DIRECTED POLYMER IN RANDOM ENVIRONMENT AND LAST PASSAGE PERCOLATION∗

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Abstract. The sequence of random probability measures \( \nu_n \) that gives a path of length \( n \), \( \frac{1}{n} \) times the sum of the random weights collected along the paths, is shown to satisfy a large deviations principle with good rate function the Legendre transform of the free energy of the associated directed polymer in a random environment. Consequences on the asymptotics of the typical number of paths whose collected weight is above a fixed proportion are then drawn.

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1. Introduction

Last passage percolation

To each site \((k,x)\) of \(\mathbb{N} \times \mathbb{Z}^d\) is assigned a random weight \(\eta(k,x)\). The \((\eta(k,x))_{k \geq 1, x \in \mathbb{Z}^d}\) are taken IID under the probability measure \(Q\).

The set of oriented paths of length \(n\) starting from the origin is

\[
\Omega_n = \{ \omega = (\omega_0, \ldots, \omega_n) : \omega_i \in \mathbb{Z}^d, \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1 \}.
\]

The weight (energy, reward) of a path is the sum of weights of visited sites:

\[
H_n = H_n(\omega, \eta) = \sum_{k=1}^{n} \eta(k, \omega_k) \quad (n \geq 1, \omega \in \Omega_n).
\]

Observe that when \(\eta(k,x)\) are Bernoulli\((p)\) distributed

\[
Q(\eta(k,x) = 1) = 1 - Q(\eta(k,x) = 0) = p \in (0,1),
\]

the quantity \(\frac{H_n}{n}(\omega, \eta)\) is the proportion of open sites visited by \(\omega\), and it is natural to consider for \(p < \rho < 1\),

\[
N_n(\rho) = \text{number of paths of length } n \text{ such that } H_n(\omega, \eta) \geq n \rho.
\]

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The problem of $\rho$-percolation, as we learnt it from Comets et al. [9] and Kesten and Sidoravicius [12], is to study the behaviour of $N_n(\rho)$ for large $n$.

**Directed polymer in a random environment**

We are going to consider fairly general environment distributions, by requiring first that they have exponential moments of any order:

$$\lambda(\beta) = \log Q\left(e^{\beta \eta(k,x)}\right) < +\infty \quad (\beta \in \mathbb{R}),$$

and second that they satisfy a logarithmic Sobolev inequality (see e.g. [2]): in particular we can apply our result to bounded support and Gaussian environments.

The polymer measure is the random probability measure defined on the set of oriented paths of length $n$ by:

$$\mu_n(\omega) = (2d)^{-n} \frac{e^{\beta H_n(\omega,\eta)}}{Z_n(\beta)} \quad (\omega \in \Omega_n),$$

with $Z_n(\beta)$ the partition function

$$Z_n(\beta) = Z_n(\beta, \eta) = (2d)^{-n} \sum_{\omega \in \Omega_n} e^{\beta H_n(\omega,\eta)} = P\left(e^{\beta H_n(\omega,\eta)}\right),$$

where $P$ is the law of simple random walk on $\mathbb{Z}^d$ starting from the origin.

Bolthausen [3] proved the existence of a deterministic limiting free energy

$$p(\beta) = \lim_{n \to +\infty} \frac{1}{n} Q(\log Z_n(\beta)) = Q \text{ a.s.} \lim_{n \to +\infty} \frac{1}{n} \log Z_n(\beta).$$

Thanks to Jensen’s inequality, we have the upper bound $p(\beta) \leq \lambda(\beta)$ and it is conjectured (and partially proved, see [6,7]) that the behaviour of a typical path under the polymer measure is diffusive iff $\beta \in C_\eta$ the critical region

$$C_\eta = \{ \beta \in \mathbb{R} : p(\beta) = \lambda(\beta) \}.$$ \hspace{1cm}

In dimension $d = 1$, $C_\eta = \{0\}$ and in dimensions $d \geq 3$, $C_\eta$ contains a neighborhood of the origin (see [3,8]).

