ESAIM: PS 15 (2011) 217–232
DOI: 10.1051/ps/2009015

ESAIM: Probability and Statistics
www.esaim-ps.org

LARGE DEVIATIONS FOR DIRECTED PERCOLATION ON A THIN RECTANGLE

Jean-Paul Ibrahim[1,2]

Abstract. Following the recent investigations of Baik and Suidan in [*Int. Math. Res. Not.* (2005) 325–337] and Bodineau and Martin in [*Electron. Commun. Probab.* **10** (2005) 105–112 (electronic)], we prove large deviation properties for a last-passage percolation model in $\mathbb{Z}_+^2$ whose paths are close to the axis. The results are mainly obtained when the random weights are Gaussian or have a finite moment-generating function and rely, as in [J. Baik and T.M. Suidan, *Int. Math. Res. Not.* (2005) 325–337] and [T. Bodineau and J. Martin, *Electron. Commun. Probab.* **10** (2005) 105–112 (electronic)], on an embedding in Brownian paths and the KMT approximation. The study of the subexponential case completes the exposition.

Mathematics Subject Classification. 60F10.

Received December 18, 2007. Revised February 13, 2009.

Introduction

Random matrix theory has developed extensively in the last several decades following the pioneering results by E. Wigner in the fifties. Gaussian models attracted a lot of attention, among them the Gaussian Unitary Ensemble (GUE). In this example, the knowledge of the joint distribution of the eigenvalues allowed for a rather complete understanding of both their global and local behaviors. In particular, the limiting behavior of the largest eigenvalue gave rise to the famous Tracy-Widom distribution [15, 21, 30]. Several random growth models, such as the longest increasing subsequence and the corner growth models, have been shown to develop a similar behavior relying on a common determinantal structure [16,18]. In particular, the last-passage percolation or so-called corner growth model (see below for a precise description) has been deeply studied by Johansson in [16]. For geometric or exponential random variables (the only cases leading to a determinantal description), Johansson established both fluctuations and large deviation asymptotics similar to the ones for the GUE random matrix model. Following the recent investigations by Baik and Suidan [1] and Bodineau and Martin [5] at the level of fluctuations, the present paper deals with large deviations for the random growth model for more general random variables but on rectangles such that one side is asymptotically negligible with respect to the other at a given rate. The main results concern Gaussian random variables and random weights having a finite moment-generating function. Somewhat surprisingly, the rate may be shown to be larger than the one for the fluctuations. The comparison method used in this work is basically inspired from [1] and [5] and relies similarly on an embedding in Brownian paths.

Keywords and phrases. Large deviations, random growth model, Skorokhod embedding theorem.

[1] Université de Toulouse, Université Paul Sabatier, Institut de Mathématiques de Toulouse, 31062 Toulouse, France.

[2] CNRS, Institut de Mathématiques de Toulouse UMR 5219, 31062 Toulouse, France. `jibrahim@math.univ-toulouse.fr`

Recall first the basic corner growth model under study. It can be described as directed paths in the lattice $\mathbb{Z}_+^2$ going from $(1,1)$ to $(N,k) \in \mathbb{Z}_+^2$ where only up and right steps are allowed. More precisely, denoting by $\Pi(N,k)$ the set of all such paths, a path $\pi \in \Pi(N,k)$ is called an *up/right* path and is defined as a collection of sites $\{(i_l, j_l)\}_{l=1}^{N+k-1}$ satisfying $(i_1, j_1) = (1,1)$, $(i_{N+k-1}, j_{N+k-1}) = (N,k)$ and $(i_{l+1}, j_{l+1}) - (i_l, j_l)$ is either $(1,0)$ or $(0,1)$. The main random variable under consideration is the last-passage time defined by

$$G(N,k) = \max_{\pi \in \Pi(N,k)} \left\{ \sum_{(i,j)\in\pi} X_i^{(j)} \right\}$$

where the $X_i^{(j)}$'s are *i.i.d.* random variables. As an alternate description, set $U(N,k)$ as the subset of $\mathbb{R}_+^{k+1}$ given by

$$U(N,k) = \left\{ u = (u_0, u_1, \ldots, u_k) \in \mathbb{R}_+^{k+1}; 0 = u_0 \leq u_1 \leq \ldots \leq u_k = N \right\}.$$

Then

$$G(N,k) = \sup_{u \in U(N,k)} \left\{ \sum_{r=1}^{k} \left[S_{\lfloor u_r \rfloor}^{(r)} - S_{\lfloor u_{r-1} \rfloor - 1}^{(r)} \right] \right\}$$

where $S_m^{(r)} = \sum_{i=1}^m X_i^{(r)}$ with the convention $S_{-1}^{(r)} = S_0^{(r)} = 0$. This follows from the fact that $U(N,k) \equiv \Pi(N,k)$ when $u \in \mathbb{Z}_+^{k+1}$. Actually, every $u \in U(N,k) \cap \mathbb{Z}_+^{k+1}$ maps to a unique path $\pi \in \Pi(N,k)$ whose ith up-jump occurs on u_i. On the other hand, each path π is characterized by its up-step sites. It will be more appropriate to adopt the second form of $G(N,k)$ in order to compare it later with the Brownian last-passage percolation model.

1. Results

Having introduced the model, recall briefly some of its properties established by Johansson in [16] in the particular case of geometric or exponential distributions. The key of the results in this case relies on the explicit description of the last-passage time distribution $G(N,k)$. When weights are *i.i.d.* geometric with parameter $q \in (0,1)$, we have

$$\mathbb{P}[G(N,k) \leq t] = \frac{1}{Z_{N,k}} \sum_{\substack{h \in \mathbb{N}^k \\ \max\{h_i\} \leq t+k-1}} \prod_{1 \leq i < j \leq k} (h_i - h_j)^2 \prod_{i=1}^{k} \binom{h_i + N - k}{h_i} q^{h_i},$$

where $N \geq k$ and $Z_{N,k}$ is the normalizing constant. Using results from logarithmic potential theory, Johansson described in [16] the large deviation behaviors of $G(N,k)$ when k and N are of the same order. Namely, he obtained that, for $\gamma \geq 1$, there exist two rate functions $i(\varepsilon)$ and $l(\varepsilon)$ such that for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}[G([\gamma k], k) \geq k(\omega(\gamma, q) + \varepsilon)] = -i(\varepsilon) \tag{1.1}$$

and

$$\lim_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}[G([\gamma k], k) \leq k(\omega(\gamma, q) - \varepsilon)] = -l(\varepsilon). \tag{1.2}$$

The functions $l(x)$ and $i(x)$ are positive for every $x > 0$ and

$$\omega(\gamma, q) := \lim_{k\to\infty} \frac{1}{k} \mathbb{E}[G([\gamma k], k)] = \frac{(1 + \sqrt{\gamma q})^2}{1 - q} - 1.$$

Using the asymptotics of the Meixner orthogonal polynomial ensemble, Johansson [16] further established the fluctuations of $G(N,k)$ at the Tracy-Widom GUE rate. He proved that for $\gamma \geq 1$ and $s \in \mathbb{R}$,

$$\lim_{k\to\infty} \mathbb{P}\left[\frac{G([\gamma k],k) - k\omega(\gamma,q)}{\sigma(\gamma,q)k^{1/3}} \leq s\right] = F_2^{\mathrm{TW}}(s),$$

where

$$\sigma(\gamma,q) = \frac{q^{1/6}\gamma^{-1/6}}{1-q}(\sqrt{\gamma}+\sqrt{q})^{2/3}(1+\sqrt{\gamma q})^{2/3}$$

and $F_2^{\mathrm{TW}}(s)$ is the distribution function of the Tracy-Widom law (see [30]). Replacing geometric weights with exponential ones gives similar results since an exponential distribution can be seen as the limit of a rescaled geometric one. See [16] for the precise formulas.

