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ENHANCED GAUSSIAN PROCESSES AND APPLICATIONS

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Abstract. We propose some construction of enhanced Gaussian processes using Karhunen-Loeve expansion. We obtain a characterization and some criterion of existence and uniqueness. Using rough-path theory, we derive some Wong-Zakai Theorem.

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1. Generalities

In [13] Lyons developed a general theory of differential equations of the form

$$dy_t = f(y_t)dx_t. \tag{1.1}$$

Classical integration/ODE theory gives a meaning to such differential equations when x has bounded variation. Lyons extended this notion to the case when x is a path with values in a Banach space B, and of finite p-variation, $p \geq 1$. To do so, one needs first to lift x to a path of finite p-variation in the free nilpotent group of B. In other words, one needs to define and make a choice for the "iterated integrals" of order less than or equal to $[p]$ of x. We refer the reader to, for example, [11,13,14].

In this paper, our aim is to work towards the study of a "natural" p-rough path process lying above an arbitrary Gaussian process. We simplify the problem by only looking at lift in the free nilpotent group of step 2, *i.e.* we are just looking at the Lévy area of Gaussian processes. This was already done by Lévy in 1950 for Brownian motion, see [12] or more recently [10] and [6], and for fractional Brownian motion, see [5]or [15]. Moreover, Biane and Yor, in [1] have constructed the Lévy area using the expansion of Brownian motion in the basis of Legendre polynoms.

Karhunen-Loeve expansion Theorem provide a natural way to approximate paths of a Gaussian process by a smooth process. This paper is devoted to study how its expansion allow to lift $\mathbb{R}^d$-valued Gaussian process x to a path $\mathbf{x}$ with values in some free nilpotent of step 2 group over $\mathbb{R}^d$ (or in other words, how to construct the Lévy area of x, *i.e.* the second iterated integral of x). We also show that if the process x with some area process satisfies some quite natural conditions, then $\mathbf{x}$ will be the limit of the lift of the Karhunen-Loeve approximations of x.

The proof of the convergence of Karhunen-Loeve expansion Theorem or of some properties on Gaussian processes relies on the convex property of the vector spaces. The free nilpotent group of step 2 do not share this property. In the first part of this paper, we give a proof of a weak version the Karhunen-Loeve expansion

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Theorem using a discrete martingale. Some basic results on the free nilpotent group of step 2 are given. Then, in the second part, using again some martingales, we lift the process x to a path $\mathbf{x}$ with values in some free nilpotent of step 2 group over $\mathbb{R}^d$. A characterisation and a result of uniqueness is also given. For the Brownian motion and the fractional Brownian motion, this definition coincides with the one obtained by dyadic linear approximation as in [5]. In the third part the case of Volterra Gaussian processes is studied. We conclude with a Wong-Zakai Theorem.

1.1. Gaussian processes

We define on the measure space $\Omega = C_0\left([0,1],\mathbb{R}^d\right)$ and its Borel σ-algebra denoted by $\mathbb{F}$, the probability measure $\mathbb{P}$ corresponding to the law of a d-dimensional centered Gaussian process with covariance function C. We let $(H,\langle,\rangle)$ the associated Cameron-Martin space associated to $\mathbb{P}$. We assume that the process has continuous sample paths, then it is continuous in $L^2(\Omega,\mathbb{F},\mathbb{P})$ and the covariance function is continuous. Following [9] Theorem 2.8.2, the space $(H,\langle,\rangle)$ is separable. Let $e=(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis on $(H,\langle,\rangle)$. One can always represents X under $\mathbb{P}$ with the formula

$$X^k = \sum_{i=0}^{\infty}(N_i^e)^k e_i^k \tag{1.2}$$

where $N_i^e = \langle X, e_i\rangle$ are independent standard d-dimensional normal random variables. Here $\langle,\rangle$ is the duality bracket. We let $\mathcal{F}_n^e = \sigma\left(N_i^e, 0\le i\le n\right)$.

We warm up with the following two propositions. Their results (and stronger results) are well known, see Theorem 2.4.2 of [9], but the proof given here allow us to generalize in the next section to the "natural lift" of X to a process with values in some free nilpotent group.

Proposition 1. *For all* $t\in[0,1]$,

$$X_n^e(t) := E\left(X(t)\,|\mathcal{F}_n^e\right) = \sum_{i=0}^{n} N_i^e e_i(t).$$

Proof. It is just the observation that $\sum_{i=n+1}^{\infty} N_i^e e_i$ is mean 0 and independent of $\mathcal{F}_n^e$. □

Proposition 2. *For all* $q\ge 1$, *and for all* $t\in[0,1]$, $X_n^e(t)$ *converges to* $X(t)$ *almost surely and in* L^q.

Proof. Since X has continuous sample paths, then almost surely $\|X\|_\infty := \sup_{t\in[0,1]}\|X(t)\| < \infty$. Note that $\|X\|_\infty<\infty$ a.s. implies that the r.v. $\|X\|_\infty$ has a Gaussian tail (from Borell's inequality), and therefore is in L^q for all $1\le q<\infty$. For all $t\in[0,1]$

$$\begin{aligned} |X_n^e(t)| &= \left|E\left(X(t)\,|\mathcal{F}_n^e\right)\right| \\ &\le E\left(\|X\|_\infty\,|\mathcal{F}_n^e\right). \end{aligned}$$

Taking the supremum over all t, we obtain that $\|X_n^e\|_\infty \le E\left(\|X\|_\infty\,|\mathcal{F}_n^e\right)$. Therefore, by Doob's inequality, $\sup_n\|X_n^e\|_\infty$ is in L^q for all $1\le q<\infty$. By the martingale convergence theorem, $X_n^e(t)\to X(t)$ for all t, where the convergence is in L^q and a.s. □

1.2. Free nilpotent group of step 2

1.2.1. *Definitions*

We define $G^2\left(\mathbb{R}^d\right)$ to be the space $\left\{(x,y)\in\mathbb{R}^d\oplus M_d(\mathbb{R}),\ y^{i,j}+y^{j,i}=x^ix^j\right\}$ together with the product

$$(x_1,y_1)\otimes(x_2,y_2) = \left(x_1+x_2, y_1+y_2+\left(x_1^ix_2^j\right)_{i,j}\right).$$

Indeed, $\left(G^2\left(\mathbb{R}^d\right), \otimes\right)$ is the free nilpotent group of step 2 over $\mathbb{R}^d$.

