

## MINIMUM VARIANCE IMPORTANCE SAMPLING VIA POPULATION MONTE CARLO

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**Abstract.** Variance reduction has always been a central issue in Monte Carlo experiments. Population Monte Carlo can be used to this effect, in that a mixture of importance functions, called a  $D$ -kernel, can be iteratively optimized to achieve the minimum asymptotic variance for a function of interest among all possible mixtures. The implementation of this iterative scheme is illustrated for the computation of the price of a European option in the Cox-Ingersoll-Ross model. A Central Limit theorem as well as moderate deviations are established for the  $D$ -kernel Population Monte Carlo methodology.

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### 1. INTRODUCTION

The main bulk of the literature on Monte Carlo methods concentrates on the approximation of integrals

$$\pi(h) = \int_{\Omega} h(x)\pi(x) \mu(dx),$$

where  $\mu$  is a measure on  $\Omega$ ,  $\pi$  a density and  $h$  a particular  $\pi$ -measurable function of interest on the same set  $\Omega$ . In particular, the focus of many studies is to reduce the variance of estimators of  $\pi(h)$ , whether *locally*, that is, for a given model and a given function, or *globally*, as in for instance Rao–Blackwellization, control and antithetic variates or quasi-random techniques (see, *e.g.*, [17, 20]). An illustration of this focus is provided in mathematical finance by the numerous improvements brought upon the computation of option prices (see, *e.g.*, [13, 15, 16]).

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In the particular case of importance sampling estimators, that is,

$$\hat{\pi}_{g,N}^{IS}(h) = N^{-1} \sum_{i=1}^N h(x_i)\pi(x_i)/g(x_i), \quad x_1, \dots, x_N \stackrel{\text{iid}}{\sim} g,$$

where  $g$  is a distribution dominating  $\pi$  (with density denoted by  $g$ ), the variance is equal to

$$g \left[ (h\pi/g - \pi(h))^2 \right] / N,$$

if  $\pi(h^2\pi/g) < \infty$ . A puzzling feature of this class of estimators is the well-known optimality of the importance distribution

$$g^*(x) = |h(x)|\pi(x) / \int |h(y)|\pi(y) \mu(dy)$$

when aiming at minimizing the variance of  $\hat{\pi}_{g,N}^{IS}$ . This result [20] is paradoxical in that it produces a zero variance estimator when  $h$  is either positive or negative (indeed, in both cases,  $\hat{\pi}_{g^*,N}^{IS} = \pi(h)$ ). The paradox is only superficial, though, in that it points out the fact that, in Monte Carlo settings, there is no ultimate importance function when there is no restriction on the choice of these functions (and when the costs of constructing and simulating these distributions are not taken into account). In particular,  $g^*$  cannot be used in practice because it depends on the integral  $\int |h(y)|\pi(y) \mu(dy)$ . This result is thus rather understood as providing a goal for choosing a importance function  $g$  tailored for the approximation of  $\pi(h)$ .

If the normalizing constants of either the target distribution  $\pi$  or the importance function  $g$  are unknown, an alternative to  $\hat{\pi}_{g,N}^{IS}$  is the self-normalized importance sampling estimator, that is

$$\hat{\pi}_{g,N}^{SNIS}(h) = \sum_{i=1}^N h(x_i)\pi(x_i)/g(x_i) \Big/ \sum_{i=1}^N \pi(x_i)/g(x_i). \quad x_1, \dots, x_N \stackrel{\text{iid}}{\sim} g,$$

where the sum of the weights normalizes the weighted sum. If  $g \left( (1 + h^2)(\pi/g)^2 \right) < \infty$ , the asymptotic variance of  $\hat{\pi}_{g,N}^{SNIS}(h)$  is given by  $\pi \left[ (h - \pi(h))^2 \pi/g \right]$ . In this case,  $g^*$  is no longer the best choice: rather,

$$g^\sharp(x) = |h(x) - \pi(h)|\pi(x) \Big/ \int |h(y) - \pi(h)|\pi(y) \mu(dy)$$

minimizes (in  $g$ ) the asymptotic variance of  $\hat{\pi}_{g,N}^{SNIS}(h)$ . This second optimum is not available either, because it still depends on  $\pi(h)$  [8].

The formal aspect of this optimality result may explain why there is little in the literature besides general recommendations that the support of  $g$  should be the support of  $|h(x)|\pi(x)$  or of  $|h(y) - \pi(h)|\pi(y)$ , or yet that the tails of  $g$  should be at least as thick as those of  $|h(x)|\pi(x)$ . Note however that a recent reference is the *cross-entropy* method of [21] where the parameter of a family of proposals is optimized, either directly or by an iterative process, to reach minimal variance or maximal entropy against the target  $|h(x)|\pi(x)$ , the function  $h$  being of the specific rare event shape  $h(x) = \mathbb{I}(S(x) \leq \gamma)$ . The population Monte Carlo methodology studied in this paper encompasses cross-entropy as a special case.

The current paper establishes that the population Monte Carlo (PMC) technique of Cappé *et al.* [7] and Douc *et al.* [12] can easily be adapted to this purpose and can result in considerable variance reduction. We recall that [7] introduced this method, following the denomination of Iba [14], to advertise the availability of universal adaptive sampling machines that do not encounter the formidable difficulties of designing adaptive MCMC algorithms. Douc *et al.* [12] showed in addition that those PMC algorithms can accommodate a progressive adaption to a given target distribution with a diminishing Kullback divergence. We now explain why this is also the case for variance diminution and optimal importance function approximation.

There is a large bulk of literature on adaptive Monte Carlo methods but we refrain from presenting a survey of these methods because, while the related papers all aim at higher efficiency, their focus is mainly on sequential problems (or targets), mostly with varying dimensions, and their degree of adaptivity is often limited (as for instance in particle filters. A reference worth mentioning however is Del Moral *et al.* [10] since these authors focus on a specific adaptive importance sampling strategy for moving and fixed targets. The former case includes optimization via simulated annealing and tempering. As in Population Monte Carlo, the choice of the transition kernels is not restricted in the sequential Monte Carlo method of Del Moral *et al.* [10]. But the adaptivity in this paper is understood in a different way: the target density is approximated by a sequence  $\pi_n$  of other densities and kernels  $K_n$  are used to move the simulations from  $\pi_{n-1}$  towards  $\pi_n$ . Even though this is not a requirement of Del Moral *et al.* [10], their favorite choice for  $K_n$  is an MCMC kernel associated with  $\pi_n$ . At last, there is no learning mechanism: if  $K_n$  is poorly chosen, there is no clear correction within the methodology, while the current paper mostly aims at this goal, namely to pick the most efficient kernel within a class of possible kernels, while cutting the MCMC connection. Using Markov kernels does not mean we are implementing an MCMC technology.

In Section 2, we recall the main features of the PMC algorithm, including the expressions for the asymptotic variances of the unnormalized and self-normalized versions of the PMC estimator. In Section 3, we establish that our updating scheme for the mixture weights in the PMC algorithm does induce a decrease in the asymptotic variance at each step. Section 4 provides an additional improvement through the cumulated estimation of  $\pi(h)$ . The following Section 5 reinforces our theoretical assessment of the methodology by establishing moderate deviations for the normalized PMC estimator, thus strengthening the CLT obtained previously. In Section 6, we illustrate the variance reduction for a toy example before launching into the evaluation of a European option price for the Cox-Ingersoll-Ross model, which provides an interesting and realistic high-dimensional setting. We also improve on earlier importance sampling strategies for this model.

## 2. POPULATION MONTE CARLO

### 2.1. Monte Carlo setting

We suppose that the target distribution  $\pi$  is *at least* known up to a normalizing constant,  $\pi(x) \propto \tilde{\pi}(x)$  with  $\tilde{\pi}$  known. For the importance sampling estimation of  $\pi(h)$ , the quality of both the unnormalized and the self-normalized approximations to  $\pi(h)$  strongly depends on the choice of the proposal distribution  $g$ , a choice that is quite delicate for complex distributions like those that occur in high dimensional problems.

We first recall that sampling importance resampling (SIR) [18,19] can be used to reset a given weighted sample from  $g$  to a sample from the target distribution  $\pi$ . Once the importance weights are derived,  $\omega_i \propto \pi(x_i)/g(x_i)$ , a (non-iid) sample from  $\pi$ ,  $\tilde{x}_1, \dots, \tilde{x}_M$  can be derived from the instrumental sample  $x_1, \dots, x_N$  by resampling using the importance weights in  $\{x_1, \dots, x_N\}$ , that is,

$$\tilde{x}_i = x_{J_i}, \quad 1 \leq i \leq M,$$

where the random variables  $J_1, \dots, J_M$  are distributed as

$$\mathbb{P}[J_l = i | x_1, \dots, x_N] = \left( \sum_{j=1}^N \frac{\pi(x_j)}{g(x_j)} \right)^{-1} \frac{\pi(x_i)}{g(x_i)} = \bar{\omega}_{i,t}$$

(see, *e.g.*, [17] Sect. 14.3.5). Multinomial sampling, *i.e.*

$$(J_l)_{1 \leq l \leq M} \stackrel{\text{iid}}{\sim} \mathcal{M}(1, (\bar{\omega}_{i,t})_{1 \leq i \leq N})$$

is a possible implementation of the SIR methodology but more efficient alternatives that reduce the variance of the resulting estimators are also available ([17], Chap. 14). However, to keep the description of the algorithms as simple as possible, we will use multinomial sampling in the following.

