbb{R}_+^r$, the set of probability measures: $\{\Pi_{\tau_r}^{(r,n)};\ n \in \mathbb{N}\}$ is tight.*

Proof. As in Lemma 3.3, for $p > 0$,

$$\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p} \sum_{j=1}^{r} \mu_{t_j}^{(n)}(|x|),$$

and by property 2 in Lemma 4.3,

$$\lim_{p\to\infty} \sup_{n\geq 0} \Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) = 0.$$ □

Let now $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which refines Fréchet's filter. As a consequence of the previous lemma, for every $r \in \mathbb{N}$ and every $\tau_r \in \mathbb{R}_+^r$, $\lim_{\mathcal{U}} \Pi_{\tau_r}^{(r,n)}$ exists for the weak convergence and we denote this limit by $\Pi_{\tau_r}^{(r)}$. By property 4 in Lemma 4.3, $\Pi_t^{(1)} = \mu_t$. There exists a process $(M_t,\ t \geq 0)$ such that, for every $r \in \mathbb{N}$ and every $\tau_r = (t_1, \ldots, t_r) \in \mathbb{R}_+^r$, the law of $(M_{t_1}, \ldots, M_{t_r})$ is $\Pi_{\tau_r}^{(r)}$. In particular, this process $(M_t,\ t \geq 0)$ is associated to $(\mu_t,\ t \geq 0)$.

Lemma 4.5. *The process $(M_t,\ t \geq 0)$ is a martingale.*

Proof. The proof is quite similar to that of Lemma 3.4, but we give the details for the sake of completeness. We recall the notation:

$$\forall p > 0,\ \ \forall x \in \mathbb{R}^d, \quad \varphi_p(x) = \left(1 \vee \frac{|x|}{p}\right)^{-1} x.$$

Let $0 \leq s_1 < s_2 < \ldots < s_r \leq s \leq t$ be elements of $\mathbb{R}_+$, and let $f \in C_b((\mathbb{R}^d)^r)$. We set: $\| f \|_\infty = \sup\{|f(x)|;\ x \in (\mathbb{R}^d)^r\}$. Then, for every n,

$$\mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)})\, M_t^{(n)}] = \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)})\, M_s^{(n)}].$$

On the other hand,

$$|\mathbb{E}[f(M_{s_1}, \ldots, M_{s_r})\, \varphi_p(M_t)] - \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r})\, M_t]| \leq \| f \|_\infty\, \mu_t \left(|x|\, 1_{\{|x|\geq p\}}\right), \quad \text{for every } p > 0,$$

$$\left|\mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)})\, \varphi_p(M_t^{(n)})] - \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)})\, M_t^{(n)}]\right| \leq \| f \|_\infty\, \mu_t^{(n)} \left(|x|\, 1_{\{|x|\geq p\}}\right), \quad \text{for every } n \text{ and every } p > 0,$$

and likewise, replacing t by s. Moreover,

$$\lim_{\mathcal{U}} \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)})\, \varphi_p(M_t^{(n)})] = \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r})\, \varphi_p(M_t)],$$

and likewise, replacing t by s. Finally, we obtain, for $p > 0$,

$$|\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, X_t] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r})\, X_s]| \leq 2\, \| f \|_\infty \sup_{n\geq 0} \left[\mu_t^{(n)} \left(|x|\, 1_{\{|x|\geq p\}}\right) + \mu_s^{(n)} \left(|x|\, 1_{\{|x|\geq p\}}\right)\right],$$

and, by property 3 in Lemma 4.3, the desired result follows, letting p go to ∞. □

This lemma completes the proof of Theorem 4.1. □

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References

[1] P. Cartier, J.M.G. Fell and P.-A. Meyer, Comparaison des mesures portées par un convexe compact. *Bull. Soc. Math. France* **92** (1964) 435–445.

[2] C. Dellacherie and P.-A. Meyer, Probabilités et potentiel, in *Théorie des martingales*, Chapitres V à VIII. Hermann (1980).

[3] J.L. Doob, Generalized sweeping-out and probability. *J. Funct. Anal.* **2** (1968) 207–225.

[4] F. Hirsch and B. Roynette, A new proof of Kellerer's theorem. *ESAIM: PS* **16** (2012) 48–60.

[5] F. Hirsch, C. Profeta, B. Roynette and M. Yor, *Peacocks and associated martingales, with explicit constructions.* Bocconi & Springer Series **3** (2011).

[6] H.G. Kellerer, Markov-Komposition und eine Anwendung auf Martingale. *Math. Ann.* **198** (1972) 99–122.

[7] G. Lowther, Fitting martingales to given marginals. `http://arxiv.org/abs/0808.2319v1` (2008).

[8] V. Strassen, The existence of probability measures with given marginals. *Ann. Math. Stat.* **36** (1965) 423–439.