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FIXED-α AND FIXED-β EFFICIENCIES

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Abstract. Consider testing $H_0 : F \in \omega_0$ against $H_1 : F \in \omega_1$ for a random sample $X_1, \ldots, X_n$ from F, where ω_0 and ω_1 are two disjoint sets of cdfs on $\mathbb{R} = (-\infty, \infty)$. Two non-local types of efficiencies, referred to as the fixed-α and fixed-β efficiencies, are introduced for this two-hypothesis testing situation. Theoretical tools are developed to evaluate these efficiencies for some of the most usual goodness of fit tests (including the Kolmogorov–Smirnov tests). Numerical comparisons are provided using several examples.

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1. Introduction

Let F_n denote the empirical cdf of a random sample $X_1, \ldots, X_n$ from a distribution function F on $\mathbb{R} = (-\infty, \infty)$. Let $\dot{\eta}(\cdot)$ denote the first derivative of $\eta(\cdot)$, a function with a single argument. Let F_0 denote some hypothesized cdf for F and assume throughout that F_0 is absolutely continuous. Let ψ denote a non-negative function on $[0, 1]$. Define

$$T_{nm}(\psi) = || \, |F_n - F_0| \, \psi\left(F_0\right) ||_{F_0,m}, \quad 0 \le m \le \infty \tag{1.1}$$

$$T^+_{nm}(\psi) = || \left(F_n - F_0\right) \psi\left(F_0\right) ||_{F_0,m}, \quad m = 1, 3, 5, \ldots \tag{1.2}$$

$$D_n(\psi) = || \, |F_n - F_0| \, \psi\left(F_0\right) ||_{F_0,\infty}, \tag{1.3}$$

$$D^+_n(\psi) = \sup\left(F_n - F_0\right) \psi\left(F_0\right), \; D^-_n(\psi) = \sup\left(F_0 - F_n\right) \psi\left(F_0\right), \tag{1.4}$$

$$D^+_{F_0}(F) = \sup\left(F - F_0\right) \psi\left(F_0\right), \; D^-_{F_0}(F) = \sup\left(F_0 - F\right) \psi\left(F_0\right), \tag{1.5}$$

$$V_n(\psi) = D^+_n(\psi) + D^-_n(\psi), \tag{1.6}$$

$$||G||_m = ||G||_{U,m}, \tag{1.7}$$

where

$$||G||_{F,m} = \begin{cases} \left(\displaystyle\int G^m \mathrm{d}F \right)^{1/m}, & \text{if } 0 < m < \infty, \\ \sup G, & \text{if } G \ge 0, m = \infty, \end{cases}$$

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and

$$U(x) = \begin{cases} 0, \ x < 0, \\ x, \ 0 \le x \le 1, \\ 1, \ 1 < x. \end{cases}$$

We wish to test whether $H_0 : F \in \omega_0$ against $H_1 : F \in \omega_1$, where ω_0 and ω_1 are two disjoint sets of cdfs on $\mathbb{R}$. For example, $\omega_0 = \{F_0\}$ and $\omega_1 = \{F \mid F \neq F_0\}$. As candidates for the test, we consider the class of statistics given by (1.1)–(1.7). This class consists of the integral, Kolmogorov–Smirnov and Kuiper statistics, $T_{n2}(1)$, $T_{n\infty}$ (1) and V_n (1) whose asymptotic null distributions are given in Anderson and Darling [2], Kolmogorov [15] and Stephens [21][3].

This paper is related to Withers and Nadarajah [23], where we showed how the asymptotic power (AP) of $T_{n2}(\psi)$ may be computed. Withers and Nadarajah [23] also compared the AP of $T_{n2}(\psi)$ with the AP of $T_{n2}(1)$, $D_n(1)$, $V_n(1)$ for the envelope power function of a particular example, the double-exponential shift family.

This paper deals with exact non-local types of efficiencies for the general two-hypothesis testing situation. There are generally three different strategies to try and approximate such efficiencies: taking alternatives close to the null hypothesis leads to Pitman efficiency; small levels are related to Bahadur efficiency [3]; consideration of high powers results in the Hodges–Lehmann [11] efficiency. There are also other strategies due to Chernoff, Kallenberg, Borovkov and Mogulskiy.

Hodges–Lehmann and Bahadur efficiencies for comparing the performance of gof tests are very much related to large deviation results. Pitman's efficiency is more connected to the notion of contiguity and is nicely studied in the framework given by Le Cam's theory of statistical experiments.

However, Pitman and Hodges–Lehmann efficiencies are not appropriate when test statistics have non-normal limiting distributions, for example, Cramer–von Mises and Watson statistics have degenerate kernels with non-normal limiting distributions. Furthermore, Hodges–Lehmann efficiency cannot discriminate between two-sided tests like Kolmogorov and Cramer–von Mises tests that are asymptotically optimal.

Bahadur efficiencies are not easy to compute. Besides, approximate Bahadur efficiencies are of "little value as measures of performance of tests since monotone transformations of a test statistic may lead to entirely different approximate Bahadur slopes" [14]. So, there is a need for variations of these efficiencies.

In this paper, we introduce two new efficiencies that are "intermediate" between the Hodges–Lehmann and Bahadur efficiencies. We provide some tools from the calculus of variations to compute them in some of the most usual nonparametric gof tests: integral and Kolmogorov–Smirnov tests. For a review of results related to this paper, we refer the readers to Wieand [22], Kallenberg and Ledwina [14], Kallenberg and Koning [13], Litvinova and Nikitin [16], and the most excellent book by Nikitin [17].

