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# BINOMIAL-POISSON ENTROPIC INEQUALITIES AND THE M/M/$\infty$ QUEUE

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**Abstract.** This article provides entropic inequalities for binomial-Poisson distributions, derived from the two point space. They appear as local inequalities of the M/M/$\infty$ queue. They describe in particular the exponential dissipation of $\Phi$-entropies along this process. This simple queueing process appears as a model of "constant curvature", and plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for Brownian Motion. Some of the inequalities are recovered by semi-group interpolation. Additionally, we explore the behaviour of these entropic inequalities under a particular scaling, which sees the Ornstein-Uhlenbeck process as a fluid limit of M/M/$\infty$ queues. Proofs are elementary and rely essentially on the development of a "$\Phi$-calculus".

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## 1. Introduction

We consider in this article the M/M/$\infty$ queueing process. This elementary continuous time Markov process on $\mathbb{N}$ plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for Brownian motion. In particular, its law at time $t$ is explicitly given by a binomial-Poisson Mehler like formula, and the associated semi-group commutes with the discrete gradient operator, up to a time decreasing exponential factor. We derive general entropic inequalities for binomial-Poisson measures from the two points space, essentially by convexity. They hold in particular for the law at fixed time of the process, as for Ornstein-Uhlenbeck. In particular, these entropic inequalities contain as special cases Poincaré inequalities and various modified logarithmic Sobolev inequalities, which appear for instance in [1, 5, 6, 16].

It is known that the lack of a chain rule and of a good notion of curvature in discrete space settings make difficult the derivation of entropic inequalities for discrete space Markov processes. Poincaré inequalities are exceptional, due to their Hilbertian nature. Their derivation does not need the diffusion property. Lévy processes and Poisson space are also exceptional, since their i.i.d. underlying structure makes them "flat" in a way. This nature is translated on the infinitesimal Markov generator as a commutation with translations. The M/M/$\infty$ queue has non-homogeneous independent increments, and is thus beyond this framework. The reader may find various entropic inequalities for finite space Markov processes in [2, 17, 33], and for infinite countable space Markov processes in [1, 5, 11–16, 21, 22, 29, 30, 35] for instance.

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Birth and death processes are the discrete space analogue of diffusion processes. However, they are not diffusions, and specific diffusion tools like Bakry-Émery $\mathbf{\Gamma_2}$ calculus are of difficult usage for such processes. It follows from our study that the M/M/$\infty$ queueing process can serve as a model of "constant curvature" on $\mathbb{N}$. It is known that convexity may serve as an alternative to the diffusion property, as presented for instance in [11, 35]. In this article, we circumvent the lack of chain rule by elementary convexity bounds for germs of discrete Dirichlet forms. This work can be seen as a continuation of [11], and was initially motivated by the time inhomogeneous M/M/$\infty$ queue which appears in the biological problem studied in [10]. The notion of queueing process is widely used in applied probability. The reader may find a modern introduction to queueing processes in the book [31] by Robert, in connection with random networks, general Markov processes, martingales, and point processes. This large family of Markov processes contains, as particular cases, the simple Poisson process, the continuous time simple random walk on $\mathbb{N}$, and more generally all continuous time birth and death processes on $\mathbb{N}$.

The approach and results of this article may be extended by various ways. The first step is to consider birth and death processes on $\mathbb{N}$ or $\mathbb{Z}$, and then on $\mathbb{N}^d$ or $\mathbb{Z}^d$ with interactions. Some versions of such models where already considered in the statistical mechanics literature, at least for Poincaré and modified logarithmic Sobolev inequalities, see for instance [5, 13–16], and references therein. These extensions concern continuous time processes on a discrete space $E$ with generator of the form

$$\mathbf{L}(f)(x) = \int_E (f(y) - f(x))\, \mathrm{d}\gamma_x(y),$$

where $\gamma_x$ is the "jump measure" at point $x$, which is a finite Borel measure on $E$. Another possibility is to consider Volterra processes driven by a simple Poisson process, possibly together with Clark-Ocone formulas as in [2, 35] for instance. We hope that some of these extensions will make the matter of forthcoming articles. We have in mind the construction of a functional bridge between discrete space Markov processes and "curved" diffusion processes, which complements, by mean of quantitative functional inequalities, the approximation in law. The recent articles [5, 8, 13–16, 19–21] may help for such a program.

