ESAIM: PS October 2010, Vol. 14, p. 263–270
DOI: 10.1051/ps:2008034 **ESAIM: Probability and Statistics**
www.esaim-ps.org

DIRECTED POLYMER IN RANDOM ENVIRONMENT AND LAST PASSAGE PERCOLATION *

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Abstract. The sequence of random probability measures ν_n that gives a path of length n, $\frac{1}{n}$ times the sum of the random weights collected along the paths, is shown to satisfy a large deviations principle with good rate function the Legendre transform of the free energy of the associated directed polymer in a random environment. Consequences on the asymptotics of the typical number of paths whose collected weight is above a fixed proportion are then drawn.

Mathematics Subject Classification. 60K37.

Received June 12, 2008.

1. Introduction

Last passage percolation

To each site (k, x) of $\mathbb{N} \times \mathbb{Z}^d$ is assigned a random weight $\eta(k, x)$. The $(\eta(k, x))_{k\geq 1, x\in\mathbb{Z}^d}$ are taken IID under the probability measure $\mathbf{Q}$.

The set of oriented paths of length n starting from the origin is

$$\Omega_n = \left\{\omega = (\omega_0, \ldots, \omega_n) : \omega_i \in \mathbb{Z}^d, \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1\right\}.$$

The weight (energy, reward) of a path is the sum of weights of visited sites:

$$H_n = H_n(\omega, \eta) = \sum_{k=1}^{n} \eta(k, \omega_k) \quad (n \geq 1, \omega \in \Omega_n).$$

Observe that when $\eta(k, x)$ are Bernoulli(p) distributed

$$\mathbf{Q}(\eta(k, x) = 1) = 1 - \mathbf{Q}(\eta(k, x) = 0) = p \in (0, 1),$$

the quantity $\frac{H_n}{n}(\omega, \eta)$ is the proportion of *open* sites visited by ω, and it is natural to consider for $p < \rho < 1$,

$$N_n(\rho) = \text{number of paths of length } n \text{ such that } H_n(\omega, \eta) \geq n\rho.$$

Keywords and phrases. Directed polymer, random environment, partition function, last passage percolation.

* *The author acknowledges the support of the French Ministry of Education through the ANR BLAN07-2184264 grant.*

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Article published by EDP Sciences

The problem of ρ-percolation, as we learnt it from Comets et al. [9] and Kesten and Sidoravicius [12], is to study the behaviour of $N_n(\rho)$ for large n.

Directed polymer in a random environment

We are going to consider fairly general environment distributions, by requiring first that they have exponential moments of any order:

$$\lambda(\beta) = \log \mathbf{Q}\Big(\mathrm{e}^{\beta\eta(k,x)}\Big) < +\infty \quad (\beta \in \mathbb{R}),$$

and second that they satisfy a logarithmic Sobolev inequality (see *e.g.* [2]): in particular we can apply our result to bounded support and Gaussian environments.

The polymer measure is the random probability measure defined on the set of oriented paths of length n by:

$$\mu_n(\omega) = (2d)^{-n}\,\frac{\mathrm{e}^{\beta H_n(\omega,\eta)}}{Z_n(\beta)} \qquad (\omega \in \Omega_n),$$

with $Z_n(\beta)$ the partition function

$$Z_n(\beta) = Z_n(\beta,\eta) = (2d)^{-n}\sum_{\omega\in\Omega_n} \mathrm{e}^{\beta H_n(\omega,\eta)} = \mathbf{P}\left(\mathrm{e}^{\beta H_n(\omega,\eta)}\right),$$

where $\mathbf{P}$ is the law of simple random walk on $\mathbb{Z}^d$ starting from the origin.

