ESAIM: PS October 2010, Vol. 14, p. 299–314
DOI: 10.1051/ps:2008030

ESAIM: Probability and Statistics
www.esaim-ps.org

CENTRAL LIMIT THEOREM FOR SAMPLED SUMS OF DEPENDENT RANDOM VARIABLES

Nadine Guillotin-Plantard[1] and Clémentine Prieur[2]

Abstract. We prove a central limit theorem for linear triangular arrays under weak dependence conditions. Our result is then applied to dependent random variables sampled by a $\mathbb{Z}$-valued transient random walk. This extends the results obtained by [N. Guillotin-Plantard and D. Schneider, *Stoch. Dynamics* **3** (2003) 477–497]. An application to parametric estimation by random sampling is also provided.

Mathematics Subject Classification. Primary 60F05, 60G50, 62D05; Secondary 37C30, 37E05.

Received December 12, 2007. Revised June 3, 2008.

1. Introduction

Let $\{\xi_i\}_{i\in\mathbb{Z}}$ be a sequence of centered, non essentially constant and square integrable real valued random variables. Let $\{a_{n,i},\ -k_n \le i \le k_n\}$ be a triangular array of real numbers such that for all $n \in \mathbb{N}$, $\sum_{i=-k_n}^{k_n} a_{n,i}^2 > 0$. We are interested in the behaviour of linear triangular arrays of the form

$$X_{n,i} = a_{n,i}\,\xi_i,\ n = 0, 1, \ldots,\ i = -k_n, \ldots, k_n, \tag{1.1}$$

where $(k_n)_{n\ge1}$ is a nondecreasing sequence of positive integers satisfying $\lim_\infty k_n = \infty$. We work under a weak dependence condition introduced in [8]. We first prove a central limit theorem for linear triangular arrays of type (1.1) (Thm. 3.1 of Sect. 3). Applying this result, we then prove a central limit theorem for the partial sums of weakly dependent sequences sampled by a transient $\mathbb{Z}$-valued random walk (Thm. 4.1 of Sect. 4). This result extends the results obtained by Guillotin-Plantard and Schneider [11]. Peligrad and Utev [20] derive a central limit theorem for triangular arrays of type (1.1) under mixing conditions. Unfortunately, mixing is a rather restrictive condition, and many simple Markov chains are not mixing. For any $y \in \mathbb{R}$, let $[y]$ denote the integer part of y. It is known since a long time that the stationary Markov chain associated to the dynamical system generated by the map $T(x) = 2x - [2x]$ on $[0, 1]$ *via* the Perron-Frobenius operator is not α-mixing, in the sense that $\alpha(\sigma(T), \sigma(T^n))$ does not tend to zero as n tends to infinity. Withers [27] proves triangular central limit theorems under a so-called l-mixing condition, which generalizes the classical notions of mixing (such as

Keywords and phrases. Random walks, weak dependence, central limit theorem, dynamical systems, random sampling, parametric estimation.

[1] Université Claude Bernard – Lyon I, Institut Camille Jordan, Bâtiment Braconnier, 43 avenue du 11 Novembre 1918, 69622 Villeurbanne Cedex, France; nadine.guillotin@univ-lyon1.fr

[2] INSA Toulouse, Institut Mathématique de Toulouse, Équipe de Statistique et Probabilités, 135 avenue de Rangueil, 31077 Toulouse Cedex 4, France; Clementine.Prieur@insa-toulouse.fr

Article published by EDP Sciences

strong mixing, absolute regularity, uniform mixing introduced respectively by Rosenblatt [22], Rozanov and Volkonskii [23] and Ibragimov [13]). The idea of l-mixing requires the asymptotic decoupling of the "past" and the "future". The dependence setting used in the present paper (introduced in Dedecker *et al.*, [8]) follows the same idea. In Section 5 we give examples satisfying our dependence conditions. Coulon-Prieur and Doukhan [5] proves a triangular central limit theorem under a weaker dependence condition. However, they assume that the random variables ξ_i are uniformly bounded. Their proof is a variation of the Lindeberg method developed in Rio [21]. Also using a variation of this method, Bardet *et al.* [2] prove a triangular central limit theorem, requiring moments of order $2+\delta$, $\delta > 0$. In Section 2, we introduce the dependence setting under which we work in the sequel. Models for which we can compute bounds for our dependence coefficients are presented in Section 5. Finally, we give an application to parametric estimation by random sampling in Section 6.

2. Definitions

In this section, we recall the definition of the dependence coefficients which we will use in the sequel. They have first been introduced in [8].

On the Euclidean space $\mathbb{R}^m$, we define the metric

$$d_1(x,y) = \sum_{i=1}^{m} |x_i - y_i|. \tag{2.1}$$

Let $\Lambda = \bigcup_{m\in\mathbb{N}^*} \Lambda_m$ where Λ_m is the set of Lipschitz functions $f : \mathbb{R}^m \to \mathbb{R}$ with respect to the metric d_1. If $f \in \Lambda_m$, we denote by $\mathrm{Lip}(f) := \sup_{x\neq y} |f(x) - f(y)|/d_1(x,y)$ the Lipschitz modulus of f. The set of functions $f \in \Lambda$ such that $\mathrm{Lip}(f) \le 1$ is denoted by $\tilde{\Lambda}$.

Definition 2.1. Let ξ be a $\mathbb{R}^m$-valued random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, assumed to be square integrable. For any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, we define the θ_2-dependence coefficient

$$\theta_2(\mathcal{M}, \xi) = \sup\{\|\mathbb{E}(f(\xi)|\mathcal{M}) - \mathbb{E}(f(\xi))\|_2,\ f \in \tilde{\Lambda}\}. \tag{2.2}$$

We now define the coefficient $\theta_{k,2}$ for a sequence of σ-algebras and a sequence of $\mathbb{R}$-valued random variables.

Definition 2.2. Let $(\xi_i)_{i\in\mathbb{Z}}$ be a sequence of square integrable random variables valued in $\mathbb{R}$. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be a sequence of σ-algebras of $\mathcal{A}$. For any $k \in \mathbb{N}^* \cup \{\infty\}$ and $n \in \mathbb{N}$, we define

$$\theta_{k,2}(n) = \max_{1\le l\le k} \frac{1}{l} \sup\{\theta_2(\mathcal{M}_p, (\xi_{j_1}, \ldots, \xi_{j_l})), p+n \le j_1 < \ldots < j_l\}$$

and

$$\theta_2(n) = \theta_{\infty,2}(n) = \sup_{k\in\mathbb{N}^*} \theta_{k,2}(n)\,.$$

Definition 2.3. Let $(\xi_i)_{i\in\mathbb{Z}}$ be a sequence of square integrable random variables valued in $\mathbb{R}$. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be a sequence of σ-algebras of $\mathcal{A}$. The sequence $(\xi_i)_{i\in\mathbb{Z}}$ is said to be θ_2-weakly dependent with respect to $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ if $\lim_\infty \theta_2(n) = 0$.

Remark 2.1. Replacing the $\|\cdot\|_2$ norm in (2.2) by the $\|\cdot\|_1$ norm, we get the θ_1 dependence coefficient first introduced by Doukhan and Louhichi [10]. This weaker coefficient is the one used in [5].

3. Central limit theorem for triangular arrays of dependent random variables

Let $\{X_{n,i},\ n \in \mathbb{N},\ -k_n \le i \le k_n\}$ be a triangular array of type (1.1). We are interested in the asymptotic behaviour of the following sum

$$\Sigma_n = \sum_{i=-k_n}^{k_n} X_{n,i} = \sum_{i=-k_n}^{k_n} a_{n,i}\, \xi_i\,.$$

Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be the sequence of σ-algebras of $\mathcal{A}$ defined by

$$\mathcal{M}_i = \sigma\left(\xi_j\,,\ j \le i\right)\,,\ i \in \mathbb{Z}\,.$$

In the sequel, the dependence coefficients are defined with respect to the sequence of σ-algebras $(\mathcal{M}_i)_{i\in\mathbb{Z}}$. We denote by σ_n^2 the variance of Σ_n.