The contents of this paper are organized as follows. In Section 2, two non-local types of efficiency (e_α, e_β) are introduced. These are computed in Sections 3 and 4 for gof tests of the type $T_{nm}(\psi)$ or $V_n(\psi)$ for parametric and non-parametric alternatives. It is argued that locally $T_{n\infty}(\psi)$ is preferable to $T_{nm}(\psi)$ if $m < \infty$, in testing $F = F_0$ against "F is not close to F_0". For α-level tests the Hodges–Lehmann efficiency or its generalization the fixed-α efficiency (Sect. 2) is appropriate, but is shown in Section 3 to tend to one under suitable conditions, for the statistics we consider, when testing $F \in \omega_0$ against $F \in \omega_1$ as ω_0 shrinks to $\{F_0\}$. Section 4 gives the Bahadur efficiency for some common parametric examples, using large deviation results derived in part from the work of Hoadley [10] and Abrahamson [1]. More interesting is a comparison of the statistics when testing whether F_0 is close to F ($\sup|F - F_0| = a_0$, say) or distant from F ($\sup|F - F_0| = a_1$, say). This is carried out by computing e_β in Section 4 when $a_0 = 0$, for the statistics $T_{n1}(1)$, $T_{n2}(1)$, $V_n(1)$ and $D_n(\psi)$ for certain ψ. The values of e_β for these statistics are compared using several examples: a normal with shift alternative example, a logistic with shift alternative example, a double-exponential with shift alternative example and others. Section 5 establishes a local inefficiency of $T_{nm}(\psi)$. The proofs of all results are given in Section 6.

[3] Their percentiles are conveniently given for *all* n in a table of Pearson and Hartley [18].

2. Two types of efficiency

Let $X_1, \ldots, X_n$ be independent and identically distributed according to F, a cdf on $\mathbb{R}$. Let ω_0 and ω_1 be two disjoint sets of cdfs on $\mathbb{R}$. Suppose we test $H_0 : F \in \omega_0$ against $H_1 : F \in \omega_1$, rejecting H_0 when $T_n(F_n) > r_n$ for some functional $T_n(\cdot)$. For simplicity of presentation we exclude randomized tests. Suppose T_n is such that $T_n(F_n) = T(F) + o_p(1)$ and $r_n \to r \in [\mu_0,\ \mu_1]$ as $n \to \infty$, where $\mu_0 = \mu(\omega_0) = \sup_{F\in\omega_0} T(F)$ and $\mu_1 = \mu(\omega_1) = \inf_{F\in\omega_1} T(F)$. We assume that $\mu_0 < \mu_1$. If $\mu_0 > \mu_1$ the statistic cannot discriminate between ω_0 and ω_1. Set

$$
\begin{aligned}
&\alpha_n(r_n, F) = P_F\{T_n(F_n) > r_n\},\\
&\beta_n(r_n, F) = P_F\{T_n(F_n) \le r_n\},\\
&\alpha_n(r_n) = \sup_{F\in\omega_0} \alpha_n(r_n, F),\ \text{the maximum type 1 error},\\
&\beta_n(r_n) = \sup_{F\in\omega_1} \beta_n(r_n, F),\ \text{the maximum type 2 error},\\
&\Omega_r = \{cdfs\ Q\ \ \text{on}\ \ \mathbb{R} : T(Q) > r\},\\
&\Omega_r^c = \{cdfs\ Q\ \ \text{on}\ \ \mathbb{R} : T(Q) \le r\},\\
&I(F,G) = \int \ln(\mathrm{d}F/\mathrm{d}G)\,\mathrm{d}F\ \text{if}\ F,\ G\ \text{are absolutely continuous cdfs},\\
&I(A,B) = \inf_{F\in A}\ \inf_{G\in B}\ I(F,G)\ \text{for sets of cdfs}\ A\ \text{and}\ B,\\
&I_1(r,\omega_1) = I(\Omega_r^c,\omega_1),\\
&I_0(r,\omega_0) = I(\Omega_r,\omega_0),\\
&I_i(r,F) = I_i(r,\{F\}),\ i = 0,1.
\end{aligned}
$$

Note that we have assumed that both F and G are absolutely continuous cdfs. A weaker condition is to assume F is absolutely continuous with respect to G and then define $I(F,G) = \infty$ otherwise.

Hoadley [10] has shown that for continuous F and "regular" $T_n(\cdot)$ (in particular for $T_n(\cdot) \equiv T(\cdot)$ uniformly continuous with respect to the "usual" metric),

$$\alpha_n(r_n, F) = \exp\{-nI_0(r,F) + o(n)\}\ \text{for}\ F \in \omega_0, \tag{2.1}$$

and

$$\beta_n(r_n, F) = \exp\{-nI_1(r,F) + o(n)\}\ \text{for}\ F \in \omega_1, \tag{2.2}$$

if $I_0(r,\ F)$ and $I_1(r,\ F)$ are continuous at r. Suppose now that (2.1) and (2.2) hold uniformly. This certainly follows from (2.1) and (2.2) if ω_0 and ω_1 are finite sets. Then

$$\alpha_n(r_n) = \exp\{-nI_0(r,\omega_0) + o(n)\}$$

and

$$\beta_n(r_n) = \exp\{-nI_1(r,\omega_1) + o(n)\}.$$

Without uniformity we only have

$$\limsup_n -\frac{1}{n}\ln\alpha_n(r_n) \le I_0(r,\omega_0)$$

and

$$\limsup_n -\frac{1}{n}\ln\beta_n(r_n) \le I_1(r,\omega_1).$$

If the maximum type 1 error is fixed, that is, $\alpha_n(r_n) \equiv \alpha$, or if $0 < \alpha_1 \le \alpha_n(r_n) \le \alpha_2$ for all n then $I_0(r, \omega_0) = 0$ so, assuming continuity of T, we have $r \le \mu_0$. If $r \le \mu_0$ then $r = \mu_0$ minimizes asymptotically the maximum type 2 error $\beta_n(r_n)$, so that if (2.2) is uniform in $F \in \omega_1$, $\beta_n(r_n) = \exp\{-nI_1(\mu_0,\ \omega_1) + o(n)\}$. Now