The *Population Monte Carlo* (PMC) method introduced in Cappé *et al.* [7] intrinsically is a form of iterated sampling importance resampling with dynamically adapted importance functions. We refer to Cappé *et al.* [7], and to Robert and Casella ([17] Chap. 14) for details on the motivations and foundations of this method, and we simply recall the essential feature of the method: At iteration  $t$  of the PMC algorithm,  $N$  values are simulated from a proposal distribution and this proposal distribution is based on the  $N \times (t - 1)$  past realizations, with basically *no constraint* on the form of dependence on the past. This allows for a wide variety of adaptive scenarios, a fact somehow missed in Cappé *et al.* [7] where the proposal was fixed.

If we define *renormalized* importance weights associated with weights  $\omega_{j,t}$  ( $1 \leq j \leq N$ ) as

$$\bar{\omega}_{i,t} = \omega_{i,t} / \sum_{j=1}^N \omega_{j,t},$$

the generic PMC algorithm reads as follows:

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### Generic PMC algorithm

At time 0,

- a) Generate  $(x_{i,0})_{1 \leq i \leq N} \stackrel{\text{iid}}{\sim} g_0$  and compute  $\omega_{i,0} = \pi(x_{i,0})/g_0(x_{i,0})$ ;
- b) Generate  $(J_{i,0})_{1 \leq i \leq N} \stackrel{\text{iid}}{\sim} \mathcal{M}(1, (\bar{\omega}_{i,0})_{1 \leq i \leq N})$  and set  $\tilde{x}_{i,0} = x_{J_{i,0},0}$  ( $1 \leq i \leq N$ ).

At time  $1 \leq t \leq T$

- a) Conditionally on past  $x_{i,j}$ 's and  $\tilde{x}_{i,j}$ 's, generate independently  $x_{i,t} \sim g_{i,t}$  and compute  $\omega_{i,t} = \pi(x_{i,t})/g_{i,t}(x_{i,t})$ ;
  - b) Generate  $(J_{i,t})_{1 \leq i \leq N} \stackrel{\text{iid}}{\sim} \mathcal{M}(1, (\bar{\omega}_{i,t})_{1 \leq i \leq N})$  and set  $\tilde{x}_{i,t} = x_{J_{i,t},t}$  ( $1 \leq i \leq N$ ).
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Obviously, the quasi-total freedom in the construction of the above  $g_{i,t}$ 's has drawbacks, namely that some proposals do not necessarily lead to improvements in terms of variance reduction or of target approximation. Therefore, we now restrict the family of proposals from which to select the new  $g_{i,t}$ 's to mixture of fixed proposals and we establish in the next section that variance improvement does occur within this family. This particular type of algorithm was already shown in [12] to lead to a reduction in the asymptotic Kullback-Leibler distance between the target and the proposal, for a specific update in the mixture weights.

## 2.2. $D$ -kernel PMC

We assume from now on that we use in parallel  $D$  fixed kernels  $Q_d(\cdot, \cdot)$  with densities  $q_d$  and that the proposal is a mixture of those kernels

$$g_{i,t}(x) = \sum_{d=1}^D \alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x), \quad \sum_d \alpha_d^{t,N} = 1,$$

where the weights  $\alpha_d^{t,N} > 0$  can be modified at each iteration. The amount of adaptivity we allow in this version of PMC is thus restricted to a possible modification of the weights  $\alpha_d^{t,N}$ . (The  $Q_d$ 's may correspond to random walk steps with different scales or to different blocking strategies of the Gibbs sampler.) The importance weight associated with this mixture proposal is

$$\pi(x_{i,t}) / \sum_{d=1}^D \alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x_{i,t}),$$

while simulation from  $g_{i,t}$  can be decomposed in the two usual mixture steps: first pick the component  $d$  then simulate from the corresponding kernel  $Q_d$ . The resulting algorithm is then a specific case of PMC algorithm where the weights  $\alpha_d^{t,N}$  are updated as follows:

**Generic  $D$ -kernel PMC algorithm**

At time 0, produce the sample  $(\tilde{x}_{i,0}, J_{i,0})_{1 \leq i \leq N}$  and set  $\alpha_d^{1,N} = 1/D$  for all  $1 \leq d \leq D$ .

At time  $1 \leq t \leq T$

- a) Conditionally on the  $\alpha_d^{t,N}$ 's, generate

$$(K_{i,t})_{1 \leq i \leq N} \stackrel{\text{iid}}{\sim} \mathcal{M}(1, (\alpha_d^{t,N})_{1 \leq d \leq D})$$

- b) Conditionally on  $(\tilde{x}_{i,t-1}, K_{i,t})_{1 \leq i \leq N}$ , generate independently

$$(x_{i,t})_{1 \leq i \leq N} \sim Q_{K_{i,t}}(\tilde{x}_{i,t-1}, \cdot)$$

and set  $\omega_{i,t} = \pi(x_{i,t}) / \sum_{d=1}^D \alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x_{i,t})$ ;

- c) Conditionally on  $(\tilde{x}_{i,t-1}, K_{i,t}, x_{i,t})_{1 \leq i \leq N}$ , generate

$$(J_{i,t})_{1 \leq i \leq N} \stackrel{\text{iid}}{\sim} \mathcal{M}(1, (\bar{\omega}_{i,t})_{1 \leq i \leq N})$$

and set  $(1 \leq i \leq N, 1 \leq d \leq D)$

$$\tilde{x}_{i,t} = x_{J_{i,t},t}, \quad \alpha_d^{t+1,N} = \Psi_d((\tilde{x}_{i,t-1}, x_{i,t}, K_{i,t})_{1 \leq i \leq N})$$

such that  $\sum_{d=1}^D \alpha_d^{t+1,N} = 1$ .

In the above algorithm,  $\Psi_d$  ( $1 \leq d \leq D$ ) denotes an update function that depends upon the past iteration. We assume that the individual kernel importance weights are almost surely finite, that is,

$$\forall d \in \{1, \dots, D\}, \bar{\pi} \{q_d(x, x') = 0\} = 0, \tag{A1}$$

where  $\bar{\pi} = \pi \otimes \pi$ . Under (A1), Douc *et al.* [12] proved that the updates  $\Psi_d$  of the mixture weights given by

$$\alpha_d^{t+1,N} = \sum_{i=1}^N \bar{\omega}_{i,t} \mathbb{1}_d(K_{i,t})$$

guarantee a systematic decrease of the Kullback-Leibler distance between the target and the  $D$ -kernel mixture, a long-term run of the algorithm providing the mixture that is (entropy-) closest to the target. Moreover, Theorem 5.1 of [12] leads to a LLN (in the number of simulations at a given iteration) for the output of the generic  $D$ -kernel PMC algorithm.

**Theorem 2.1.** *Under (A1), for any function  $h$  in  $L^1_\pi$  and for all  $t \geq 0$ , both the unnormalized and the self-normalized PMC estimators are convergent,*

$$\hat{\pi}_{t,N}^{PMC}(h) = \frac{1}{N} \sum_{i=1}^N \omega_{i,t} h(x_{i,t}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \pi(h) \quad \text{and} \quad \hat{\pi}_{t,N}^{SPMC}(h) = \sum_{i=1}^N \bar{\omega}_{i,t} h(x_{i,t}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \pi(h).$$

As noted earlier, the unnormalized PMC estimator can only be used when  $\pi$  is completely known and even in those instances it is not necessarily improving upon the self-normalized PMC estimator.

A CLT can also be established in this setting, under the additional following integrability condition

$$\bar{\pi} \left\{ (1 + h^2(x')) \frac{\pi(x')}{q_{\bar{d}}(x, x')} \right\} < \infty \text{ for a } \bar{d} \in \{1, \dots, D\}. \tag{A2}$$

Note that this condition must only hold for *one*  $1 \leq \bar{d} \leq D$  rather than for *all*  $d$ 's. Theorem 5.2 of Douc *et al.* [12] then provides a CLT for the generic  $D$ -kernel PMC algorithm.

**Theorem 2.2.** *Under (A1) and (A2), if for all  $t \geq 1$ ,*

$$\forall 1 \leq d \leq D, \quad \alpha_d^{t,N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \alpha_d^t > 0,$$

then both

$$\sqrt{N} \left( \sum_{i=1}^N \bar{\omega}_{i,t} h(x_{i,t}) - \pi(h) \right) \quad \text{and} \quad \sqrt{N} \left( \frac{1}{N} \sum_{i=1}^N \omega_{i,t} h(x_{i,t}) - \pi(h) \right) \tag{1}$$

converge in distribution as  $n$  goes to infinity to normal distributions with variances

$$\sigma_{1,t}^2 = \bar{\pi} \left( (h(x') - \pi(h))^2 \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^t q_d(x, x')} \right)$$

and

$$\sigma_{2,t}^2 = \bar{\pi} \left\{ \left( \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^t q_d(x, x')} h(x') - \pi(h) \right)^2 \frac{\sum_{d=1}^D \alpha_d^t q_d(x, x')}{\pi(x')} \right\}.$$

The additional condition in Theorem 2.2 is necessary to ensure a stabilization of the weights as the number of simulations increases. It is guaranteed in cases like those of Douc *et al.* [12] updating scheme and we will show below that it also holds for our updating scheme. The quantities  $\sigma_{1,t}^2$  and  $\sigma_{2,t}^2$  exhibited in this result are thus associated with the limiting set of weights  $(\alpha_1^t, \dots, \alpha_D^t)$ , defined on the simplex set of  $\mathbb{R}^D$ ,

$$\mathcal{S}_D = \left\{ \alpha = (\alpha_1, \dots, \alpha_D); \forall d \in \{1, \dots, D\}, \alpha_d \geq 0 \quad \text{and} \quad \sum_{d=1}^D \alpha_d = 1 \right\}.$$

We now proceed to exhibit an updating scheme on the weights  $\alpha_d^t$  such that the asymptotic variances  $\sigma_{1,t}^2$  and  $\sigma_{2,t}^2$  are decreasing at each iteration of the  $D$ -kernel PMC algorithm.