We define for a $\mathbb{R}^d$-valued path x of finite q-variation for $q < 2$, the canonical lift of x to a $G^2\left(\mathbb{R}^d\right)$-valued path:

$$S(x)_t = \left(x_t, \int_0^t x_u^i dx_u^j\right), \ t \in [0,1].$$

Observe that $G^2\left(\mathbb{R}^d\right) = \left\{S(x)_1, \ x \text{ smooth } \mathbb{R}^d\text{-valued path}\right\}$. That allows us to define a homogeneous norm on $G^2\left(\mathbb{R}^d\right)$:

$$\|g\| = \sup_{\substack{x \text{ smooth} \\ S(x)_1 = g}} \int_0^1 \left|\dot{x}_u\right| du,$$

and from this homogeneous norm, a left invariant distance on $G^2\left(\mathbb{R}^d\right)$:

$$d(g,h) = \left\|g^{-1} \otimes h\right\|.$$

If $g = (x,y) \in G^2\left(\mathbb{R}^d\right)$, we define $\pi_i(g)$ to be the projection of x on the ith component of $\mathbb{R}^d$, and $\pi_{j,k}(g)$ the (j,k)th component of y on $\mathbb{R}^d \times \mathbb{R}^d$. In particular, if x is a smooth $\mathbb{R}^d$-valued path then $\pi_i\left(S(x)_1\right) = x_1^i$ and $\pi_{j,k}\left(S(x)_1\right) = \int_0^1 x_u^j dx_u^k$.

We have an equivalence of homogeneous norm result: there exists some constant $c, C > 0$ such that for all $g \in G^2(\mathbb{R}^d)$,

$$c\|g\| \le \max_{i,j,k}\left\{\left|\pi_i(g)\right|, \sqrt{\left|\pi_{j,k}(g)\right|}\right\} \le C\|g\|. \tag{1.3}$$

1.2.2. *Paths with values in* $G^2\left(\mathbb{R}^d\right)$

When x is a path in $C_0\left([0,1], G^2(\mathbb{R}^d)\right)$, the space of continuous functions from $[0,1]$ into $G^2\left(\mathbb{R}^d\right)$ starting at 0, we let as a notation

$$x_{s,t} = x_s^{-1} \otimes x_t, \quad (s,t) \in [0,1]^2.$$

On $C_0\left([0,1], G^2(\mathbb{R}^d)\right)$, we define the following distances:

$$\begin{aligned} d_\infty(x,y) &= \sup_{0 \le t \le 1} d(x_t, y_t), \\ \|x\|_\infty &= d_\infty(0,x). \end{aligned}$$

For a given control[1] ω and $p \ge 1$, we define

$$\begin{aligned} d_{p,\omega}(x,y) &= \sup_{0 \le s < t \le 1} \frac{d(x_{s,t}, y_{s,t})}{\omega(s,t)^{1/p}}, \\ \|x\|_{p,\omega} &= d_{p,\omega}(0,x). \end{aligned}$$

The applications $\|.\|_\infty$ and $\|.\|_{p,\omega}$ are not some pseudo norms since $x, y \in C_0\left([0,1], G^2(\mathbb{R}^d)\right)$ does not imply $x \otimes y \in C_0\left([0,1], G^2(\mathbb{R}^d)\right)$.

[1] *I.e.*, a continuous map from $\{s \le t, s, t \in [0,1]\}$ such that $\omega(t,s) + \omega(s,u) \le \omega(t,u), \ \ \forall\, t \le s \le u$, null on the diagonal.

1.2.3. *Translation operator on path space*

We will want to "add" two paths with values in $G^2(\mathbb{R}^d)$. This can be done when between these two paths make sense. This addition or translation operation will be denoted T.

The map T can be understood in the following way: for some smooth paths x and h, we define

$$T_h(S(x)) := S(x+h).$$

Then if $(S(x_n))$ converges to X in the uniform topology associated to d_∞ and h_n converges to h in bounded variation, then $(T_{h_n}(S(x_n)))$ converges in the uniform topology to a continuous $G^2(\mathbb{R}^d)$-valued path denoted $T_h(X)$. One can check that $T_h(X)$ satisfies

$$\begin{aligned} \pi_i(T_h(X)) &= h^i + x^i, \\ \pi_{i,j}(T_h(X)) &= \pi_{i,j}(X) + \int_0^{\cdot} h_u^i dh_u^j + \int_0^{\cdot} x_u^i dh_u^j \\ &\quad + h_\cdot^i x_\cdot^j - h_0^i x_0^j - \int_0^{\cdot} x_u^j dh_u^i. \end{aligned} \tag{1.4}$$

2. Natural lift of a Gaussian process to a $G^2(\mathbb{R}^d)$-valued process

2.1. Definition and first property

We denote by BV the set of continuous paths of bounded variation.

Assumption 1. (1) *There exists an orthonormal basis $e = (e_i)_{i\geq 0}$ of $(H, \langle , \rangle)$ which is in $H \cap BV$;*
(2) *the components of X are independent.*

Example 1. *Point (1) of Assumption 1 is fulfilled if C is continuous for the usual distance on $[0,1]^2$, and for all $t \in [0,1]$, $C(t,.) \in BV$. Indeed, the vector space generated by $\{C(t,.), \quad t \in [0,1]\}$ is dense in $(H, \langle , \rangle)$ and Assumption 1 follows from an orthonormalisation procedure.*

Example 2. *In particular, Assumption 1 is satisfied for fractional Brownian motion, for any Hurst parameter $h > 0$.*

All the orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H that we will consider will be implicitly assumed to be in BV.

Definition 1. *We say that $\mathbf{X} : [0,1] \to G^2(\mathbb{R}^d)$ defines* ***a natural lift of the Gaussian process*** *X*, ***associated to the orthonormal basis*** *e (to a $G^2(\mathbb{R}^d)$-valued process) if*
(1) *$S\left(X_{0,n}^e\right)_t$ converges in probability to $\mathbf{X}_t$ for all $t \in [0,1]$;*
(2) *$\mathbf{X}$ has a continuous sample paths.*

Note from the definition of the canonical lift on smooth $\mathbb{R}^d$ valued path and Proposition 2, that

$$\begin{aligned} \pi_i(\mathbf{X}) &= X^i, \quad i \in \{1, ..., d\} \\ \pi_{i,i}(\mathbf{X}) &= \frac{(X^i)^2}{2}\cdot \end{aligned} \tag{2.1}$$