Recently, Bodineau and Martin [5] and Baik and Suidan [1] studied the same model when paths are close to the axis, *i.e.* $k = o(N^\alpha)$ for some $\alpha < 1$, but allowing more general distributions. The authors used a coupling with the Brownian trajectories through the following Brownian last-passage percolation. Letting $(B_t^{(r)})_{r\geq 1}$ be a sequence of independent Brownian motions, set

$$L(N,k) = \sup_{u\in U(N,k)} \left\{\sum_{r=1}^{k}\left[B_{u_r}^{(r)} - B_{u_{r-1}}^{(r)}\right]\right\}.$$

It has been proved in [2,14,23] that $L(1,k)$ has the same distribution as the largest eigenvalue of a $k\times k$ rescaled GUE random matrix. As a consequence of the fluctuation result for the GUE model, it follows that

$$k^{1/6}\left[L(1,k) - 2\sqrt{k}\right] \xrightarrow{d} \Theta,$$

where Θ is a random variable having the distribution function $F_2^{\mathrm{TW}}(s)$. Using this result and a comparison between the continuous model with Brownian paths and the discrete one with random weights, the authors of [1] and [5] deduced fluctuation properties of the corner growth model for rather general random variables. However, the embedding in the Brownian paths requires to restrict the paths on small rectangles. For example, in [5], the discrete and the continuous models were coupled using the Komlós-Major-Tusnády (KMT) approximation which couples random walks with Brownian motion. The authors proved that if the weights satisfy $\mathbb{E}|X_i^{(j)}|^p < \infty$ for some $p > 2$, setting $\mu = \mathbb{E}X_i^{(j)}$ and $\sigma^2 = \mathrm{var}(X_i^{(j)})$, then for all $\alpha < \frac{6}{7}(\frac{1}{2} - \frac{1}{p})$,

$$\frac{G(N,\lfloor N^\alpha\rfloor) - N\mu - 2\sigma N^{\frac{1+\alpha}{2}}}{\sigma N^{\frac{1}{2}-\frac{\alpha}{6}}} \xrightarrow{d} \Theta.$$

If the random variables $X_i^{(j)}$ have all moments, *i.e.* $p = \infty$, then α is lower than $3/7$. This is true when the weights are Gaussian or are bounded for example. It is not known how optimal this rate could be: the authors in [5] think that such a result might hold, for some independence reasons, when $\alpha < 3/4$. However, they do not give a complete proof. In [1], the authors compared the discrete and continuous model *via* the Skorokhod embedding theorem in order to obtain almost the same results. Lately, Suidan in [29] produced another proof of the last theorem when the variables have a third moment. He compared two discrete directed percolation models using a theorem of Chatterjee [8]. The fluctuation properties for the first (with geometric distribution) lead him to similar ones for the second.

In this paper, we follow the comparison methods of [5] and [1] to establish large deviations limit theorems for directed percolation models on thin rectangles. The results mainely concern Gaussian weights and random weights with finite moment-generating function aroud zero. We rely similarly on the corresponding results for the Brownian percolation model. Namely, as a consequence of the GUE random matrix interpretation [14,23], for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}\left[L(1,k) \geq 2\sqrt{k}(1+\varepsilon)\right] = -J_{\mathrm{GUE}}(\varepsilon) \tag{1.3}$$

and

$$\lim_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}\left[L(1,k) \leq 2\sqrt{k}(1-\varepsilon)\right] = -I_{\mathrm{GUE}}(\varepsilon). \tag{1.4}$$

The two functions $J_{\mathrm{GUE}}(x)$ and $I_{\mathrm{GUE}}(x)$ are both positive for every positive x. J_{GUE} can be computed explicitly (see [3]) as

$$J_{\mathrm{GUE}}(\varepsilon) = 4\int_0^{\varepsilon} \sqrt{x(x+2)}\, \mathrm{d}x.$$

To the best of our knowledge, there is no explicit form for I_{GUE}. This function appears in the logarithmic potential theory and it represents physically the minimal potential energy of charges on a one-dimension conductor exposed to an external field (see [25]). In this work, we do not need the explicit form of I_{GUE}. However, its continuity, proved at the end of Section 2, will be necessary for the proof.

The following three theorems are the main results of this paper. Despite some similarity in their proofs, the second one requires more work. One surprising feature is that the rate α in the Gaussian case can be taken arbitrary less than 1 (compared to $\alpha < 3/7$ for the fluctuation result).

Throughout the article, k and N are two integers and N depends on k, *i.e.* $N = N(k) := N_k$. We assume further that $k = \mathrm{o}(N_k)$ and we let k goes to infinity. For simplicity, we write N instead of N_k throughout the proofs.

Theorem 1.1. *Assume that the variables $(X_i^{(j)})_{i,j=1}^{\infty}$ are i.i.d. standard normal random variables. Assume further that $k = \mathrm{o}(\frac{N_k}{\log N_k})$. Then, for all $\varepsilon > 0$,*

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}\left[G(N_k,k) \geq 2\sqrt{N_k k}(1+\varepsilon)\right] = -J_{\mathrm{GUE}}(\varepsilon).$$

On the left of the mean, we have for $k = \mathrm{o}(N_k^{\frac{1}{2}})$,

$$\lim_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}\left[G(N_k,k) \leq 2\sqrt{N_k k}(1-\varepsilon)\right] = -I_{\mathrm{GUE}}(\varepsilon).$$

In the second statement, we replace Gaussian variables with weights having finite exponential moments. Loosing the Gaussian assumption will complicate the coupling and reduce the size of the rectangles. We denote by X a random variable having the common law of the *i.i.d.* variables in the sequence $(X_i^{(j)})_{i,j=1}^{\infty}$.

Theorem 1.2. *Assume that the variables $(X_i^{(j)})_{i,j=1}^{\infty}$ are i.i.d. random variables such that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. Assume further that there exit $\mu_0 > 0$ such that for all $\mu < \mu_0$,*

$$\mathbb{E}\exp(\mu|X|) < +\infty. \tag{1.5}$$

If $k = \mathrm{o}(\frac{N_k}{(\log N_k)^2})$, then for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}\left[G(N_k,k) \geq 2\sqrt{N_k k}(1+\varepsilon)\right] = -J_{\mathrm{GUE}}(\varepsilon).$$

Similarly, if $k = \mathrm{o}(N_k^{\frac{1}{3}})$, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}\left[G(N_k,k) \leq 2\sqrt{N_k k}(1-\varepsilon)\right] = -I_{\mathrm{GUE}}(\varepsilon).$$

The proof of Theorem 1.2 relies on the Komlós-Major-Tusnády approximation for sums of *i.i.d* centered random variables with finite exponential moments, see [17]. The following theorem deals with a particular class of subexponential weights. We make use of the Skorokhod embedding theorem to obtain this result.

Theorem 1.3. *Assume that the variables* $(X_i^{(j)})_{i,j=1}^{\infty}$ *are i.i.d. random variables satisfying* $\mathbb{E}X = 0$ *and* $\mathbb{E}X^2 = 1$. *Furthermore, assume that there exit* $\mu > 0$ *and* $0 < \gamma < 1$ *such that*

$$\mathbb{E}\exp\left(\mu|X|^{\gamma}\right) < +\infty. \tag{1.6}$$

If $k = \mathrm{o}(N_k^{\alpha})$ *with* $\alpha < \frac{\gamma}{2\gamma+2}$, *then, for all* $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}\left[G(N_k, k) \geq 2\sqrt{N_k k}(1+\varepsilon)\right] = -J_{\mathrm{GUE}}(\varepsilon).$$

Similarly, if $k = \mathrm{o}(N_k^{\alpha})$ *with* $\alpha < \frac{\gamma}{5\gamma+4}$, *for all* $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}\big[G(N_k, k) \leq 2\sqrt{N_k k}(1-\varepsilon)\big] = -I_{\mathrm{GUE}}(\varepsilon).$$

The results in Theorem 1.3 cover in particular the examples of Weibull and Lévy distributions. Notice that in Theorem 1.3 we can take $\gamma \geq 1$. However, the result is worthless because of Theorem 1.2. Actually, the KMT approximation is more efficient than the Skorokhod embedding theorem for *i.i.d.* random variables with finite exponential moments.