$$I_1(\mu_0, \omega_1) \le I(\omega_0, \omega_1) \tag{2.3}$$

with equality if $\omega_0 = \{F \mid T(F) \le a_0\}$ for some a_0. Further, in the parametric case when $T(F_n)$ is the fixed α-level LR (likelihood-ratio) test, under suitable conditions (*) (see below),

$$\beta_n(r_n, F) = \exp\{-nI(\omega_0, F) + o(n)\},$$

so that, if this holds uniformly for $F \in \omega_1$, then equality is obtained in (2.3). These considerations lead us to define the *fixed-α efficiency* of $T_n(F_n)$ as

$$e_\alpha = \frac{I_1(\mu_0, \omega_1)}{I(\omega_0, \omega_1)}\cdot$$

For similar reasons we define the *fixed-β efficiency* of $T_n(F_n)$ as

$$e_\beta = \frac{I_0(\mu_1, \omega_0)}{I(\omega_1, \omega_0)}\cdot$$

Bahadur [4,5] and Brown [6] show that under suitable conditions for the LR test in the parametric case, (2.2) holds at $r = \mu_1$ and $I_1(\mu_1,\ F) = I(\omega_1,\ F)$; (*) above is the dual of these conditions.

When ω_0 and ω_1 are simple, e_α is the Hodges–Lehmann efficiency relative to the LR test, and e_β is the exact Bahadur efficiency relative to the LR test, *cf.* Appendix 1 of Bahadur [3]. We note in passing that Bahadur's definition of e_β in terms of 'the level attained', extends to ω_0 and ω_1 composite.

Between the two extremes of fixing the maximum type 1 error and fixing the maximum type 2 error, is the middle course of choosing r_n to minimize $l_n = \alpha_n(r_n) + \lambda\beta_n(r_n)$ for some $\lambda > 0$. In either case, uniformity in (2.1) and (2.2) implies that independently of λ

$$l_n = \exp\left[-n \min\{I_0(r, \omega_0), I_1(r, \omega_1)\} + o(n)\right],$$

so that the optimal $r_n \to \mu_2$, the root of $I_0(r, \omega_0) = I_1(r, \omega_1)$, which exists and is unique if $\{I_i(r,\ \omega_i),\ i = 0, 1\}$ are continuous and strictly monotone in $[\mu_0, \mu_1]$.

In the parametric case, one can show from Brown [6] and Lemma 8 of Chernoff [7] that under suitable conditions (2.1) holds in ω_0, (2.2) holds in ω_1, and

$$I_0(\mu_2, \omega_0) = I_1(\mu_2, \omega_1) \le J(\omega_0, \omega_1),$$

where

$$J(\omega_0, \omega_1) = \inf_{F_0 \in \omega_0} \inf_{F_1 \in \omega_1} \sup_{0<t<1} -\ln \int (\mathrm{d}F_0/\mathrm{d}\nu)^{1-t} (\mathrm{d}F_1/\mathrm{d}\nu)^t \,\mathrm{d}\nu$$

with equality for the LR test of ω_0 against ω_1.

3. Fixed-α efficiency

Here we show that under suitable conditions for many gof tests $e_\alpha \to 1$ as $\omega_0 \to \{F_0\}$ (which means that F approaches F_0 in distribution for every $F \in \omega_0$). Consider testing the hypothesis $H_0 : F \in \omega_0$, a set of cdfs containing a cdf F_0, against the alternative $H_1 : F \in \omega_1$, another set of cdfs. Suppose that we consider statistics $T(F_n)$ such that

(a) $T(F_0) = 0$;
(b) $T(F) \leq 0 \Rightarrow F = F_0$.
For example, the gof tests $T_{nm}(\psi)$, $V_n(\psi)$ satisfy these conditions for $0 < m \leq \infty$, ψ positive and bounded. Suppose also that
(c) $I_1(r,\ \omega_1)$ is right-continuous at $r = 0$;
(d) $\mu_0 \to 0$ as $\omega_0 \to \{F_0\}$.
Then $\lim_{r\downarrow 0} I_1(r, \omega_1) = I_1(0, \omega_1) = \inf_{T(F)\leq 0} I(F, \omega_1) = I(F_0, \omega_1)$. Hence, $e_\alpha \to 1$ as $\omega_0 \to \{F_0\}$. However, in order for e_α to be a measure of efficiency when the type-one error is fixed, we require that (2.2) holds uniformly in $F \in \omega_1$. For example, by Hoadley's [10], Theorem 1, sufficient additional conditions are
(e) ω_1 is a finite set (since then (2.2) holds uniformly if it holds pointwise);
(f) all cdfs in ω_1 are continuous;
(g) for some $\delta > 0$ for all F in ω_1, $I_1(\cdot, F)$ is continuous in $(0, \delta)$;
(h) $T(\cdot)$ is uniformly continuous with respect to the "usual" metric, $\sup|F - G|$.

4. Fixed-β efficiency

According to the definition in Section 2, in order to calculate e_β in testing $F = F_0$ against $F \in \omega_1$ we need to find $I_0(r, F_0)$. Theorems 4.1 and 4.2 derive expressions for $I_0(r, F_0)$ for integral type statistics: compare with Sections 2.3 and 2.4 in Nikitin [17]. Theorems 4.3 and 4.4 derive expressions for $I_0(r, F_0)$ for Kolmogorov–Smirnov and Kuiper type statistics: compare with Sections 2.1 and 2.2 in Nikitin [17]. Figures 1 to 6 provide a comparison of the values of e_β for these statistics using several examples.