Lemma 1. *Let e be an orthonormal basis on H, such that $e_n \in BV$ for $n \in \mathbb{N}$. Let $t \in [0,1]$, the random variable $(S(X_n^e)_t)$ converges almost surely if and only if*

$$\max_{i,j} \sum_{0\leq l<k<\infty} \left[\int_0^t \left(e_l^i(s)\dot{e}_k^j(s) - e_k^j(s)\dot{e}_l^i(s)\right) \mathrm{d}s\right]^2 < +\infty.$$

Proof. From equality (2.1) and Proposition 2, we only have to study the convergence of $((\pi_{i,j}-\pi_{j,i})\,(S(X_n^e)_t))$. First observe that $((\pi_{i,j}-\pi_{j,i})\,(S(X_n^e)_t))$ is $\mathcal{F}_n^e$-martingale. Moreover, because its belongs to the second Wiener chaos, the convergence of $((\pi_{i,j}-\pi_{j,i})\,S(X_n^e)_t)$ in probability is equivalent to the convergence in L^2. By martingale convergence theorem, $((\pi_{i,j}-\pi_{j,i})\,S(X_n^e)_t)$ converges in L^2 and almost surely if and only if $\lim_{n\to\infty} E\left(|(\pi_{i,j}-\pi_{j,i})\,S(X_n^e)_t|^2\right)<\infty$. But

$$E\left(|(\pi_{i,j}-\pi_{j,i})\,(S(X_n^e)_t)|^2\right) = E\left(\left|\sum_{0\le l,k\le n}\left(N_l^iN_k^j\right)\int_0^t\left(e_l^i(s)\dot{e}_k^j(s)-e_k^j(s)\dot{e}_l^i(s)\right)\mathrm{d}s\right|^2\right)$$
$$=\sum_{0\le l,k\le n}\left[\int_0^t\left(e_l^i(s)\dot{e}_k^j(s)-e_k^j(s)\dot{e}_l^i(s)\right)\mathrm{d}s\right]^2.$$

Observe that we have used the independence of the coordinates of the Gaussian process X. □

A kind of $0-1$ law is also available.

Lemma 2. *Let e be an orthonormal basis on H, such that $e_n\in BV$ for $n\in\mathbb{N}$. Let $t\in[0,1]$. If $\mathbb{P}(\{\omega,\ (S(X_n^e)_t(\omega))_{n\in\mathbb{N}}$ converges $\})>0$, then $\mathbb{P}(\{\omega,\ (S(X_n^e)_t(\omega))_{n\in\mathbb{N}}$ converges $\})=1$.*

Proof. Assume that $\mathbb{P}(\{\omega,\ (S(X_n^e)_t(\omega))_{n\in\mathbb{N}}$ converges $\})>0$, and denote for $i,j\in\{1,...,d\}$

$$\Gamma^{i,j}=\{\omega,\ \ (S(X_n^e)_t^{i,j}(\omega)-S(X_n^e)_t(\omega)^{j,i})_{n\in\mathbb{N}}\ \ \text{converges}\ \}.$$

For $i\in\{1,..,d\}$, observe that $S(X_n^e)_t^i=\frac{(X_n^e(t)^i)^2}{2}$. Theorem 1.1.1 of [9] applied to the Gaussian vector $(X_n^e(t)^i)_{n\in\mathbb{N}}$ yields

$$\mathbb{P}(\{(X_n^e(t)^i)_{n\in\mathbb{N}}\quad\text{and}\ \ (X_n^e(t)^{i,i})_{n\in\mathbb{N}}\ \ \text{converge}\})=1.$$

For $i\neq j$, conditionally to $\sigma(N_l^i,\ \ l\in\mathbb{N})$, $(S(X_n^e)_t^{i,j}-S(X_n^e)_t^{j,i})_{n\in\mathbb{N}}$ is a Gaussian vector, and using the same arguments, almost surely

$$\mathbb{E}(\mathbf{1}_{\Gamma^{i,j}}/\sigma(N_l^i,\ \ l\in\mathbb{N}))=\mathbf{1}_{\Gamma^{i,j}}.$$

But the role of i and j in the conditioning are symmetric and the following equality holds

$$\mathbb{E}(\mathbf{1}_{\Gamma^{i,j}}/\sigma(N_l^i,\ \ l\in\mathbb{N}))=\mathbf{1}_{\Gamma^{i,j}}=\mathbb{E}(\mathbf{1}_{\Gamma^{i,j}}/\sigma(N_l^j,\ \ l\in\mathbb{N})).\tag{2.2}$$

Since the σ fields $\sigma(N_l^j,\ \ l\in\mathbb{N})$ and $\sigma(N_l^i,\ \ l\in\mathbb{N})$ are independent, conditioning all terms of equality (2.2) by $\sigma(N_l^j,\ \ l\in\mathbb{N})$ yields

$$\mathbf{1}_{\Gamma_{i,j}}=\mathbb{P}(\Gamma_{i,j})>0.$$

Then, $\mathbf{1}_{\Gamma_{i,j}}=1$ almost surely. This achieves the proof, since

$$S(X_n^e)^{i,j}=\frac{1}{2}\left[(X_n^e)^i(X_n^e)^j+S(X_n^e)^{i,j}-S(X_n^e)^{j,i}\right].$$ □

2.2. A characterization of a natural lift, and a uniqueness result

We will use the maps

$$\begin{array}{rcl}\phi_i & : & C_0\left([0,1],\mathbb{R}^d\right)\to C_0\left([0,1],\mathbb{R}^d\right)\\ (x_1,...,x_d) & \to & (x_1,...,x_{i-1},-x_i,x_{i+1},...,x_d)\,.\end{array}$$

Observe that $\mathbb{P}\circ\phi_i=\mathbb{P}$ for all i. ($\mathbb{P}$ is the probability measure introduced in the previous section.)

Theorem 1. *Assume that Assumption 1 is fulfilled.*

The path $\mathbf{X}:[0,1]\to G^2\left(\mathbb{R}^d\right)$ *is a natural lift of* X *for some orthonormal basis* e*, if and only if there exists a measurable map*

$$\Psi : C_0\left([0,1],\mathbb{R}^d\right)\to C_0\left([0,1],G^2\left(\mathbb{R}^d\right)\right)\cup\{\delta\}$$

(where δ *is a cemetery point) such that*

Definition 2. (1) $\mathbf{X}=\Psi(X)$ *a.s. and* $\Psi(X)\neq\delta$ *a.s.*

(2) *The projection of* $\Psi(X)$ *onto* $\mathbb{R}^d$ *is equal to* X *a.s. (Lifting property.)*

(3) $\begin{cases}\pi_{j,k}\Psi\left(\phi_i(X)\right)=-\pi_{j,k}\Psi(X) & \text{if } i\in\{j,k\},\\ \pi_{j,k}\Psi\left(\phi_i(X)\right)=\pi_{j,k}\Psi(X) & \text{if } i\notin\{j,k\}.\end{cases}$ *(Symmetry property.)*

(4) *For all path* $h\in H\cap BV$*,* $\Psi(h)=S(h)$*. (Definition on "smooth" paths.)*

(5) *For all path* $h\in H\cap BV$*,* $\Psi(X+h)=T_h\Psi(X)$ *almost surely. (Stability of translations property.)*

(6) *The r.v.* $\|\mathbf{X}_{s,t}\|$ *is in* $L^2(\Omega,\mathbb{F},\mathbb{P})$ *for* $0\le s<t\le 1$*. (Integrability property.)*

Proof. If $\mathbf{X}$ is a natural lift (associated to an orthonormal basis f), it is easy to check that it satisfies properties (1) to (6).