Let us notice here that if the weights have a finite pth moment with $p \geq 2$, we still obtain the same deviation results but for $k = \mathrm{o}(\log N^{p-1})$. In this case, we use again the KMT approximation for random weights with finite pth moment, see [17, 26].

It should be noted also that the previous results hold for geometric and exponential weights as soon as $k = \mathrm{o}(N_k)$, with the same GUE rate functions. This is a consequence of the large deviations (1.1) and (1.2) and the fact that the Laguerre ensemble converges to the GUE on the scaling $k = \mathrm{o}(N_k)$. The reader can see [16] and [19] for rigorous results.

Non-asymptotic bounds for the preceding models can be deduced from the previous theorems proofs. The rectangle width for small deviations matches in this case the fluctuation results. For this we use analogous deviation inequalities to the right of the mean obtained for the largest eigenvalue of the GUE, see [19]. To the left of the mean, we use recent deviation results for the largest eigenvalue of the GUE obtained by Ledoux and Rider [20].

Theorem 1.4. *Assume that the variables* $(X_i^{(j)})_{i,j=1}^{\infty}$ *are i.i.d. standard normal random variables, and that* $k = N_k^{\alpha}$ *with* $\alpha < \frac{3}{7}$. *Then, there exists a positive constant* C_α *depending only on* α *such that, for all* $0 < \varepsilon < 1$,

$$\mathbb{P}\left[G(N,k) \geq 2\sqrt{Nk}(1+\varepsilon)\right] \leq C_\alpha \exp\left(-\frac{k\varepsilon^{3/2}}{C_\alpha}\right)\cdot \tag{1.7}$$

On the left of the mean, we have

$$\mathbb{P}\left[G(N,k) \leq 2\sqrt{Nk}(1-\varepsilon)\right] \leq C_\alpha \exp\left(-\frac{k^2\varepsilon^{3}}{C_\alpha}\right)\cdot \tag{1.8}$$

In the Gaussian case, large deviation inequalities for large $\varepsilon > 1$ hold for the optimal rate $\alpha = 1$ using some concentration arguments. We refer to [19] for the results and the proofs. For random weights with finite moment-generating function, we have similar results.

Theorem 1.5. *Assume that the variables $(X_i^{(j)})_{i,j=1}^{\infty}$ are i.i.d. random variables such that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. Assume further that there exit $\mu_0 > 0$ such that for all $\mu < \mu_0$,*

$$\mathbb{E}\exp\left(\mu|X|\right) < +\infty. \tag{1.9}$$

If $k = N_k^{\alpha}$ with $\alpha < \frac{3}{7}$, then there exists a positive constant C_α depending α and the distribution of X such that, for all $0 < \varepsilon < 1$,

$$\mathbb{P}\left[G(N_k,k) \geq 2\sqrt{N_k k}(1+\varepsilon)\right] \leq C_\alpha \exp\left(-\frac{k\varepsilon^{3/2}}{C_\alpha}\right)\cdot \tag{1.10}$$

Similarly, if $k = N_k^{\alpha}$ with $\alpha < \frac{1}{3}$, then

$$\mathbb{P}\left[G(N_k,k) \leq 2\sqrt{N_k k}(1-\varepsilon)\right] \leq C_\alpha \exp\left(-\frac{k^2\varepsilon^{3}}{C_\alpha}\right)\cdot \tag{1.11}$$

As the reader can notice, if X satisfies the condition (1.6), the same exponential inequalities as in Theorem 1.5 can be obtained. In this case, the positive constant will depend on α, μ and γ. A smaller α will be also necessary. The precise calculations are left to the reader.

We strongly believe that the preceding results hold on a wider rectangle. However, the method used here does not allow us to improve the rectangle width. Theorem 1.1 will be proved in Section 2 while Theorem 1.2 will be proved in Section 3. The case of the subexponential weights given in Theorem 1.3 will be discussed in Section 4. Theorem 1.4 and Theorem 1.5 will be addressed in Section 5 on the basis of the preceding results and proofs.

2. Proof of Theorem 1.1

Throughout the rest of the paper, we write N instead of N_k for simplicity. As claimed before, to prove Theorem 1.1, we compare $G(N,k)$ and $L(N,k)$. To do so, let for any $\varepsilon > 0$,

$$A = \left\{G(N,k) \geq 2\sqrt{Nk}(1+\varepsilon)\right\}$$

and

$$B = \left\{\left|G(N,k) - L(N,k)\right| \geq 2\sqrt{Nk}(\varepsilon - \varepsilon_1)\right\}$$

where $0 < \varepsilon_1 < \varepsilon$. Clearly,

$$\mathbb{P}[A] \leq \mathbb{P}\left[L(N,k) \geq 2\sqrt{Nk}(1+\varepsilon_1)\right] + \mathbb{P}[B] \tag{2.1}$$

and

$$\mathbb{P}[A] \geq \mathbb{P}\left[L(N,k) \geq 2\sqrt{Nk}(1+2\varepsilon-\varepsilon_1)\right] - \mathbb{P}[B]. \tag{2.2}$$

Moreover, for every $\eta > 0$,

$$\mathbb{P}\left[L(N,k) \geq 2\sqrt{Nk}(1+\eta)\right] = \mathbb{P}\left[L(1,k) \geq 2\sqrt{k}(1+\eta)\right] \tag{2.3}$$

as a consequence of the Brownian scaling $\sqrt{N}L(1,k) \overset{d}{=} L(N,k)$. To evaluate $\mathbb{P}[B]$, we couple $G(N,k)$ and $L(N,k)$ by letting $X_i^{(j)} = B_i^{(j)} - B_{i-1}^{(j)}$ for all $i,j \geq 1$ so that the sequence $(X_i^{(j)})_{i,j=1}^{\infty}$ is *i.i.d.* with standard normal distribution. When comparing $G(N,k)$ and $L(N,k)$, it is obvious that most of the variables will vanish.

More precisely, repeating the computation done by Bodineau and Martin in Section 2 of [5], we get, by letting $B_{-1}^{(r)} = 0$,

$$
\begin{aligned}
\big|G(N,k) - L(N,k)\big| &= \left| \sup_{u \in U(N,k)} \sum_{r=1}^{k} \left[S_{\lfloor u_r \rfloor}^{(r)} - S_{\lfloor u_{r-1} \rfloor - 1}^{(r)} \right] - \sup_{u' \in U(N,k)} \sum_{r=1}^{k} \left[B_{u'_r}^{(r)} - B_{u'_{r-1}}^{(r)} \right] \right| \\
&\le \sup_{u \in U(N,k)} \sum_{r=1}^{k} \Big[\Big| S_{\lfloor u_r \rfloor}^{(r)} - B_{\lfloor u_r \rfloor}^{(r)} \Big| + \Big| S_{\lfloor u_{r-1} \rfloor - 1}^{(r)} - B_{\lfloor u_{r-1} \rfloor - 1}^{(r)} \Big| \\
&\quad + \Big| B_{\lfloor u_r \rfloor}^{(r)} - B_{u_r}^{(r)} \Big| + \Big| B_{\lfloor u_{r-1} \rfloor - 1}^{(r)} - B_{u_{r-1}}^{(r)} \Big| \Big] \\
&\le 2 \sum_{r=1}^{k} \left(\max_{i=1,\dots,N} \Big| S_i^{(r)} - B_i^{(r)} \Big| \right) + 2 \sum_{r=1}^{k} \left(\sup_{\substack{0 \le s,t \le N \\ |s-t| < 2}} \Big| B_s^{(r)} - B_t^{(r)} \Big| \right).
\end{aligned} \tag{2.4}
$$

Denote by Y_k and Z_k respectively the two terms on the right-hand side of the last line in (2.4). Then we have,

$$
\begin{aligned}
\mathbb{P}[B] &\le \mathbb{P}\Big[Y_k + Z_k \ge 2\sqrt{Nk}(\varepsilon - \varepsilon_1) \Big] \\
&\le \mathbb{P}\Big[Y_k \ge \sqrt{Nk}(\varepsilon - \varepsilon_1) \Big] + \mathbb{P}\Big[Z_k \ge \sqrt{Nk}(\varepsilon - \varepsilon_1) \Big].
\end{aligned} \tag{2.5}
$$