**The main theorem**

The connection between Last passage percolation and Directed polymer in random environment is made by the family $(\nu_n)_{n \in \mathbb{N}}$ of random probability measures on the real line:

$$\nu_n(A) = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} 1_{\{H_n(\omega,\eta) \in A\}} = P\left(\frac{H_n}{n}(\omega, \eta) \in A\right).$$

Indeed,

$$N_n(\rho) = \sum_{\omega \in \Omega_n} 1_{\{H_n(\omega,\eta) \geq n\rho\}} = (2d)^n \nu_n([\rho, +\infty)).$$

The main result of the paper is

**Theorem 1.1.** $Q$ almost surely, the family $(\nu_n)_{n \in \mathbb{N}}$ satisfies a large deviations principle with good rate function $I = p^*$ the Legendre transform of the free energy of the directed polymer.

Let $m = Q(\eta(k, x))$ be the average weight of a path $m = Q(\frac{H_n}{n}(\omega, \eta))$. It is natural to consider the quantities:

$$N_n(\rho) = \begin{cases} \sum_{\omega \in \Omega_n} 1_{\{H_n(\omega,\eta) \geq n\rho\}} & \text{if } \rho \geq m, \\ \sum_{\omega \in \Omega_n} 1_{\{H_n(\omega,\eta) \leq n\rho\}} & \text{if } \rho < m. \end{cases}$$
A simple exchange of limits $\beta \to \pm \infty$, and $n \to +\infty$, yields the following

$$
\rho^\pm = Q \text{ a.s. } \lim_{n \to +\infty} \max_{\omega \in \Omega_n} \pm \frac{H_n(\omega, \eta)}{n} = \lim_{\beta \to +\infty} \frac{p(\pm \beta)}{\beta} \in [0, +\infty].
$$

Repeating the proof of Theorem 1.1 of [9] gives

**Corollary 1.2.** For $-\rho^- < \rho < \rho^+$, we have $Q$ almost surely,

$$
\lim_{n \to +\infty} (N_n(\rho))^\pm = (2d)e^{-I(\rho)}.
$$

We can then translate our knowledge of the critical region $C_\eta$, into the following remark. Let $V_\eta = \{ \rho \in \mathbb{R} : I(\rho) = \lambda^*(\rho) \}$.

In dimension $d = 1$, $V_\eta = \{m\}$ and in dimensions $d \geq 3$, $V_\eta$ contains a neighbourhood of $m$.

This means that in dimensions $d \geq 3$, the typical large deviation of $\frac{H_n}{n}(\omega, \eta)$ close to its mean is the same as the large deviation of $\frac{1}{n}(\eta_1 + \cdots + \eta_n)$ close to its mean, with $\eta_i$ IID. There is no influence of the path $\omega$: this gives another justification to the name weak-disorder region given to the critical set $C_\eta$.

### 2. PROOF OF THE MAIN THEOREM

Observe that for any $\beta \in \mathbb{R}$ we have:

$$
\int e^{\beta n} d\nu_n(x) = P\left(e^{\beta H_n(\omega, \eta)}\right) = Z_n(\beta) \text{ Q.a.s. (2.1)}
$$

Consequently, since $e^u + e^{-u} \geq 2|u|$, we obtain for any $\beta > 0$, 

$$
\limsup_{n \to +\infty} \frac{1}{n} \log \left( \int e^{\beta n|x|} d\nu_n(x) \right) \leq p(\beta) + p(-\beta) < +\infty,
$$

and the family $(\nu_n)_{n \geq 0}$ is exponentially tight (see the Appendix, Lem. A.1). We only need to show now that for a lower semicontinuous function $I$, and for $x \in \mathbb{R}$

$$
\lim_{\delta \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log \nu_n((x - \delta, x + \delta)) = -I(x),
$$

$$
\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \nu_n([x - \delta, x + \delta]) = -I(x).
$$

From these, we shall infer that $(\nu_n)_{n \geq 0}$ follows a large deviations principle with good rate function $I$. Eventually, equation (2.1) and

$$
\lim_{n \to +\infty} \frac{1}{n} \log Z_n(\beta) = p(\beta)
$$

will imply, by Varadhan’s lemma that $I$ and $p$ are Legendre conjugate:

$$
I(x) = p^*(x) = \sup_{\beta \in \mathbb{R}} (x\beta - p(\beta)).
$$

The strategy of proof finds its origin in Varadhan’s seminal paper [13], and has already successfully been applied in [5]. Let us define for $\lambda > 0$, $x \in \mathbb{Z}$, $a \in \mathbb{R}$

$$
V_n^{(\lambda)}(x, a; \eta) = \log P_x \left( e^{-\lambda H_n(\omega, \eta) - a} \right) = V^{(\lambda)}(0, a; \tau_{o,x} \circ \eta).
$$
with $\tau_{k,x}$ the translation operator on the environment defined by:

$$\tau_{k,x} \circ \eta(i, y) = \eta(k + i, x + y),$$

and $P^x$ the law of simple random walk starting from $x$.