Theorem 4.1. *For $T_{nm}(1)$ and $T^+_{nm}(1)$,*

$$I_0(r, F_0) = m^{-1}\lambda^{-1/m}\int_\epsilon^\gamma y\exp(y)\{\mu + y - \exp(y)\}^{1/m-1}\,\mathrm{d}y$$

for $r > 0$, where λ, μ, ϵ, γ are determined by

$$\begin{aligned}
\mu &= \exp(\epsilon) - \epsilon = \exp(\gamma) - \gamma,\ \ \epsilon < \gamma,\\
m\lambda^{1/m} &= \int_\epsilon^\gamma \{\mu + y - \exp(y)\}^{1/m-1}\,\mathrm{d}y,\\
\lambda^{1+1/m}\ mr^m &= \int_\epsilon^\gamma \{\mu + y - \exp(y)\}^{1/m}\ \mathrm{d}y.
\end{aligned}$$

Theorem 4.2. *For $T_{n2}(\psi_0)$ with $r > 0$ and $\psi_0(t) = (t - t^2)^{-1/2}$, $I_0(r, F_0) \equiv r^2$.*

Theorem 4.3. *Suppose $[(x - x^2)\psi(x)]^{1/\psi(x)} \to 0$ as $x \to 0, 1$, and that ψ is positive and continuous in $(0, 1)$. Then for $V_n(\psi)$ if $\psi = 1$, and for $D_n(\psi)$, $I_0(r,\ F_0)$ is continuous for $0 \leq r < \max(\delta_1,\ \delta_2)$ and*

$$I_0(r, F_0) = \inf_x\ \min\{a(x, r/\psi(x)), a(1 - x,\ r/\psi(x))\},$$

where $\delta_1 = \sup x\psi(x)$, $\delta_2 = \sup(1 - x)\psi(x)$ and

$$a(x, r) = \begin{cases} (x + r)\ln\left(1 + \dfrac{r}{x}\right) + (1 - x - r)\ln\left(1 - \dfrac{r}{1 - x}\right), & 0 < x < 1 - r,\\ \infty, & \textit{otherwise.}\end{cases}$$

For $D^+_n(\psi)$,

$$I_0(r, F_0) = \inf a(x, r/\psi(x)),\quad 0 \leq r < \delta_2$$

and $I_0(r, F_0)$ is continuous in this range. For $D_n^-(\psi)$,

$$I_0\left(r, F_0\right)=\inf_x a\left(1-x, r/\psi(x)\right), \quad 0 \leq r<\delta_1,$$

and $I_0(r, F_0)$ is continuous in this range.

So, $I_0(r, F_0)$ for $D_n(\psi)$ is the minimum of $I_0(r, F_0)$ for $D_n^+(\psi)$ and $I_0(r, F_0)$ for $D_n^-(\psi)$. If ψ is symmetric about 1/2, $I_0(r, F_0)$ is the same for $D_n^+(\psi)$, $D_n^-(\psi)$ and $D_n(\psi)$. So, for $F_0 \in \omega_0$ and $\omega_1 \subset \{F \mid F \neq F_0\}$, e_β for $D_n(\psi) \leq e_\beta$ for $D_n^+(\psi)$ with equality when ψ is symmetric about 1/2.

Theorem 4.4. *Define $a(x, r)$ as in Theorem 4.3. We have the following.*
(i) Let ψ be non-negative piecewise continuous and bounded in $[0,1)$. Let $(1-x)\psi(x) \to 0$ as $x \to 1$. Set $G^{-1}(x)=\sup\{y : G(y)=x\}$. Suppose F/F_0 is bounded and

$$(I): f=F_0\left(F^{-1}\right) \text{ is continuous and } f(0)=0, \quad f(1)=1$$

or

$$(II): g=F\left(F_0^{-1}\right) \text{ is continuous and } g(0)=0, \quad g(1)=1.$$

Then for $D_n^+(\psi)$, $I_0(r, F)$ is continuous provided $D_{F_0}^+(F)<r<\sup_{(0,1)}(1-x)\psi(x)$ and $I_0(r,F)=\inf a(F, r/\psi(F_0)+F_0-F)$, where $D_{F_0}^+(F)=\sup(F-F_0)\psi(F_0)$.

(ii) Let ψ be non-negative, piecewise-continuous and bounded in $(0,1]$. Let $x\psi(x) \to 0$ as $x \to 0$. Suppose $(1-F)/(1-F_0)$ is bounded and (I) or (II). Then for $D_n^-(\psi)$, $I_0(r, F)$ is continuous provided $D_{F_0}^-(F)<r<\sup_{(0,1)} x\psi(x)$ and $I_0(r,F)=\inf a(1-F, r/\psi(F_0)-F_0+F)$, where $D_{F_0}^-(F)=\sup(F_0-F)\psi(F_0)$.