Bolthausen [3] proved the existence of a deterministic limiting free energy

$$p(\beta) = \lim_{n\to+\infty}\frac{1}{n}\mathbf{Q}(\log Z_n(\beta)) = \mathbf{Q}\,a.s.\lim_{n\to+\infty}\frac{1}{n}\log Z_n(\beta).$$

Thanks to Jensen's inequality, we have the upper bound $p(\beta) \le \lambda(\beta)$ and it is conjectured (and partially proved, see [6,7]) that the behaviour of a typical path under the polymer measure is diffusive iff $\beta \in \mathcal{C}_\eta$ the critical region

$$\mathcal{C}_\eta = \{\beta \in \mathbb{R} : p(\beta) = \lambda(\beta)\}.$$

In dimension $d = 1$, $\mathcal{C}_\eta = \{0\}$ and in dimensions $d \ge 3$, $\mathcal{C}_\eta$ contains a neighborhood of the origin (see [3,8]).

The main theorem

The connection between Last passage percolation and Directed polymer in random environment is made by the family $(\nu_n)_{n\in\mathbb{N}}$ of random probability measures on the real line:

$$\nu_n(A) = \frac{1}{|\Omega_n|}\sum_{\omega\in\Omega_n}\mathbf{1}_{(\frac{H_n}{n}(\omega,\eta)\in A)} = \mathbf{P}\left(\frac{H_n}{n}(\omega,\eta) \in A\right).$$

Indeed,

$$N_n(\rho) = \sum_{\omega\in\Omega_n}\mathbf{1}_{(H_n(\omega,\eta)\ge n\rho)} = (2d)^n\nu_n([\rho,+\infty)).$$

The main result of the paper is

Theorem 1.1. $\mathbf{Q}$ *almost surely, the family* $(\nu_n)_{n\in\mathbb{N}}$ *satisfies a large deviations principle with good rate function* $I = p^*$ *the Legendre transform of the free energy of the directed polymer.*

Let $m = \mathbf{Q}(\eta(k,x))$ be the average weight of a path $m = \mathbf{Q}\big(\frac{H_n}{n}(\omega,\eta)\big)$. It is natural to consider the quantities:

$$N_n(\rho) = \begin{cases}\sum_{\omega\in\Omega_n}\mathbf{1}_{(H_n(\omega,\eta)\ge n\rho)} & \text{if } \rho \ge m,\\ \sum_{\omega\in\Omega_n}\mathbf{1}_{(H_n(\omega,\eta)\le n\rho)} & \text{if } \rho < m.\end{cases}$$

A simple exchange of limits $\beta \to \pm\infty$, and $n \to +\infty$, yields the following

$$\rho^{\pm} = \mathbf{Q}\, a.s. \lim_{n\to+\infty} \max_{\omega\in\Omega_n} \pm \frac{H_n}{n}(\omega,\eta) = \lim_{\beta\to+\infty} \frac{p(\pm\beta)}{\beta} \in [0,+\infty].$$

Repeating the proof of Theorem 1.1 of [9] gives

Corollary 1.2. *For $-\rho^- < \rho < \rho^+$, we have $\mathbf{Q}$ almost surely,*

$$\lim_{n\to+\infty} (N_n(\rho))^{\frac{1}{n}} = (2d)\mathrm{e}^{-I(\rho)}.$$

We can then translate our knowledge of the critical region $\mathcal{C}_\eta$, into the following remark. Let

$$\mathcal{V}_\eta = \{\rho \in \mathbb{R} : I(\rho) = \lambda^*(\rho)\}\,.$$

In dimension $d = 1$, $\mathcal{V}_\eta = \{m\}$ and in dimensions $d \geq 3$, $\mathcal{V}_\eta$ contains a neighbourhood of m.

This means that in dimensions $d \geq 3$, the typical large deviation of $\frac{H_n}{n}(\omega,\eta)$ close to its mean is the same as the large deviation of $\frac{1}{n}(\eta_1 + \cdots + \eta_n)$ close to its mean, with η_i IID. There is no influence of the path ω: this gives another justification to the name weak-disorder region given to the critical set $\mathcal{C}_\eta$.