Theorem 3.1. *Assume that the following conditions are satisfied:*

(A_1) (i) $\liminf_{n\to+\infty}\left(\sum_{i=-k_n}^{k_n} a_{n,i}^2\right)^{-1}\sigma_n^2 > 0$,
(ii) $\lim_{n\to+\infty}\sigma_n^{-1}\ \max_{-k_n\le i\le k_n}|a_{n,i}| = 0$.
(A_2) $\{\xi_i^2\}_{i\in\mathbb{Z}}$ *is an uniformly integrable family.*
(A_3) $\theta_2^{\xi}(\cdot)$ *is bounded above by a non-negative function* $g(\cdot)$ *such that*
$x \mapsto x^{3/2}\ g(x)$ *is non-increasing,*
$\sum_{i=0}^{\infty} 2^{3i/2} g(2^{i\varepsilon}) < \infty$ *for some* $\varepsilon \in]0,1[$.

Then, as n tends to infinity, $\sigma_n^{-1}\Sigma_n$ *converges in distribution to* $\mathcal{N}(0,1)$.

Remark 3.1. Theorem 2.2 (c) in [20] yields a central limit theorem for strongly mixing linear triangular arrays of type (1.1). They assume that $\{|\xi_i|^{2+\delta}\}$ is uniformly integrable for a certain $\delta > 0$. Such an assumption is also required for Theorem 2.1 in [27] for l-mixing arrays. In [5], the random variables ξ_i are assumed to be uniformly bounded. The proof of Theorem 2.2 (c) in [20] relies on a variation on Theorem 4.1 in [25] (see Theorem B in [20]). The proof of Theorem 3.1, which is postponed to the Appendix, also makes use of a variation on Theorem 4.1 in [25] (see also [26]).

Remark 3.2. If $\theta_2^{\xi}(n) = \mathcal{O}\left(n^{-a}\right)$ for some positive a, condition (A_3) holds for $a > 3/2$.

4. Central limit theorem for the sum of dependent random variables sampled by a transient random walk

4.1. The main result

Let $(E, \mathcal{E}, \mu)$ be a probability space, and $T : E \mapsto E$ a bijective bimeasurable transformation preserving the probability μ. We define the stationary sequence $(\zeta_i)_{i\in\mathbb{Z}} = (T^i)_{i\in\mathbb{Z}}$ from (E,μ) to E. Let $(X_i)_{i\ge1}$ be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{Z}$ and

$$S_n = \sum_{i=1}^{n} X_i,\ n \ge 1, \qquad S_0 \equiv 0.$$

For $f \in \mathbb{L}^1(\mu)$ and $\omega \in \Omega$, we are interested in the sampled ergodic sum

$$\sum_{k=0}^{n-1} f \circ \zeta_{S_k(\omega)}.$$

By applying Birkhoff's ergodic theorem to the skew-product:

$$\begin{aligned} U \quad : \quad & \Omega \times E \to \Omega \times E \\ & (\omega, x) \mapsto (\sigma\omega, T^{\omega_1} x) \end{aligned}$$

where σ is the shift on the path space $\Omega = \mathbb{Z}^{\mathbb{N}}$, we obtain that for every function $f \in \mathbb{L}^1(\mu)$, the sampled ergodic sum converges $\mathbb{P} \otimes \mu$-almost surely. A natural question is to know if the random walk is universally representative for $\mathbb{L}^p$, $p > 1$ in the following sense: there exists a subset Ω_0 of Ω of probability one such that for every $\omega \in \Omega_0$, for every dynamical system $(E, \mathcal{E}, \mu, T)$, for every $f \in \mathbb{L}^p$, $p > 1$, the sampled ergodic average converges μ-almost surely. The answer can be found in [16] if the X_i's are square integrable: the random walk is universally representative for $\mathbb{L}^p$, $p > 1$ if and only if the expectation of X_1 is not equal to 0 which corresponds to the case where the random walk is transient. In that case, it seems natural to study the fluctuations of the sampled ergodic averages around the limit. From Lacey's theorem [15], for any $H \in (0, 1)$, there exists some function $f \in \mathbb{L}^2(\mathbb{P} \otimes \mu)$ such that the finite-dimensional distributions of the process

$$\frac{1}{n^H} \sum_{k=0}^{[nt]-1} f \circ U^k(\omega, x)$$

converge to the finite dimensional distributions of a self-similar process. Unfortunately, this convergence on the product space does not imply the convergence in distribution for a given path of the random walk. A first answer to this question is given in [11] where the technique of martingale differences is used. Let us recall that this method consists (under convenient conditions) of decomposing the function f as the sum of a function g generating a sequence of martingale differences and a cocycle $h - h \circ T$. In the standard case, the central limit theorem for the ergodic sum is deduced from central limit theorems for the sums of martingale differences, the term corresponding to the cocycle being negligeable in probability. In [11], only functions f generating a sequence of martingale differences are considered. In this section, in which we prove a central limit theorem for θ_2-weakly dependent random variables sampled by a transient random walk, this argument does not hold anymore. We apply Theorem 3.1 of Section 3.

In the sequel, the random walk $(S_n)_{n\geq 0}$ is assumed to be transient. In particular, for every $x \in \mathbb{Z}$, the Green function

$$G(0, x) = \sum_{k=0}^{+\infty} \mathbb{P}(S_k = x)$$

is finite. For example, it is the case if the random variable X_1 is assumed with finite absolute mean and nonzero mean. It is also possible to choose the random variables $(X_i)_{i\geq 1}$ symmetric and for every $x \in \mathbb{R}$,

$$\mathbb{P}(n^{-1/\alpha} S_n \leq x) \xrightarrow[n\to+\infty]{} F_\alpha(x),$$

where F_α is the distribution function of a stable law with index $\alpha \in (0, 1)$. Stone [24] has proved a local limit theorem for this kind of random walks from which the transience can be deduced. The expectation with respect to the measure μ (resp. with respect to $\mathbb{P}$, $\mathbb{P} \otimes \mu$) will be denoted in the sequel by $\mathbb{E}_\mu$ (resp. by $\mathbb{E}_{\mathbb{P}}$, $\mathbb{E}$).

For every function $f \in \mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$, we define

$$\sigma^2(f) = 2 \sum_{x\in\mathbb{Z}} G(0, x) \mathbb{E}_\mu(f f \circ T^x) - \mathbb{E}_\mu(f^2).$$

Let us now state our main result whose proof is deferred to Section 4.3.

Theorem 4.1. *Let f be a function in $\mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$. Assume that the sequence $(\xi_x)_{x\in\mathbb{Z}} := (f \circ T^x)_{x\in\mathbb{Z}}$ satisfies assumption (A_3) of Theorem 3.1. Assume that $\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu|f f \circ T^x|$ is finite.*

If $\sigma^2(f)$ is positive, then for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f \circ T^{S_k(\omega)} \xrightarrow[n\to+\infty]{} \mathcal{N}(0, \sigma^2(f)) \quad \textit{in distribution.}$$

Remark 4.1. In the particular case where $(f \circ T^x)_{x\in\mathbb{Z}}$ is a sequence of martingale differences, we recognize Theorem 3.2 of [11]. Indeed, assumptions are satisfied using orthogonality of the $f \circ T^x$'s and then, $\sigma^2(f) = (2G(0,0) - 1)\mathbb{E}_\mu(f^2)$.