(iii) Under the assumptions of (i) and (ii), for $D_n(\psi)$, $I_0(r, F)$ is continuous for $D_{F_0}(F)<r<\max\{\sup x\psi(x), \sup(1-x)\psi(x)\}$ and

$$I_0(r, F)=\min\left\{I_0(r, F) \text{ for } D_n^+(\psi), I_0(r, F) \text{ for } D_n^-(\psi)\right\},$$

where $D_{F_0}(F)=|||F-F_0|\psi(F_0)||_{F_0,\infty}$.
(iv) For $V_n(1)$, $V_{F_0}(F)<r<1$, $I_0(r, F)$ is continuous and

$$\begin{aligned} I_0(r, F)=\inf_{x>y} \min \Big\{ & a\left(F(x)-F(y), r-F(x)+F(y)+F_0(x)-F_0(y)\right), \\ & a\left(1-F(x)+F(y), r+F(x)-F(y)-F_0(x)+F_0(y)\right)\Big\}, \end{aligned}$$

where $V_{F_0}(F)=D_{F_0}^+(F)+D_{F_0}^-(F)$.
(v) Under the conditions of (iii), for $V_n(\psi)$, $I_0(r, F)$ is continuous for $V_{F_0}(F)<r<\sup_{cdf\,G}\ V_{F_0}(G)$, and $I_0(r, F)=-\ln\max[\rho_V(r), \rho_V(-r)]$ for $\rho_V(r)=\sup_{x>y}\ G(T(x,y,r),x,y,r)$, $G(t,x,y,r)=\exp[-t(r+F_0(x)\phi(F_0(x))-F_0(y)\phi(F_0(y)))]\phi(t,x,y)$, $T(x,y,r)$ is the root of $r=(\partial/\partial t)\ln\phi(t,x,y)$ for $x>y$ when the root exists and of $\phi(t,x,y)=E\exp(tZ)$ for $x>y$, where

$$Z=\begin{cases}\psi\left(F_0(x)\right)-\psi\left(F_0(y)\right) & w.p. F(y), \\ \psi\left(F_0(x)\right) & w.p. F(x)-F(y), \\ 0 & w.p. 1-F(x).\end{cases}$$

When ψ equals ψ_0 one can show that $I_0(r, F_0) \equiv 0$, $\mu_1=\lim T(F_n)=\infty$ for $D_n(\psi_0)$, $V_n(\psi_0)$, $D_n^+(\psi_0)$, $D_n^-(\psi_0)$, so that e_β cannot be calculated using these methods; however, e_β becomes arbitrarily small as ψ remains bounded but approaches ψ_0.

Figure 1 shows the variation of $I_0(r, F_0)$ *versus* r for $T_{n1}(1)$ (and so for $T_{n1}^+(1)$); for $T_{n2}(1)$; for $D_n(1)$ (and so for $D_n^+(1)$, $D_n^-(1)$ and $V_n(1)$); and for $D_n(\psi_1)$ (and so for $D_n^+(\psi_1)$ and $D_n^-(\psi_1)$), where $\psi_1=\psi_0$ in

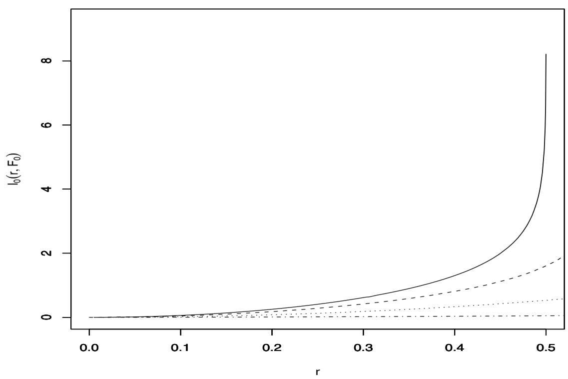

FIGURE 1. $I_0(r, F_0)$ *versus* r for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$ (curve of dots) and $D_n(\psi_1)$ (curve of dots and dashes).

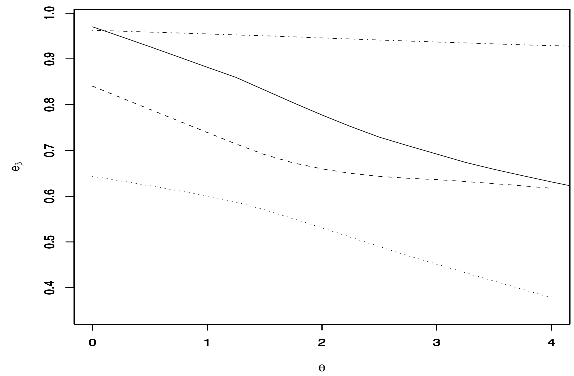

FIGURE 2. Fixed β-efficiency *versus* θ for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$, $V_n(1)$ (curve of dots) and $T_{n2}(\psi_0)$ (curve of dots and dashes), where $F_\theta(x) = \Phi(x - \theta)$, the normal with shift alternative.

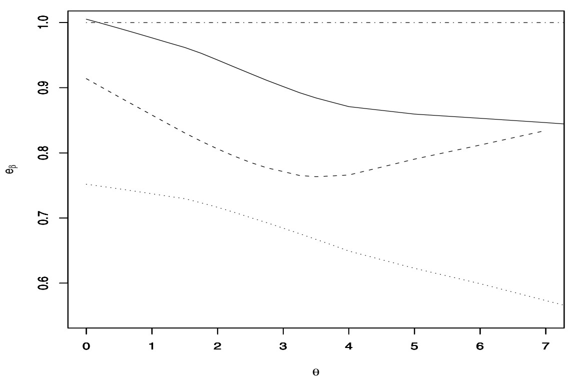

FIGURE 3. Fixed β-efficiency *versus* θ for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$, $V_n(1)$ (curve of dots) and $T_{n2}(\psi_0)$ (curve of dots and dashes), where $F_\theta(x) = 1/\{1 + \exp(-x + \theta)\}$, the logistic with shift alternative.