### 3. PMC AS A VARIANCE REDUCTION SCHEME

#### 3.1. Self-normalized PMC estimator

For the estimator  $\sum_{i=1}^N \bar{\omega}_{i,t} h(x_{i,t})$ , we first introduce notations that simplify the study of its asymptotic variance. If  $\nu_h$  denotes the measure on  $\Omega \times \Omega$  defined by

$$\nu_h(dx, dx') = \pi(x') (h(x') - \pi(h))^2 \pi(dx) \pi(dx'), \tag{2}$$

which naturally appears in  $\sigma_{1,t}^2$ , we define two functions  $\sigma_1^2$  and  $F_1$  on  $\mathcal{S}_D$  such that

$$\sigma_1^2(\alpha) = \nu_h \left( 1 / \sum_{d=1}^D \alpha_d q_d(x, x') \right) \quad \text{and} \quad F_1(\alpha) = \left( \nu_h \left( \frac{\alpha_d q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right) / \sigma_1^2(\alpha) \right)_{1 \leq d \leq D}.$$

Clearly,  $\sigma_1^2$  is then the asymptotic variance associated with a given set of weights, while  $F_1$  takes its values in  $\mathcal{S}_D$  and is thus a transform (or update) of the weights. The central result of this paper is that this particular choice of update induces a reduction of the asymptotic variance at each step of the PMC algorithm:

**Proposition 3.1.** *Under (A1) and (A2), for all  $\alpha \in \mathcal{S}_D$ , we have*

$$\sigma_1^2(F_1(\alpha)) \leq \sigma_1^2(\alpha).$$

*Proof.* We have

$$\begin{aligned} \sigma_1^2(F_1(\alpha)) &= \nu_h \left( \frac{1}{\sum_{d=1}^D \alpha_d q_d(y, y') \nu_h \left( \frac{q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right)} \right) \sigma_1^2(\alpha) \\ &= \nu_h \left( \frac{1}{\sum_{l=1}^D \alpha_l q_l(y, y')} \frac{1}{\sum_{d=1}^D \frac{\alpha_d q_d(y, y')}{\sum_{l=1}^D \alpha_l q_l(y, y')} \nu_h \left( \frac{q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right)} \right) \sigma_1^2(\alpha) \\ &\leq \nu_h \left( \frac{1}{\sum_{l=1}^D \alpha_l q_l(y, y')} \sum_{d=1}^D \frac{\alpha_d q_d(y, y')}{\sum_{l=1}^D \alpha_l q_l(y, y')} \frac{1}{\nu_h \left( \frac{q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right)} \right) \sigma_1^2(\alpha) \\ &= \sum_{d=1}^D \alpha_d \nu_h \left( \frac{q_d(y, y') / (\sum_{l=1}^D \alpha_l q_l(y, y'))^2}{\nu_h \left( q_d(x, x') / (\sum_{l=1}^D \alpha_l q_l(x, x'))^2 \right)} \right) \sigma_1^2(\alpha) = \sigma_1^2(\alpha), \end{aligned}$$

the inequality following from Jensen’s inequality. □

We thus take advantage of the diminution of the asymptotic variance to construct a sequence on  $\mathcal{S}_D$  such that

$$\begin{cases} \alpha^{1,1} = (1/D, \dots, 1/D) \\ \alpha^{1,t+1} = F_1(\alpha^{1,t}) \quad \text{for } t \geq 1. \end{cases} \tag{3}$$

At each step of the PMC algorithm, the asymptotic variance is therefore decreasing. Since  $\sigma_1^2$  is a convex function on the connected compact set  $\mathcal{S}_D$ , it thus has a unique minimum. If we denote this minimum by

$$\alpha^{1,min} = \arg \min_{\alpha \in \mathcal{S}_D} \sigma_1^2(\alpha),$$

we then have the convergence result for this updating mechanism (whose proof is given in Appendix 7.1).

**Proposition 3.2.** *Under (A1) and (A2),*

$$\lim_{t \rightarrow \infty} \alpha^{1,t} = \alpha^{1,min}.$$

Propositions 3.1 and 3.2 together establish the convergence to the minimal variance solution of the ideal algorithm, that is, the one using the update mechanism  $\alpha^{1,t+1} = F_1(\alpha^{1,t})$ . To complete the validation of a practical algorithm, we now have to replace the ideal updating with a practical updating and to show that the substitution does not jeopardize convergence. In other words, we need to establish the convergence of the mixture weights to the  $\alpha^{1,t}$ ’s and this is sufficient for Theorem 2.2 to apply, *i.e.*, for the asymptotic variance to be a valid assessment of our algorithm.

We thus define, as a substitute to  $F_1$ , the following update of the mixture weights

$$\alpha_d^{t+1,N} = \frac{\sum_{i=1}^N \bar{\omega}_{i,t}^2 \left( h(x_{i,t}) - \sum_{j=1}^N \bar{\omega}_{j,t} h(x_{j,t}) \right)^2 \mathbb{I}_d(K_{i,t})}{\sum_{i=1}^N \bar{\omega}_{i,t}^2 \left( h(x_{i,t}) - \sum_{j=1}^N \bar{\omega}_{j,t} h(x_{j,t}) \right)^2}, \tag{4}$$

which also holds when  $\bar{\omega}_{i,t}^2$  is replaced with  $\omega_{i,t}^2$  in both the numerator and the denominator and is thus independent of the normalizing constant to some extent. The convergence of this updating scheme is then ensured by the following result, whose proof is deferred to Appendix 7.2

**Proposition 3.3.** *Under (A1) and (A2), for all  $t \geq 1$  and  $\forall 1 \leq d \leq D$ ,*

$$\alpha_d^{t,N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \alpha_d^t \tag{5}$$

where the  $\alpha_d^{t,N}$ 's are defined by equation (4) and the  $\alpha_d^t$ 's are given in (3).

Note that in the proof of Proposition 3.2 (see Appendix 7.2), we prove in addition that

$$\hat{\sigma}_{1,t}^2 = N \sum_{i=1}^N \bar{\omega}_{i,t}^2 \left( h(x_{i,t}) - \sum_{j=1}^N \bar{\omega}_{j,t} h(x_{j,t}) \right)^2 \tag{6}$$

is a consistent estimator of  $\sigma_{1,t}^2(\alpha^t)$ .

### 3.2. Unnormalized PMC estimator

The same sequence of results holds for the unnormalized estimator  $\frac{1}{N} \sum_{i=1}^N \omega_{i,t} h(x_{i,t})$ . We first define a measure on  $\Omega \times \Omega$

$$\rho_h(dx, dx') = \pi(x') h(x')^2 \pi(dx) \pi(dx'),$$

which relates to  $\sigma_{2,t}^2$  since

$$\sigma_{2,t}^2 = \bar{\pi} \left( h^2(x') \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^t q_d(x, x')} \right) - 2\pi(h) + \pi(h)^2.$$

We also consider two functions  $\sigma_2^2$  and  $F_2$  on  $\mathcal{S}_D$  as

$$\sigma_2^2(\alpha) = \rho_h \left( 1 / \sum_{d=1}^D \alpha_d q_d(x, x') \right) - 2\pi(h) + \pi(h)^2$$

and

$$F_2(\alpha) = \left\{ \rho_h \left( \frac{\alpha_d q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right) / \sigma_2^2(\alpha) \right\}_{1 \leq d \leq D}.$$

Then we can use the same steps as in Section 3.1 and derive convergence from the results there. First, as a corollary to Proposition 3.1, we establish the decrease in the asymptotic variance for the ideal weights:

**Proposition 3.4.** *Under (A1) and (A2), for all  $\alpha \in \mathcal{S}_D$ , we have*

$$\sigma_2^2(F_2(\alpha)) \leq \sigma_2^2(\alpha).$$



Second, if we set the sequence of ideal weights as

$$\begin{cases} \alpha^{2,1} = (1/D, \dots, 1/D) \\ \alpha^{2,t+1} = F_2(\alpha^{2,t}) \end{cases} \quad \text{for } t \geq 1 \tag{7}$$

then we deduce from Proposition 3.2 that this ideal sequence converges to the optimal set of weights since, as  $\sigma_2^2$  is convex on  $\mathcal{S}_D$ , it also has a unique minimum

$$\alpha^{2,min} = \arg \min_{\alpha \in \mathcal{S}_D} \sigma_2^2(\alpha).$$

**Proposition 3.5.** *Under (A1) and (A2),*

$$\lim_{t \rightarrow \infty} \alpha^{2,t} = \alpha^{2,min}.$$

Third, we now exhibit the empirical version of the updating scheme which ensures that the practical version of the algorithm also converges, by virtue of Theorem 2.2 and Propositions 3.4 and 3.5. In the unnormalized case, it is now given by

$$\alpha_d^{t+1,N} = \frac{\sum_{i=1}^N \omega_{i,t}^2 h^2(x_{i,t}) \mathbb{I}_d(K_{i,t})}{\sum_{i=1}^N \omega_{i,t}^2 h^2(x_{i,t})}. \tag{8}$$

Finally, as a corollary to Proposition 3.3, we then have the overall convergence guarantee:

**Proposition 3.6.** *Under (A1) and (A2), for all  $t \geq 1$  and  $\forall 1 \leq d \leq D$ ,*

$$\alpha_d^{t,N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \alpha_d^t \tag{9}$$

where the  $\alpha_d^{t,N}$ 's and the  $\alpha_d^t$ 's are defined by equations (8) and (7), respectively.

Note again that

$$\frac{1}{N} \sum_{i=1}^N \left( \omega_{i,t} h(x_{i,t}) - N^{-1} \sum_{j=1}^N \omega_{j,t} h(x_{j,t}) \right)^2$$

is a consistent estimator of  $\sigma_{2,t}^2$ .