**Step 1.** The functions $v^{(\lambda)}_n(a) = Q(V^{(\lambda)}(0, a; \eta))$ satisfy the inequality

$$v^{(\lambda)}_{n+m}(a + b) \geq v^{(\lambda)}_n(a) + v^{(\lambda)}_m(b) \quad (n, m \in \mathbb{N}; a, b \in \mathbb{R}).$$

**Proof.** Since $|H_{n+m} - (a + b)| \leq |H_n - b| + |(H_{n+m} - H_n) - a|$ we have

$$V^{(\lambda)}_{n+m}(x, a + b; \eta) \geq \log P^x \left( e^{-\lambda|H_n - b|} e^{-\lambda(H_{n+m} - H_n) - a} \right)$$

$$= \log P^x \left( e^{-\lambda|H_n - b|} e^{v^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta)} \right)$$

$$= \log \sum_y P^x \left( e^{-\lambda|H_n - b|} 1_{(S_n = y)} e^{v^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta)} \right)$$

$$= V^{(\lambda)}_n(x, b; \eta) + \log \left( \sum_y \sigma_n(y) e^{v^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta)} \right)$$

$$\geq V^{(\lambda)}_n(x, b; \eta) + \sum_y \sigma_n(y) V^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta)$$

(Jensen’s inequality),

with $\sigma_n$ the probability measure on $\mathbb{Z}^d$:

$$\sigma_n(y) = \frac{1}{V^{(\lambda)}_n(x, b; \eta)} P^x \left( e^{-\lambda|H_n - b|} 1_{(S_n = y)} \right) \quad (y \in \mathbb{Z}^d).$$

Observe that the random variables $\sigma_n(y)$ are measurable with respect to the sigma field $G_n = \sigma(\eta(i, x) : i \leq n, x \in \mathbb{Z}^d)$, whereas the random variables $V^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta)$ are independent from $G_n$. Hence, by stationarity,

$$v^{(\lambda)}_{n+m}(a + b) = Q \left( v^{(\lambda)}_{n+m}(0, a + b; \eta) \right)$$

$$\geq v^{(\lambda)}_n(b) + \sum_y Q(\sigma_n(y)) Q \left( v^{(\lambda)}_m(0, a; \tau_{n,y} \circ \eta) \right)$$

$$= v^{(\lambda)}_n(b) + \sum_y Q(\sigma_n(y)) v^{(\lambda)}_m(a)$$

$$= v^{(\lambda)}_n(b) + v^{(\lambda)}_m(a) Q \left( \sum_y \sigma_n(y) \right)$$

$$= v^{(\lambda)}_n(b) + v^{(\lambda)}_m(a). \square$$

**Step 2.** There exists a function $I^{(\lambda)} : \mathbb{R} \to \mathbb{R}^+$ convex, non negative, Lipschitz with constant $\lambda$, such that

$$- \lim_{n \to \infty} \frac{1}{n} v^{(\lambda)}_n(a_n) = I^{(\lambda)}(\xi) \quad \text{(if} \quad \frac{a_n}{n} \to \xi \in \mathbb{R}) \quad \text{(2.5)}.$$

**Proof.** From $|H_n - a| \leq |H_n - b| + |a - b|$ we infer that

$$V^{(\lambda)}_n(0, a; \eta) \geq V^{(\lambda)}_n(0, b; \eta) - \lambda|a - b|. $$
Therefore the functions $v_n^{(\lambda)}$ are all Lipschitz continuous with the same constant $\lambda$ and we combine this fact with standard subadditivity arguments (see e.g. Varadhan [13] or Alexander [1]). For sake of completeness, we give a detailed proof in the Appendix Lemma A.2. □