The entropic inequalities that we consider in this article can be called "$\Phi$-Sobolev inequalities" since they involve a $\Phi$-entropy and a $\Phi$-Dirichlet form. They contain in particular Poincaré inequalities and "$\mathrm{L}^1$"-logarithmic-Sobolev inequalities. As presented in [11], they hold, under convexity assumptions on $\Phi$, for log-concave measures on $\mathbb{R}^d$, for diffusions on manifolds with positive bounded below curvature, for many Wiener measures, for Poisson space, and for many Lévy processes. Their genericity on $\Phi$ is particularly interesting in discrete space settings for which no chain rule is available. The aim of this article is to extend these entropic inequalities to discrete space processes, beyond the i.i.d. increments case, in particular, beyond Lévy processes and Poisson space.

This work goes beyond many results of [6], in terms of entropies, Dirichlet forms, and measures. We show how the entropic inequalities are scaled when the discrete space curved process (M/M/$\infty$) approximates a curved diffusion process (Ornstein-Uhlenbeck). This work can thus be seen as a precise and instructive case study. Many aspects are still valid for more general birth and death processes, and we believe that the entropic inequalities that we consider here hold for interacting birth and death processes. However, a lot of work is still needed to achieve this objective. In particular, and to our knowledge, a good notion of curvature is still lacking for interacting birth and death processes. Viewed as a unidimensional (*e.g.* single site) particle system, the M/M/$\infty$ queue is not conservative. It can be viewed as a particular unidimensional case of the processes considered in [13–16]

**Outline of the rest of the article.** In the introduction, the definition of the M/M/$\infty$ queueing process is followed by the presentation of links and analogies with the Ornstein-Uhlenbeck process, and then by the introduction of the $\Phi$-entropy together with the $A - B - C$ transforms of $\Phi$. Section two is a two point space approach to binomial-Poisson entropic inequalities. In Section three, we address the exponential decay of

$\Phi$-entropy functionals along the M/M/$\infty$ queue, we give various proofs of entropic inequalities by using semi-group interpolations, and we use a scaling limit to recover Gaussian inequalities for the Ornstein-Uhlenbeck process. The fourth and last section is devoted to key convexity properties related to the $A-B-C$ transforms.

## 1.1. The M/M/$\infty$ queueing process

The M/M/$\infty$ queue with input rate $\lambda \geqslant 0$ and service rate $\mu \geqslant 0$ is a particular space-inhomogeneous and time-homogeneous birth and death process on $\mathbb{N}$. Let $X_t$ be the number of customers in the queue – i.e. the length of the queue – at time $t$. The name "M/M/$\infty$" comes from Kendall's classification of simple queueing processes. It corresponds to an infinite number of servers with random memoryless inter-arrivals (first M) and service times (second M), see for instance [31], p. xiii. Since the number of servers is infinite, each client gets immediately his own dedicated server, and the length of the queue is exactly the number of busy servers. The infinitesimal Markov generator $\mathbf{L}$ of the M/M/$\infty$ queue $(X_t)_{t\geqslant 0}$ is given for any $f : \mathbb{N} \to \mathbb{R}$ and any $n \in \mathbb{N}$ by

$$\mathbf{L}(f)(n) = n\mu \mathrm{D}^*(f)(n) + \lambda \mathrm{D}(f)(n), \tag{1}$$

where the discrete gradients D and $\mathrm{D}^*$ are defined respectively for any $f : \mathbb{Z} \to \mathbb{R}$ and any $n \in \mathbb{N}$ by

$$\mathrm{D}(f)(n) := f(n+1) - f(n) \quad \text{and} \quad \mathrm{D}^*(f)(n) := f(n-1) - f(n). \tag{2}$$

The operators D and $\mathrm{D}^*$ commute with translations, but $\mathbf{L}$ does not. Notice that $f(-1)$ does not need to be defined in (1) since it is multiplied by 0. The stared notation for $\mathrm{D}^*$ comes from the fact that $\mathrm{D}^*$ is the adjoint of D with respect to the counting measure on $\mathbb{Z}$. The identity $\mathrm{DD}^* = \mathrm{D}^*\mathrm{D} = -(\mathrm{D} + \mathrm{D}^*)$ leads to a polarised version of the infinitesimal generator (1),

$$\mathbf{L}(f)(n) = -\lambda(\mathrm{DD}^*)(f)(n) + (n\mu - \lambda)\mathrm{D}^*(f)(n),$$

for any $n \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}$. The finite difference operator $\mathrm{DD}^*$ is the discrete Laplacian, given by $(\mathrm{DD}^*)(f)(n) = 2f(n) - f(n-1) - f(n+1)$ for any $f : \mathbb{Z} \to \mathbb{R}$ and any $n \in \mathbb{Z}$. Consider the process conditional to the event $\{X_s = n\}$. Let $T := \min\{t > s : X_t \neq X_s\} - s$ be the waiting time before next jump. Then $T$ follows an exponential law $\mathcal{E}(\lambda + n\mu)$ of mean $1/(\lambda + n\mu)$. The transition matrix $\mathbf{J}$ of the embedded discrete time jump Markov chain is given for any $m, n \in \mathbb{N}$ by