Remark 4.2. The stationarity assumption can be relaxed to a stationarity assumption of order 2 on the sequence $(\xi_i)_{i\in\mathbb{Z}}$, if we assume furthermore that the latter sequence is uniformly integrable.

4.2. Computation of the variance

The random walk $(S_n)_{n\geq 0}$ is defined as in the previous section. The local time of the random walk is then defined for every $x \in \mathbb{Z}$ by

$$N_n(x) = \sum_{i=0}^{n} \mathbf{1}_{\{S_i = x\}}.$$

The self-intersection local time is defined for every $x \in \mathbb{Z}$ by

$$\alpha(n,x) = \sum_{i,j=0}^{n} \mathbf{1}_{\{S_i - S_j = x\}}$$

and can be rewritten using the definition of the local time as

$$\alpha(n,x) = \sum_{y\in\mathbb{Z}} N_n(y+x)\ N_n(y).$$

Let f be a function in $\mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$. For every $\omega \in \Omega$,

$$\sum_{k=0}^{n} f \circ T^{S_k(\omega)} = \sum_{x\in\mathbb{Z}} N_n(x)(\omega) f \circ T^x.$$

In order to apply results of Theorem 3.1, we need to study, for any fixed $\omega \in \Omega$, the asymptotic behaviour of the variance of this sum, namely

$$\sigma_n^2(f) = \mathbb{E}_\mu\left(\left|\sum_{k=0}^{n} f \circ T^{S_k(\omega)}\right|^2\right).$$

The variable ω will be omitted in the next calculations. We have

$$\begin{aligned}\sigma_n^2(f) &= \mathbb{E}_\mu\left|\sum_{x\in\mathbb{Z}} N_n(x) f\circ T^x\right|^2\\ &= \sum_{x,y\in\mathbb{Z}} N_n(x)N_n(y)\mathbb{E}_\mu(f\circ T^{x-y}\ f)\\ &= \sum_{y,z\in\mathbb{Z}} N_n(y+z)N_n(y)\mathbb{E}_\mu(f\circ T^{z}\ f)\\ &= \sum_{z\in\mathbb{Z}} \alpha(n,z)\mathbb{E}_\mu(f\circ T^{z}\ f).\end{aligned}$$

We are now able to prove the following proposition:

Proposition 4.1. *If $\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu\,|f\ f\circ T^x|$ is finite, then*

$$\frac{\sigma_n^2(f)}{n} \xrightarrow[n\to+\infty]{\mathbb{P}\text{-}a.s.} \sigma^2(f).$$

Proof of Proposition 4.1. We first prove the following result: if $(b(x))_{x\in\mathbb{Z}}$ is a sequence of positive reals such that $\sum_{x\in\mathbb{Z}} G(0,x)b(x)$ is finite, then

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\alpha_n(0,x)b(x) \xrightarrow[n\to+\infty]{\mathbb{P}\text{-a.s.}} 2\sum_{x\in\mathbb{Z}} G(0,x)b(x) - b(0). \tag{4.1}$$

For every $0\le m<n$, we denote by $W_{m,n}$ the random variable

$$-\sum_{i,j=m}^{n}\sum_{x\in\mathbb{Z}} 1_{\{S_i-S_j=x\}}b(x).$$

Then, due to the positivity of the $b(x)$'s, for every k,m,n such that $0\le k<m<n$,

$$W_{k,n}\le W_{k,m}+W_{m,n},$$

that is $(W_{m,n})_{m,n\ge 0}$ is a subadditive sequence. Then,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(W_{0,n}) &= -\sum_{i,j=0}^{n}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_i-S_j=x)b(x), \quad \text{by Fubini theorem}\\ &= -\Big((n+1)b(0)+2\sum_{i=1}^{n}\sum_{j=0}^{i-1}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_{i-j}=x)b(x)\Big)\\ &= -\Big((n+1)b(0)+2\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_{j}=x)b(x)\Big).\end{aligned}$$

Now, using that

$$\lim_{i\to+\infty}\sum_{j=1}^{i}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_j=x)b(x)=\sum_{x\in\mathbb{Z}} G(0,x)b(x)-b(0),$$

we conclude that

$$\lim_{n\to+\infty} \frac{\mathbb{E}_{\mathbb{P}}(W_{0,n})}{n} = b(0) - 2\sum_{x\in\mathbb{Z}} G(0,x)b(x) < \infty.$$

So the sequence $(W_{m,n})_{m,n\geq 0}$ satisfies all the conditions of Theorem 5 in [14]. Hence

$$\frac{W_{0,n}}{n} \xrightarrow[n\to+\infty]{\mathbb{P}\text{-a.s.}} b(0) - 2\sum_{x\in\mathbb{Z}} G(0,x)b(x).$$

By remarking that $W_{0,n} = -\sum_{x\in\mathbb{Z}} \alpha(n,x)b(x)$, (4.1) follows. For every $x \in \mathbb{Z}$, the function $f\ f\circ T^x$ can be decomposed as

$$f\ f\circ T^x = (f\ f\circ T^x)1_{\{f\ f\circ T^x\geq 0\}} - (-f\ f\circ T^x)1_{\{f\ f\circ T^x<0\}}.$$

By applying the above result to both positive terms of the right-hand side, Proposition 4.1 follows. □

Remark 4.3. Let us consider the simple random walk with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = q$ with $p > q$. Then, for $x \geq 0$,

$$G(0,x) = (p-q)^{-1},$$

and for $x \leq -1$,

$$G(0,x) = (p-q)^{-1}\left(\frac{p}{q}\right)^x.$$

If we assume that $\sum_{x\in\mathbb{N}} \mathbb{E}\,|h\ h\circ T^x| < +\infty$, a simple calculation gives

$$\sigma^2(h - h\circ T) = 2\sum_{x\in\mathbb{Z}} \left[2G(0,x) - G(0,x+1) - G(0,x-1)\right]\mathbb{E}_\mu(h\ h\circ T^x) - 2\mathbb{E}_\mu(h^2) + 2\mathbb{E}_\mu(h\ h\circ T)$$

$$= -2\frac{p-1}{p}\mathbb{E}_\mu(h^2) + 2\mathbb{E}_\mu(h\ h\circ T) - 2\frac{(p-q)}{pq}\sum_{x\geq 1}\left(\frac{q}{p}\right)^x \mathbb{E}_\mu(h\ h\circ T^x).$$

4.3. **Proof of Theorem 4.1**

Let us define $M_n = \max_{0\leq k\leq n} |S_k|$. First note that

$$\sum_{k=0}^{n} f\circ T^{S_k} = \sum_{|x|\leq M_n} N_n(x) f\circ T^x.$$

We want to apply Theorem 3.1 to the triangular array

$$\left\{X_{n,i} = \frac{N_n(i)}{\sqrt{n}} f\circ T^i,\ n\in\mathbb{N},\ -M_n\leq i\leq M_n\right\}. \tag{4.2}$$

The family $\left\{\left(f\circ T^i\right)^2\right\}_{i\in\mathbb{Z}}$ is uniformly integrable since f belongs to $\mathbb{L}^2(\mu)$ and since the sequence $(\zeta_i)_{i\in\mathbb{Z}}$ is stationary. It remains to prove that assumption (A_1) of Theorem 3.1 is satisfied for the triangular array defined by (4.3).