Step 3. Q almost surely, for any $\xi \in \mathbb{R}$, if $\frac{a_n}{n} \to \xi$, then

$$\lim_{n \to \infty} -\frac{1}{n} \log P\left(e^{-\lambda|H_n - a_n|}\right) = I^{(\lambda)}(\xi). \quad (2.6)$$

Proof. Since the functions are Lipschitz, it is enough to prove that for any fixed $\xi \in \mathbb{Q}$, (2.6) holds a.s. This is where we use the restrictive assumptions made on the distribution of the environment. If the distribution of $\eta$ is with bounded support, or Gaussian, or more generally satisfies a logarithmic Sobolev inequality with constant $c > 0$, then it has the Gaussian concentration of measure property (see [2]): for any 1-Lipschitz function $F$ of independent random variables distributed as $\eta$,

$$P\left(|F - P(F)| \geq r\right) \leq 2e^{-\frac{r^2}{2c}} (r > 0).$$

It is easy to prove, as in Proposition 1.4 of [4], that the function

$$(\eta(k, x), k \leq n, |x| \leq n) \to \log P\left(e^{-\lambda[H_n(\omega, \eta) - a]}\right)$$

is Lipschitz, with respect to the Euclidean norm, with Lipschitz constant at most $\lambda \sqrt{n}$. Therefore, the Gaussian concentration of measure yields

$$P\left(|V_n^{(\lambda)}(0, a; \eta) - v_n^{(\lambda)}(a)| \geq u\right) \leq 2e^{-\frac{\lambda^2 u^2}{8n}}.$$

We conclude by a Borel Cantelli argument combined with (2.5).

Observe that for fixed $\xi \in \mathbb{R}$, the function $\lambda \to I^{(\lambda)}(\xi)$ is increasing; we shall consider the limit:

$$I(\xi) = \lim_{\lambda \to +\infty} I^{(\lambda)}(\xi)$$

which is by construction non negative, convex and lower semi continuous.

Step 4. The function $I$ satisfy (2.2) and (2.3).

Proof. Given, $\xi \in \mathbb{R}$ and $\lambda > 0$, $\delta > 0$, we have

$$P\left(|H_n(\omega, \eta) - \xi| \leq \delta\right) = P\left(e^{-\lambda n|H_n(\omega, \eta) - \xi|} \geq e^{-\lambda n \delta}\right) \leq e^{\lambda n \delta} P\left(e^{-\lambda[H_n(\omega, \eta) - n \xi]}\right).$$

Therefore,

$$\limsup_{\delta \to 0} \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq \lambda \delta - I^{(\lambda)}(\xi)$$

and we obtain by letting $\lambda \to +\infty$,

$$\limsup_{\delta \to 0} \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq -I(\xi).$$
Given $\xi \in \mathbb{R}$ such that $I(\xi) < +\infty$, and $\delta > 0$, we have for $\lambda > 0$,

$$P \left( \left| \frac{H_n}{n} - \xi \right| < \delta \right) \geq P \left( e^{-\lambda|n_n - \xi|} \right) - e^{-\lambda \delta n}.$$ 

Hence, if we choose $\lambda > 0$ large enough such that $\lambda \delta > I(\xi) \geq I(\lambda)(\xi)$, we obtain

$$\liminf_{n \to +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I(\lambda)(\xi) \geq -I(\xi)$$

and therefore

$$\liminf_{\delta \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I(\xi). \quad \Box$$

### Appendix A

Exponential tightness plays the same role in Large Deviations theory as tightness in weak convergence theory; in particular it implies that the Large Deviations Property holds along a subsequence with a good rate function (see Thm. 3.7 of Feng and Kurtz [11], or Lem. 4.1.23 of Dembo and Zeitouni [10]). Therefore, once exponential tightness is established, we only need to identify the rate function: the Weak Large Deviations Property implies the Large Deviations Property with a good rate function (see Dembo and Zeitouni [10], Lem. 1.2.18). Our strategy of proof is then clear. First we establish exponential tightness, by applying the following lemma to the probability $\nu_n$ and the Lyapunov function $x \to |x|$, then we prove a Weak Large Deviations Property by checking that the limits (2.2) and (2.3) hold.