Conversely, we want to check that if Ψ is a measurable map satisfying the above condition, then $\Psi(X)$ is the natural lift associated to e. The proof will be complete once we prove that for all n, for all $0\le s\le t\le 1$,

$$E\left(\log\Psi(X)_{s,t}|\mathcal{F}_n^e\right)=\log S\left(X_n^e\right)_{s,t}. \tag{2.3}$$

Indeed, the martingale $\log S\left(X_{0,,n}^e\right)_{s,t}$ converges to $\log\mathbf{X}_{s,t}$, and the above equality plus the fact that $\log\Psi(X)_{s,t}$ is $\mathcal{F}_{0,\infty}^e$-measurable would prove that $\mathbf{X}_{s,t}=\Psi(X)_{s,t}$. Here, log is defined by its power serie.

The first level of equality (2.3) was proved in Proposition 1. We therefore only need to prove that for $i\neq j$, $0\le s\le t\le 1$, $n\in\mathbb{N}$ using (1.4),

$$E\left((\pi_{i,j}-\pi_{j,i})\left(\Psi(X)_{s,t}\right)|\mathcal{F}_n^e\right)=(\pi_{i,j}-\pi_{j,i})\left(S\left(X_n^e\right)_{s,t}\right).$$

From the stability of translations property, $\Psi(X)=T_{X_n^e}\left(\Psi\left(X-X_n^e\right)\right)$. In particular, for all $0\le s<t\le 1$, and $1\le i<j\le d$,

$$\begin{aligned}(\pi_{i,j}-\pi_{j,i})\left(\Psi(X)_{s,t}\right) &= (\pi_{i,j}-\pi_{j,i})\left(\Psi\left(X-X_n^e\right)_{s,t}\right)+(\pi_{i,j}-\pi_{i,j})\left(S\left(X_n^e\right)_{s,t}\right)\\ &\quad+\int_s^t\left(X-X_n^e\right)_{s,u}^i d\left(X_n^e\right)_u^j-\int_s^t\left(X-X_n^e\right)_{s,u}^j d\left(X_n^e\right)_u^i\\ &\quad-\int_s^t\left(X-X_n^e\right)_{s,u}^j d\left(X_n^e\right)_u^i+\int_s^t\left(X-X_n^e\right)_{s,u}^i d\left(X_n^e\right)_u^j\\ &\quad+\left(X_n^e\right)_{s,t}^i\left(X-X_n^e\right)_{s,t}^j-\left(X_n^e\right)_{s,t}^j\left(X-X_n^e\right)_{s,t}^i.\end{aligned} \tag{2.4}$$

It is easy to check that all the expressions in equality (2.4) are in $L^2(\Omega,\mathbb{F},\mathbb{P})$. As $(X-X_n^e)$ is independent of $\mathcal{F}_n^e$ while N_k^e is $\mathcal{F}_n^e$-measurable,

$$\begin{aligned}E\left(\int_s^t\left(X-X_n^e\right)_{s,u}^i d\left(X_n^e\right)^j(u)|\mathcal{F}_n^e\right) &= \sum_{k=0}^n N_k^j\int_s^t E\left(\left(X-X_n^e\right)_{s,u}^i|\mathcal{F}_n^e\right)de_k^j(u)\\ &= \sum_{k=0}^n N_k^j\int_s^t E\left(\left(X-X_n^e\right)_{s,u}^i\right)de_k^j(u)\\ &= 0.\end{aligned}$$

The same equality and argument applies to $\int_s^t (X - X_n^e)_{s,u}^j \, d(X_n^e)_u^i$ and to $(X_n^e)_{s,t}^i (X - X_n^e)_{s,t}^j$ and the reverse expressions. Therefore,

$$E\left((\pi_{i,j} - \pi_{j,i})\left(\Psi(X)_{s,t}\right)|\mathcal{F}_n^e\right) = (\pi_{i,j} - \pi_{j,i})\left(S(X_n^e)_{s,t}\right) + E\left((\pi_{i,j} - \pi_{j,i})\left(\Psi(X - X_n^e)_{s,t}\right)|\mathcal{F}_n^e\right).$$

From the symmetry assumption, $(\pi_{i,j} - \pi_{j,i})\left(\Psi(X - X_n^e)_{s,t}\right) = -(\pi_{i,j} - \pi_{j,i})\left(\Psi \circ \Phi_i (X - X_n^e)_{s,t}\right)$. Since X and $\phi_i(X)$ has the same law, $(\pi_{i,j} - \pi_{j,i})\left(\Psi(X - X_n^e)_{s,t}\right)$ is a centered random variable. Hence, as $X - X_n^e$ is independent of $\mathcal{F}_n^e$, we obtain,

$$E\left((\pi_{i,j} - \pi_{j,i})\left(\Psi(X - X_n^e)_{s,t}\right)|\mathcal{F}_n^e\right) = 0.$$

Therefore,

$$E\left((\pi_{i,j} - \pi_{j,i})\left(\Psi(X)_{s,t}\right)|\mathcal{F}_n^e\right) = (\pi_{i,j} - \pi_{j,i})\left(S(X_n^e)_{s,t}\right).$$ □

As a simple corollary of the previous result, we obtain the important result of uniqueness of the natural lift.

