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ALMOST SURE FUNCTIONAL LIMIT THEOREM FOR THE PRODUCT OF PARTIAL SUMS

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Abstract. We prove an almost sure functional limit theorem for the product of partial sums of i.i.d. positive random variables with finite second moment.

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1. Introduction and main result

Limiting distributions of the product of partial sums of positive random variables have been widely studied in recent years. Arnold and Villaseñor [1] proved the limit theorem for the partial sum of a sequence of exponential random variables. Rempała and Wesołowski [4] proved it for any independent and identically distributed (i.i.d.) random variables with finite variance. Later, Qi [5] considered a sequence of random variables with α-stable distribution and established the limit distribution of the product of the partial sums when $1 < \alpha \leq 2$.

Recently, Zhang and Huang [6] proved the following invariance principle of the product of partial sums of i.i.d. positive random variables with mean $\mu > 0$ and variance σ^2:

$$\left(\prod_{k=1}^{[nt]} \frac{S_k}{\mu k}\right)^{\frac{\mu}{\sigma\sqrt{n}}} \xrightarrow{\mathcal{D}} \exp\left(\int_0^t \frac{W(s)}{s}\mathrm{d}s\right) \text{ as } n \to \infty. \tag{1.1}$$

The goal of this paper is to obtain an almost sure version of the above invariance principle which can also be a functional version of the almost sure limit theorem obtained by Gonchigdanzan and Rempała [3]. Here is the result:

Theorem 1.1. *Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. positive random variables with mean $\mu > 0$ and variance σ^2 and let $S_n = X_1 + \cdots + X_n$. Define a process $\{\xi_n(t) : 0 \leq t \leq 1\}$ by*

$$\xi_n(t) := \left(\prod_{k=1}^{[nt]} \frac{S_k}{\mu k}\right)^{\frac{\mu}{\sigma\sqrt{n}}}.$$

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Let F_t be the distribution function of the random variable on the right-hand side of (1.1). Then

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{I}\left(\xi_k(t)\le x\right)\overset{a.s.}{\longrightarrow}F_t(x)\ \text{as}\ n\to\infty \tag{1.2}$$

if and only if

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{P}\left(\xi_k(t)\le x\right)\longrightarrow F_t(x)\ \ \text{as}\ \ n\to\infty. \tag{1.3}$$

Corollary 1.1. *Let $(X_k)_{k\ge1}$ be a sequence of i.i.d. positive random variables with mean $\mu>0$ and variance σ^2. Then we have*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{I}\left(\left(\prod_{j=1}^{[kt]}\frac{S_j}{\mu j}\right)^{\frac{\mu}{\sigma\sqrt{k}}}\le x\right)\overset{a.s.}{\longrightarrow}F_t(x)\ \text{as}\ n\to\infty.$$

2. Auxiliary results and proofs

Throughout the paper $\log x$ and $\log\log x$ stand for $\ln(e\vee n)$ and $\ln\ln(n\vee e^e)$ respectively, and "$\ll$" is meant for the big "O" notation.

2.1. Auxiliary results

Lemma 2.1. *Let $(Y_n)_{n\ge1}$ be a sequence of random variables. Set $S_n=Y_1+\cdots+Y_n$. Then we have*

$$E\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}\log\left(\frac{n+1}{j}\right)Y_j\right|\right)\le 3\log(n+1)\,E\left(\max_{1\le k\le n}|S_k|\right).$$

Proof. Observe that

$$\left|\sum_{j=1}^{k}\log\left(\frac{n+1}{j}\right)Y_j\right|\le\left|\sum_{j=1}^{k}\log(n+1)\,Y_j\right|+\left|\sum_{j=2}^{k}\log j\,Y_j\right|=T_1+T_2.$$

Obviously, $T_1\le\log(n+1)|S_k|$ which implies

$$\max_{1\le k\le n}T_1\le\log(n+1)\max_{1\le k\le n}|S_k|.$$

For the second term T_2 we have

$$\begin{aligned}\max_{2\le k\le n}T_2&=\max_{2\le k\le n}\left|\sum_{j=2}^{k}\log j\,Y_j\right|=\max_{2\le k\le n}\left|\sum_{j=2}^{k}(Y_j+Y_{j+1}+\cdots+Y_k)(\log j-\log(j-1))\right|\\&\le\max_{2\le k\le n}\sum_{j=2}^{k}|Y_j+Y_{j+1}+\cdots+Y_k|\,(\log j-\log(j-1))\\&\le2\max_{2\le k\le n}|Y_1+Y_2+\cdots+Y_k|\sum_{j=2}^{n}(\log j-\log(j-1))\le2\log(n+1)\max_{1\le k\le n}|S_k|.\end{aligned}$$