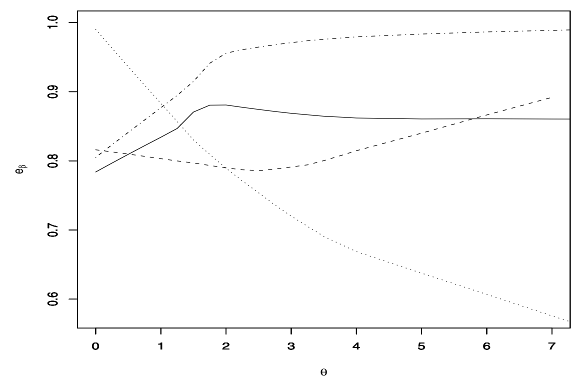

FIGURE 4. Fixed β-efficiency *versus* θ for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$, $V_n(1)$ (curve of dots) and $T_{n2}(\psi_0)$ (curve of dots and dashes), where $F_\theta(x) = F_0(x - \theta)$ and $\dot{F}_0(x) = \exp(-|x|)/2$, the double-exponential with shift alternative.

$[0.005, 0.995]$ and $\psi_1 = \psi_0(0.005)$ otherwise. Figure 2 shows the variation of e_β *versus* θ for $T_{n1}(1)$, $T_{n2}(1)$, $D_n(1)$, $V_n(1)$ and $T_{n2}(\psi_0)$, where $F_\theta(x) = \Phi(x - \theta)$, the normal with shift alternative. Figure 3 shows the same for $F_\theta(x) = 1/\{1+\exp(-x+\theta)\}$, the logistic with shift alternative. Figure 4 shows the same for $F_\theta(x) = F_0(x-\theta)$ and $\dot{F}_0(x) = \exp(-|x|)/2$, the double-exponential with shift alternative. Figure 5 shows the same for $F_\theta(x) = F_0(x)^{\theta+1}$, the Lehmann alternative. Finally, Figure 6 shows the same for $F_\theta = \{\exp(\theta F_0) - 1\}/\{\exp(\theta) - 1\}$.

Figures 2 to 4 show that $T_{n2}(\psi_0)$ exhibits the highest e_β efficiencies. Figures 5 and 6 show that $T_{n1}(1)$ exhibits the highest e_β efficiencies. So, $T_{n1}(1)$ and $T_{n2}(\psi_0)$ exhibit the highest e_β efficiencies. The lowest e_β efficiencies in each figure are for $V_n(1)$.

When $T_{n2}(\psi_0)$ exhibits the highest e_β efficiencies, the second and third largest efficiencies are those by $T_{n1}(1)$ and $T_{n2}(1)$, respectively. When $T_{n1}(1)$ exhibits the highest e_β efficiencies, the second and third largest efficiencies are those by $T_{n2}(\psi_0)$ and $T_{n2}(1)$, respectively.

Furthermore, in the case of $F_\theta(x) = 1/\{1 + \exp(-x + \theta)\}$, $T_{n2}(\psi_0)$ is just as good as the LR test for all θ. In the case of $F_\theta = \{\exp(\theta F_0) - 1\}/\{\exp(\theta) - 1\}$, $T_{n1}(1)$ is just as good as the LR test for all θ, which is not surprising since the LR test is equivalent to $T_{n1}^{-}(1)$.

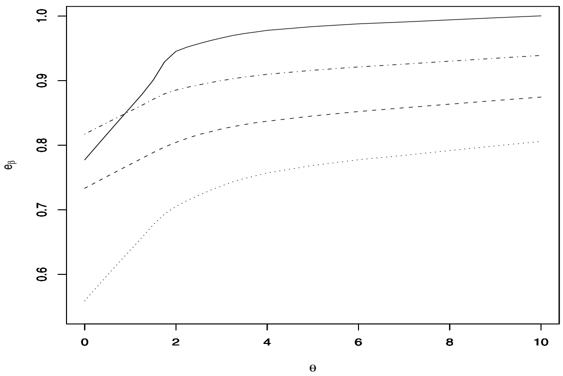

FIGURE 5. Fixed β-efficiency *versus* θ for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$, $V_n(1)$ (curve of dots) and $T_{n2}(\psi_0)$ (curve of dots and dashes), where $F_\theta(x) = F_0(x)^{\theta+1}$, the Lehmann alternative.

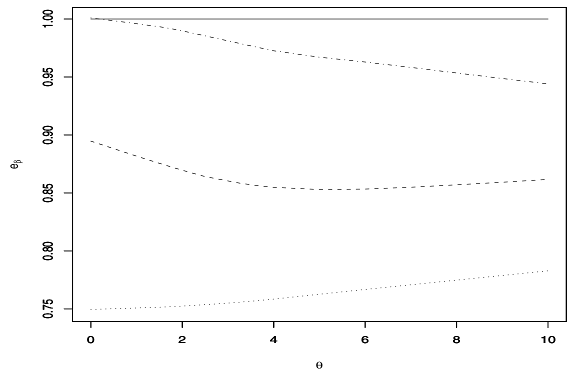

FIGURE 6. Fixed β-efficiency *versus* θ for $T_{n1}(1)$ (solid curve), $T_{n2}(1)$ (curve of dashes), $D_n(1)$, $V_n(1)$ (curve of dots) and $T_{n2}(\psi_0)$ (curve of dots and dashes), where $F_\theta = \{\exp(\theta F_0) - 1\}/\{\exp(\theta) - 1\}$.

These figures suggest $T_{nm}(\psi)$ are the most powerful statistics and $V_n(1)$ are the least powerful statistics in terms of e_β efficiencies.

5. Local inefficiency of $T_{nm}(\psi)$

Suppose that we test $F = F_0$ against the alternative that F is not close to F_0, in the sense that

$$F \in \omega_1 = \{F \text{ continuous cdf on } \mathbb{R}, ||\ |F - F_0|\, \psi_A\,(F_0)\,||_{k,F_0} \geq a_1\},$$

where $a_1 > 0$ and ψ_A is some non-negative function, and $0 \leq k \leq \infty$.