Corollary 1. *Let X be a Gaussian process, and assume that there exists a natural lift $\mathbf{X}$ of X associated to some orthonormal basis e of H in BV. Then, for all orthonormal basis f of H in BV, there exists a natural lift $\mathbf{X}^f$ associated to X. Moreover, almost surely, for all such orthonormal basis, $\mathbf{X}^f = \mathbf{X}^e$.*

2.3. **Other constructions**

Theorem 2. *Assume that there exists linear measurable maps $\Delta_n : C_0\left([0,1], \mathbb{R}^d\right) \to H \cap BV$ such that*

(1) *almost surely, $S \circ \Delta_n(X)$ converges in uniform topology;*
(2) *$\Delta_n(h)$ converges pointwise to h and $\sup_n |\Delta_n(h)|_{BV} < \infty \ \forall h \in H \cap BV$;*
(3) *for all $1 \le i \le d$, $\Delta_n \circ \phi_i = \phi_i \circ \Delta_n$.*

Then, there exists a (unique up to indistingability) natural lift of X, and it is $\mathbf{X} := \lim_{n\to\infty} S \circ \Delta_n(X)$.

Proof. We define $\Psi(X) = \lim_{n\to\infty} S \circ \Delta_n(X)$. Condition 1 clearly implies that $\Psi(X)$ has almost surely continuous paths. Hence, $T_h(\Psi(X))$ exists for all $h \in BV$. Moreover,

$$\begin{aligned} S \circ \Delta_n(X+h) &= S\left(\Delta_n(X) + \Delta_n(h)\right) \\ &= T_{\Delta_n(h)}\left(S \circ \Delta_n(X)\right), \end{aligned}$$

and by property of the translation operator, we see that $T_{\Delta_n(h)}\left(S \circ \Delta_n(X)\right)$ converges in uniform topology to variation topology to $T_h\left(\Psi(X)\right)$. Hence, $\Psi(X+h)$ is well defined a.s. and equal a.s. to $T_h(\Psi(X))$. The other conditions of Theorem 1 are easily checked to be true. □

Corollary 2. *The level n dyadic piecewise linear approximation of a continuous path, i.e.*

$$\Delta_n(x)_t = x_{\frac{k}{2^n}} + (2^n t - k)\left(x_{\frac{k+1}{2^n}} - x_{\frac{k}{2^n}}\right) \text{ for } t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right].$$

Assume that $S \circ \Delta_n(X)$ converges almost in uniform topology. Then, $\mathbf{X} := \lim_{n\to\infty} S \circ \Delta_n(X)$ is the unique natural lift associated to X.

The above corollary is obvious. It proves in particular that the lift of fractional Brownian motion constructed in [5] is a natural one.

2.4. Convergence in $d_{p,\omega}$ topology

Theorem 3. *Assume that there exists a natural lift associated* $\mathbf{X}$ *to a Gaussian process* X*, and assume that for some control* ω*, such that* $\|\mathbf{X}\|_{p,\omega}$ *is in* $\mathbf{L}^q(\Omega,\mathbb{F},\mathbb{R})$ *for* $q \geq 2$*. Let us fix a orthonormal basis* e *of* H *which in* BV*. Then,*

$$\sup_n \left\| S(X^e_{0,n}) \right\|_{p,\omega}$$

is in L^q*. In particular, for all* $p' > p$*,* $d_{p',\omega}\left(S(X^e_{0,n}), \mathbf{X}\right)$ *converges to* 0 *almost surely and in* L^q*.*

Proof. We define $A = ((\pi_{i,j} - \pi_{j,i})(\mathbf{X}))_{(i,j)\in\{1,\dots,d\}}$ to be the area of $\mathbf{X}$. For all $s < t \in [0,1]$,

$$\begin{aligned}
\left\| S(X^e_{0,n})_{s,t} \right\| &\leq C \left| E\left(X_{s,t} | \mathcal{F}^e_n\right) \right| + C\sqrt{\left| E\left(A_{s,t} | \mathcal{F}^e_n\right) \right|} \\
&\leq 2C\sqrt{E\left(\left\| \mathbf{X}_{s,t} \right\|^2 | \mathcal{F}^e_n \right)} \\
&\leq 2\omega(s,t)^{1/p}\sqrt{E\left(\left\| \mathbf{X} \right\|^2_{p,\omega} | \mathcal{F}^e_n \right)}.
\end{aligned}$$

Hence, since $q \geq 2$,

$$\sup_n \left\| S(X^e_{0,n}) \right\|_{p,\omega} \leq C \sup_n E\left(\left\| \mathbf{X} \right\|^q_{p,\omega} | \mathcal{F}^e_{0,n} \right)^{1/q},$$

which in L^q by Doob's inequality. By interpolation, we obtain the convergence of $d_{p',\omega}\left(S(X^e_{0,n}), \mathbf{X}\right)$ to 0 both almost surely and in L^q. □

3. The particular case of a Volterra Gaussian process

This section is devoted to apply the previous results to Volterra Gaussian processes. There are a lot of work about integration with respect to these processes see [3,7] or [4] for more details and references therein. Since we are only interesting in the construction of enhanced Gaussian Volterra processes, we work in a more simpler framework.

Let K be a measurable kernel $K : [0,1]^2 \to \mathbb{R}$ such that for all $t \in [0,1]$, $K(t,.) \in L^2([0,1],\mathbb{R},\mathrm{d}r)$, and for all $0 \leq t \leq s \leq 1$, $K(t,s) = 0$. Let $B = (B^1, \dots, B^d)$ be a d-dimensional Brownian motion, then the Gaussian Volterra process associated to B and K is the process $(X(t),\ t \in [0,1])$ defined by:

$$X^i(t) = \int_0^t K(t,s)\mathrm{d}B^i_s, \quad t \in [0,1], \quad i = 1,\dots,d.$$

Its covariance function is

$$C(t,s) = c(t,s)I^d_{\mathbb{R}}, \quad (s,t) \in [0,1]$$

where $I^d_{\mathbb{R}}$ is the identity matrix and

$$c(t,s) = \int_0^1 K(t,u)K(s,u)\mathrm{d}u.$$

In order to construct the natural lift we may assume the following.

Assumption 2. (1) *There exists* $\alpha > 0$ *such that the map* $t \mapsto K(t,.)$ *is* α *Hölder continuous from* $[0,1]$ *to* $L^2([0,1],\mathbb{R},\mathrm{d}r)$*;*

(2) *the function* $t \mapsto \int_0^t K(t,s)\mathrm{d}s$ *is of bounded variation;*

(3) *the function* $t \mapsto K(t,s)$ *has a differential with respect to* t *on* $]s,1]$ *denoted by* $\partial K(t,s)$, $\partial K(t,.)$ *belongs to* $L^1_{loc}(]0,t[,\mathbb{R},du)$ *and* $\sup_{0\leq s<t\leq 1}|\partial K(t,s)|(t-s)^{\frac{3}{2}} < +\infty$.