Proof of (A_1) (i). First, by Proposition 3.1. in [11], $\sum_{i=-M_n}^{M_n} a_{n,i}^2 = \alpha(n,0)/n$ converges $\mathbb{P}$-almost surely to $2G(0,0) - 1$ as n goes to infinity. Then, by Proposition 4.1, we know that $\sigma_n^2(f)/n$ converges to $\sigma^2(f)$, which is assumed to be positive. Hence $(A_1)(i)$ is satisfied. □

Proof of (A_1) (ii). Now, by Proposition 3.2. in [11], we know that for every $\rho > 0$,

$$\max_{-M_n \le i \le M_n} |a_{n,i}| = \frac{1}{\sqrt{n}} \max_{i\in\mathbb{Z}} N_n(i) = o\left(n^{\rho-\frac{1}{2}}\right) \mathbb{P} - \text{almost surely.}$$

So $\sigma_n^{-1}(f)\sqrt{n}\ \max_{-M_n\le i\le M_n} |a_{n,i}|$ tends to zero $\mathbb{P}-$almost surely and assumption $(A_1)(ii)$ is satisfied. □

Hence Theorem 3.1 applied to $\sum_{i=-M_n}^{M_n} a_{n,i}\ f\circ T^i$, Proposition 4.1 and Slutsky lemma yield the result.

5. Examples

In this section, we present examples for which we can compute upper bounds for $\theta_2(n)$ for any $n \ge 1$. We refer to chapter 3 in [8] and references therein for more details.

5.1. **Example 1: causal functions of stationary sequences**

Let $(E, \mathcal{E}, \mathbb{Q})$ be a probability space. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with values in a measurable space $\mathcal{S}$. Assume that there exists a real valued function H defined on a subset of $\mathcal{S}^{\mathbb{N}}$, such that $H(\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \ldots,)$ is defined almost surely. The stationary sequence $(\xi_n)_{n\in\mathbb{Z}}$ defined by $\xi_n = H(\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots)$ is called a causal function of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Assume that there exists a stationary sequence $({\varepsilon_i}')_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$. Define $\xi_n^* = H({\varepsilon_n}', {\varepsilon_{n-1}}', {\varepsilon_{n-2}}', \ldots)$. Clearly, ξ_n^* is independent of $\mathcal{M}_0 = \sigma(\xi_i\,,\ i\le 0)$ and distributed as ξ_n. Let $(\delta_2(i))_{i>0}$ be a non increasing sequence such that

$$\|\mathbb{E}\left(|\xi_i - \xi_i^*|\ |\ \mathcal{M}_0\right)\|_2 \le \delta_2(i). \tag{5.1}$$

Then the coefficient θ_2 of the sequence $(\xi_n)_{n\ge 0}$ satisfies

$$\theta_2(i) \le \delta_2(i). \tag{5.2}$$

Let us consider the particular case where the sequence of innovations $(\varepsilon_i)_{i\in\mathbb{Z}}$ is absolutely regular in the sense of Rozanov and Volkonskii [23]. Then, according to Theorem 4.4.7 in [3], if E is rich enough, there exists $(\varepsilon_i')_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$ such that

$$\mathbb{Q}(\varepsilon_i \ne \varepsilon_i' \text{ for some } i \ge k\,|\,\mathcal{F}_0) = \frac{1}{2}\left\|\mathbb{Q}_{\tilde{\varepsilon}_k|\mathcal{F}_0} - \mathbb{Q}_{\tilde{\varepsilon}_k}\right\|_v,$$

where $\tilde{\varepsilon}_k = (\varepsilon_k, \varepsilon_{k+1}, \ldots)$, $\mathcal{F}_0 = \sigma(\varepsilon_i\,,\ i \le 0)$, and $\|\cdot\|_v$ is the variation norm. In particular if the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is independent and identically distributed, it suffices to take $\varepsilon_i' = \varepsilon_i$ for $i > 0$ and $\varepsilon_i' - \varepsilon_i''$ for $i \le 0$, where $(\varepsilon_i'')_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Application to causal linear processes:
In that case, $\xi_n = \sum_{j\ge 0} a_j \varepsilon_{n-j}$, where $(a_j)_{j\ge 0}$ is a sequence of real numbers. We can choose

$$\delta_2(i) \ge \|\varepsilon_0 - \varepsilon_0'\|_2 \sum_{j\ge i} |a_j| + \sum_{j=0}^{i-1} |a_j| \|\varepsilon_{i-j} - \varepsilon_{i-j}'\|_2\,.$$

From Proposition 2.3 in [18], we obtain that

$$\delta_2(i) \le \|\varepsilon_0 - \varepsilon_0'\|_2 \sum_{j\ge i} |a_j| + \sum_{j=0}^{i-1} |a_j| \left(2^2 \int_0^{\beta(\sigma(\varepsilon_k, k\le 0), \sigma(\varepsilon_k, k\ge i-j))} Q_{\varepsilon_0}^2(u)\right)^{1/2} \mathrm{d}u,$$

where Q_{ε_0} is the generalized inverse of the tail function $x \mapsto \mathbb{Q}(|\varepsilon_0| > x)$. In that latter case, notice that assumption (A_3) of Theorem 3.1 is satisfied if the sequence $(|a_j|)_{j\geq 0}$ decreases fast enough to zero and if ε_0 is square integrable. If the particular case where the innovations are i.i.d., we can choose $\delta_2(i) = \|\varepsilon_0 - \varepsilon_0'\|_2 \sum_{j\geq i} |a_j| \leq 2\|\varepsilon_0\|_2 \sum_{j\geq i} |a_j|$. Hence (A_3) is satisfied as soon as $|a_i| = \mathcal{O}\left(i^{-b}\right)$, with $b > 5/2$.

5.2. Example 2: iterated random functions

Let $(\xi_n)_{n\geq 0}$ be a real valued stationary Markov chain, such that $\xi_n = F(\xi_{n-1}, \varepsilon_n)$ for some measurable function F and some independent and identically distributed sequence $(\varepsilon_i)_{i>0}$ independent of ξ_0. Let ξ_0^* be a random variable distributed as ξ_0 and independent of $(\xi_0, (\varepsilon_i)_{i>0})$. Define $\xi_n^* = F(\xi_{n-1}^*, \varepsilon_n)$. The sequence $(\xi_n^*)_{n\geq 0}$ is distributed as $(\xi_n)_{n\geq 0}$ and independent of ξ_0. Let $\mathcal{M}_i = \sigma(\xi_j, 0 \leq j \leq i)$. As in example 1, define the sequence $(\delta_2(i))_{i>0}$ by (5.1). The coefficient θ_2 of the sequence $(\xi_n)_{n\geq 0}$ satisfies the bound (5.2) of example 1.

Let μ be the distribution of ξ_0 and $(\xi_n^x)_{n\geq 0}$ be the chain starting from $\xi_0^x = x$. With these notations, we can choose $\delta_2(i)$ such that

$$\delta_2(i) \geq \|\xi_i - \xi_i^*\|_2 = \left(\int\int \| |\xi_i^x - \xi_i^y\|_2^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y)\right)^{1/2}.$$

For instance, if there exists a sequence $(d_2(i))_{i\geq 0}$ of positive numbers such that

$$\|\xi_i^x - \xi_i^y\|_2 \leq d_2(i)|x-y|,$$

then we can take $\delta_2(i) = d_2(i)\|\xi_0 - \xi_0^*\|_2$. For example, in the usual case where $\|F(x,\varepsilon_0) - F(y,\varepsilon_0)\|_2 \leq \kappa|x-y|$ for some $\kappa < 1$, we can take $d_2(i) = \kappa^i$.

An important example is $\xi_n = f(\xi_{n-1}) + \varepsilon_n$ for some κ-Lipschitz function f. If ξ_0 has a moment of order 2, then $\delta_2(i) \leq \kappa^i \|\xi_0 - \xi_0^*\|_2$.