2. Proof of the main theorem

Observe that for any $\beta \in \mathbb{R}$ we have:

$$\int \mathrm{e}^{\beta n x}\mathrm{d}\nu_n(x) = \mathbf{P}\left(\mathrm{e}^{\beta H_n(\omega,\eta)}\right) = Z_n(\beta) \quad \mathbf{Q}\, a.s. \tag{2.1}$$

Consequently, since $\mathrm{e}^u + \mathrm{e}^{-u} \geq \mathrm{e}^{|u|}$, we obtain for any $\beta > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log\left(\int \mathrm{e}^{\beta n|x|}\mathrm{d}\nu_n(x)\right) \leq p(\beta) + p(-\beta) < +\infty,$$

and the family $(\nu_n)_{n\geq 0}$ is exponentially tight (see the Appendix, Lem. A.1). We only need to show now that for a lower semicontinuous function I, and for $x \in \mathbb{R}$

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \nu_n((x-\delta, x+\delta)) = -I(x), \tag{2.2}$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \nu_n([x-\delta, x+\delta]) = -I(x). \tag{2.3}$$

From these, we shall infer that $(\nu_n)_{n\in\mathbb{N}}$ follows a large deviations principle with good rate function I. Eventually, equation (2.1) and

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) = p(\beta)$$

will imply, by Varadhan's lemma that I and p are Legendre conjugate:

$$I(x) = p^*(x) = \sup_{\beta\in\mathbb{R}} (x\beta - p(\beta)).$$

The strategy of proof finds its origin in Varadhan's seminal paper [13], and has already successfully been applied in [5]. Let us define for $\lambda > 0, x \in \mathbb{Z}, a \in \mathbb{R}$

$$V_n^{(\lambda)}(x,a;\eta) = \log \mathbf{P}^x\left(\mathrm{e}^{-\lambda|H_n(\omega,\eta)-a|}\right) = V^{(\lambda)}(0,a;\tau_{o,x}\circ\eta),$$

with $\tau_{k,x}$ the translation operator on the environment defined by:

$$\tau_{k,x} \circ \eta(i,y) = \eta(k+i, x+y),$$

and $\mathbf{P}^x$ the law of simple random walk starting from x.

Step 1. The functions $v_n^{(\lambda)}(a) = \mathbf{Q}\big(V^{(\lambda)}(0,a;\eta)\big)$ satisfy the inequality

$$v_{n+m}^{(\lambda)}(a+b) \geq v_n^{(\lambda)}(a) + v_m^{(\lambda)}(b) \qquad (n,m \in \mathbb{N};\ a,b \in \mathbb{R}). \tag{2.4}$$

Proof. Since $|H_{n+m} - (a+b)| \leq |H_n - b| + |(H_{n+m} - H_n) - a|$ we have

$$\begin{aligned}
V_{n+m}^{(\lambda)}(x, a+b;\eta) &\geq \log \mathbf{P}^x\left(\mathrm{e}^{-\lambda|H_n-b|}\mathrm{e}^{-\lambda|(H_{n+m}-H_n)-a|}\right)\\
&= \log \mathbf{P}^x\left(\mathrm{e}^{-\lambda|H_n-b|}\mathrm{e}^{V_m^{(\lambda)}(0,a;\tau_{n,S_n}\circ\eta)}\right)\\
&= \log \sum_y \mathbf{P}^x\left(\mathrm{e}^{-\lambda|H_n-b|}\,\mathbf{1}_{(S_n=y)}\right)\mathrm{e}^{V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)}\\
&= V_n^{(\lambda)}(x,b;\eta) + \log\left(\sum_y \sigma_n(y)\mathrm{e}^{V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)}\right)\\
&\geq V_n^{(\lambda)}(x,b;\eta) + \sum_y \sigma_n(y) V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta) \qquad \text{(Jensen's inequality)},
\end{aligned}$$

with σ_n the probability measure on $\mathbb{Z}^d$:

$$\sigma_n(y) = \frac{1}{V_n^{(\lambda)}(x,b;\eta)}\mathbf{P}^x\left(\mathrm{e}^{-\lambda|H_n-b|}\,\mathbf{1}_{(S_n=y)}\right) \quad (y \in \mathbb{Z}^d).$$