□

Lemma 2.2. *Let* $(X_k)_{k\geq 1}$ *be a sequence of i.i.d. positive random variables with mean* μ *and variance* σ^2*. Then setting* $S_n = X_1 + \cdots + X_n$ *we have*

$$\max_{0\leq t\leq 1}\left|\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\log\frac{S_k}{\mu k} - \frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)\right| \xrightarrow{a.s.} 0 \quad \text{as} \quad n\to\infty.$$

Proof. Note that $\log(x+1) = x - r(x)$ where $r(x)/x^2 \to \frac{1}{2}$ as $x\to 0$. By the strong law of large numbers we have $S_k/k - \mu \xrightarrow{a.s.} 0$ as $k\to\infty$.

Thus by the law of iterated logarithm we get

$$\left|\frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\log\frac{S_k}{\mu k} - \frac{\mu}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{\mu k}-1\right)\right| \overset{a.s.}{\ll} \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(\frac{S_k}{k}-\mu\right)^2$$
$$\overset{a.s.}{\ll} \max_{0\leq t\leq 1}\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\frac{1}{k}\log\log k \ll \frac{1}{\sigma\sqrt{n}}\log n\log\log n \to 0.$$

□

2.2. Proof of Theorem 1.1

We use Berkes and Dehling's [2] technique to prove our theorem. Observe that

$$\sum_{k=1}^{n}\left(\frac{S_k}{k}-\mu\right) = \sum_{k=1}^{n} b_{k,n}(X_k-\mu)$$

where $b_{k,n} = \sum_{j=k}^{n} 1/j$. Hence by Lemma 2.2 it suffices to show that for any Borel-subset A of $D[0,1]$

$$\frac{1}{\log n}\sum_{k=2}^{n}\frac{1}{k}\mathbf{I}\left(\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\in A\right) - \mathbf{P}\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\in A\right)\right) \xrightarrow{a.s.} 0 \text{ as } n\to\infty, \tag{2.1}$$

where $\hat{s}_n = \sum_{i=1}^{[nt]} b_{i,[nt]}(X_i-\mu)$. From Berkes and Dehling [2] (p. 1647), to prove (2.1) it suffices to show that for any bounded Lipschitz function f on $D[0,1]$ we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\zeta_k \xrightarrow{a.s.} 0 \text{ as } n\to\infty \tag{2.2}$$

where $\zeta_k = f(\hat{s}_k/\sigma\sqrt{k}) - \mathbf{E}f(\hat{s}_k/\sigma\sqrt{k})$. In fact, the following implies (2.2) (see p. 1648, Berkes and Dehling [2] for the proof):

$$\mathbf{E}\left(\sum_{k=1}^{n}\frac{1}{k}\zeta_k\right)^2 \ll \log^2 n(\log\log n)^{-\varepsilon} \text{ for some } \varepsilon>0. \tag{2.3}$$

Therefore, showing (2.3) would be sufficient for the proof of Theorem 1.1. Observing

$$\hat{s}_l - \hat{s}_k = b_{[kt]+1,[lt]}(S_{[kt]} - [kt]\mu) + \big(b_{[kt]+1,[lt]}(X_{[kt]+1}-\mu) + \cdots + b_{[lt],[lt]}(X_{[lt]}-\mu)\big)$$

for $l\geq k$ we find that $\hat{s}_l - \hat{s}_k - b_{[kt]+1,[lt]}(S_{[kt]} - \mu[kt])$ is independent of $\hat{s}_{[kt]}$ which implies that

$$\mathrm{Cov}\left(f\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\right), f\left(\frac{\hat{s}_l - \hat{s}_k - b_{[kt]+1,[lt]}(S_{[kt]}-\mu[kt])}{\sigma\sqrt{l}}\right)\right) = 0 \text{ for } l\geq k.$$

By the Lipschitz property of f

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right| &\ll \left|\mathrm{Cov}\left(f\left(\frac{\hat{s}_k}{\sigma\sqrt{k}}\right), f\left(\frac{\hat{s}_l}{\sigma\sqrt{l}}\right) - f\left(\frac{|\hat{s}_l - \hat{s}_k - b_{[kt]+1,[lt]}(S_{[kt]} - \mu[kt])|}{\sigma\sqrt{l}}\right)\right)\right| \\ &\ll \mathbf{E}\left(\max_{0\le t\le 1} \frac{|\hat{s}_k + b_{[kt]+1,[lt]}(S_{[kt]} - \mu[kt])|}{\sigma\sqrt{l}}\right) \\ &\ll \mathbf{E}\left(\max_{0\le t\le 1} \frac{|\hat{s}_k|}{\sigma\sqrt{l}}\right) + \mathbf{E}\left(\max_{0\le t\le 1} \frac{|b_{[kt]+1,[lt]}(S_{[kt]} - \mu[kt])|}{\sigma\sqrt{l}}\right) \\ &\ll \left(\frac{k}{l}\right)^{1/2} \mathbf{E}\left(\max_{0\le t\le 1} \frac{|\hat{s}_k|}{\sigma\sqrt{k}}\right) + \left(\frac{k}{l}\right)^{1/2} \mathbf{E}\left(\max_{0\le t\le 1} b_{[kt]+1,[lt]} \frac{|S_{[kt]} - \mu[kt]|}{\sigma\sqrt{k}}\right).\end{aligned}$$