5.3. Example 3: dynamical systems on $[0,1]$

Let $I = [0,1]$, T be a map from I to I and define $X_i = T^i$. If μ is invariant by T, the sequence $(X_i)_{i\geq 0}$ of random variables from (I,μ) to I is strictly stationary.

For any finite measure ν on I, we use the notations $\nu(h) = \int_I h(x)\nu(dx)$. For any finite signed measure ν on I, let $\|\nu\| = |\nu|(I)$ be the total variation of ν. Denote by $\|g\|_{1,\lambda}$ the $\mathbb{L}^1$-norm with respect to the Lebesgue measure λ on I.

Covariance inequalities. In many interesting cases, one can prove that, for any BV function h and any k in $\mathbb{L}^1(I,\mu)$,

$$|\operatorname{Cov}(h(X_0), k(X_n))| \leq a_n \|k(X_n)\|_1 (\|h\|_{1,\lambda} + \|dh\|), \tag{5.3}$$

for some nonincreasing sequence a_n tending to zero as n tends to infinity.

Spectral gap. Define the operator $\mathcal{L}$ from $\mathbb{L}^1(I,\lambda)$ to $\mathbb{L}^1(I,\lambda)$ *via* the equality

$$\int_0^1 \mathcal{L}(h)(x)k(x)\mathrm{d}\lambda(x) = \int_0^1 h(x)(k\circ T)(x)\mathrm{d}\lambda(x) \quad \text{where } h \in \mathbb{L}^1(I,\lambda) \text{ and } k \in \mathbb{L}^\infty(I,\lambda).$$

The operator $\mathcal{L}$ is called the Perron-Frobenius operator of T. In many interesting cases, the spectral analysis of $\mathcal{L}$ in the Banach space of BV-functions equiped with the norm $\|h\|_v = \|dh\| + \|h\|_{1,\lambda}$ can be done by using the theorem of Ionescu-Tulcea and Marinescu (see [17] and [12]). Assume that 1 is a simple eigenvalue of $\mathcal{L}$ and that the rest of the spectrum is contained in a closed disk of radius strictly smaller than one. Then there exists a unique T-invariant absolutely continuous probability μ whose density f_μ is BV, and

$$\mathcal{L}^n(h) = \lambda(h)f_\mu + \Psi^n(h) \quad \text{with} \quad \|\Psi^n(h)\|_v \leq K\rho^n\|h\|_v. \tag{5.4}$$

for some $0 \leq \rho < 1$ and $K > 0$. Assume moreover that:

$$I_* = \{f_\mu \neq 0\} \text{ is an interval, and there exists } \gamma > 0 \text{ such that } f_\mu > \gamma^{-1} \text{ on } I_*. \tag{5.5}$$

Without loss of generality assume that $I_* = I$ (otherwise, take the restriction to I_* in what follows). Define now the Markov kernel associated to T by

$$P(h)(x) = \frac{\mathcal{L}(f_\mu h)(x)}{f_\mu(x)}. \tag{5.6}$$

It is easy to check (see for instance [1]) that $(X_0, X_1, \ldots, X_n)$ has the same distribution as $(Y_n, Y_{n-1}, \ldots, Y_0)$ where $(Y_i)_{i\geq 0}$ is a stationary Markov chain with invariant distribution μ and transition kernel P. Since $\|fg\|_\infty \leq \|fg\|_v \leq 2\|f\|_v\|g\|_v$, we infer that, taking $C = 2K\gamma(\|\mathrm{d}f_\mu\| + 1)$,

$$P^n(h) = \mu(h) + g_n \quad \text{with} \quad \|g_n\|_\infty \leq C\rho^n\|h\|_v. \tag{5.7}$$

This estimate implies (5.3) with $a_n = C\rho^n$ (see [7]).

Expanding maps: Let $([a_i, a_{i+1}[)_{1\leq i\leq N}$ be a finite partition of $[0, 1[$. We make the same assumptions on T as in [4].

(1) For each $1 \leq j \leq N$, the restriction T_j of T to $]a_j, a_{j+1}[$ is strictly monotonic and can be extented to a function $\overline{T}_j$ belonging to $C^2([a_j, a_{j+1}])$.
(2) Let I_n be the set where $(T^n)'$ is defined. There exists $A > 0$ and $s > 1$ such that $\inf_{x\in I_n} |(T^n)'(x)| > As^n$.
(3) The map T is topologically mixing: for any two nonempty open sets U, V, there exists $n_0 \geq 1$ such that $T^{-n}(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

If T satisfies 1, 2 and 3, then (5.4) holds. Assume furthermore that (5.5) holds (see [19] for sufficient conditions). Then, arguing as in example 4 in Section 7 of [7], we can prove that for the Markov chain $(Y_i)_{i\geq 0}$ and the σ-algebras $\mathcal{M}_i = \sigma(Y_j, j \leq i)$, there exists a positive constant C such that $\theta_2(i) \leq C\rho^i$.

Remark 5.1. In examples 2 and 3, the sequences are indexed by $\mathbb{N}$ and not by $\mathbb{Z}$. However, using existence theorem of Kolmogorov (see Thm. 0.2.7 in [6]), if $(X_i)_{i\in\mathbb{N}}$ is a stationary process indexed by $\mathbb{N}$, there exists a stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that for any $k \leq l \in \mathbb{Z}$, both marginals $(Y_k, \ldots, Y_l)$ and $(X_0, \ldots, X_{l-k})$ have the same distribution. Moreover, in examples 2 and 3, the sequences are Markovian, hence $\theta_2^Y(n) = \theta_2^X(n)$ for any $n \geq 1$. We then apply Theorem 4.1 to the sequence $(Y_i)_{i\in\mathbb{Z}}$. The limit variance can be rewritten as

$$\sigma^2(f) = 2\sum_{x\in\mathbb{Z}} G(0, x)\,\mathrm{Cov}(f(X_0), f(X_{|x|})) - \mathrm{Var}(f(X_0)).$$

6. Application to parametric estimation by random sampling

We investigate in this section the problem of parametric estimation by random sampling for second order stationary processes. We assume that we observe a stationary process $(\xi_i)_{i\in\mathbb{N}}$ at random times S_n, $n \geq 0$, where $(S_n)_{n\geq 0}$ is a non negative increasing random walk satisfying the assumptions of Section 4. In the case where the marginal expectation of the process $(\xi_i)_{i\in\mathbb{N}}$, m, is unknown, Deniau *et al.* [9] estimate it using the sampled empirical mean $\hat{m}_n = \frac{1}{n}\sum_{i=1}^n \xi_{S_i}$. They measure the quality of this estimator by considering the following quadratic criterion function:

$$a(S) = \lim_{n\to+\infty} (n\,\mathrm{Var}\,\hat{m}_n).$$

In the case where $(\mathrm{Cov}(\xi_1, \xi_{n+1}))_{n\in\mathbb{N}}$ is in l^1, we have

$$a(S) = \sum_{k=-\infty}^{+\infty} \mathrm{Cov}(\xi_{S_1}, \xi_{S_{|k|+1}}) < \infty.$$

We then get Corollary 6.1 below, which gives the asymptotic behaviour of the estimate $\hat{m}_n$ after centering and normalization.

Corollary 6.1. *Let us keep the assumptions of Section 4 on the random walk $(S_n)_{n\in\mathbb{N}}$ and on the process $(\xi_i)_{i\in\mathbb{N}}$. Assume moreover that $S_0 = 0$ and that $(S_{n+1} - S_n)_{n\in\mathbb{N}}$ takes its values in $\mathbb{N}^*$. Then, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\sqrt{n}\,(\hat{m}_n - m) \xrightarrow[n\to+\infty]{} \mathcal{N}(0, a(S)).$$

Proof of Corollary 6.1. Corollary 6.1 can be deduced from Theorem 4.1 of Section 4 applied to $f(x) = x - m$. We have indeed $\sigma^2(f) = a(S)$. □

7. Appendix

This section is devoted to the proof of Theorem 3.1 of Section 3.