Under point (1) of Assumption 2, X has a modification with β Hölder continuous sample path for any $\beta < \alpha$. In the sequel, we only consider this modification. Indeed, the variance of the increments is

$$\sum_{i=1}^{d} \mathbb{E}(|X^i(t) - X^i(s)|^2) = d\int_0^1 [K(t,u) - K(s,u)]^2 \mathrm{d}u.$$

Therefore using (1) of Assumption 2 there exists a constant C_α such that

$$\sum_{i=1}^{d} E(|X^i_t - X^i_s|^2) \leq C_\alpha |t-s|^{2\alpha},$$

and the existence of a continuous modification is a consequence of the Kolmogorov Theorem.

The Cameron-Martin space associated to X, is

$$H = \{h, \quad h(t) = \int_0^t K(t,s)\dot{h}(s)\mathrm{d}s, \quad t\in[0,1], \quad \dot{h} \in L^2([0,1],\mathbb{R}^d,\mathrm{d}s)\},$$

endowed with the scalar product $\langle h,g\rangle = \langle \dot{h},\dot{g}\rangle_{L^2([0,1],\mathbb{R}^d,\mathrm{d}s)}$. Let us recall the proof given in [8]. Indeed, in one hand, if $h(t) = \int_0^t K(t,s)\dot{h}(s)\mathrm{d}s$, $\dot{h} \in L^2([0,1],\mathbb{R}^d,\mathrm{d}s)$, $t \in [0,1]$, then for any $n\in\mathbb{N}^*$, $\alpha_i \in \mathbb{R}$, $t_i \in [0,1]$, $i = 1,...,n$,

$$\begin{aligned}\left\|\sum_{i=1}^{n}\alpha_i h(t_i)\right\|^2 &= \left\|\int_{[0,1]}\sum_{i=1}^{n}\alpha_i K(t_i,s)\dot{h}(s)\mathrm{d}s\right\|^2\\ &\leq \|\dot{h}\|^2_{L^2([0,1],\mathbb{R}^d,\mathrm{d}s)}\left[\sum_{i,j=1}^{n}\alpha_i\alpha_j c(t_i,t_j)\right],\end{aligned}$$

that means that h belongs to H and $|h|_H \leq \|\dot{h}\|_{L^2([0,1],\mathbb{R}^d,\mathrm{d}s)}$. 0n the other hand, let $h,g\in H$, there exists two Gaussian random vectors Φ_h, Φ_g, belonging to the Gaussian space associated to X such that for all $t\in[0,1]$, $j = 1,...,d$, $h^j(t) = \mathbb{E}(\Phi^j_h X^j(t))$ and $g^j(t) = \mathbb{E}(\Phi^j_g X^j(t))$. Then, Φ_h and Φ_g belong to the Gaussian space associated to B and there exists $\dot{h}$ and $\dot{g}$ in $L^2([0,1],\mathbb{R}^d,\mathrm{d}s)$ such that $\Phi^j_h = \int_0^1 \dot{h}^j(s)\mathrm{d}B^j_s$ and $\Phi^j_g = \int_0^1 \dot{g}^j(s)\mathrm{d}B^j_s$ for $j = 1,...,d$. We derive that, for $t\in[0,1]$, $h(t) = \int_0^1 K(t,s)\dot{h}(s)\mathrm{d}s$, $g(t) = \int_0^t \dot{g}(s)K(t,s)\mathrm{d}s$ and

$$\langle h,g\rangle = E(\langle \Phi_h,\Phi_g\rangle_{\mathbb{R}^d}) = \langle \dot{h},\dot{g}\rangle_{L^2([0,1],\mathbb{R}^d,\mathrm{d}s)}.$$

Let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,1],\mathbb{R},\mathrm{d}r)$ belonging to $C^\infty([0,1],\mathbb{R})$, and set

$$e_n(t) = \int_0^t K(t,s)h_n(s)\mathrm{d}s, \quad t\in[0,1]. \tag{3.1}$$

Then under Assumption 2, (2) and (3), (e_n) is an orthonormal basis of $(H,\langle,\rangle)$ which is $H \cap BV$. Indeed, for $n\in\mathbb{N}$, $t\in[0,1]$,

$$e_n(t) = \int_0^1 K(t,s)\mathrm{d}s h_n(t) + \int_0^1 K(t,s)(h_n(s) - h_n(t))\mathrm{d}s$$

is the sum of a function in BV and a function absolutely continuous with respect to the Lebesgue measure with derivative given by

$$\int_0^t \partial K(t,s)(h_n(s)-h_n(t))\mathrm{d}s - \int_0^t K(t,s)\mathrm{d}s\dot{h}_n(t), \quad t\in[0,1].$$

Let us introduce some notation: for $\Pi=(t_i)_{i=0}^{|\Pi|}$ a subdivision of $[0,1]$, and $t\in[0,1]$,

$$K_2^{\Pi}(t,u,v)=\sum_{t_i\in\Pi,\quad t_i\le t}\left(K(t_i,u)[K(t_{i+1},v)-K(t_i,v)]-K(t_i,v)[K(t_{i+1},u)-K(t_i,u)]\right), \tag{3.2}$$

and

$$\begin{aligned} K_2(t,u,v) &= 2\int_u^t K(r,u)\partial K(r,v)\mathrm{d}r - K(t,u)K(t,v) \quad \text{if} \quad u>v,\\ &= -2\int_v^t K(r,v)\partial K(r,u)\mathrm{d}r + K(t,u)K(t,v) \quad \text{if} \quad v>u. \end{aligned}$$

Lemma 3. *Let K be a measurable kernel fulfilling Assumption 2. The sequel $(S_t(X_n^e))_{n\in\mathbf{N}}$ converges in probability if and only if $K_2(t,.,.)$ belongs to $L^2([0,1],\mathrm{d}u\mathrm{d}v)$.*

Proof. Let $k,l\in\mathbb{N}$, since e_k and e_l have finite variation, the integral of e_l with respect to e_k is limit of the Riemann sums. Then using the integral representation given in (3.1), Fubini's Lemma and the definition of $K_2^{\Pi^n}(t,.,.)$ given in (3.2), we have

$$\int_0^t (e_l(s)\dot{e}_k(s)-e_k(s)\dot{e}_l(s))\,\mathrm{d}s = \lim_{n\to\infty}\langle h_l\otimes h_k, K_2^{\Pi^n}(t,.,.)\rangle. \tag{3.3}$$