#### 4. A CUMULATED ESTIMATOR

While each iteration of the PMC algorithm leads to an asymptotic decrease in the variance of the PMC estimator when compared with the previous iteration estimator, provided updates (4) or (8) are used, those previous iterations can obviously be recycled into a weighted cumulated estimator of the PMC estimators over iterations. Since, as noted in Theorem 2.2, each PMC sample is asymptotically independent of the others, the asymptotic variance of the weighted estimator is simply the sum of the cumulated weighted variances and the weights in this cumulated estimator can thus be directly optimized, as explained below.

The cumulated self-normalized PMC estimator,  $\hat{\pi}_\beta^{CSN}(h)$ , of  $\pi(h)$  is chosen as

$$\hat{\pi}_\beta^{CSN}(h) = \sum_{t=0}^T \beta_t \left( \sum_{i=1}^N \bar{\omega}_{i,t} h(x_{i,t}) \right), \quad \text{with } \beta = (\beta_1, \dots, \beta_T) \in \mathcal{S}_{T+1}. \tag{10}$$

Under assumptions (A1–A2), Theorem 2.2, implies that

$$\sqrt{N} \{ \hat{\pi}_\beta^{CSN}(h) - \pi(h) \} \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left( 0, \sum_{t=0}^T \beta_t^2 \sigma_{1,t}^2 \right).$$

We can therefore minimize this asymptotic variance by picking the weights as ( $1 \leq t \leq T$ )

$$\beta_t^{min} = \sigma_{1,t}^{-2} / \left( \sum_{t=0}^T \sigma_{1,t}^{-2} \right) \quad \text{and} \quad \sum_{t=0}^T (\beta_t^{min})^2 \sigma_{1,t}^2 = \left( \sum_{t=0}^T \sigma_{1,t}^{-2} \right)^{-1}.$$

Furthermore, this optimal  $\beta_t^{min}$  can be consistently estimated by

$$\hat{\beta}_t^{min} = \frac{\hat{\sigma}_{1,t}^{-2}}{\sum_{t=0}^T \hat{\sigma}_{1,t}^{-2}}, \tag{11}$$

where  $\hat{\sigma}_{1,t}^2$  is defined in (6). Therefore,

**Proposition 4.1.** *For an arbitrary  $h$  in  $L_\pi^1$ , for any  $T$ ,*

- (i) *under **(A1)**,  $\hat{\pi}_{\hat{\beta}^{min}}^{CSN}(h) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \pi(h)$ ;*
- (ii) *under **(A1–A2)**,  $\sqrt{N} \left\{ \hat{\pi}_{\hat{\beta}^{min}}^{CSN}(h) - \pi(h) \right\} \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left\{ 0, \left( \sum_{t=0}^T \sigma_{1,t}^{-2} \right)^{-1} \right\}$ .*

Note that a corresponding result also holds for the unnormalized PMC estimator.

### 5. MODERATE DEVIATIONS

Theorem 2.2 has established that the self-normalized PMC algorithm enjoys asymptotic normality. We can however prove a much more detailed evaluation of its asymptotic properties since we are able to derive a Moderate Deviation Principle (MDP) associated with this algorithm. The main interest of an MDP is that the rate function is expressed in terms of the variance, and thus a decrease in the variance leads to improved exponential bounds.

Consider a family of bounded measurable functions  $\psi_0, \dots, \psi_T$ . Now, recall that by application of Theorem 2.2, for any  $t \in \{0, \dots, T\}$ ,

$$\sqrt{N} \left( \sum_{i=1}^N \bar{\omega}_{i,t} \psi_t(x_{i,t}) - \pi(\psi_t) \right) \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left( 0, \sigma_{1,t}^2(\psi_t) \right)$$

where  $\sigma_{1,t}^2(h)$  is now defined as a function of  $h$  by

$$\sigma_{1,t}^2(h) = \bar{\pi} \left( (h(x') - \pi(h))^2 \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^t q_d(x, x')} \right).$$

Denote by  $V_T(\psi_{0:T})$  the diagonal matrix with diagonal elements  $\sigma_{1,t}^2(\psi_t)$  ( $0 \leq t \leq T$ ) and define

$$M_N^t(\psi_t) = \frac{\sqrt{N}}{b_N} \sum_{i=1}^N (\bar{\omega}_{i,t} \psi_t(x_{i,t}) - \pi(\psi_t))$$

and  $\mathbf{M}_N^T(\psi_{0:T}) = (M_N^0(\psi_0), \dots, M_N^T(\psi_T))$ , where  $b_N$  satisfies

$$\lim_{N \rightarrow \infty} b_N = \lim_{N \rightarrow \infty} \sqrt{N} b_N^{-1} = \infty.$$

The Moderate Deviation Principle then holds under the following additional assumption: there exists  $\lambda > 0$  such that

$$\forall d \in \{1, \dots, D\}, \quad \sup_x \int \exp \left\{ \lambda \frac{\pi(x')}{q_d(x, x')} \right\} \pi(dx') < \infty. \tag{A3}$$

This results complement the Central Limit Theorem obtained in Theorem 2.2. For applications of Moderate Deviation Principle, see for example Bucklew [6] or Dembo and Zeitouni [11].

**Theorem 5.1.** *Under assumptions (A1–A2–A3), the function defined for all  $\mathbf{x}_T \in \mathbb{R}^{T+1}$  by*

$$\begin{aligned} I_T(\mathbf{x}_T) &= \sup_{\boldsymbol{\lambda}_T \in \mathbb{R}^{T+1}} \left\{ \langle \mathbf{x}_T, \boldsymbol{\lambda}_T \rangle - \frac{\langle \boldsymbol{\lambda}_T, V_T(\psi_{0:T}) \cdot \boldsymbol{\lambda}_T \rangle}{2} \right\}, \\ &= \frac{\langle \mathbf{x}_T, V_T^{-1}(\psi_{0:T}) \cdot \mathbf{x}_T \rangle}{2} \end{aligned}$$

is a good rate function and the family of random variables  $(\mathbf{M}_N^T(\psi_{0:T}))_{N \geq 1}$  satisfies the Moderate Deviation Principle with speed  $b_N^2$  and good rate function  $I_T$ , i.e.

$$\begin{aligned} - \inf_{\text{int}(\Gamma)} I_T &\leq \liminf_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{P}(\mathbf{M}_N^T(\psi_{0:T}) \in \Gamma) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{P}(\mathbf{M}_N^T(\psi_{0:T}) \in \Gamma) \leq \inf_{\bar{\Gamma}} I_T \end{aligned}$$

for  $\Gamma \in \mathcal{B}(\mathbb{R}^{T+1})$ .

In the above bounds,  $\text{int}(\Gamma)$  denotes the interior of  $\Gamma$  and  $\bar{\Gamma}$  the closure of  $\Gamma$ . For example, considering only the last generation of particles, we obtain at time  $T$  the exact rate for typical deviations as for all positive  $r$ 's

$$\lim_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{P}(|M_N^T(\psi_T)| > r) = -\frac{r^2}{2\sigma_{1,T}^2(\psi_T)}$$

which is decreasing as iterations increase, by our result of Section 3, reinforcing the appeal of our adaptation procedure. Proof of Theorem 5.1 is provided in Appendix 7.3. It relies mainly on Gärtner-Ellis theorem, using a recurrence procedure and suited conditioning.

Note that this result is stated under the assumption that the functions  $\psi_0, \dots, \psi_T$  are bounded. This is quite a restriction, obviously, and its only justification is to provide a short and self-contained proof of the MDP. If we suppose instead (A1–A2–A3) and that for all  $t = 0, \dots, T$

$$\forall d \in \{1, \dots, D\}, \quad \sup_x \int \psi_t(x') \exp \left\{ \lambda \frac{\pi(x')}{q_d(x, x')} \psi_t(x') \right\} \pi(dx') < \infty \tag{A4}$$

then the conclusion of Theorem 5.1 still holds. (The corresponding lengthy if straightforward modifications of the proof are left to the reader.)

Note at last that a similar MDP for the cumulated estimator (10) can be established using the main theorem of this section.

## 6. APPLICATIONS

### 6.1. A normal toy example

We first consider a toy example where the optimal solution is known: using the  $\mathcal{N}(0, 1)$  density and  $h(x) = x$ , the optimal importance distribution which minimizes the variance of the unnormalized importance sampling estimator is  $g^*(x) \propto |x| \exp -x^2/2$ . It actually corresponds to the distribution of the root of an exponential

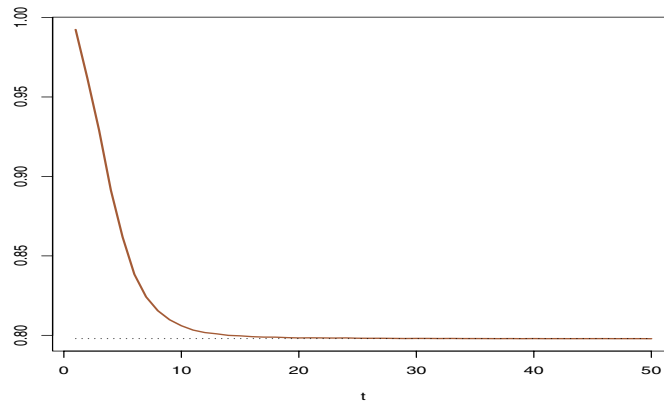


FIGURE 1. Estimation of  $\mathbb{E}[X] = 0$  for a normal variate: decrease of the standard deviation to its optimal value.

TABLE 1. Estimation of  $\mathbb{E}[X] = 0$  for a normal variate using the  $D$ -kernel PMC algorithm with  $D = 3$ ,  $N = 100\,000$  and normal, Cauchy and transformed Gamma independent kernels: evolution of the PMC estimates, kernel weights and asymptotic standard deviation estimates over 20 iterations.