Of course, $Y_k = 0$ in this Gaussian example, but (2.5) will be used below for more general variables. Applying the Markov inequality gives for all $\lambda > 0$,

$$
\begin{aligned}
\mathbb{P}\Big[Z_k \ge \sqrt{Nk}(\varepsilon - \varepsilon_1) \Big] &\le \mathbb{E}\big[\exp\big(\lambda Z_k^2\big) \big] \cdot \exp\big(-\lambda (\varepsilon - \varepsilon_1)^2 Nk \big) \\
&\le \mathbb{E}\left[\exp\left(4\lambda k \left(\sup_{\substack{0 \le s,t \le N \\ |s-t| < 2}} \Big| B_s^{(1)} - B_t^{(1)} \Big| \right)^2 \right) \right]^k \exp\big(-\lambda (\varepsilon - \varepsilon_1)^2 Nk \big) \\
&\le \left(\int_0^\infty 8\lambda k u \, \exp\big(4\lambda k u^2\big) \, \mathbb{P}\left[\sup_{\substack{0 \le s,t \le N \\ |s-t| < 2}} \Big| B_s^{(1)} - B_t^{(1)} \Big| \ge u \right] \mathrm{d}u \right)^k \exp\big(-\lambda (\varepsilon - \varepsilon_1)^2 Nk \big).
\end{aligned} \tag{2.6}
$$

However, for every $u \ge 0$,

$$
\begin{aligned}
\mathbb{P}\left[\sup_{\substack{0 \le s,t \le N \\ |s-t| < 2}} |B_s^{(1)} - B_t^{(1)}| \ge u \right] &\le \sum_{i=0}^{N-3} \mathbb{P}\left[\sup_{i \le t \le i+3} B_t - \inf_{i \le t \le i+3} B_t \ge u \right] \\
&\le N \mathbb{P}\left[\sup_{0 \le t \le 3} |B_t| \ge u/2 \right].
\end{aligned}
$$

By the Brownian motion reflection principle (see for example [24]), $\sup_{0 \le t \le a} B_t \overset{d}{=} |B_a|$. Thus,

$$
\begin{aligned}
\mathbb{P}\left[\sup_{\substack{0 \le s,t \le N \\ |s-t| < 2}} \Big| B_s^{(1)} - B_t^{(1)} \Big| \ge u \right] &\le C_1 N \mathbb{P}\Big[B_3 \ge u/2 \Big] \\
&\le C_1 N \exp\left(-\frac{u^2}{C_1} \right).
\end{aligned} \tag{2.7}
$$

where C_1 is a numerical positive constant. Now, insert (2.7) in the integral in (2.6) and choose $\lambda = \frac{c}{k}$ were c is a positive constant smaller than $\frac{2}{C_1}$. Then we get for some constant $C_2 > 0$,

$$\mathbb{P}\Big[Z_k \geq \sqrt{Nk}(\varepsilon - \varepsilon_1)\Big] \leq C_2 \exp\left(\frac{k \log N}{C_2}\right) \exp\left(-\frac{(\varepsilon - \varepsilon_1)^2 N}{C_2}\right). \tag{2.8}$$

The last bound leads to the condition $k = \mathrm{o}(N/\log N)$. Combining (2.1), (2.2), (2.3) and (2.8) then leads to

$$\mathbb{P}[A] \leq \mathbb{P}\Big[L(1,k) \geq 2\sqrt{k}(1+\varepsilon_1)\Big] + C_2 \exp\left(-\frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_2}\right) \tag{2.9}$$

and

$$\mathbb{P}[A] \geq \mathbb{P}\Big[L(1,k) \geq 2\sqrt{k}(1+2\varepsilon-\varepsilon_1)\Big] - C_2 \exp\left(-\frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_2}\right). \tag{2.10}$$

Dividing (2.9) by $\mathrm{e}^{-kJ_{\mathrm{GUE}}(\varepsilon_1)}$ and (2.10) by $\mathrm{e}^{-kJ_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}$, taking their logarithm and then dividing the results by k, we get for $\alpha < \frac{1}{2}$,

$$\frac{1}{k}\log\left(\frac{\mathbb{P}[A]}{\mathrm{e}^{-kJ_{\mathrm{GUE}}(\varepsilon_1)}}\right) \leq \frac{1}{k}\log\left(\frac{\mathbb{P}[L(1,k) \geq 2\sqrt{k}(1+\varepsilon_1)]}{\mathrm{e}^{-kJ_{\mathrm{GUE}}(\varepsilon_1)}} + g_k(\varepsilon_1, \varepsilon)\right) \tag{2.11}$$

and

$$\frac{1}{k}\log\left(\frac{\mathbb{P}[A]}{\mathrm{e}^{-kJ_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}}\right) \geq \frac{1}{k}\log\left(\frac{\mathbb{P}[L(1,k) \geq 2\sqrt{k}(1+2\varepsilon-\varepsilon_1)]}{\mathrm{e}^{-kJ_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}} - g'_k(\varepsilon_1, \varepsilon)\right) \tag{2.12}$$

where

$$g_k(\varepsilon_1,\varepsilon) = C_2 \exp\left(-\frac{(\varepsilon-\varepsilon_1)^2 N}{C_2} + k\left(\frac{\log N}{C_2} + J_{\mathrm{GUE}}(\varepsilon_1)\right)\right)$$

and

$$g'_k(\varepsilon_1,\varepsilon) = C_2 \exp\left(-\frac{(\varepsilon-\varepsilon_1)^2 N}{C_2} + k\left(\frac{\log N}{C_2} + J_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)\right)\right),$$

are two positive functions. Moreover, for k large enough, $g_k(\varepsilon_1,\varepsilon)$ and $g'_k(\varepsilon_1,\varepsilon)$ are negligible with respect to $\mathrm{e}^{-\eta k}$ for every $\eta > 0$ since $k = \mathrm{o}(N/\log N)$. Thus, using (1.3), a straightforward computation shows that the right-hand sides of (2.11) and (2.12) both converge to zero when $k \to \infty$. In other words, for $k = \mathrm{o}(N/\log N)$ and $\varepsilon_1 < \varepsilon$,

$$\limsup_{k\to\infty} \frac{1}{k}\log \mathbb{P}[A] \leq -J_{\mathrm{GUE}}(\varepsilon_1) \tag{2.13}$$

and

$$\liminf_{k\to\infty} \frac{1}{k}\log \mathbb{P}[A] \geq -J_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1). \tag{2.14}$$

Finally, notice that $J_{\mathrm{GUE}}(\varepsilon)$ is a continuous function of $\varepsilon > 0$. It therefore follows from (2.13) and (2.14) that for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k}\log \mathbb{P}\Big[G(N,k) \geq 2\sqrt{Nk}(1+\varepsilon)\Big] = -J_{\mathrm{GUE}}(\varepsilon).$$

The proof of the large deviation formula on the left of the mean is similar. Set now, for all $\varepsilon > 0$ and $\varepsilon_1 < \varepsilon$,

$$E = \Big\{G(N,k) \leq 2\sqrt{Nk}(1-\varepsilon)\Big\}.$$

By the same arguments as before, we get

$$\mathbb{P}[E] \leq \mathbb{P}\Big[L(N,k) \leq 2\sqrt{Nk}(1-\varepsilon_1)\Big] + \mathbb{P}[B] \tag{2.15}$$

and

$$\mathbb{P}[E] \geq \mathbb{P}\Big[L(N,k) \leq 2\sqrt{Nk}(1-2\varepsilon+\varepsilon_1)\Big] - \mathbb{P}[B]. \tag{2.16}$$