Proof of Theorem 3.1. Let us assume, without loss of generality, that $\sigma_n = 1$. In a first time, we state second moment inequalities (see Lem. 7.1 below). □

Lemma 7.1. *Assume that $(\eta_i)_{i\in\mathbb{Z}}$ is centered and satisfies conditions (A_2) and (A_3) of Theorem 3.1, then for any reals $-k_n \le a \le b \le k_n$,*

$$\operatorname{Var}\left(\sum_{i=a}^{b} a_{n,i}\eta_i\right) \le C \sum_{i=a}^{b} a_{n,i}^2\,,$$

with $C = \sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right) + 2\sqrt{\sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right)}\ \sum_{l=1}^{\infty}\theta_{1,2}(l)$.

Proof of Lemma 7.1.

$$\begin{aligned}\operatorname{Var}\Big(\sum_{j=a}^{b} a_{n,j}\eta_j\Big) &= \sum_{j=a}^{b} a_{n,j}^2 \operatorname{Var}(\eta_j) + \sum_{i=a}^{b}\sum_{j=a;j\neq i}^{b} a_{n,i}\, a_{n,j} \operatorname{Cov}(\eta_i,\eta_j)\\ &\le \sum_{j=a}^{b} a_{n,j}^2 \operatorname{Var}(\eta_j) + \sum_{i=a}^{b} a_{n,i}^2 \sum_{j=a;j\neq i}^{b} |\operatorname{Cov}(\eta_i,\eta_j)|\end{aligned}$$

by remarking that $|a_{n,i}|\ |a_{n,j}| \le \frac{1}{2}(a_{n,i}^2 + a_{n,j}^2)$.
Then for any $j > i$, using Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}|\operatorname{Cov}(\eta_i,\eta_j)| &= |\mathbb{E}\,(\eta_i\, \mathbb{E}\,(\eta_j|\mathcal{M}_i))|\\ &\le ||\eta_i||_2\ ||\mathbb{E}\,(\eta_j|\mathcal{M}_i)||_2\\ &\le ||\eta_i||_2\ \theta_{1,2}(j-i).\end{aligned}$$

As $(\eta_i)_{i\in\mathbb{Z}}$ is centered, and as $(\eta_i^2)_{i\in\mathbb{Z}}$ is uniformly integrable, we deduce that

$$\operatorname{Var}\Big(\sum_{j=a}^{b} a_{n,j}\eta_j\Big) \le C\sum_{j=a}^{b} a_{n,j}^2,$$

with $C = \sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right) + 2\sqrt{\sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right)}\ \sum_{l=1}^{\infty}\theta_{1,2}(l)$ which is finite from assumptions (A_2) and (A_3). □

First, for any $M > 0$, we define:

$$\varphi_M\ :\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \varphi_M(x) = (x\wedge M)\vee(-M).\end{cases}$$

Using the moment inequalities stated in Lemma 7.1, we can now use a classical truncation argument to reduce the problem to the study of a triangular array $\{Z_{n,i},\ -k_n \le i \le k_n,\ i \in \mathbb{N}\}$ with assumptions:

- $Z_{n,i} = \tilde{a}_{n,i} g_{n,i}(\xi_i)$ with $g_{n,i}$ Lipschitz satisfying $\mathrm{Lip}(g_{n,i}) \le 1$;
- $\max_{-k_n \le i \le k_n} |Z_{n,i}| \le \delta_n$ with $\lim_\infty \delta_n = 0$;
- $\limsup_{n\to+\infty} \sum_{i=-k_n}^{k_n} \tilde{a}_{n,i}^2 < \infty$ and
- $\mathrm{Var}\left(\sum_{i=-k_n}^{k_n} Z_{n,i}\right) = 1$,

with $g_{n,i}(x) = \varphi_{\varepsilon_n/|a_{n,i}|}(x) - \mathbb{E}\left(\varphi_{\varepsilon_n/|a_{n,i}|}(\xi_i)\right)$ where $(\varepsilon_n)_{n\ge 1}$ is a sequence of positive numbers such that $\lim_\infty \varepsilon_n = 0$ and such that

$$\mathrm{Var}\left(\sum_{i=-k_n}^{k_n} a_{n,i} g_{n,i}(\xi_i)\right) \sim_{n\to+\infty} \mathrm{Var}\left(\sum_{i=-k_n}^{k_n} a_{n,i}\xi_i\right) = 1.$$

Remark that the $g_{n,i}$'s satisfy $|g_{n,i}(x)| \le |x| + \sqrt{\sup_{i\in\mathbb{Z}}(\mathbb{E}\xi_i^2)}$, and it implies that the triangular array $\{g_{n,i}(\xi_i),$ $-k_n \le i \le k_n,\ n \in \mathbb{N}\}$ is square uniformly integrable by assumption (A_2) of Theorem 3.1.
We then take $\tilde{a}_{n,i} = a_{n,i}/\sqrt{\mathrm{Var}\left(\sum_{i=-k_n}^{k_n} a_{n,i} g_{n,i}(\xi_i)\right)}$.

Let us prove now that the truncated array satisfies the central limit theorem:

$$\sum_{i=-k_n}^{k_n} Z_{n,i} \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}(0,1). \tag{7.1}$$

The proof is a variation on the proof of Theorem 4.1 in [25]. Let

$$d_t(X,Y) = \left|\mathbb{E}e^{\mathrm{i}tX} - \mathbb{E}e^{\mathrm{i}tY}\right|.$$

To prove Theorem 3.1, it is enough to prove that for all t,

$$d_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i}, \eta\right) \xrightarrow[n\to+\infty]{} 0,$$

with η the standard normal distribution. We first need some simple properties of the distance d_t. Let X, X_1, X_2, Y_1, Y_2 be random variables with zero means and finite second moments. We assume that the random variables Y_1, Y_2 are independent. We define $A_t(X) = d_t\left(X, \eta\sqrt{\mathbb{E}X^2}\right)$. We have then the following inequalities:

Lemma 7.2 (Lem. 4.3 in [25])**.**

$$A_t(X) \le \frac{2}{3}|t|^3\mathbb{E}|X|^3,$$

$$A_t(Y_1+Y_2) \le A_t(Y_1) + A_t(Y_2),$$

$$d_t(X_1+X_2, X_1) \le \frac{t^2}{2}\left(\mathbb{E}X_2^2 + (\mathbb{E}X_1^2\mathbb{E}X_2^2)^{1/2}\right),$$

$$d_t(\eta a, \eta b) \le \frac{t^2}{2}|a^2 - b^2|.$$

We next need the following lemma:

Lemma 7.3. *Let $0 < \varepsilon < 1$. There exists some positive constant $C(\varepsilon)$ such that for all $a \in \mathbb{Z}$, for all $v \in \mathbb{N}^*$, $A_t\left(\sum_{i=a+1}^{a+v} Z_{n,i}\right)$ is bounded by*

$$C(\varepsilon)\left(|t|^3 h^{2/\varepsilon} \sum_{i=a+1}^{a+v} \mathbb{E}\left(|Z_{n,i}|^3\right) + t^2\left(h^{(\varepsilon-1)/2} + \sum_{j\ :2^j \geq h^{1/\varepsilon}} 2^{3j/2} g(2^{j\varepsilon})\right) \sum_{i=a+1}^{a+v} \tilde{a}_{n,i}^2\right),$$

where h is an arbitrary positive natural number and with g introduced in Assumption (A_3) of Theorem 3.1.