$t$	$\hat{\pi}_{t,N}^{PMC}(x)$	$\alpha_1^{t+1,N}$	$\alpha_2^{t+1,N}$	$\alpha_3^{t+1,N}$	$\sigma_{1,t}$
1	0.0000	0.1000	0.8000	0.1000	0.9524
2	-0.0030	0.1144	0.7116	0.1740	0.9192
3	-0.0017	0.1191	0.6033	0.2776	0.8912
4	-0.0006	0.1189	0.4733	0.4078	0.8608
5	-0.0035	0.1084	0.3545	0.5371	0.8394
10	0.0065	0.0519	0.0622	0.8859	0.8016
15	0.0033	0.0305	0.0136	0.9559	0.7987
20	-0.0042	0.0204	0.0041	0.9755	0.7984

$\mathcal{E}(1/2)$  random variable with random sign, that is,  $(-1)^s \sqrt{\mathcal{E}(1/2)}$  where  $s \sim \mathcal{B}(0.5)$ , a Bernoulli distribution with parameter 1/2. We then choose  $g^*$  as one of  $D = 3$  independent kernels, the other kernels being the  $\mathcal{N}(0, 1)$  and the  $\mathcal{C}(0, 1)$  (Cauchy) distributions. Note that the fact that the proposals are independent does not modify the validity of the above results. In particular, conditions **(A1–A2)** do hold in that case. (The only modification in the algorithm is that the resampling step is no longer necessary.) Remark also that conditions **(A3–4)** are verified here and the MDP then holds.

Table 6.1 presents the results of the variance  $D$ -kernel PMC scheme with  $N = 100\,000$  and  $T = 20$ . At each iteration, the (estimated) asymptotic variance of the self-normalized PMC estimator decreases and the weights of the mixture proposal correctly concentrate around the correct optimal kernel. The optimal standard deviation in that case is equal to  $2/\sqrt{2\pi} = 0.7979$ . Figure 1 represents the convergence to this optimal value over 50 iterations.

### 6.2. The Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model (Cox *et al.* [9]) is a diffusion process used to model interest rate changes. The fundamental stochastic differential equation of the CIR model is

$$dr_t = (\eta - kr_t)dt + \sigma\sqrt{r_t}dW_t, \tag{12}$$

where  $(W_t)_{[0,T]}$  is a Brownian motion under the risk neutral measure  $\mathbb{P}$ . In the financial application,  $(r_t)_{[0,T]}$  represents the short term rate over the measurement period. A quantity of interest is the so-called *European caplet* option, which is an option written on the interest rate with the following payoff function at deadline (or maturity)  $T$ :

$$M \max(r_T - K, 0),$$

where  $K$  is called the strike rate and  $M$  the nominee amount. The actualized price of the caplet at time 0 is therefore given by

$$\mathbb{E}_{\mathbb{P}} \left[ \exp \left( - \int_0^T r_t dt \right) M \max(r_T - K, 0) \right]. \tag{13}$$

The explicit calculation of (13) is obviously intractable even though the transition density of the diffusion is available (Cox *et al.* [9]).

The standard approach to processing diffusions is to use a symmetrized Euler approximation scheme, which consists in discretizing the time interval  $[0, T]$  into  $n$  steps and in studying instead the discrete time process  $(r_{pT/n}^n)_{0 \leq p \leq n}$  with  $r_0^n = r_0$  and

$$r_{(p+1)T/n}^n = \left| r_{pT/n}^n + (\eta - kr_{pT/n}^n) \frac{T}{n} + \sigma \sqrt{r_{pT/n}^n} (W_{(p+1)T/n} - W_{pT/n}) \right|, \tag{14}$$

since the differences  $W_{(p+1)T/n} - W_{pT/n}$  are iid  $\mathcal{N}(0, T/n)$ . The quantity of interest (13) is then approximated by

$$\mathfrak{P} = \mathbb{E} \left[ \exp \left\{ - (T/n) \left( (r_0^n + r_T^n) / 2 + \sum_{p=1}^{n-2} r_{(p+1)T/n}^n \right) \right\} M \max(r_T^n - K, 0) \right],$$

where  $r_{(p+1)T/n}^n \sim \left| \mathcal{N} \left( r_{pT/n}^n + (T/n)(\eta - kr_{pT/n}^n), (T/n)\sigma^2 r_{pT/n}^n \right) \right|$ . Details on the symmetrized Euler approximation can be found in Glasserman [13] as well as Bally and Bossy [3, 4] and Talay *et al.* [5]. (Note that the true distribution of the sequence  $(r_{pT/n}^n)_{0 \leq p \leq n}$  is known and does not require the Euler scheme.)

Even when using the Euler approximation, the explicit derivation of  $\mathfrak{P}$  is impossible and we need to use Monte Carlo simulation to approximate this  $T + 1$  dimensional integral. Some importance sampling techniques have already been proposed by Arouna [1, 2], Su and Fu [22] and we now study the behavior of our  $D$ -kernel PMC scheme. While the exact distribution in (14) can be used in the Monte Carlo approximation, it seems rather natural to force the process to end up as much as possible above  $K$  for  $r_T - K$  to be positive.

Our two alternatives to the Euler distribution in (14) are both based on the introduction of a location drift on the Brownian motion  $(W_t)_t$ . More precisely, for  $\theta \in \mathbb{R}$ , we define the family of all equivalent probability measures  $Q(\theta)$  with respect to  $\mathbb{P}$  that follow from introducing a drift of  $\theta$  in  $(W_t)$ . By Girsanov's theorem, we know that, under the measure  $Q(\theta)$ ,

$$dr_t = (\eta - kr_t + \theta\sigma\sqrt{r_t})dt + \sigma\sqrt{r_t}d\tilde{W}_t,$$

where  $(\tilde{W}_t)_{[0,T]}$  is a Brownian motion and the change of measure process is given by

$$\frac{d\mathbb{P}}{dQ(\theta)} = \exp(-\theta W_T - 0.5\theta^2 T). \tag{15}$$

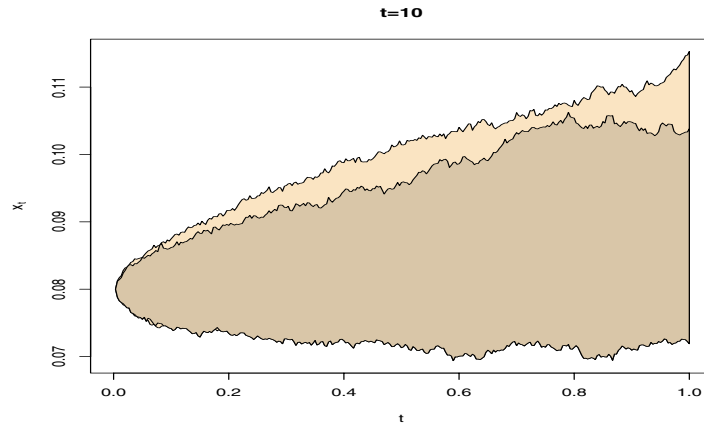


FIGURE 2. Cox-Ingersoll-Ross European option simulation: range of  $N$  simulated process  $(r_{pT/n}^n)_{1 \leq p \leq n}$  (lighter background) compared with the corresponding  $N$  resampled process  $(\tilde{r}_{pT/n}^n)_{1 \leq p \leq n}$  (darker background) for  $K = 0.09$ , using a  $D = 3$ -kernel PMC algorithm with  $\theta = 0$ ,  $\theta = 1$  and  $\theta = 2$ ,  $T = 10$  iterations and  $N = 10,000$  simulations.

Reverting to the symmetrized Euler approximation, we can then define (and simulate) the associated process

$$r_{(p+1)T/n}^n = \left| r_{pT/n}^n + \left( \eta - kr_{pT/n}^n + \theta\sigma\sqrt{r_{pT/n}^n} \right) \frac{T}{n} + \sigma\sqrt{r_{pT/n}^n}\epsilon_p \right|, \tag{16}$$

where the  $\epsilon_p$ 's are iid  $\mathcal{N}(0, T/n)$ , and compute the importance sampling weight associated with the simulation. (In the discrete case, the change of measure is the same as (15).) Obviously, the importance weights are based on the comparison between (14) and (16) and do not take into account the Euler approximation. Note that the idea of a location drift is already present in the literature (see, e.g., [1,2,22]), with [22] deriving optimal choices of  $\theta$  towards variance minimizing. As in the toy example, we are using independent proposals and thus do not require the resampling step in the algorithm.

The choice of  $\theta$  being open, a  $D$ -kernel scheme can be used to select efficient values within a finite number of values of  $\theta$ , towards the approximation of  $\mathfrak{P}$ . Let us stress once more that the underlying idea of the alternative proposals is to force  $r_T^n$  to be larger than  $K$  in order to decrease the variance of the Monte Carlo estimator and so positive values of  $\theta$  are called for. Note that the case  $\theta = 0$  corresponds to a crude Monte Carlo approximation. Figure 2 compares the observed range (that is, the area between the minimum and the maximum values of  $r_{pT/n}$  for each  $p$ ) of the proposed processes  $(r_{pT/n}^n)$  with the observed range of 1000 resampled processes using the importance weights. While the range of the proposed values is obviously larger, the decrease in the range due to resampling is quite limited, which signals a good enough fit between the target distribution and the optimized mixture.