Since $\max_{0\le t\le 1} b_{[kt]+1,[lt]} = \log(l/k) \ll (l/k)^{\gamma}$ (we choose $0 < \gamma < 1/2$)

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right| &\ll \left(\frac{k}{l}\right)^{1/2} \mathbf{E}\left(\max_{0\le t\le 1} \frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{[kt]} b_{i,k}(X_i - \mu)\right|\right) + \left(\frac{k}{l}\right)^{1/2-\gamma} \mathbf{E}\left(\max_{0\le t\le 1} \frac{|S_{[kt]} - \mu[kt]|}{\sigma\sqrt{k}}\right) \\ &= \left(\frac{k}{l}\right)^{1/2} \mathbf{E}\left(\max_{0\le j\le k} \frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{j} b_{i,k}(X_i - \mu)\right|\right) + \left(\frac{k}{l}\right)^{1/2-\gamma} \mathbf{E}\left(\max_{0\le j\le k} \frac{|S_j - \mu j|}{\sigma\sqrt{k}}\right) \\ &= M_1 + M_2.\end{aligned}$$

Now applying Lemma 2.1 to M_1 we obtain

$$\begin{aligned}\left|\mathbf{E}(\zeta_k\zeta_l)\right| &\ll \left(\frac{k}{l}\right)^{1/2} \log k\, \mathbf{E}\left(\max_{1\le j\le k} \frac{1}{\sigma\sqrt{k}}\left|\sum_{i=1}^{j}(X_i - \mu)\right|\right) + \left(\frac{k}{l}\right)^{1/2-\gamma} \mathbf{E}\left(\max_{1\le j\le k} \frac{|S_j - \mu j|}{\sigma\sqrt{k}}\right) \\ &\ll \left|\mathbf{E}(\zeta_k\zeta_l)\right| \ll \log k\, \left(\frac{k}{l}\right)^{1/2-\gamma} \mathbf{E}\left(\max_{1\le j\le k} \frac{|S_j - \mu j|}{\sigma\sqrt{k}}\right) \ll \log k\, \left(\frac{k}{l}\right)^{\gamma'}\end{aligned}$$

where $0 < \gamma' < 1/2 - \gamma$.

On the other hand $\mathbf{E}(\zeta_k\zeta_l) \ll 1$ because ζ_k is bounded . Hence we have the following estimate for $\mathbf{E}(\zeta_k\zeta_l)$:

$$\mathbf{E}(\zeta_k\zeta_l) \ll \begin{cases} 1, & \text{if } l/k \le \exp\big((\log n)^{1-\varepsilon}\big) \\ (k/l)^{\gamma'} \log k, & \text{if } l/k \ge \exp\big((\log n)^{1-\varepsilon}\big) \end{cases}$$

where ε is any positive number. Hence,

$$\sum_{\substack{1\,\le\, k\,\le\, l\,\le\, n \\ l/k\,\le\,\exp\,(\log n)^{1-\varepsilon}}} \frac{\mathbf{E}(\zeta_k\zeta_l)}{kl} \le \sum_{1\le k\le n} \frac{1}{k} \sum_{k\le l\le k e^{(\log n)^{1-\varepsilon}}} \frac{1}{l} \ll \sum_{k=1}^{n} \frac{1}{k} \log^{1-\varepsilon} n \ll \log^{2-\varepsilon} n \tag{2.4}$$

and

$$\sum_{\substack{1\,\le\, l\,\le\, k\,\le\, n \\ l/k\,\ge\,\exp\,(\log n)^{1-\varepsilon}}} \frac{\mathbf{E}(\zeta_k\zeta_l)}{kl} \le \mathrm{e}^{-\gamma'(\log n)^{1-\varepsilon}} \log n \sum_{1\le k\le l\le n} \frac{1}{kl} \ll \mathrm{e}^{-\gamma'(\log n)^{1-\varepsilon}} \log^3 n \ll \log^{2-\varepsilon} n. \tag{2.5}$$

Thus (2.4) and (2.5) immediately imply (2.3) which completes the proof of Theorem 1.1. □

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