Lemma 5.1. *Set*

$$f(x) = \begin{cases} x & in\ \ [0, 1/2], \\ 1 - x & in\ \ [1/2, 1]. \end{cases}$$

Then, we have the following:
(*i*) *If* $||F\psi_A||_k < \infty$, $1 \leq k \leq m$ *then, for* $T_{nm}(\psi)$,

$$a_1 \inf \psi/\psi_A \leq \mu_1 \leq a_1 ||f\psi_A||_m/||f\psi_A||_k.$$

(*ii*) *If* $0 < \inf \psi_A$, $1 \leq m \leq k$ *then, for* $T_{nm}(\psi)$,

$$\mu_1 \leq \sup \psi \left(\frac{a_1}{\inf \psi_A}\right)^{k/(k+1)(m+1)/m} (k+1)^{(m+1)/(k+1)/m}(m+1)^{-1/m}.$$

(*iii*) *If* $\omega_1 = \{F : \sup |F - F_0| \geq a_1\}$ *then, for* $T_{nm}(1)$, $\mu_1 = a_1^{1+1/m}(m+1)^{-1/m}$ *(that is, equality is obtained in (ii)). The same is true for* $\omega_1 = \{F \leq F_0 : \sup(F - F_0) \geq a_1\}$.
(*iv*) *For* $V_n(\phi)$, *if* $k = \infty$, $\inf \psi/\psi_A a_1 \leq \mu_1 \leq \sup \psi/\psi_A a_1$.

Theorem 5.1. *Let* ψ, ψ_A *be bounded away from zero and* ∞. *Suppose* $\alpha_n(r_n, F_0)$, $\beta_n(r_n)$ *are* $O^*(1)$, *by allowing* a_1 *to decrease to zero as* $n \to \infty$, *where by* $f = O^*(g)$ *we mean that* f/g *is bounded away from zero and* ∞. *Then* $T_{nm}(\psi)$ *needs* $O^*(a_1^{-c})$ *observations, where*

$$c = \begin{cases} 2, & k \leq m \leq \infty, \\ \dfrac{k}{k+1}\ \dfrac{m+1}{m}, & m < k \leq \infty \end{cases}$$

while $V_n(\psi)$ needs $O^*(a_1^{-2})$ observations.

Hence, in order to ensure that the local efficiency (in the sense implicit in the theorem − a generalization of the idea of Pitman efficiency) is positive for *all* k and ψ_A bounded, one must take $m=\infty$ when using $T_{nm}(\psi)$.

6. Proofs

The following lemma is needed.

Lemma *For*

$$\begin{aligned}(i)\ & T^+_{nm}(\psi),\ \psi \text{ bounded},\\ (ii)\ & T_{nm}(\psi),\ m=2,4,6,\dots,\psi \text{ bounded},\\ (iii)\ & T_{nm}(\psi), 0<m<\infty,\ \psi=1,\end{aligned}$$

we have

$$I_0(r,F_0)=\int_0^1 \dot H \ln \dot H,$$

where $H=H(x)$ is an absolutely continuous cdf on [0, 1] such that

$$\ddot H/\dot H + m\lambda(H-x)^{m-1}\psi(x)^m=0,$$
$$\int_0^1 (H-x)^m\psi(x)^m \mathrm{d}x = r^m$$

and for (iii), $H(x)\geq x$.

Proof. For (i) and (ii),

$$I_0\left(r,F_0\right)=\left\{\inf I\left(Q,F_0\right): Q \text{ a cdf such that } \left|\left|\left(Q-F_0\right)\psi\left(F_0\right)\right|\right|_{F_0,m}>r\right\}=\inf\int_0^1 \dot H\ln\dot H,$$

where the inf is taken over cdfs on $[0,1]$, H, such that $\int_0^1 (H-x)^m\psi(x)^m=r^m$ since the closer $H(x)$ is to x, the smaller is $\int \dot H\ln\dot H$ if $\dot H$ exists. For

$$V=V\left(H,\dot H,x\right)=\dot H\ln\dot H-\lambda(H-x)^m\psi(x)^m,$$

by the method of Lagrange multipliers, we seek a function H on [0, 1] giving an extremal of $\int_0^1 V\mathrm{d}x$ subject to $H(0)=0$ and $H(1)=1$. By the calculus of variation (for example, Courant and Hilbert [8]) H satisfies Euler's equation:

$$\frac{\partial V}{\partial H}=\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\partial V}{\partial \dot H}\cdot$$

For (iii) we wish to find an extremal of $\int_0^1 V\mathrm{d}x$, where $V=\dot H\ln\dot H-\lambda|H-x|^m$. Suppose H_2 gives an extremal. Then H_2 is a.c. For all intervals such that $H_2(x)-x<0$ in (a,b) and $H_2(x)-x=0$ at a,b, let $H_1(x)=a+b-H_2(a+b-x)\in(a,b)$. Let $H_1(x)=H_2(x)$ when $H_2(x)\geq x$. Then

$$\int_0^1 \dot H_2\ln\dot H_2=\int_0^1 \dot H_1\ln\dot H_1$$

and

$$\int_0^1 \left|H_2-x\right|^m=\int_0^1\left|H_1-x\right|^m.$$

It is now straightforward to see that the solution of Euler's equation is the minimizing cdf. □