Furthermore, by (1.4),

$$\lim_{k\to\infty} \frac{1}{k^2} \log\left(\frac{\mathbb{P}[L(1,k) \leq 2\sqrt{k}(1-\varepsilon_1)]}{\mathrm{e}^{-k^2 I_{\mathrm{GUE}}(\varepsilon_1)}}\right) = 0 \tag{2.17}$$

and

$$\lim_{k\to\infty} \frac{1}{k^2} \log\left(\frac{\mathbb{P}[L(1,k) \leq 2\sqrt{k}(1-2\varepsilon+\varepsilon_1)]}{\mathrm{e}^{-k^2 I_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}}\right) = 0. \tag{2.18}$$

Using the same upper bound on $\mathbb{P}[B]$ and combining (2.15), (2.16), (2.17) and (2.18), one can easily deduce that, for $k = \mathrm{o}(N^{1/2})$ and $\varepsilon_1 < \varepsilon$,

$$\limsup_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}[E] \leq -I_{\mathrm{GUE}}(\varepsilon_1)$$

and

$$\liminf_{k\to\infty} \frac{1}{k^2} \log \mathbb{P}[E] \geq -I_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1).$$

At this stage, let us assume that $I_{\mathrm{GUE}}(\varepsilon)$ is a continuous function of ε. Then, for $k = \mathrm{o}(N^{1/2})$,

$$\lim_{N\to\infty} \frac{1}{k^2}\ \log \mathbb{P}\Big[G(N,k) \leq 2\sqrt{Nk}(1-\varepsilon)\Big] = -I_{\mathrm{GUE}}(\varepsilon),$$

which is the result.

We are left with the proof of the continuity of $I_{\mathrm{GUE}}(\varepsilon)$. Set $\mathcal{M}((-\infty,t])$, the set of all probability measures on $(-\infty,t]$ when $t \in \mathbb{R}$. For a given distribution $\mu \in \mathcal{M}((-\infty,t])$, define the corresponding potential energy, as in [25], by

$$I_\mu(t) = 2\int_{-\infty}^{t} x^2 \mathrm{d}\mu(x) - \int_{-\infty}^{t}\int_{-\infty}^{t} \log|x-y| \mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

The minimal energy

$$I(t) = \inf_{\mu\in\mathcal{M}((-\infty,t])} I_\mu(t)$$

precisely allows us to compute the rate function I_{GUE} *via* the formula $I_{\mathrm{GUE}}(\varepsilon) = I(1-\varepsilon) - I(\infty)$. The last equality could be found in [11]. For $t \geq 1$, $I(t)$ is a constant function, the extremal measure is the so-called semi-circular law supported on $[-1,1]$ and the energy $I(t) = \log(2) + 3/4$, (*cf.* [4, 25]). For each $t \in \mathbb{R}$, there is a unique measure $\nu_t \in \mathcal{M}((-\infty,t])$, with no mass point, achieving the infimum (*cf.* [25]). Furthermore, ν_t is compactly supported and the corresponding energy is finite. Since $I(t)$ is an infimum and a non-increasing function of t, for any $\eta > 0$,

$$I(t) \leq I(t-\eta) \leq \frac{I_{\nu_t}(t-\eta)}{\nu_t^2((-\infty,t-\eta])}\cdot \tag{2.19}$$

It is obvious that the right-hand side of (2.19) converges to $I(t)$ when η converges to zero. This proves the left-continuity of $I(t)$.

To show the right-continuity, notice that by a simple change of variable,

$$I(t) = \inf_{\mu\in\mathcal{M}((-\infty,t+\eta])} I_\mu^\eta(t)$$

where

$$I_\mu^\eta(t) = 2\int_{-\infty}^{t+\eta} (x-\eta)^2 \mathrm{d}\mu(x) - \int_{-\infty}^{t+\eta}\int_{-\infty}^{t+\eta} \log|x-y| \mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Consequently,

$$\begin{aligned} I(t) - I(t+\eta) &\le I^{\eta}_{\nu_{t+\eta}}(t) - I(t+\eta) \\ &\le 2\eta^2 + 4\eta \int_{-\infty}^{t+\eta} |x| \, \mathrm{d}\nu_{t+\eta}(x). \end{aligned} \tag{2.20}$$

For $|x| \ge |y|$, we have $\log|x-y| \le \log|2x|$. Moreover, there is a positive constant C_3 such that $|x| \le C_3(2x^2 - 2\log|2x|)$. In view of (2.20), a straightforward calculation leads to

$$I(t) - I(t+\eta) \le 2\eta^2 + 4\eta\, C_3 I(t). \tag{2.21}$$

Since $I(t)$ is finite, the right-hand side of (2.21) converges to zero when $\eta \to 0$. Thus, the continuity of $I(t)$ is proved, and that of $I_{\mathrm{GUE}}(\varepsilon)$ as well. The proof of Theorem 1.1 is now complete.

3. Exponential-tailed distribution and the KMT approximation

In this section, we replace the standard normal variables with weights having a finite moment-generating function around zero. When comparing $G(N,k)$ to the Brownian last-passage percolation model, Y_k will not vanish as in the Gaussian case where the coupling was "perfect". In the non-Gaussian case, a new coupling is required, and the things become more complicated. Following [5], we make use of the KMT approximation: a powerful tool to couple a partial sum of *i.i.d.* random variables and a Wiener process, both constructed on the same probability space. The KMT approximation was first introduced in 1975 by Komlós, Major and Tusnády in their famous work [17]. The basic version deals with a partial sum of *i.i.d.* random variables reconstructed in a way to be "close" to a partial sum of *i.i.d.* standard normal random variables. Later versions of this strong approximation do not require a common distribution, see [26]. The reader can also consult [9] for a complete survey.

Let $(X_i)_{i\ge1}$ be a sequence of independent random variables and denote by S_N the corresponding partial sum. Let $(B_t)_{t\ge0}$ be a Brownian motion built on the same probability space. The following theorem is an immediate consequence of Theorem 1 in [17].

Theorem 3.1 (Komlós-Major-Tusnády)**.** *Assume that $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Assume further that there exit $\mu_0 > 0$ such that for all $\mu < \mu_0$,*

$$\mathbb{E}\exp\left(\mu|X_1|\right) < +\infty.$$

Then for every $N \ge 1$, the sequence $(X_i)_{i\ge1}$ and the Brownian motion $(B_t)_{t\ge0}$ can be constructed in such a way that for all $x > 0$,

$$\mathbb{P}\Big[\max_{i=1,\dots,N} \big|S_i - B_i\big| > \Theta \log N + x\Big] \le C\exp\left(-\theta x\right).$$

The positive constants Θ, C and θ depend only on the distribution of X_1 and θ can be taken as large as desired by choosing Θ large enough.

We now establish, with this tool, Theorem 1.2. According to the notation and the steps in Section 2, recall from (2.4) that

$$|G(N,k) - L(N,k)| \le Y_k + Z_k.$$

We control Z_k as in (2.8) and we want Y_k to be as small as possible. To this purpose, we construct the sequence $(X_i^{(j)})_{i,j=1}^{\infty}$ and the independent Brownian motions $(B_t^{(r)})_{t\ge0}$ in the sense of Theorem 3.1. By the Markov inequality, we have for all $\varepsilon > 0$, $\varepsilon_1 < \varepsilon$ and $\lambda > 0$,

$$\begin{aligned} \mathbb{P}\Big[Y_k \ge \sqrt{Nk}(\varepsilon - \varepsilon_1)\Big] &\le \mathbb{E}\big[\exp\left(\lambda Y_k\right)\big] \cdot \exp\Big(-\lambda(\varepsilon-\varepsilon_1)\sqrt{Nk}\Big) \\ &\le \mathbb{E}\Big[\exp\Big(2\lambda \max_{i=1,\dots,N}\Big|S_i^{(1)} - B_i^{(1)}\Big|\Big)\Big]^k \exp\Big(-\lambda(\varepsilon-\varepsilon_1)\sqrt{Nk}\Big) \\ &\le \bigg(\int_0^{\infty} 2\lambda\exp\left(2\lambda u\right) \cdot \mathbb{P}\Big[\max_{i=1,\dots,N}\Big|S_i^{(1)} - B_i^{(1)}\Big| \ge u\Big]\mathrm{d}u\bigg)^k \exp\Big(-\lambda(\varepsilon-\varepsilon_1)\sqrt{Nk}\Big). \end{aligned}$$