Observe that the random variables $\sigma_n(y)$ are measurable with respect to the sigma field $\mathcal{G}_n = \sigma(\eta(i,x) : i \leq n, x \in \mathbb{Z}^d)$, whereas the random variables $V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)$ are independent from $\mathcal{G}_n$. Hence, by stationarity,

$$\begin{aligned}
v_{n+m}^{(\lambda)}(a+b) &= \mathbf{Q}\Big(V_{n+m}^{(\lambda)}(0,a+b;\eta)\Big)\\
&\geq v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y))\mathbf{Q}\Big(V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)\Big)\\
&= v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y)) v_m^{(\lambda)}(a)\\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a)\mathbf{Q}\left(\sum_y \sigma_n(y)\right)\\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a).
\end{aligned}$$

□

Step 2. There exists a function $I^{(\lambda)} : \mathbb{R} \to \mathbb{R}^+$ convex, non negative, Lipschitz with constant λ, such that

$$-\lim_{n\to\infty} \frac{1}{n} v_n^{(\lambda)}(a_n) = I^{(\lambda)}(\xi) \qquad \left(\text{if } \frac{a_n}{n} \to \xi \in \mathbb{R}\right). \tag{2.5}$$

Proof. From $|H_n - a| \leq |H_n - b| + |a - b|$ we infer that

$$V_n^{(\lambda)}(0,a;\eta) \geq V_n^{(\lambda)}(0,b;\eta) - \lambda|a-b|.$$

Therefore the functions $v_n^{(\lambda)}$ are all Lipschitz continuous with the same constant λ and we combine this fact with standard subadditivity arguments (see *e.g.* Varadhan [13] or Alexander [1]). For sake of completeness, we give a detailed proof in the Appendix Lemma A.2. □

Step 3. $\mathbf{Q}$ almost surely, for any $\xi \in \mathbb{R}$, if $\frac{a_n}{n} \to \xi$, then

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbf{P}\left(\mathrm{e}^{-\lambda|H_n - a_n|}\right) = I^{(\lambda)}(\xi). \tag{2.6}$$

Proof. Since the functions are Lipschitz, it is enough to prove that for any fixed $\xi \in \mathbb{Q}$, (2.6) holds a.s. This is where we use the restrictive assumptions made on the distribution of the environment. If the distribution of η is with bounded support, or Gaussian, or more generally satisfies a logarithmic Sobolev inequality with constant $c > 0$, then it has the Gaussian concentration of measure property (see [2]): for any 1-Lipschitz function F of independent random variables distributed as η,

$$\mathbf{P}\left(|F - \mathbf{P}(F)| \geq r\right) \leq 2\mathrm{e}^{-r^2/c} \quad (r > 0).$$

It is easy to prove, as in Proposition 1.4 of [4], that the function

$$(\eta(k,x), k \leq n, |x| \leq n) \to \log \mathbf{P}\left(\mathrm{e}^{-\lambda|H_n(\omega,\eta) - a|}\right)$$

is Lipschitz, with respect to the Euclidean norm, with Lipschitz constant at most $\lambda\sqrt{n}$. Therefore, the Gaussian concentration of measure yields

$$\mathbf{Q}\left(\left|V_n^{(\lambda)}(0,a;\eta) - v_n^{(\lambda)}(a)\right| \geq u\right) \leq 2\mathrm{e}^{-\frac{\lambda^2 u^2}{cn}}.$$

We conclude by a Borel Cantelli argument combined with (2.5). □

Observe that for fixed $\xi \in \mathbb{R}$, the function $\lambda \to I^{(\lambda)}(\xi)$ is increasing; we shall consider the limit:

$$I(\xi) = \lim_{\lambda\uparrow+\infty} \uparrow I^{(\lambda)}(\xi)$$

which is by construction non negative, convex and lower semi continuous.