Before proving Lemma 7.3, we achieve the proof of Theorem 3.1. By Lemma 7.3, we have

$$d_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i}, \eta\right) = A_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i}\right) \leq C(t,\varepsilon)\left(h^{2/\varepsilon} \sum_{i=-k_n}^{k_n} \mathbb{E}(|Z_{n,i}|^3) + \delta(h) \sum_{i=-k_n}^{k_n} \tilde{a}_{n,i}^2\right),$$

with $\delta(h) = h^{(\varepsilon-1)/2} + \sum_{j\ :2^j \geq h^{1/\varepsilon}} 2^{3j/2} g(2^{j\varepsilon})$.

Now, using assumption (A_3), we get $\delta(h) \xrightarrow[h\to+\infty]{} 0$.

On the other hand we have

$$\sum_{i=-k_n}^{k_n} \mathbb{E}(|Z_{n,i}|^3) \leq \delta_n \sum_{i=-k_n}^{k_n} \operatorname{Var}(Z_{n,i}). \tag{7.2}$$

Then, arguing as for the proof of Lemma 7.1, using assumptions (A_2) and (A_3) of Theorem 3.1 and the fact that the $g_{n,i}$'s are 1-Lipschitz, we get the existence of a finite constant C such that for any reals $-k_n \leq a \leq b \leq k_n$,

$$\operatorname{Var}\left(\sum_{i=a}^{b} Z_{n,i}\right) \leq C \sum_{i=a}^{b} \tilde{a}_{n,i}^2. \tag{7.3}$$

Hence the right hand term of (7.2) is bounded by $C\ \delta_n \sum_{i=-k_n}^{k_n} \tilde{a}_{n,i}^2$, which tends to zero as n tends to infinity.

Consequently

$$\inf_{h\geq 1}\left(h^{2/\varepsilon} \sum_{i=-k_n}^{k_n} \mathbb{E}(|Z_{n,i}|^3) + \delta(h) \sum_{i=-k_n}^{k_n} \tilde{a}_{n,i}^2\right) \xrightarrow[n\to+\infty]{} 0.$$

It achieves the proof of Theorem 3.1.

Proof of Lemma 7.3. Let $h \in \mathbb{N}^*$. Let $0 < \varepsilon < 1$. In the following, C denotes some constant which may vary from line to line. Let κ_ε be a positive constant greater than 1 which will be precised further. Let $v < \kappa_\varepsilon\ h^{1/\varepsilon}$. We have

$$A_t\left(\sum_{i=a+1}^{a+v} Z_{n,i}\right) \leq \frac{2}{3}|t|^3 \mathbb{E}\left|\sum_{i=a+1}^{a+v} Z_{n,i}\right|^3 \leq \frac{2}{3}\ \kappa_\varepsilon^2\ |t|^3\ h^{2/\varepsilon} \sum_{i=a+1}^{a+v} \mathbb{E}(|Z_{n,i}|^3) \tag{7.4}$$

since $|x|^3$ is a convex function.

Let now $v \geq \kappa_\varepsilon\ h^{1/\varepsilon}$. Without loss of generality, assume that $a = 0$. Let $\delta_\varepsilon = (1 - \varepsilon^2 + 2\varepsilon)/2$. Define then

$$m = [v^\varepsilon],\ B = \left\{u \in \mathbb{N}\ :\ 2^{-1}(v - [v^{\delta_\varepsilon}]) \leq um \leq 2^{-1}v\right\},$$

$$A = \left\{u \in \mathbb{N}\ : 0 \leq u \leq v,\ \sum_{i=um+1}^{(u+1)m} \tilde{a}_{n,i}^2 \leq (m/v)^\varepsilon \sum_{i=1}^{v} \tilde{a}_{n,i}^2\right\}.$$

Following [26] we prove that, for $0<\varepsilon<1$, $A\cap B$ is not void for v greater than κ_ε. We have indeed

$$|A\cap B| = |B| - |\overline{A}\cap B| \geq |B| - |\overline{A}| \geq \frac{v^{(1-\varepsilon^2)/2}}{2}\left(1-4v^{-(1-\varepsilon)^2/2}\right) - \frac{3}{2},$$

where $\overline{A}$ denotes the complementary of the set A. We can find κ_ε large enough so that $|A\cap B|$ be positive. □

Let $u\in A\cap B$. We start from the following simple identity

$$\begin{aligned} Q &\equiv \sum_{i=1}^{v} Z_{n,i} \\ &= \sum_{i=1}^{um} Z_{n,i} + \sum_{i=um+1}^{(u+1)m} Z_{n,i} + \sum_{i=(u+1)m+1}^{v} Z_{n,i} \\ &\equiv Q_1+Q_2+Q_3. \end{aligned} \tag{7.5}$$

By Lemma 7.2,

$$d_t(Q,Q_1+Q_3) = d_t(Q,Q-Q_2) \leq \frac{t^2}{2}\left(\mathbb{E}Q_2^2 + (\mathbb{E}Q_2^2\mathbb{E}Q^2)^{1/2}\right). \tag{7.6}$$

Using (7.6) and (7.3), we get

$$d_t(Q,Q_1+Q_3) \leq Ct^2 v^{(\varepsilon-1)\varepsilon/2}\sum_{i=1}^{v}\tilde{a}_{n,i}^2. \tag{7.7}$$

Now, given the random variables Q_1 and Q_3, we define two independent random variables g_1 and g_3 such that the distribution of g_i coincides with that of Q_i, $i=1,3$. We have

$$\begin{aligned} d_t(Q_1+Q_3,g_1+g_3) &= \left|\mathbb{E}(\mathrm{e}^{itQ_1}-1)(\mathrm{e}^{itQ_3}-1) - \mathbb{E}(\mathrm{e}^{itQ_1}-1)\mathbb{E}(\mathrm{e}^{itQ_3}-1)\right| \\ &\leq \left\|\mathrm{e}^{itQ_1}-1\right\|_2 \left\|\mathbb{E}\left(\mathrm{e}^{itQ_3}-1-\mathbb{E}(\mathrm{e}^{itQ_3}-1)\mid \mathcal{M}_{um}\right)\right\|_2 \\ &\leq 2|t|\,\|\textstyle\sum_{i=1}^{um} Z_{n,i}\|_2\, v\,|t|\left(\sum_{i=(u+1)m+1}^{v}|\tilde{a}_{n,i}|\right)\theta_2^{\xi}(m+1) \\ &\leq C\ t^2\ v^{3/2}\left(\textstyle\sum_{i=1}^{v}\tilde{a}_{n,i}^2\right) g(v^\varepsilon), \end{aligned}$$

by (7.3), Definition 2.2 and Assumption (A_3) of Theorem 3.1. Hence

$$d_t(Q_1+Q_3,g_1+g_3) \leq Ct^2 f(v)\sum_{i=1}^{v}\tilde{a}_{n,i}^2, \tag{7.8}$$

where $f(v)=v^{3/2}\ g(v^\varepsilon)$ is non-increasing by assumption (A_3) of Theorem 3.1.