We thus ran a simulation experiment with the following parameters:  $\eta = 0.016$ ,  $k = 0.2$ ,  $\sigma = 0.02$ ,  $M = 1000$ ,  $r_0 = 0.08$ ,  $T = 1$  and  $n = 299$ . For simplicity's sake, we only took three values of  $\theta$ ,  $\theta = 0, 1, 2$ , the value  $\theta = 0$  acting as a stabilizing factor in the importance weight (since it ensures that assumptions **(A1–A2)** hold). Obviously, a finer grid of values of  $\theta$  with  $D$  larger than 3 could have been chosen as well, at the cost of more simulations. Tables 2–4 present the results of this experiment over  $N = 100\,000$  simulations and  $T = 10$  iterations of the weight update for  $K = 0.07$ ,  $K = 0.08$  and  $K = 0.09$ . The larger the bound  $K$ , the larger the weight on the larger value of  $\theta$ . In the three cases, the decrease in variance from the equally weighted  $D$ -kernel proposal is quite appreciable. (A phenomenon that is quite common in practice is the quick decrease of the variance in the very first iterations, followed by a much slower decline that explains why we stopped at  $T = 10$

TABLE 2. Cox-Ingersoll-Ross European option: approximation of the price  $\mathfrak{P}$  for  $K = 0.07$  using a  $D$ -kernel PMC algorithm with  $\theta = 0$ ,  $\theta = 1$  and  $\theta = 2$ .

$t$	$\widehat{\mathfrak{P}}_{t,N}^{PMC}$	$\alpha_1^{t,N}$	$\alpha_2^{t,N}$	$\alpha_3^{t,N}$	$\sigma_{1,t}^2$
1	9.2635	0.3333	0.3333	0.3334	27.0664
2	9.2344	0.4748	0.3703	0.1549	13.4474
3	9.2785	0.5393	0.3771	0.0836	9.7458
4	9.2495	0.5672	0.3835	0.0493	8.5258
5	9.2444	0.5764	0.3924	0.0312	7.8595
6	9.2400	0.5780	0.4014	0.0206	7.5471
7	9.2621	0.5765	0.4098	0.0137	7.2214
8	9.2435	0.5727	0.4183	0.0090	7.1354
9	9.2553	0.5682	0.4260	0.0058	7.0289
10	9.2602	0.5645	0.4320	0.0035	6.8854

TABLE 3. Same table as Table 2 for  $K = 0.08$ .

$t$	$\widehat{\mathfrak{P}}_{t,N}^{PMC}$	$\alpha_1^{t,N}$	$\alpha_2^{t,N}$	$\alpha_3^{t,N}$	$\sigma_{1,t}^2$
1	1.8784	0.3333	0.3333	0.3334	2.1781
2	1.8791	0.2458	0.4187	0.3355	1.9287
3	1.8793	0.1797	0.5078	0.3125	1.7329
4	1.8848	0.1279	0.5924	0.2797	1.5670
5	1.8877	0.0878	0.6704	0.2418	1.4374
6	1.8881	0.0589	0.7340	0.2071	1.3303
7	1.8892	0.0359	0.7873	0.1768	1.2530
8	1.8853	0.0229	0.8275	0.1496	1.2010
9	1.8860	0.0137	0.8613	0.1250	1.1593
10	1.8879	0.0079	0.8883	0.1038	1.1262

TABLE 4. Same table as Table 2 for  $K = 0.09$ .

$t$	$\widehat{\mathfrak{P}}_{t,N}^{PMC}$	$\alpha_1^{t,N}$	$\alpha_2^{t,N}$	$\alpha_3^{t,N}$	$\sigma_{1,t}^2$
1	0.0555	0.3333	0.3333	0.3334	0.0114
2	0.0559	0.0333	0.2474	0.7193	0.0053
3	0.0554	0.0026	0.1108	0.8866	0.0043
4	0.0558	0.0001	0.0443	0.9556	0.0039
5	0.0557	0.0000	0.0164	0.9836	0.0038
6	0.0559	0.0000	0.0059	0.9941	0.0038
7	0.0559	0.0000	0.0028	0.9972	0.0038
8	0.0555	0.0000	0.0010	0.9990	0.0038
9	0.0558	0.0000	0.0003	0.9997	0.0038
10	0.0556	0.0000	0.0002	0.9998	0.0037

iterations. In fact, more iterations would see complete disappearance of one component for Tables 2 and 3, and of two components for Table 4. This feature is also found in the EM algorithm, whose links with the  $D$ -kernel PMC are discussed in [12].)

The gain compared with the naive Monte Carlo approximation to  $\mathfrak{P}$  is quite important: for  $K = 0.07$ ,  $K = 0.08$  and  $K = 0.09$ , the variances are 21.59, 7.914 and 0.1937, respectively. Note that Su and Fu [22] derived optimal values for  $\theta$  in exactly the same setting, obtaining  $\theta = 0.487$ ,  $\theta = 1.077$  and  $\theta = 1.234$  in

the three cases, respectively. An interesting byproduct of our simulation experiment is that, while  $\theta = 0.487$  does lead to a smaller variance when  $K = 0.07$ , namely 2.318, compared with 6.88, the second case leads to almost the same variance, 1.126 versus 0.9939, when  $K = 0.08$  since the optimal value is  $\theta = 1.077$ , and, even more surprisingly, the case  $K = 0.09$  produces a much smaller variance, 0.0037 versus 0.0112. We have thus uncovered an improvement over their proposed value. (The reason for this discrepancy is that Su and Fu [22] ran a fixed number of iterations of their optimization algorithm, rather than to wait for the minimum,  $\theta = 2$ , which then produces a much smaller variance than the proposed value  $\theta = 1.234$ .)

As a final comment, we want to stress that, while  $\mathfrak{P}$  is a real quantity  $\pi(h)$ , the function  $h$  in the (discretized) expectation involves the whole vector of  $r_t$ 's. Therefore this example provides furthermore an interesting illustration of the ability of PMC to also function in high dimensional problems (even though it obviously takes advantage of the special features of the problem).

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## 7. PROOFS

### 7.1. Proof of Proposition 3.2

The proof of this result follows the same lines as the proof of Proposition 4.3 in Douc *et al.* [12]. The only condition to check is the equivalent of Proposition 4.2 in [12]. For every  $\alpha \in \mathcal{S}_D, \alpha \neq \alpha^{1,min}$ , we now show that there exists a neighborhood  $V_\alpha$  of  $\alpha$  such that if  $\alpha^{t_0} \in V_\alpha$  then  $(\alpha_t)_{t \geq t_0}$  leaves  $V_\alpha$  within a finite time. Then, by continuity of  $\sigma_1^2(\alpha)$ , there exists  $\epsilon > 0$  such that

$$\begin{aligned} -\epsilon \sigma_1^2(\alpha) &\geq \sigma_1^2(\alpha^{1,min}) - \sigma_1^2(\alpha) = \nu_h \left( \frac{1}{\sum_{d=1}^D \alpha_d^{1,min} q_d(x, x')} - \frac{1}{\sum_{d=1}^D \alpha_d q_d(x, x')} \right) \\ &\geq \nu_h \left( \frac{\sum_{d=1}^D (\alpha_d - \alpha_d^{1,min}) q_d(x, x')}{(\sum_{d=1}^D \alpha_d q_d(x, x'))^2} \right) = \sum_{d=1}^D \alpha_d^{1,min} \left[ \sigma_1^2(\alpha) - \nu_h \left( \frac{q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right) \right]. \end{aligned}$$

Thus, there exists  $1 \leq d \leq D$  such that

$$\nu_h \left( \frac{q_d(x, x')}{(\sum_{l=1}^D \alpha_l q_l(x, x'))^2} \right) \geq (1 + \epsilon) \sigma_1^2(\alpha)$$

which implies that  $[F_1(\alpha)]_d \geq (1 + \epsilon) \alpha_d$ . Since  $F_1(\cdot)$  is continuous, this implies that there exists  $V_\alpha$  a neighborhood of  $\alpha$  such that for all  $\alpha' \in V_\alpha$ ,  $[F_1(\alpha')]_d \geq (1 + \epsilon/2) \alpha'_d$ . Since  $0 \leq \alpha_d \leq 1$ , it follows that if  $\alpha^{t_0} \in V_\alpha$  then  $(\alpha_t)_{t \geq t_0} = (F_1^{t-t_0}(\alpha_{t_0}))_{t \geq t_0}$  will leave  $V_\alpha$  within a finite time. The proof is completed.

### 7.2. Proof of Proposition 3.3

The case  $t = 1$  is obvious. Now, assume (5) holds for some  $t \geq 1$ . Recall that  $\nu_h$  is defined in (2). We now prove that the following convergence results

$$\frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 \left( h(x_{i,t}) - \sum_{j=1}^N \bar{\omega}_{j,t} h(x_{j,t}) \right)^2 \mathbb{I}_d(K_{i,t}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \nu_h \left( \frac{\alpha_d^t q_d(x, x')}{(\sum_{l=1}^D \alpha_l^t q_l(x, x'))^2} \right), \tag{17}$$

$$\frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 \left( h(x_{i,t}) - \sum_{j=1}^N \bar{\omega}_{j,t} h(x_{j,t}) \right)^2 \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \sigma_1^2(\alpha^t). \tag{18}$$



Only the first convergence needs be considered since the latter can be easily deduced from the former by summing over  $d$ . To prove Eq. (17), we will show that

$$\frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}) \xrightarrow{N \rightarrow \infty} \nu_H \left( \frac{\alpha_d^t q_d(x, x')}{(\sum_{l=1}^D \alpha_l^t q_l(x, x'))^2} \right)$$

for any function  $H$  satisfying  $\pi(H) = 0$  and

$$\bar{\pi} \{ (1 + H(x')) \pi(x') / q_{\bar{d}}(x, x') \} < \infty \quad \text{for some } \bar{d} \in \{1, \dots, D\}. \tag{19}$$

We apply Theorem A.1 of [12] with

$$\mathcal{G}_N = \sigma \left( (\tilde{x}_{i,t-1})_{1 \leq i \leq N}, (\alpha_d^{t,N})_{1 \leq d \leq D} \right) \quad \text{and} \quad U_{N,i} = N^{-1} \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}).$$