Step 4. The function I satisfy (2.2) and (2.3).

Proof. Given, $\xi \in \mathbb{R}$ and $\lambda > 0, \delta > 0$, we have

$$\mathbf{P}\left(\left|\frac{H_n}{n}(\omega,\eta) - \xi\right| \leq \delta\right) = \mathbf{P}\left(\mathrm{e}^{-\lambda n\left|\frac{H_n}{n}(\omega,\eta) - \xi\right|} \geq \mathrm{e}^{-\lambda n\delta}\right) \leq \mathrm{e}^{\lambda n\delta}\mathbf{P}\left(\mathrm{e}^{-\lambda|H_n - n\xi|}\right).$$

Therefore,

$$\limsup \frac{1}{n} \log \nu_n([\xi-\delta, \xi+\delta]) \leq \lambda\delta - I^{(\lambda)}(\xi)$$
$$\limsup_{\delta\to 0} \limsup \frac{1}{n} \log \nu_n([\xi-\delta, \xi+\delta]) \leq -I^{(\lambda)}(\xi)$$

and we obtain by letting $\lambda \to +\infty$,

$$\limsup_{\delta\to 0} \limsup \frac{1}{n} \log \nu_n([\xi-\delta, \xi+\delta]) \leq -I(\xi).$$

Given $\xi \in \mathbb{R}$ such that $I(\xi) < +\infty$, and $\delta > 0$, we have for $\lambda > 0$,

$$\mathbf{P}\left(\left|\frac{H_n}{n} - \xi\right| < \delta\right) \geq \mathbf{P}\left(\mathrm{e}^{-\lambda|H_n - n\xi|}\right) - \mathrm{e}^{-\lambda\delta n}.$$

Hence, if we choose $\lambda > 0$ large enough such that $\lambda\delta > I(\xi) \geq I^{(\lambda)}(\xi)$, we obtain

$$\liminf_{n\to+\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I^{(\lambda)}(\xi) \geq -I(\xi)$$

and therefore

$$\liminf_{\delta\to 0} \liminf_{n\to+\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I(\xi). \qquad \square$$

Appendix A

Exponential tightness plays the same role in Large Deviations theory as tightness in weak convergence theory; in particular it implies that the Large Deviations Property holds along a subsequence with a good rate function (see Thm. 3.7 of Feng and Kurtz [11], or Lem. 4.1.23 of Dembo and Zeitouni [10]). Therefore, once exponential tightness is established, we only need to identify the rate function: the Weak Large Deviations Property implies the Large Deviations Property with a good rate function (see Dembo and Zeitouni [10], Lem. 1.2.18). Our strategy of proof is then clear. First we establish exponential tightness, by applying the following lemma to the probability ν_n and the Lyapunov function $x \to |x|$, then we prove a Weak Large Deviations Property by checking that the limits (2.2) and (2.3) hold.

Lemma A.1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on a Polish space X. Assume that there exists a (Lyapunov) function $F : X \to \mathbb{R}_+$ such that the level sets $\{F \leq C\}_{C>0}$ are compacts, and*

$$\limsup_{n\to+\infty} \frac{1}{n} \log\left(\int \mathrm{e}^{nF(x)} \mathrm{d}\mu_n(x)\right) < +\infty.$$

The $(\mu_n)_{n\in\mathbb{N}}$ is exponentially tight, i.e. for each $A > 0$, there exists a compact K_A such that:

$$\limsup_{n\to+\infty} \log \mu_n(K_A^C) \leq -A.$$

Proof. Let $M = \limsup_{n\to+\infty} \frac{1}{n} \log\left(\int \mathrm{e}^{nF(x)} \mathrm{d}\mu_n(x)\right)$. There exists n_0 such that for $n \geq n_0$,

$$\frac{1}{n} \log\left(\int \mathrm{e}^{nF(x)} \mathrm{d}\mu_n(x)\right) \leq 2M.$$