In order to apply Theorem 3.1, we make the simple variable change $t = s - \Theta \log N$ and we choose $\lambda < \theta/2$. Therefore, there exist two positive constant c_4 and C_4 such that,

$$\mathbb{P}\Big[Y_k \geq \sqrt{Nk}(\varepsilon - \varepsilon_1)\Big] \leq C_4 \exp\left(- \frac{(\varepsilon - \varepsilon_1)\sqrt{Nk} - k \log N}{C_4}\right). \tag{3.1}$$

Now, putting (2.1), (2.2), (2.8) and (3.1) together gives

$$\begin{aligned}\mathbb{P}[A] \leq \mathbb{P}\Big[L(N,k) \geq 2\sqrt{Nk}(1+\varepsilon_1)\Big] + C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_5}\right) \\ + C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)\sqrt{Nk} - k \log N}{C_5}\right)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathbb{P}[A] \geq \mathbb{P}\Big[L(N,k) \geq 2\sqrt{Nk}(1+2\varepsilon-\varepsilon_1)\Big] - C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_5}\right) \\ - C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)\sqrt{Nk} - k \log N}{C_5}\right).\end{aligned} \tag{3.3}$$

On the left of the mean, we have

$$\begin{aligned}\mathbb{P}[E] \leq \mathbb{P}\Big[L(N,k) \leq 2\sqrt{Nk}(1-\varepsilon_1)\Big] + C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_5}\right) \\ + C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)\sqrt{Nk} - k \log N}{C_5}\right)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathbb{P}[E] \geq \mathbb{P}\Big[L(N,k) \leq 2\sqrt{Nk}(1-2\varepsilon+\varepsilon_1)\Big] - C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)^2 N - k \log N}{C_5}\right) \\ - C_5 \exp\left(- \frac{(\varepsilon - \varepsilon_1)\sqrt{Nk} - k \log N}{C_5}\right).\end{aligned} \tag{3.5}$$

Proceeding like in Section 2, we divide (3.2) by $\mathrm{e}^{-kJ_{\mathrm{GUE}}(\varepsilon_1)}$, (3.3) by $\mathrm{e}^{-kJ_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}$, (3.4) by $\mathrm{e}^{-k^2 I_{\mathrm{GUE}}(\varepsilon_1)}$ and (3.5) by $\mathrm{e}^{-k^2 I_{\mathrm{GUE}}(2\varepsilon-\varepsilon_1)}$. To handle the remaining parts when $k \to \infty$, we take k negligible with respect to N. On the right of the mean, we need $k = \mathrm{o}\big(\frac{N}{(\log N)^2}\big)$ and on the left, we take $k = \mathrm{o}(N^{1/3})$. Finally we conclude as in section 2 using the continuity of J_{GUE} and I_{GUE}.

4. Subexponential weights and the Skorokhod embedding

In this section, we consider a particular category of subexponential weights such that $\mathbb{E}\exp(\mu|X|^{\gamma})$ is finite for some $\mu > 0$ and $\gamma \in (0,1)$. We say that these variables have a Weibull-like tails because of the similarity with the right-tail of the Weibull distribution with a shape parameter lower than 1. Such weights are considered to be heavy-tailed and then, do not have finite moment-generating functions. However, $G(N,k)$ still satisfies an LDP principle on a very thin rectangle. The precise definition of a subexponential distribution and large deviations for a partial sum of *i.i.d.* such weights can be found in [22].

We do not know a KMT strong approximation version for random variables satisfying the moment condition above. We therefore follow [1] by using the Skorokhod embedding theorem instead. The Skorokhod embedding theorem is another tool to couple a sum of *i.i.d.* random variables with a Brownian motion, see (*cf.* [6, 27, 28]) for more details concerning this theorem.

Theorem 4.1 (Skorokhod)**.** *Let* $(B_t)_{t\geq 0}$ *be a standard one-dimensional Brownian motion and* X *a real valued random variable satisfying* $\mathbb{E}X = 0$ *and* $\mathbb{E}X^2 = 1$*. Then, there is a stopping time* T *for the Brownian motion such that* $B_T \overset{d}{=} X$ *and* $\mathbb{E}T = 1$*.*

An immediate consequence of this theorem allows to embed sums of real independent random variables into the Brownian motion. Applying the strong Markov property to the Brownian motion, Theorem 4.1 yields the following classical corollary.

Corollary 4.2. *Let* $X_1, X_2, \dots, X_N, \dots$ *be i.i.d. satisfying* $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$ *and set* $S_N = X_1 + X_2 + \dots + X_N$, $N \geq 1$*. There is a sequence of i.i.d. stopping times* $\tau_0 = 0, \tau_1, \dots, \tau_N, \dots$ *such that*

$$S_N \overset{d}{=} B_{\tau_1+\dots+\tau_N}$$

and $(B_{\tau_1+\dots+\tau_{N+1}} - B_{\tau_1+\dots+\tau_N})_{N\geq 0}$ *is a sequence of i.i.d. random variables having the same distribution as* X_1*.*

In our context, an application of the last corollary allows to claim that there exists *i.i.d.* stopping times for the Brownian motion $\tau_0 = 0, \tau_1, \dots, \tau_N, \dots$ such that $\mathbb{E}\tau_1 = 1$ and $S_i^{(r)} \overset{d}{=} B^{(r)}_{\tau_1+\dots+\tau_i}$ for $i \geq 1$ and $r \geq 1$. Consequently, choose

$$X_i^{(r)} = B^{(r)}_{\tau_1+\dots+\tau_i} - B^{(r)}_{\tau_1+\dots+\tau_{i-1}},$$

in order to have

$$S_i^{(r)} = B^{(r)}_{\tau_1+\dots+\tau_i} \qquad a.s.$$

Thus, for any $t \geq 0$,

$$\mathbb{P}\Big[\max_{i=1,\dots,N}\Big|S_i^{(1)} - B_i^{(1)}\Big| \geq t\Big] = \mathbb{P}\left[\max_{i=1,\dots,N}\Big|B^{(1)}_{\tau_1+\dots+\tau_i} - B_i^{(1)}\Big| \geq t; \max_{i=1,\dots,N}\left|\sum_{l=1}^{i}(\tau_l - 1)\right| \geq t^\beta\right]$$
$$+ \mathbb{P}\left[\max_{i=1,\dots,N}\Big|B^{(1)}_{\tau_1+\dots+\tau_i} - B_i^{(1)}\Big| \geq t; \max_{i=1,\dots,N}\left|\sum_{l=1}^{i}(\tau_l - 1)\right| < t^\beta\right].$$

Hence

$$\mathbb{P}\Big[\max_{i=1,\dots,N}\Big|S_i^{(1)} - B_i^{(1)}\Big| \geq t\Big] \leq \mathbb{P}\left[\sup_{\substack{0\leq s,t\leq N\\ |s-t|<N^\beta}}\Big|B_s^{(1)} - B_t^{(1)}\Big| \geq t\right]$$
$$+\mathbb{P}\left[\max_{i=1,\dots,N}\left|\sum_{l=1}^{i}(\tau_l - 1)\right| \geq t^\beta\right]. \tag{4.1}$$

We evaluate each term of (4.1) separately. First,

$$\mathbb{P}\left[\sup_{\substack{0\leq s,t\leq N\\ |s-t|<t^\beta}}\Big|B_s^{(1)} - B_t^{(1)}\Big| \geq t\right] \leq \sum_{i=0}^{N-t^\beta}\mathbb{P}\left[\sup_{i\leq t\leq i+t^\beta+1} B_t - \inf_{i\leq t\leq i+t^\beta+1} B_t \geq t\right]$$
$$\leq N\mathbb{P}\left[\sup_{0\leq t\leq t^\beta+1}|B_t| \geq t/2\right].$$

Applying the reflection principle as in Section 2, we get

$$\mathbb{P}\left[\sup_{\substack{0\leq s,t\leq N\\ |s-t|<t^\beta}}\Big|B_s^{(1)} - B_t^{(1)}\Big| \geq t\right] \leq 4N\mathbb{P}\Big[B_{t^\beta+1} \geq t/2\Big]$$
$$\leq 4N\exp\left(-\frac{t^{2-\beta}}{8}\right). \tag{4.2}$$

To find an upper bound for the second term on the right-hand side of (4.1), we need a connection between the weight moments and those of the stopping times obtained by the Skorokhod embedding. Furthermore, we need to control the sum of the independent stopping times to reach an exponentially decaying inequality. When the weights are bounded for exemple, we can construct a stopping time with finite exponential moments. The sum is then controlled by the Bernstein inequality.