Note that $(u,v)\mapsto K_2^{\Pi}(t,u,v)$ is antisymmetric, so we deal only with $u>v$. Then using a change of variable, with $t_{i_t}\le t<t_{i_t+1}$,

$$\begin{aligned} K_2^{\Pi}(t,u,v) = &-K(t_{i_t+1},u)K(t_{i_t+1},v) + \sum_{t_i\in\Pi,\quad t_i\le t} 2K(t_i,u)[K(t_{i+1},v)-K(t_i,v)]\\ &+\sum_{t_i\in\Pi,\quad t_i\le t}[K(t_{i+1},u)-K(t_i,u)]\,[K(t_{i+1},v)-K(t_i,v)]\,. \end{aligned}$$

Since $(h_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2([0,1],\mathbb{R},\mathrm{d}u)$, then $(h_k\otimes h_l)_{(l,k)\in\mathbb{N}^2}$ is an orthonormal basis of $L^2([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)$. According Lemma 1, the sequence of random variables $(S\,(X_n^e)_t)_n$ converges almost surely if and only if

$$\sum_{0\le l<k<\infty}\left[\int_0^t (e_l(s)\dot{e}_k(s)-e_k(s)\dot{e}_l(s))\,\mathrm{d}s\right]^2<+\infty.$$

In other words,

$$\sum_{0\le k<l<\infty}\lim_{n\to\infty}\langle h_k\otimes h_l, K_2^{\Pi^n}(t,.,.)\rangle^2<\infty.$$

If under Assumption 2, $(K_2^{\Pi^n}(t,.,.))_n$ converges to $K_2(t,.,.)$ in $L^1([0,1]^2,\mathrm{d}u\mathrm{d}v)$ then using the fact that the function h_k are bounded on $[0,1]$ for all $k\in\mathbb{N}$, $\lim_{n\to\infty}\langle h_k\otimes h_l, K_2^{\Pi^n}(t,.,.)\rangle=\langle h_k\otimes h_l, K_2(t,.,.)\rangle$ for all $l,k\in\mathbb{N}$; and $(S\,(X_n^e)_t)_n$ converges almost surely if and only if $K_2(t,.,.)$ belongs to $L^2([0,1],\mathrm{d}u\mathrm{d}v)$.

Then, it remains to prove that $(K_2^{\Pi^n}(t,.,.))_n$ converge to $K_2(t,.,.)$ in $L^1([0,1]^2, \mathrm{d}u\mathrm{d}v)$.

We split $K_2^{\Pi^n}(t,.,.) - K_2(t,.,.) = S_1^{\Pi^n}(t,.,.) + 2S_2^{\Pi^n}(t,.,.) + S_3^{\Pi^n}(t,.,.)$, where for $0 \le v < u \le 1$,

$$S_1^{\Pi^n}(t,u,v) = K(t,v)K(t,u) - K(t_{i_t+1},u)K(t_{i_t},v),$$

$$S_2^{\Pi^n}(t,u,v) = \sum_{t_i \in \Pi^n,\ t_i \le t} K(t_i,u)\left(K(t_{i+1},v) - K(t_i,v)\right) - \int_u^t K(r,u)\partial K(r,v)\mathrm{d}r,$$

and

$$S_3^{\Pi^n}(t,.,.) = \sum_{t_i \in \Pi,\ \ t_i \le t} \left[K(t_{i+1},u) - K(t_i,u)\right]\left[K(t_{i+1},v) - K(t_i,v)\right].$$

First, observe that for $v < u$

$$\left|S_1^{\Pi^n}(t,u,v)\right| \le |K(t_{i_t+1},u)|\,|K(t_{i_t},v) - K(t,v)| + |K(t,u) - K(t_{i_t+1},u)|\,|K(t,v)|$$

and use Fubini's Theorem and Cauchy Schwartz inequality to derive:

$$\|S_1^{\Pi^n}(t,.,.)\|_{L^1([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)} \le 2\left(\sqrt{\mathbb{E}((X^1_{t_{i_t+1}})^2)\mathbb{E}\left(\left(X^1_{t_{i_t}} - X^1_t\right)^2\right)} + \sqrt{\mathbb{E}((X^1_t)^2)\mathbb{E}\left(\left(X^1_{t_{i_t+1}} - X^1_t\right)^2\right)}\right).$$

Since X is a Gaussian process with continuous sample path $\|S_1^{\Pi^n}(t,.,.)\|_{L^1([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)}$ converge to 0 when n goes to infinity.

Second, we observe that for $0 \le v < u \le 1$,

$$\begin{aligned} S_2^{\Pi^n}(t,u,v)) = &\int_{t_{i_{u+1}}}^t \left(K(t_{i_r},u) - K(r,u)\right)\partial K(r,v)\mathrm{d}r \\ &- \int_u^{t_{i_u+1}} K(r,u)\partial K(r,v)\mathrm{d}r + K(t_{i_t},u)\left(K(t_{i_t+1},v) - K(t,v)\right). \end{aligned} \tag{3.4}$$

The convergence of the last term of the right member of (3.4) to 0 in $L^1([0,1]^2, \mathrm{d}u\mathrm{d}v)$ follows the same lines as the convergence of $\|S_1^{\Pi^n}(t,.,.)\|_{L^1([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)}$ to 0.

For the first term of the right member of (3.4) note that

$$\begin{aligned} \int_{0\le v\le u\le 1} \mathrm{d}u\mathrm{d}v \int_{t_{i_u+1}}^t |K(t_{i_r},u) - K(r,u)||\partial K(r,v)|\mathrm{d}r = \\ \int_{t_1}^t \mathrm{d}r \int_0^{t_{i_r}} \mathrm{d}v|\partial K(r,v)| \int_v^{t_{i_r}} |K(t_{i_r},u) - K(r,u)|\mathrm{d}u. \end{aligned}$$

Using Cauchy Schwarz inequality in the integral with respect to du and

$$\mathbb{E}((X_z - X_y)^2) = \int_0^1 [K(z,r) - K(y,r)]^2\mathrm{d}r$$

we derive

$$\int_{0\le v\le u\le 1} \mathrm{d}u\mathrm{d}v \int_{t_{i_u+1}}^{t} |K(t_{i_r},u)-K(r,u)||\partial K(r,v)|\mathrm{d}r \le \sqrt{C_\alpha}\int_0^1 \mathrm{d}r \int_0^{t_{i_r}} |\partial K(r,v)|\sqrt{t_{i_r}-v}|r-t_{i_r}|^\alpha \mathrm{d}v.$$