**Lemma A.1.** Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on a Polish space $X$. Assume that there exists a (Lyapunov) function $F: X \to \mathbb{R}_+$ such that the level sets \{ $F \leq C$ \}$_{C > 0}$ are compacts, and

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left( \int e^{nF(x)} d\mu_n(x) \right) < +\infty.$$ 

The $(\mu_n)_{n \in \mathbb{N}}$ is exponentially tight, i.e. for each $A > 0$, there exists a compact $K_A$ such that:

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mu_n(K_A^C) \leq -A.$$ 

**Proof.** Let $M = \limsup_{n \to +\infty} \frac{1}{n} \log \left( \int e^{nF(x)} d\mu_n(x) \right)$. There exists $n_0$ such that for $n \geq n_0$,

$$\frac{1}{n} \log \left( \int e^{nF(x)} d\mu_n(x) \right) \leq 2M.$$ 

Thanks to Markov inequality, for $C > 0$ and $n \geq n_0$,

$$\mu_n(F > C) = \mu_n(e^{nF} > e^{nC}) \leq e^{-nC} \int e^{nF} d\mu_n \leq e^{-n(C-2M)}.$$ 

Hence, if $C > 2M + A$, then for the compact set $K_A = \{ F \leq C \}$, and $n \geq n_0$,

$$\frac{1}{n} \log \mu_n(K_A^C) \leq -(C - 2M) < -A. \quad \Box$$

In Step 2 of the proof of the main theorem, we apply the following lemma to the family of functions $u_n = -v_n(\lambda)$. 

**Lemma A.2.** Assume that the non negative functions $u_n : \mathbb{R} \rightarrow \mathbb{R}_+$ are Lipschitz with the same constant $C > 0$, that is
\[ \forall n, x, y, \quad |u_n(x) - u_n(y)| \leq C|x - y|.\]
Assume furthermore the subadditivity:
\[ \forall x, y, n, m, \quad u_{n+m}(x + y) \leq u_n(x) + u_m(y).\]
Then there exists a non negative function $I : \mathbb{R} \rightarrow \mathbb{R}_+$, Lipschitz with constant $C$, that satisfies:
(i) if $\frac{a_n}{n} \rightarrow x$, then $\frac{1}{n}u_n(a_n) \rightarrow I(x)$.
(ii) $I$ is convex.

**Proof.** For fixed $x \in \mathbb{R}$, the sequence $z_n = u_n(nx)$ is subadditive and non negative:
\[ z_{n+m} \leq z_n + z_m.\]
Therefore, by the standard subadditive theorem for sequences of real numbers, we can consider the limit
\[ I(x) = \inf_{n \geq 1} \frac{1}{n}z_n = \lim_{n \rightarrow +\infty} \frac{1}{n}z_n = \lim_{n \rightarrow +\infty} \frac{1}{n}u_n(nx).\]
If we take limits in the inequality
\[ \left| \frac{1}{n}u_n(nx) - \frac{1}{n}u_n(ny) \right| \leq C|x - y|\]
we obtain $|I(x) - I(y)| \leq C|x - y|$.
(i) Assume $\frac{a_n}{n} \rightarrow x$, then
\[ \left| \frac{1}{n}u_n(nx) - \frac{1}{n}u_n(a_n) \right| \leq C\left| x - \frac{a_n}{n} \right| \rightarrow 0.\]
Hence, $\frac{1}{n}u_n(a_n) \rightarrow I(x)$.
(ii) We have, $[y]$ denoting the integer part of the real number $y$, for any $x, y$ and $0 < t < 1$,
\[ u_{[t]}(ntx + n(1-t)y) \leq u_{[tn]}(ntx) + u_{[(1-t)n]}(n(1-t)y).\]
Since $\frac{1}{n}([tn] + (1-t)n) \rightarrow 1$, we have by (i)
\[ \frac{1}{n}u_{[tn]}(ntx + n(1-t)y) \rightarrow I(tx + (1-t)y).\]
Furthermore, since $\frac{1}{n}[tn] \rightarrow t$, we have by (i),
\[ \frac{1}{n}u_{[tn]}(ntx) \rightarrow tI(x)\]
and similarly,
\[ \frac{1}{n}u_{[(1-t)n]}(n(1-t)y) \rightarrow (1-t)I(y).\]
Combining these limits with the preceding inequality yields,
\[ I(tx + (1-t)y) \leq tI(x) + (1-t)I(y)\]
that is $I$ is convex.

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