However, when X only satisfies (1.6), the Skorokhod stopping time does not necessarily have a finite exponential moment and thus the Bernstein inequality can not be applied. For example, in [10], Davis found the best universal constant connecting the stopping time moments to those of the stopped Brownian motion. More precisely, if $(B_t)_{t>0}$ is a Brownian motion and τ is a stopping time, then there is a universal constant a_p such that, when $1 < p < \infty$ and $\mathbb{E}\tau^{p/2} < +\infty$,

$$a_p \mathbb{E}\tau^{p/2} \leq \mathbb{E}|B_\tau|^p. \tag{4.3}$$

Moreover, Davis proved that the best constant for $p = 2n$ ($n \in \mathbb{N}^*$) is ${z_{2n}^*}^{2n}$ which is the smallest positive zero of the Hermite polynomial of order $2n$. In [7], this constant is shown to be $O((2n)^{-n})$). So unless $X = B_\tau$ is a bounded variable, τ can not have finite exponential moments.

The constant above is universal but it could be sharpened for some particular stopping times. For example, considering the stopping time of the Skorokhod representation [6, 28], Sawyer improved the constant a_p and established, in [27], the following inequality.

Theorem 4.3. *Let X be a centered random variable such that*

$$\mathbb{E}\exp(\mu|X|^\gamma) < +\infty$$

for some $\gamma > 0$ and $\mu > 0$, and let τ be the corresponding stopping time of the Skorokhod representation. Set $\theta = \frac{\gamma}{2+\gamma}$ and $\nu = \mu^{1-\theta}$. Then,

$$\mathbb{E}\exp(\nu\tau^\theta) \leq \Phi_\gamma\, \mathbb{E}\exp(\mu|X|^\gamma),$$

for some positive constant Φ_γ depending only on γ.

Note that a similar exponential bound may be obtained from (4.3). However the cost is a worse constant μ. Under the assumption (1.6) and in view of Theorem 4.3, the Bernstein inequality can not be applied to the sum of the independent stopping times because $\theta = \gamma/(2+\gamma) < 1$. To avoid this obstacle, we introduce the Fuk-Nagaev inequality [13] which requires less restrictive assumptions.

Theorem 4.4 (Fuk-Nagaev)**.** *Let $X_1, \cdots, X_N$ be a sequence of real i.i.d. random variables satisfying $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = \sigma^2$. Then, for all $x > 0$ and $y > 0$,*

$$\mathbb{P}\Big[\max_{1\leq i\leq N}\Big|\sum_{l=1}^{i} X_l\Big| \geq x\Big] \leq N\mathbb{P}\Big[|X_1| > y\Big] + 2\exp\Big(-\frac{x^2}{2(N\sigma^2 + xy/3)}\Big). \tag{4.4}$$

We refer to [12] for more detail on this inequality. Recall now the second term of the right-hand side of (4.1) and consider the stopping times of the Skorokhod representation. Choosing $x = t^\beta$ and $y = t^\delta$ in Theorem 4.4 and applying Markov inequality to the first term of the right-hand side of (4.4), one has

$$\mathbb{P}\Bigg[\max_{i=1,\ldots,N}\Bigg|\sum_{l=1}^{i}(\tau_l - 1)\Bigg| \geq t^\beta\Bigg] \leq C_6 N\exp\Big(-\frac{\nu t^{\theta\delta}}{C_6}\Big) + C_6\exp\Big(-\frac{t^{2\beta}}{C_6\max\{N, t^{\beta+\delta}\}}\Big). \tag{4.5}$$

Now, we apply Markov inequality to Y_k as in Section 3 and we get for some $\lambda > 0$ and $0 < \eta < 1$,

$$\begin{aligned}\mathbb{P}\Big[Y_k \geq \sqrt{Nk}(\varepsilon - \varepsilon_1)\Big] &\leq \mathbb{E}\big[\exp\big(\lambda Y_k^\eta\big)\big].\exp\big(-\lambda(\varepsilon-\varepsilon_1)^\eta(Nk)^{\eta/2}\big)\\ &\leq \Big(\int_0^\infty 2\eta\lambda u^{\eta-1}\mathrm{e}^{2\lambda u^\eta}\mathbb{P}\Big[\max_{i=1,\ldots,N}\Big|S_i^{(1)} - B_i^{(1)}\Big| \geq u\Big]du\Big)^k \exp\big(-\lambda(\varepsilon-\varepsilon_1)^\eta(Nk)^{\eta/2}\big).\end{aligned} \tag{4.6}$$

Inserting (4.1) and (4.5) in (4.6) and choosing λ very small yield the following constraints on η, β, δ and θ.

$$\begin{cases} \eta < 2-\beta \\ \eta < \beta - \delta \\ \eta < \theta\delta. \end{cases}$$

Straightforward computations lead us to choose $\beta = \frac{4\gamma+4}{3\gamma+2}$ and $\delta = \frac{2\gamma+4}{3\gamma+2}$ since $\theta = \frac{\gamma}{\gamma+2}$. Consequently, we obtain $\eta < \frac{2\gamma}{3\gamma+2}$. To get large deviation asymptotic formulas on the right and the left of the mean, we respectively need $\alpha < \frac{\eta}{2-\eta}$ and $\alpha < \frac{\eta}{4-\eta}$. This completes the proof of Theorem 1.3.

5. Small and large deviations inequalities

Non-asymptotic bound on the right and the left of the mean is an immediate consequence of the corresponding bound for the GUE and the arguments developed in Sections 2 and 3. In particular, we use (*cf.* [19]) that there exists a positive constant C_7 such that, for any $\varepsilon > 0$,

$$\mathbb{P}\Big[L(1,k) \geq 2\sqrt{k}(1+\varepsilon)\Big] \leq \exp\big(-kJ_{\mathrm{GUE}}(\varepsilon)\big) \leq C_7 \exp\left(-\frac{k\max(\varepsilon^2, \varepsilon^{3/2})}{C_7}\right)\cdot \tag{5.1}$$

On the left of the mean, deviation inequalities for the largest eigenvalue of the GUE for a given k are quite more complicated to prove. Ledoux and Rider obtained in a recent paper [20], that the leftmost charge of the largest eigenvalue of a large set of random matrices behaves like the left tail of the corresponding Tracy-Widom law. More precisely, they get for all $0 < \varepsilon \leq 1$,

$$\mathbb{P}\Big[L(1,k) \leq 2\sqrt{k}(1-\varepsilon)\Big] \leq C_7 \exp\left(-\frac{k^2\varepsilon^3}{C_7}\right)\cdot \tag{5.2}$$

As we mentioned before, when $\varepsilon > 1$, we have Gaussian behavior for both left and right tails. This follows from concentration arguments dealing with Lipschitz functions of independent standard normal variables. Once more, two cases will be tackled: standard normal weights and finite exponential moments ones.