Then $\int_{0\le v\le u\le 1}\mathrm{d}u\mathrm{d}v\int_{t_{i_u+1}}^{t}|K(t_{i_r},u)-K(r,u))||\partial K(r,v)|\mathrm{d}r$ converges to 0 when n goes to ∞ since $t_r \le r$. For the second term of the right member of (3.4), the same kind of computations yields

$$\begin{aligned}\int_{0\le v\le u\le 1}\mathrm{d}u\mathrm{d}v\int_{u}^{t_{i_u+1}}|K(r,u)||\partial K(r,v)|\mathrm{d}r &\le \int_0^1\mathrm{d}r\int_0^r|\partial K(r,v)|\mathrm{d}v\int_{\max(v,t_r)}^{r}|K(r,u)|\mathrm{d}u\\ &\le \sqrt{C_\alpha}\int_0^1\mathrm{d}r\int_0^r|\partial K(r,v)||r-\max(v,t_r)|^{1/2}|r-v|^\alpha\mathrm{d}v.\end{aligned}$$

We conclude that $\|S_2^{\Pi^n}(t,.,.)\|_{L^1([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)}$ converge to 0 when n goes to infinity. Using the same arguments, $\|S_3^{\Pi^n}(t,.,.)\|_{L^1([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)}$ converge to 0 when n goes to infinity. □

Corollary 3. *Let K be a measurable kernel fulfilling Assumption 2. Assume that*

- $(K_2^{\Pi^n}(t,.,.))_{n\in\mathbf{N}}$ *converges in* $L^2([0,1]^2,\mathrm{d}u\mathrm{d}v)$ *to* $K_2(t,.,.)$;
- $t\mapsto K_2(t,.,.)$ *is* β *Hölder continuous in* $L^2([0,1]^2,\mathbb{R},\mathrm{d}u\mathrm{d}v)$.

then $\mathbf{X}$ *is natural lift of* X.

Proof. We define $A=((\pi_{i,j}-\pi_{j,i})(\mathbf{X}))_{(i,j)\in\{1,..,d\}^2}$ to be the area of $\mathbf{X}$. For all $s<t\in[0,1]$.

In order to establish the continuity of the paths of $\mathbf{X}$, according to the expression of $\pi_{i,i}(\mathbf{X})$ given in (2.1) it only remains to prove that A has a continuous version. Just observe that for $t,s\in[0,1]^2$

$$\begin{aligned}\mathbb{E}((A_t^{i,j}-A_s^{i,j})^2) &= \|K^2(t,.,.)-K^2(s,.,.)\|^2_{L^2([0,1],\mathbb{R},\mathrm{d}u\mathrm{d}v)}\\ &\le |t-s|^{2\beta}.\end{aligned}$$

Then using the fact (see [2]) that there exists a constant C_p such that for all variable Y in the second Wiener chaos of X,

$$\mathbb{E}(Y^p)\le C_p\mathbb{E}(Y^2)^{p/2}$$

and the Kolmogorov Lemma we obtain the continuity of $A^{i,j}$ and then of $\mathbf{X}$ in $G^2\left(\mathbb{R}^d\right)$. □

As it is pointed out in the pioneering paper of Decreusefond-Üstünel, [8], a now celebrate example of Volterra process which satisfies the previous assumptions is the fractional Brownian motion with Hurst parameter $h\in(0,1]$. The associated kernel is [16]: for $s<t$,

$$\begin{aligned}K_h(t,s) &= c_h s^{\frac12-h}\int_s^t(u-s)^{h-\frac32}u^{h-\frac12}\mathrm{d}u, \quad \text{for} \quad h>\frac12,\\ &= c_h\left[\frac{(t-s)^{h-\frac12}t^{h-\frac12}}{h-\frac12}-\int_s^t(u-s)^{h-\frac12}u^{h-\frac32}\mathrm{d}u\right]s^{\frac12-h}, \quad \text{for } h<\frac12,\\ &= \mathbf{1}_{[0,t]}(s) \quad \text{for} \quad h=\frac12\end{aligned}$$

where c_h is a suitable constant such that the covariance function is

$$c(t,s)=\frac12\left[s^{2h}+t^{2h}-|t-s|^{2h}\right].$$

Remark 1. *As it pointed out in Proposition 32 of [5], fractional Brownian motion fulfills the existence condition of Corollary 3 if and only if is* $h > \frac{1}{4}$.

4. Application: a generalised Wong-Zakai Theorem

For simplicity, we will work with the Gaussian process Brownian Motion B, together with its natural lift $\mathbf{B}_t = \left(B_t, \int_0^t B_u \circ \mathrm{d}B_u\right)$ (it is a natural lift from Th. 2 for example). It is clear that we can extend the following result to more general Gaussian processes. We fix a orthonormal basis e of the Cameron-Martin space of B, *i.e.* $\left(\dot{e}_n\right)_n$ is a orthonormal basis of $L^2\left([0,1], \mathrm{d}u\right)$. Then, $B_t = \sum_{i=0}^{\infty} N_i^e e_i\left(t\right)$, and define $B_{0,n}^e\left(t\right) = \sum_{i=0}^{n} N_i^e e_i\left(t\right)$. Then, from the continuity of the Ito map and the results in this paper, we obtain the following theorem:

Theorem 4. *Assume that* $p \in [2,3)$. *Let* $V = \left(V_i\right)_{1 \le i \le d}$ *be some vector fields on* $\mathbb{R}^{N\cdot}$ *which are* $C^{p+\varepsilon}$, $\varepsilon > 0$. *Define* $Y_{0,n}$ *to be the solution of the ODE*

$$\begin{cases} \mathrm{d}Y_{0,n}\left(t\right) = V\left(Y_{0,n}\left(t\right)\right)\mathrm{d}B_{0,n}\left(t\right) \\ Y_0 = y_0. \end{cases}$$

Almost surely, $Y_{0,n}$ *converges in* p*-variation topology to the solution of the Stratonovich SDE*

$$\begin{cases} dY\left(t\right) = V\left(Y\left(t\right)\right) \circ \mathrm{d}B\left(t\right) \\ Y_0 = y_0. \end{cases}$$

Observe that if we take the Haar basis for the orthonormal basis e of $L^2\left([0,1], \mathrm{d}u\right)$, then we fall back on the classical Wong-Zakai Theorem.

5. Direction of further research

It would be nice to extend our result to lift to the free nilpotent group of step 3, or to a general step n. Things there get harder, as the martingale arguments fails to work for integrals of the type $\int \left|B_u^1\right|^2 \mathrm{d}B_u^2$. The condition to check whether we can have a lift is quite neat and easy to read on Volterra process.

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