We also have by Lemma 7.2

$$A_t(g_1+g_3) \leq A_t(g_1) + A_t(g_3). \tag{7.9}$$

Finally, still by Lemma 7.2, and using Definition 2.2, we have

$$\begin{aligned} d_t\left(\eta\sqrt{\mathbb{E}(Q^2)},\eta\sqrt{\mathbb{E}\left((g_1+g_3)^2\right)}\right) &\leq \frac{t^2}{2}\left|\mathbb{E}(Q^2)-\mathbb{E}\left((g_1+g_3)^2\right)\right| \\ &\leq \frac{t^2}{2}\left|\mathbb{E}(Q_2^2)+2\mathbb{E}(Q_1Q_2)+2\mathbb{E}(Q_2Q_3)+2\mathbb{E}(Q_1Q_3)\right| \\ &\leq Ct^2\left(v^{(\varepsilon-1)\varepsilon/2}+f(v)\right)\sum_{i=1}^{v}\tilde{a}_{n,i}^2. \end{aligned} \tag{7.10}$$

Combining (7.7)-(7.10), we get the following recurrent inequality:

$$\begin{aligned} A_t\left(\sum_{i=1}^{v} Z_{n,i}\right) \quad &\leq \quad A_t\left(\sum_{i=1}^{um} Z_{n,i}\right) + A_t\left(\sum_{i=(u+1)m+1}^{v} Z_{n,i}\right) \\ &\quad + Ct^2\left(v^{(\varepsilon-1)\varepsilon/2} + f(v)\right)\sum_{i=1}^{v} \tilde{a}_{n,i}^2 \end{aligned}$$

for $v \geq \kappa_\varepsilon \; h^{1/\varepsilon} \geq \kappa_\varepsilon$.

We then need the following lemma, which is a variation on Lemma 1.2. in [26].

Lemma 7.4. *For every* $\varepsilon \in \,]0,1[$, *denote* $\delta_\varepsilon = (1-\varepsilon^2+2\varepsilon)/2$. *Let a non-decreasing sequence of non-negative numbers* $a(n)$ *be specified, such that there exist non-increasing sequences of non-negative numbers* $\varepsilon(k)$, $\gamma(k)$ *and a sequence of naturals* $T(k)$, *satisfying conditions*

$$T(k) \leq (k + [k^{\delta_\varepsilon}])/2,$$

$$a(k) \leq \max_{k_0 \leq s \leq k} (a(T(s)) + \gamma(s))$$

for any $k \geq k_0$ *with an arbitrary* $k_0 \in \mathbb{N}^*$. *Then*

$$a(n) \leq a(n_0) + 2 \sum_{k_0 \leq 2^j \leq n} \gamma(2^j),$$

for any $n \geq k_0$, *where one can take* $n_0 = 2^c$ *with* $c > (2-\delta_\varepsilon)/(1-\delta_\varepsilon)$.

Proof of Lemma 7.4. The proof follows essentially the same lines as the proof of Lemma 1.2. in [26] and therefore is omitted here. □

We now apply Lemma 7.4 above with

- ⋆ $k_0 = \kappa_\varepsilon \; h^{1/\varepsilon}$,
- ⋆ for $k \geq k_0$, $T(k) = \max\{u_k m_k, k - u_k m_k - m_k\}$ where u_k and m_k are defined from k as u and m from v (see the proof of $A \cap B$ not void),
- ⋆ $c < \ln(\kappa_\varepsilon)/\ln(2)$ (we may need to enlarge κ_ε),
- ⋆ for $s \geq k_0$, $\gamma(s) = C\; t^2 \; \left(s^{\varepsilon(\varepsilon-1)/2} + f(s)\right)$,
- ⋆ for $s \geq k_0$, $a(s) = \sup_{l \in \mathbb{Z}} \max_{k_0 \leq i \leq s} \dfrac{A_t\left(\sum_{j=l+1}^{l+i} Z_{n,j}\right)}{\sum_{j=l+1}^{l+i} \tilde{a}_{n,j}^2}$.

Applying Lemma 7.4 yields the statement of Lemma 7.3.

References

[1] A.D. Barbour, R.M. Gerrard and G. Reinert, Iterates of expanding maps. *Probab. Theory Relat. Fields* **116** (2000) 151–180.
[2] J.M. Bardet, P. Doukhan, G. Lang and N. Ragache, Dependent Linderberg central limit theorem and some applications. *ESAIM: PS* **12** (2008) 154–172.
[3] H.C.P. Berbee, Random walks with stationary increments and renewal theory. Math. Centre Tracts 112, Amsterdam (1979).
[4] P. Collet, S. Martinez and B. Schmitt, Exponential inequalities for dynamical measures of expanding maps of the interval. *Probab. Theory. Relat. Fields* **123** (2002) 301–322.
[5] C. Coulon-Prieur and P. Doukhan, A triangular CLT for weakly dependent sequences. *Statist. Probab. Lett.* **47** (2000) 61–68.
[6] D. Dacunha-Castelle and M. Duflo, Problèmes à temps mobile. Deuxième édition. Masson (1993).
[7] J. Dedecker and C. Prieur, New dependence coefficients, Examples and applications to statistics. *Probab. Theory Relat. Fields* **132** (2005) 203–236.
[8] J. Dedecker, P. Doukhan, G. Lang, J.R. Leon, S. Louhichi and C. Prieur, Weak dependence: With Examples and Applications. *Lect. Notes in Stat.* **190**. Springer, XIV (2007).
[9] C. Deniau, G. Oppenheim and M.C. Viano, Estimation de paramètre par échantillonnage aléatoire. (French. English summary) [Random sampling and parametric estimation]. *C. R. Acad. Sci. Paris Sér. I Math.* **306** (1988) 565–568.

[10] P. Doukhan and S. Louhichi, A new weak dependence condition and applications to moment inequalities. *Stochastic Process. Appl.* **84** (1999) 313–342.
[11] N. Guillotin-Plantard and D. Schneider, Limit theorems for sampled dynamical systems. *Stoch. Dynamics* **3** (2003) 477–497.
[12] F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** (1982) 119–140.
[13] I.A. Ibragimov, Some limit theorems for stationary processes. *Theory Probab. Appl.* **7** (1962) 349–382.
[14] J.F.C. Kingman, The ergodic theory of subadditive stochastic processes. *J. R. Statist. Soc. B* **30** (1968) 499–510.
[15] M. Lacey, On weak convergence in dynamical systems to self-similar processes with spectral representation. *Trans. Amer. Math. Soc.* **328** (1991) 767–778.
[16] M. Lacey, K. Petersen, D. Rudolph and M. Wierdl, Random ergodic theorems with universally representative sequences. *Ann. Inst. H. Poincaré Probab. Statist.* **30** (1994) 353–395.
[17] A. Lasota and J.A. Yorke, On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.* **186** (1974) 481–488.
[18] F. Merlevède and M. Peligrad, On the coupling of dependent random variables and applications. *Empirical process techniques for dependent data*. Birkhäuser, Boston (2002), pp. 171–193.
[19] T. Morita, Local limit theorem and distribution of periodic orbits of Lasota-Yorke transformations with infinite Markov partition. *J. Math. Soc. Jpn* **46** (1994) 309–343.
[20] M. Peligrad and S. Utev, Central limit theorem for linear processes. *Ann. Probab.* **25** (1997) 443–456.
[21] E. Rio, About the Lindeberg method for strongly mixing sequences. *ESAIM: PS* **1** (1995) 35–61.
[22] M. Rosenblatt, A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA* **42** (1956) 43–47.
[23] Y.A. Rozanov and V.A. Volkonskii, Some limit theorems for random functions I. *Theory Probab. Appl.* **4** (1959) 178–197.
[24] C.J. Stone, On local and ratio limit theorems. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* Univ. Californie (1966), pp. 217–224.
[25] S.A. Utev, Central limit theorem for dependent random variables. *Probab. Theory Math. Statist.* **2** (1990) 519–528.
[26] S.A. Utev, Sums of random variables with φ-mixing. *Siberian Adv. Math.* **1** (1991) 124–155.
[27] C.S. Withers, Central limit theorems for dependent variables. *I. Z. Wahrsch. Verw. Gebiete* **57** (1981) 509–534 (corrigendum in *Z. Wahrsch. Verw. Gebiete* **63** (1983) 555).