5.1. Standard normal variables

We now prove Theorem 1.4. Following the proof of Theorem 1.1 in Section 2, choose $\varepsilon_1 = \frac{\varepsilon}{2}$. Then, combining (2.9) and (5.1), for any $\varepsilon > 0$,

$$\mathbb{P}[A] \leq C_8 \exp\left(-\frac{k\max(\varepsilon^{3/2}, \varepsilon^2)}{C_8}\right) + C_8 \exp\left(-\frac{\varepsilon^2 N - k\log N}{C_8}\right)$$

where $C_8 > 0$. In order to reach (1.10) when $0 < \varepsilon < 1$, we need a positive constant $C(\alpha) > C_8$, depending only on α, such that

$$C_\alpha \exp\left(-\frac{k\varepsilon^{3/2}}{C_\alpha}\right) \geq C_8 \exp\left(-\frac{\varepsilon^2 N - k\log N}{C_8}\right)\cdot \tag{5.3}$$

Taking the logarithm of (5.3), C_α has to satisfy

$$\log\frac{C_8}{C_\alpha} - k\varepsilon^{3/2}\left(\frac{N\varepsilon^{1/2}}{C_8 k} - \frac{\log N}{C_8\varepsilon^{3/2}} - \frac{1}{C_\alpha}\right) \leq 0. \tag{5.4}$$

However, since $\mathbb{P}[A] \leq 1$, ε has to satisfy

$$k\varepsilon^{3/2} \geq 1. \tag{5.5}$$

Combining now (5.4) and (5.5), we finally get that C_α has to satisfy

$$\log\frac{C_8}{C_\alpha}+\frac{1}{C_\alpha}-\frac{N^{1-\frac{4\alpha}{3}}-N^\alpha\log N}{C_8}\leq 0. \tag{5.6}$$

Hence C_α exists and satisfies (5.6) only if $\alpha<\frac{3}{7}$. In that case, we make the reverse computation to conclude that

$$\mathbb{P}\Big[G(N,k)\geq 2\sqrt{Nk}(1+\varepsilon)\Big]\leq 2C_\alpha\exp\left(\frac{-k\varepsilon^{\frac{3}{2}}}{C_\alpha}\right).$$

We make the same computations for the left-tail upper bound. Here, C_α has to satisfy

$$\log\frac{C_8}{C_\alpha}-k^2\varepsilon^2\left(\frac{N}{C_8k^2}-\frac{\log N}{C_8k\varepsilon^2}-\frac{\varepsilon}{C_\alpha}\right)\leq 0,$$

which finally gives

$$\log\frac{C_8}{C_\alpha}+\frac{1}{C_\alpha}-\frac{N^{1-2\alpha}-N^{\alpha/3}\log N}{C_8}\leq 0.$$

This proves Theorem 1.4.

5.2. Finite moment-generating function case

We finally prove Theorem 1.5. Choosing $\varepsilon_1=\frac{\varepsilon}{2}$ in (3.2) and taking into consideration (5.5), the inequalities (5.1) and (5.2) imply that there exists a positive constant C_9 depending on α and the distribution of X such that, for all $\varepsilon>0$,

$$\mathbb{P}[A]\leq C_9\exp\Big(-\tfrac{k\varepsilon^{3/2}}{C_9}\Big)\bigg(1+\exp\Big(-\tfrac{N^{\frac{1}{2}-\frac{\alpha}{6}}-N^\alpha\log N-N^\alpha}{C_9}\Big)+\exp\Big(-\tfrac{N^{1-4\alpha/3}-N^\alpha\log N-N^\alpha}{C_9}\Big)\bigg)$$

and

$$\mathbb{P}[E]\leq C_9\exp\left(-\frac{k^2\varepsilon^3}{C_9}\right)\left(1+\exp\left(-\frac{N^{\frac{1-3\alpha}{2}}-1}{C_9}\right)+\exp\left(-\frac{N^{1-2\alpha}-N^{\frac{\alpha}{3}}-1}{C_9}\right)\right).$$

This means that we have a right-tail bound for $\alpha<3/7$ and a left-tail bound for $\alpha<1/3$. The proof is complete and thus Theorem 1.5 is proved.

Acknowledgements. I would like to thank my advisor Michel Ledoux for many valuable discussions on random growth and random matrices. I also thank Delphine Féral for remarks on the rate functions of the GUE.

References

[1] J. Baik and T.M. Suidan, A GUE central limit theorem and universality of directed first and last passage site percolation. *Int. Math. Res. Not.* (2005) 325–337.

[2] Yu. Baryshnikov, GUEs and queues. *Probab. Theory Relat. Fields* **119** (2001) 256–274.

[3] G. Ben Arous, A. Dembo and A. Guionnet, Aging of spherical spin glasses. *Probab. Theory Relat. Fields* **120** (2001) 1–67.

[4] G. Ben Arous and A. Guionnet, Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Relat. Fields* **108** (1997) 517–542.

[5] T. Bodineau and J. Martin, A universality property for last-passage percolation paths close to the axis. *Electron. Commun. Probab.* **10** (2005) 105–112 (electronic).

[6] L. Breiman, *Probability, Classics in Applied Mathematics* **7**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1992). Corrected reprint of the 1968 original.

[7] D.L. Burkholder, Distribution function inequalities for martingales. *Ann. Probability* **1** (1973) 19–42.

[8] S. Chatterjee, A simple invariance theorem. Preprint `arXiv:math.PR/0508213` (2005).

[9] S. Csörgő and P. Hall, The Komlós-Major-Tusnády approximations and their applications. *Austral. J. Statist.* **26** (1984) 189–218.
[10] B. Davis, On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.* **43** (1976) 697–704.
[11] D. Féral, On large deviations for the spectral measure of discrete coulomb gas, in *Séminaire de Probabilités, XLI. Lecture Notes in Math.* **1934**. Springer, Berlin (2008) 19–50.
[12] D.H. Fuk, Certain probabilistic inequalities for martingales. *Sibirsk. Mat. Ž.* **14** (1973) 185–193, 239.
[13] D.H. Fuk and S.V. Nagaev, Probabilistic inequalities for sums of independent random variables. *Teor. Verojatnost. i Primenen.* **16** (1971) 660–675.
[14] J. Gravner, C.A. Tracy and H. Widom, Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Statist. Phys.* **102** (2001) 1085–1132.
[15] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** (1998) 151–204.
[16] K. Johansson, Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** (2000) 437–476.
[17] J. Komlós, P. Major and G. Tusnády, An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrscheinlichkeitstheor. und Verw. Geb.* **34** (1976) 33–58.
[18] W. König, Orthogonal polynomial ensembles in probability theory. *Prob. Surveys* **2** (2005) 385–447 (electronic).
[19] M. Ledoux, Deviation inequalities on largest eigenvalues, in *Geometric aspects of functional analysis. Lecture Notes in Math.* **1910** (2007) 167–219.
[20] M. Ledoux and B. Rider, Small deviations for beta ensembles. Preprint (2010).
[21] M.L. Mehta, *Random matrices*, 2nd edition. Academic Press Inc., Boston, MA (1991).
[22] T. Mikosch and A.V. Nagaev, Large deviations of heavy-tailed sums with applications in insurance. *Extremes* **1** (1998) 81–110.
[23] N. O'Connell and M. Yor, A representation for non-colliding random walks. *Electron. Commun. Probab.* **7** (2002) 1–12 (electronic).
[24] D.l Revuz and M. Yor, *Continuous martingales and Brownian motion*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **293**, 3rd edition,. Springer-Verlag, Berlin (1999).
[25] E.B. Saff and V. Totik, *Logarithmic potentials with external fields*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **316**. Springer-Verlag, Berlin (1997). Appendix B by Thomas Bloom.
[26] A.I. Sakhanenko, A new way to obtain estimates in the invariance principle, in *High dimensional probability, II (Seattle, WA, 1999)*, *Progr. Probab.* **47**. Birkhäuser Boston, Boston, MA (2000) 223–245.
[27] S. Sawyer, A remark on the Skorohod representation. *Z. Wahrscheinlichkeitstheor. und Verw. Geb.* **23** (1972) 67–74.
[28] A.V. Skorokhod, *Studies in the theory of random processes*. Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass (1965).
[29] T. Suidan, A remark on a theorem of Chatterjee and last passage percolation. *J. Phys. A* **39** (2006) 8977–8981.
[30] C.A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel. *Phys. Lett. B* **305** (1993) 115–118.