Conditionally on  $\mathcal{G}_N$ ,  $(K_{i,t}, x_{i,t})_{1 \leq i \leq N}$  are independent and for all  $(d, A)$  in  $\{1, \dots, D\} \times \mathcal{A}$ ,

$$\mathbb{P}(K_{i,t} = d, x_{i,t} \in A | \mathcal{G}_N) = \alpha_d^{t,N} Q_d(\tilde{x}_{i,t-1}, A).$$

To apply Theorem A.1 of [12], we just need to check condition (iii). We have

$$\begin{aligned} & \mathbb{E} \left( \left. \frac{\sum_{i=1}^N \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t})}{N} \mathbb{I}_{\{\omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}) > C\}} \right| \mathcal{G}_N \right) \\ & \leq \sum_{j=1}^D \frac{1}{N} \sum_{i=1}^N \int \pi(dx) \frac{\pi(x) H(x) \alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x)}{(\sum_{l=1}^D \alpha_l^{t,N} q_l(\tilde{x}_{i,t-1}, x))^2} \mathbb{I}_{\left\{ \frac{\pi(x)^2 H(x)}{D^{-2} q_d^2(\tilde{x}_{i,t-1}, x)} > C \right\}} \\ & \leq \sum_{j=1}^D \frac{1}{N} \sum_{i=1}^N \int \pi(dx) \frac{\pi(x) H(x)}{\alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x)} \mathbb{I}_{\left\{ \frac{\pi(x)^2 H(x)}{D^{-2} q_d^2(\tilde{x}_{i,t-1}, x)} > C \right\}} \\ & \xrightarrow{N \rightarrow \infty} \sum_{j=1}^D \int \pi(dx') \pi(dx) \frac{\pi(x) H(x)}{\alpha_d^t q_d(x', x)} \mathbb{I}_{\left\{ \frac{\pi(x)^2 H(x)}{D^{-2} q_d^2(x', x)} > C \right\}} \end{aligned}$$

by the LLN stated in Theorem 2.1 and since the induction assumption implies that  $\alpha_d^{t,N} \xrightarrow{N \rightarrow \infty} \alpha_d^t$  which is positive by the updating formula of  $\alpha_d^t$ . The rhs converges to 0 as  $C$  gets to infinity using (19) and  $\bar{\pi} \{ H(x') = \infty \text{ or } q_j(x, x') = 0 \} = 1$ . Thus, Theorem A.1 of Douc *et al.* [12] applies and

$$\frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}) - \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}) \middle| \mathcal{G}_N \right) \xrightarrow{N \rightarrow \infty} 0.$$

To complete the proof, we only need to show that

$$\mathbb{E} \left( \frac{1}{N} \sum_{i=1}^N \omega_{i,t}^2 H(x_{i,t}) \mathbb{I}_d(K_{i,t}) \middle| \mathcal{G}_N \right) = \frac{1}{N} \sum_{i=1}^N \int \pi(dx) \frac{\pi(x) H(x) \alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x)}{(\sum_{l=1}^D \alpha_l^{t,N} q_l(\tilde{x}_{i,t-1}, x))^2} \xrightarrow{N \rightarrow \infty} \nu_H \left( \frac{\alpha_d^t q_d(x, x')}{(\sum_{l=1}^D \alpha_l^t q_l(x, x'))^2} \right). \tag{20}$$

Using again the LLN stated in Theorem 2.1,

$$\frac{1}{N} \sum_{i=1}^N \int \pi(dx) \frac{\pi(x) H(x) \alpha_d^t q_d(\tilde{x}_{i,t-1}, x)}{(\sum_{l=1}^D \alpha_l^t q_l(\tilde{x}_{i,t-1}, x))^2} \xrightarrow{N \rightarrow \infty} \nu_H \left( \frac{\alpha_d^t q_d(x, x')}{(\sum_{l=1}^D \alpha_l^t q_l(x, x'))^2} \right). \tag{21}$$

Thus, to establish (20), we use (21) and check that the difference between both terms converges to 0 in probability. To see this, first note that for all  $t \geq 1$ , for all  $d$  in  $\{1, \dots, D\}$ ,  $\alpha_d^t > 0$  and thus, by the induction assumption, for all  $d$  in  $\{1, \dots, D\}$ ,  $\frac{\alpha_d^{t,N} - \alpha_d^t}{\alpha_d^t} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$ . It has been shown in Douc *et al.* [12] that

$$\left| \frac{\alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x)}{\sum_{l=1}^D \alpha_l^{t,N} q_l(\tilde{x}_{i,t-1}, x)} - \frac{\alpha_d^t q_d(\tilde{x}_{i,t-1}, x)}{\sum_{l=1}^D \alpha_l^t q_l(\tilde{x}_{i,t-1}, x)} \right| \leq 2 \sup_{l \in \{1, \dots, D\}} \left| \frac{\alpha_l^{t,N} - \alpha_l^t}{\alpha_l^t} \right|.$$

Combining with  $\left| \frac{A}{B^2} - \frac{C}{D^2} \right| \leq \left| \frac{A}{B} \right| \left| \frac{D-B}{BD} \right| + \left| \frac{1}{D} \right| \left| \frac{A}{B} - \frac{C}{D} \right|$  yields by straightforward algebra,

$$\begin{aligned} & \left| \frac{\alpha_d^{t,N} q_d(\tilde{x}_{i,t-1}, x)}{(\sum_{l=1}^D \alpha_l^{t,N} q_l(\tilde{x}_{i,t-1}, x))^2} - \frac{\alpha_d^t q_d(\tilde{x}_{i,t-1}, x)}{(\sum_{l=1}^D \alpha_l^t q_l(\tilde{x}_{i,t-1}, x))^2} \right| \\ & \leq \frac{1}{\sum_{l=1}^D \alpha_l^{t,N} q_l(\tilde{x}_{i,t-1}, x)} \left( \sup_{l \in \{1, \dots, D\}} \left| \frac{\alpha_l^{t,N} - \alpha_l^t}{\alpha_l^t} \right| \right) \\ & \quad + \frac{1}{\sum_{l=1}^D \alpha_l^t q_l(\tilde{x}_{i,t-1}, x)} \left( 2 \sup_{l \in \{1, \dots, D\}} \left| \frac{\alpha_d^{t,N} - \alpha_d^t}{\alpha_d^t} \right| \right) \\ & \leq \left( \frac{1}{\alpha_d^{t,N}} + \frac{2}{\alpha_d^t} \right) \frac{1}{q_d(\tilde{x}_{i,t-1}, x)} \left( \sup_{l \in \{1, \dots, D\}} \left| \frac{\alpha_l^{t,N} - \alpha_l^t}{\alpha_l^t} \right| \right). \end{aligned}$$

The proof follows from  $\frac{\alpha_d^{t,N} - \alpha_d^t}{\alpha_d^t} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$  and

$$\frac{1}{N} \sum_{i=1}^N \int \pi(dx) \frac{\pi(x)H(x)}{q_d(\tilde{x}_{i,t-1}, x)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \bar{\pi} \left\{ \frac{H(x')\pi(x')}{q_d(x, x')} \right\}.$$

### 7.3. Proof of Theorem 5.1

We will need the following technical lemma:

**Lemma 7.1.** *Assume (A1). Then for all  $\delta > 0$ ,*

$$\limsup_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{P}(|M_N^T(\psi_T) - \tilde{M}_N^T(\psi_T)| > \delta) = -\infty$$

where

$$\tilde{M}_N^T(\psi_T) = \frac{1}{b_N \sqrt{N}} \sum_{i=1}^N \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} (\psi_T(x_{i,T}) - \pi(\psi_T)).$$

The proof of this lemma is deferred till the end of this appendix. We first establish the main result, namely, Theorem 5.1.

*Proof.* The proof proceeds by induction. First note that the random variables  $M_N^t(\psi_t)$  and  $\tilde{M}_N^t(\psi_t)$  are exponentially equivalent up to the speed  $b_N^2$ , by Lemma 7.1. It is thus sufficient by Gärtner-Ellis' Theorem ([11] Th. 2.3.6) to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{E} \left( \exp \left\{ b_N^2 \sum_{t=0}^{T-1} \lambda_t M_N^t(\psi_t) + b_N^2 \lambda_T \tilde{M}_N^T(\psi_T) \right\} \right) = \frac{\langle \lambda_T, V_T(\psi_{0:T}) \cdot \lambda_T \rangle}{2}.$$

Namely, we prove here a MDP for  $\tilde{M}_N^T(\psi_T)$ . More precisely, we now show that  $\lim_{N \rightarrow \infty} A_N = 0$  where

$$\begin{aligned} A_N &:= \frac{1}{b_N^2} \log \mathbb{E} \left( e^{b_N^2 \sum_{t=0}^{T-1} \lambda_t M_N^t(\psi_t) + b_N^2 \lambda_T \tilde{M}_N^T(\psi_T)} - \frac{\langle \boldsymbol{\lambda}_T, V_T(\psi_{0:T}) \cdot \lambda_T \rangle}{2} \right) \\ &= \frac{1}{b_N^2} \log \mathbb{E} \left( \exp \left\{ \lambda_T b_N^2 \tilde{M}_N^T(\psi_T) - \frac{\lambda_T^2 b_N^2 \hat{\sigma}_{1,T}^2(\psi_T)}{2} + \frac{\lambda_T^2 b_N^2}{2} \Delta_N \right. \right. \\ &\quad \left. \left. + b_N^2 \sum_{t=0}^{T-1} \lambda_t M_N^t(\psi_t) \right\} \right) - \frac{\langle \boldsymbol{\lambda}_{T-1}, V_{T-1}(\psi_{0:T-1}) \cdot \lambda_{T-1} \rangle}{2} \end{aligned}$$

where  $\Delta_N = \hat{\sigma}_{1,T}^2(\psi_T) - \sigma_{1,T}^2(\psi_T)$ ,

$$\begin{aligned} \hat{\sigma}_{1,T}^2(\psi_T) &= \text{Var} \left( b_N \tilde{M}_N^T(\psi_T) \middle| \mathcal{F}_N^{T-1} \right) = \frac{1}{N} \sum_{i=1}^N \hat{\sigma}_{1,T}^2(\tilde{x}_{i,T-1}) \quad \text{with} \\ \hat{\sigma}_{1,T}^2(x) &= \int \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^{T,N} q_d(x, x')} (\psi_T(x') - \pi(\psi_T))^2 \pi(dx'). \end{aligned}$$