Thanks to Markov inequality, for $C > 0$ and $n \geq n_0$,

$$\mu_n(F > C) = \mu_n(\mathrm{e}^{nF} > \mathrm{e}^{nC}) \leq \mathrm{e}^{-nC} \int \mathrm{e}^{nf} \mathrm{d}\mu_n \leq \mathrm{e}^{-n(C-2M)}.$$

Hence, if $C > 2M + A$, then for the compact set $K_A = \{F \leq C\}$, and $n \geq n_0$,

$$\frac{1}{n} \log \mu_n(K_A^C) \leq -(C - 2M) < -A. \qquad \square$$

In Step 2 of the proof of the main theorem, we apply the following lemma to the family of functions $u_n = -v_n^{(\lambda)}$.

Lemma A.2. *Assume that the non negative functions* $u_n : \mathbb{R} \to \mathbb{R}_+$ *are Lipschitz with the same constant* $C > 0$*, that is*

$$\forall n, x, y, \quad |u_n(x) - u_n(y)| \leq C|x - y| \, .$$

Assume furthermore the subadditivity:

$$\forall x, y, n, m, \quad u_{n+m}(x + y) \leq u_n(x) + u_m(y) \, .$$

Then there exists a non negative function $I : \mathbb{R} \to \mathbb{R}_+$*, Lipschitz with constant* C*, that satisfies:*

(i) if $\frac{a_n}{n} \to x$*, then* $\frac{1}{n} u_n(a_n) \to I(x)$.

(ii) I *is convex.*

Proof. For fixed $x \in \mathbb{R}$, the sequence $z_n = u_n(nx)$ is subadditive and non negative:

$$z_{n+m} \leq z_n + z_m \, .$$

Therefore, by the standard subadditive theorem for sequences of real numbers, we can consider the limit

$$I(x) = \inf_{n \geq 1} \frac{1}{n} z_n = \lim_{n \to +\infty} \frac{1}{n} z_n = \lim_{n \to +\infty} \frac{1}{n} u_n(nx) \, .$$

If we take limits in the inequality

$$\left| \frac{1}{n} u_n(nx) - \frac{1}{n} u_n(ny) \right| \leq C|x - y|$$

we obtain $|I(x) - I(y)| \leq C|x - y|$.

(i) Assume $\frac{a_n}{n} \to x$, then

$$\left| \frac{1}{n} u_n(nx) - \frac{1}{n} u_n(a_n) \right| \leq C \left| x - \frac{a_n}{n} \right| \to 0.$$

Hence, $\frac{1}{n} u_n(a_n) \to I(x)$.

(ii) We have, $\lfloor y \rfloor$ denoting the integer part of the real number y, for any x, y and $0 < t < 1$,

$$u_{\lfloor tn \rfloor + \lfloor (1-t)n \rfloor}(ntx + n(1-t)y) \leq u_{\lfloor tn \rfloor}(ntx) + u_{\lfloor (1-t)n \rfloor}(n(1-t)y).$$

Since $\frac{1}{n}(\lfloor tn \rfloor + \lfloor (1-t)n \rfloor) \to 1$, we have by (i)

$$\frac{1}{n} u_{\lfloor tn \rfloor + \lfloor (1-t)n \rfloor}(ntx + n(1-t)y) \to I(tx + (1-t)y).$$

Furthermore, since $\frac{1}{n} \lfloor tn \rfloor \to t$, we have by (i),

$$\frac{1}{n} u_{\lfloor tn \rfloor}(ntx) \to tI(x)$$

and similarly,

$$\frac{1}{n} u_{\lfloor (1-t)n \rfloor}(n(1-t)y) \to (1-t)I(y).$$

Combining these limits with the preceding inequality yields,

$$I(tx + (1-t)y) \leq tI(x) + (1-t)I(y)$$

that is I is convex. □

Acknowledgements. The author wants to thank an anonymous referee for suggesting useful improvements to the original manuscript.

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