Proof of Theorem 4.1. By the lemma,

$$I_0\left(r, F_0\right) = \int \dot{H}_1 \ln \dot{H}_1,$$

where $H_1 = x + J$, $J \geq 0$, $J(0) = J(1) = 0$, $\ddot{J}/(\dot{J}+1) + \lambda m J^{m-1} = 0$ and $\int_0^1 J^m = r^m$. Let $t = \dot{J} + 1$. So, $t - \ln t = \mu - \lambda J^m$, μ a constant, and

$$x = \int^J \frac{\mathrm{d}J}{t-1} = -\frac{1}{m\lambda} \int^t \left(\left(\mu - t + \ln t\right)/\lambda\right)^{-1+1/m} \mathrm{d} \ln t$$

since $\lambda > 0$ ($\lambda \leq 0$ leads to a contradiction). Set $T = \ln t$ and $g(x) = \int_\gamma^x \left\{\mu + y - \exp(y)\right\}^{1/m-1} \mathrm{d}y$ for γ a constant such that $-m\lambda^{1/m} x = g(T)$. Set $G(x) = g^{-1}(x)$. So, $T = G(-m\lambda^{1/m} x)$, $G(0) = \gamma$ and $J(0) = 0 \Rightarrow \mu = \exp(\gamma) - \gamma$. Let $\epsilon = G(-m\lambda^{1/m})$, $J(0) = 0 \Rightarrow \mu = \exp(\epsilon) - \epsilon$ and $-m\lambda^{1/m} = g(\epsilon)$. So, $\epsilon < \gamma$. The theorem follows. □

Proof of Theorem 4.2. Hoadley's [10] Theorem 2 can be used to show that the lemma extends to $T_{n2}(\psi_0)$. So,

$$I_0\left(r, F_0\right) = \int_0^1 \dot{H} \ln \dot{H},$$

where $(x - x^2)\ddot{H} + 2\lambda(\dot{H}H - \dot{H}x) = 0$. Note that

$$H(0) = 0 \Rightarrow \left(x - x^2\right)\dot{H} + \lambda H^2 - H + 2(1-\lambda)\int_0^x x \mathrm{d}H = 0$$

and $H(1) = 0 \Rightarrow \lambda = 1$ as $\int_0^1 x \mathrm{d}H \neq 1/2$. So,

$$H - x = \frac{z\left(x - x^2\right)}{1 + zx},$$

where $z + 1$ is a positive constant,

$$I_0\left(r, F_0\right) = \int \dot{H} \ln \dot{H} = \ln(1+z)(1+2/z) + 2/z - 2(1+1/z) = \int \frac{(H-x)^2}{x - x^2} = r^2.$$

The proof is complete. □

Proof of Theorem 4.3. When $\psi = 1$ this is well-known – see Theorem 1 of Sethuraman [19], Theorems 5.1 and 5.2 of Hoadley [9]. Under the different condition that

$$\int_0^1 \exp\left\{s \sup \psi(t)\right\} \mathrm{d}y < \infty \text{ for all } s,$$

the theorem follows from Theorem 1 of Sethuraman [20]. This version follows from Theorems 1 and 2 of Abrahamson [1] when corrected: for the continuity of $\rho_\psi^*(\epsilon)$, $(x - x^2)\psi(x) \to 0$ as $x \to 0, 1$ is not strong enough; the fifth line from the bottom of page 1481 of Abrahamson [1] is incorrect as the right hand side depends on n. □

Proof of Theorem 4.4. Follows easily from Abrahamson [1]. □

Proof of Lemma 5.1. For (i), choose $F \in \omega_1$ of the form $F_0 + a_1 c f(F_0)$. The result of (ii) follows from the case $\psi = \psi_A = 1$ by choosing

$$F \in \omega_1 \text{ of the form } \begin{cases} 0 & \text{in } [0, c), \\ F_0 & \text{in } [c, 1]. \end{cases}$$

For (iii), if $\dot{H}$ exists and $H(x) - x \geq 0$ in A and $H(x) - x \leq 0$ in B, then if $y \in B$, there is a y_0 in B such that $[y_0, y] \subset B$ and $H(y_0) = y_0$, so that

$$\int |H - x|^m \geq \int_B (x - H)^m \geq \int_{y_0}^{y} (x - H)^m \mathrm{d}(x - H) = \left. \frac{(y - H(y))^{m+1}}{m+1} \right|_{y_0}^{y},$$

implying that

$$\int |H - x|^m \geq \sup_B |H(y) - y|^{m+1} / (m+1).$$

Similarly, for $y \in A$. So, $\mu_1^m \geq a_1^{m+1}/(m+1)$. By (ii) equality is obtained and hence also for the one-sided case since $F \leq F_0$ in (ii).

For (iv),

$$a_1 \inf \psi/\psi_A = (\inf \psi/\psi_A) \inf \left\{ \lambda^+ + \lambda^- : F \text{ such that } \max\left(\lambda^+, \lambda^-\right) \geq a_1 \right\} \leq \mu_1,$$

where $\lambda^+ = D^+_{F_0}(F)$ at $\psi = \psi_A$ and $\lambda^- = D^-_{F_0}(F)$ at $\psi = \psi_A$. □

Proof of Theorem 5.1. This was given in Kac *et al.* [12] for k, $m = 2, \infty$ and $\psi = \psi_A = 1$. With β_n bounded away from zero,

$$\alpha_n = \exp\left[-n I_0\left(\mu_1, F_0\right) + o(n)\right],$$

so α_n is bounded away from zero, implying that $nI_0(\mu_1,\ F_0) = O^*(1)$. But $I_0(r, F_0) = O(r^2)$ as $r \downarrow 0$ and by Lemma 5.1, $\mu_1 = O^*(a_1^c)$ as $a_1 \downarrow 0$. The result follows. □

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