Write  $\tilde{\mathcal{F}}_N^T$  (resp.  $\mathcal{F}_N^T$ ) the  $\sigma$ -field generated by  $(x_{i,t}, \tilde{x}_{i,t}; 1 \leq i \leq N, 0 \leq t \leq T)$  (resp.  $(x_{i,t}; 1 \leq i \leq N, 0 \leq t \leq T)$ ). Then,

$$\begin{aligned} A_N &= \frac{1}{b_N^2} \log \mathbb{E} \left[ \mathbb{E} \left( \exp \left\{ \lambda_T b_N^2 \tilde{M}_N^T(\psi_T) - \frac{\lambda_T^2 b_N^2 \hat{\sigma}_{1,T}^2(\psi_T)}{2} \right\} \middle| \tilde{\mathcal{F}}_N^{T-1} \right) \right. \\ &\quad \left. \times \exp \left\{ \frac{\lambda_T^2 b_N^2}{2} \Delta_N + b_N^2 \sum_{t=0}^{T-1} \lambda_t M_N^t(\psi_t) \right\} \right] - \frac{\langle \boldsymbol{\lambda}_{T-1}, V_{T-1}(\psi_{0:T-1}) \cdot \lambda_{T-1} \rangle}{2}. \quad (22) \end{aligned}$$

Conditionally on  $\tilde{\mathcal{F}}_N^{T-1}$ , the variables  $(x_{i,t})$  are independent and thus,

$$\begin{aligned} B_N &:= \mathbb{E} \left( \exp \left\{ \lambda_T b_N^2 \tilde{M}_N^T(\psi_T) - \frac{\lambda_T^2 b_N^2 \hat{\sigma}_{1,T}^2}{2} \right\} \middle| \tilde{\mathcal{F}}_N^{T-1} \right) \\ &= \prod_{i=1}^N \mathbb{E} \left( \exp \left\{ \frac{\lambda_T b_N}{\sqrt{N}} \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} (\psi_T(x_{i,T}) - \pi(\psi_T)) - \frac{\lambda_T^2 b_N^2 \hat{\sigma}_{i,T}^2(\tilde{x}_{i,T-1})}{2N} \right\} \middle| \tilde{\mathcal{F}}_N^{T-1} \right) \\ &= \prod_{i=1}^N \mathbb{E} \left( 1 + \frac{\lambda_T b_N}{\sqrt{N}} \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} (\psi_T(x_{i,T}) - \pi(\psi_T)) - \frac{\lambda_T^2 b_N^2 \hat{\sigma}_{i,T}^2(\tilde{x}_{i,T-1})}{2N} \right. \\ &\quad \left. + \frac{\lambda_T^2 b_N^2}{2N} \left( \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} (\psi_T(x_{i,T}) - \pi(\psi_T)) \right)^2 + O \left( \frac{\lambda_T^3 b_N^3}{N^{3/2}} \right) \middle| \tilde{\mathcal{F}}_N^{T-1} \right) \\ &= \left( 1 + O \left( \frac{\lambda_T^3 b_N^3}{N^{3/2}} \right) \middle| \tilde{\mathcal{F}}_N^{T-1} \right)^N. \quad (23) \end{aligned}$$

Under the integrability condition **(A3)** and taking advantage of the fact that  $\psi_T$  is bounded, there exists a constant  $\gamma > 0$  such that

$$\begin{aligned} \sup_x \int e^{\gamma \frac{\pi(x')}{\sum_{d=1}^D \alpha_d^{T,N} q_d(x,x')}} |\psi_T(x) - \pi(\psi_T)| \sum_{d=1}^D \alpha_d^{T,N} Q_d(x, dx') \\ \leq e^{2\gamma \sup_x |\psi_T(x)|} + \sup_x \int e^{\frac{\pi(x')}{\sum_{d=1}^D \alpha_d^{T,N} q_d(x,x')}} 2\gamma \sup_x |\psi_T(x)| \pi(dx') < \infty. \end{aligned}$$

This implies that in Eq. (23),  $O(\lambda_T^3 b_N^3 / N^{3/2}) \leq K \lambda_T^3 b_N^3 / N^{3/2}$  where  $K$  does not depend on  $(x_{i,T})_{1 \leq i \leq T}$ . Therefore,

$$\left(1 - K \frac{\lambda_T^3 b_N^3}{N^{3/2}}\right)^N \leq B_N \leq \left(1 + K \frac{\lambda_T^3 b_N^3}{N^{3/2}}\right)^N. \quad (24)$$

Now, consider  $(\lambda_T^2 b_N^2 / 2) \Delta_N = (b_N^2 \lambda_T^2 / 2) (\hat{\sigma}_{1,T}^2(\psi_T) - \sigma_{1,T}^2(\psi_T))$ . We have that

$$|\hat{\sigma}_{1,T}^2(\psi_T) - \sigma_{1,T}^2(\psi_T)| = \left| \frac{1}{N} \sum_{i=1}^N \tilde{\sigma}_{1,T}^2(\tilde{x}_{i,T-1}) - \sigma_{1,T}^2(\psi_T) \right|.$$

Note that conditionally on  $\mathcal{F}_N^{T-1}$ , the  $(\tilde{x}_{i,T-1})_{1 \leq i \leq N}$  are iid and  $x \mapsto \tilde{\sigma}_{1,T}(x)$  is bounded under **(A3)**. Thus, by Hoeffding's inequality,

$$\mathbb{P}(|\hat{\sigma}_{1,T}^2(\psi_T) - \sigma_{1,T}^2(\psi_T)| > \epsilon) \leq 2 \exp(-2N(\epsilon / \sup_x \tilde{\sigma}_{1,T}^2(x))^2),$$

which implies that  $\limsup_{N \rightarrow \infty} (b_N)^{-2} \log \mathbb{P}(|\hat{\sigma}_{1,T}^2(\psi_T) - \sigma_{1,T}^2(\psi_T)| > \delta) = -\infty$ ,  $\forall \delta > 0$ . Combining this, equations (24) and (22) and the induction assumption yields that  $\lim_{N \rightarrow \infty} A_N = 0$ . The proof is completed.  $\square$

We now establish Lemma 7.1.

*Proof.* Note first that the proof of Theorem 5.1 focusses mostly on establishing the MDP of  $\tilde{M}_N^T(\psi_T)$ . We use this argument to deduce the exponential negligibility of  $M_N^T(\psi_T) - \tilde{M}_N^T(\psi_T)$ . We first have

$$R_n^t := M_N^T(\psi_T) - \tilde{M}_N^T(\psi_T) = \tilde{M}_N^T(\psi_T) \left( \frac{1}{\frac{1}{N} \sum_1^N \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})}} - 1 \right)$$

so that for all positive  $\delta, L, \epsilon$ ,

$$\begin{aligned} \mathbb{P}(|R_n^t| > \delta) &\leq \mathbb{P}(|\tilde{M}_N^T(\psi_T)| > \sqrt{\delta \epsilon} L) \\ &+ \mathbb{P} \left( \left| \frac{1}{N} \sum_1^N \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} - 1 \right| > \sqrt{\delta \epsilon} / L \right) \\ &+ \mathbb{P} \left( \left| \frac{1}{N} \sum_1^N \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} \right| < \epsilon \right). \end{aligned}$$

To establish the desired negligibility, we thus have to prove the negligibility (wrt the MDP) of the three terms of the right hand side of this last inequality. By the MDP for  $\tilde{M}_N^T$  proved previously, we easily get that for all

positive  $\delta$ 's, the first term is of no importance since, given the MDP upper bound and the fact that the rate function is quadratic,

$$\lim_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{b_N^2} \log \mathbb{P}(|\tilde{M}_N^T(\psi_T)| > \sqrt{\delta\epsilon}L) = -\infty.$$

It thus remains to establish negligibility for the second term (the third term can be processed in the same way). Note first that for all positive  $K$ 's, for  $N$  sufficiently large, we have

$$\begin{aligned} \mathbb{P} \left( \left| \frac{1}{N} \sum_1^N \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} - 1 \right| > \sqrt{\delta\epsilon}/L \right) \\ \leq \mathbb{P} \left( \left| \frac{1}{\sqrt{N}b_N} \sum_1^N \left( \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} - 1 \right) \right| > K\sqrt{\delta\epsilon}/L \right) \end{aligned}$$

which amounts to establish an MDP upper bound for a sequence of (conditionally on  $\tilde{\mathcal{F}}_N^{T-1}$ ) independent centered rv's satisfying a common exponential integrability property by **(A3)**. It then follows from standard calculus (like  $B_N$  in the previous proof) that there exists  $\bar{\sigma}^2$  such that

$$\limsup \frac{1}{b_N^2} \log \mathbb{P} \left( \left| \frac{1}{\sqrt{N}b_N} \sum_1^N \left( \frac{\pi(x_{i,T})}{\sum_{d=1}^D \alpha_d^{T,N} q_d(\tilde{x}_{i,T-1}, x_{i,T})} - 1 \right) \right| > K\sqrt{\delta\epsilon}/L \right) \leq -\frac{K^2}{2L^2\delta\epsilon\bar{\sigma}^2}.$$

Since  $K$  is arbitrary, this establishes the desired negligibility and thus concludes the proof.  $\square$

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