$$\mathbf{J}(n,m) := \frac{1}{\lambda + n\mu} \begin{cases} n\mu & \text{if } m = n-1 \\ \lambda & \text{if } m = n+1 \\ 0 & \text{otherwise} \end{cases},$$

where we assumed for simplicity that $\lambda + \mu > 0$. The embedded chain is recurrent irreducible as soon as $\lambda > 0$ and $\mu > 0$. The jump intensity function $n \mapsto \lambda + n\mu$ is not bounded when $\mu > 0$, however, the process is not explosive by virtue of Reuter criterion for irreducible birth and death processes, *cf.* [9], Theorem 4.5.

Defining a stochastic process corresponds to specify a law on paths space. Following [31], Chapter 6, the stochastic process $(X_t)_{t\geqslant 0}$ with $X_0 = n$ can be constructed as follows:

$$X_t = n + \mathcal{N}_\lambda(]0,t]) - \sum_{i=1}^{\infty} \int_{]0,t]} 1_{\{X_{s-} \geqslant i\}} \mathcal{N}_\mu^i(\mathrm{d}s),$$

where $\mathcal{N}_\lambda$ is a Poisson random measure on $\mathbb{R}_+$ of intensity $\lambda$ and where $(\mathcal{N}_\mu^i)_{i\in\mathbb{N}}$ is an i.i.d. collection of Poisson random measures on $\mathbb{R}_+$ of intensity $\mu$, independent of $\mathcal{N}_\lambda$. In other words, the process $(X_t)_{t\geqslant 0}$ solves the Stochastic Differential Equation

$$\mathrm{d}X_t = \mathcal{N}_\lambda(\mathrm{d}t) - \sum_{i=1}^{X_{t-}} \mathcal{N}_\mu^i(\mathrm{d}t).$$

Let us consider the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ defined for any $t\in\mathbb{R}_+$ by

$$\mathcal{F}_t := \sigma\{\mathcal{N}_\lambda(]0,s]); s\in[0,t]\}\vee\sigma\big\{\mathcal{N}_\mu^i(]0,s]); (s,i)\in[0,t]\times\mathbb{N}\big\}.$$

The process $(X_t - X_0 - \lambda t + \mu\int_0^t X_s\,\mathrm{d}s)_{t\geqslant 0}$ is a square integrable martingale with increasing process given by $\lambda t + \mu\int_0^t X_s\,\mathrm{d}s$. More generally, the process $(X_t)_{t\geqslant 0}$ is a solution of the martingale problem associated to the Markov generator $\mathbf{L}$ defined by (1). Namely, for any $f:\mathbb{N}\to\mathbb{R}$, the process

$$\left(f(X_t) - f(X_0) - \int_0^t \mathbf{L}(f)(X_s)\,\mathrm{d}s\right)_{t\geqslant 0}$$

is a local martingale. When $f(n)=n$ for any $n\in\mathbb{N}$, we get $\mathbf{L}(f)(n)=\lambda-\mu n$. The Markov semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ of $(X_t)_{t\geqslant 0}$ is defined for any bounded $f:\mathbb{N}\to\mathbb{R}$ by

$$\mathbf{P}_t(f)\,(n) := \mathbf{E}(f(X_t)\,|X_0=n)\,,$$

in such a way that $\mathbf{P}_t(\mathrm{I}_A)\,(n) = \mathbb{P}(X_t\in A\,|\,X_0=n)$ for any $A\subset\mathbb{N}$. We have $\mathbf{P}_t(\cdot)\,(n) = \mathcal{L}(X_t\,|X_0=n)$ for any $n\in\mathbb{N}$. In particular, $\mathbf{P}_t\circ\mathbf{P}_s = \mathbf{P}_{t+s}$, and $\mathbf{P}_0=\mathrm{Id}$, and $\mathbf{L}f := \partial_{t=0}\mathbf{P}_t(f)$. The coefficient $\rho$ of the M/M/$\infty$ queue is defined by

$$\rho := \frac{\lambda}{\mu}\cdot$$

In the sequel, we denote by $\mathbf{E}_Q(f)$ or by $\mathbf{E}_Q f$ the mean of function $f$ with respect to the probability measure $Q$, and by $\mathrm{L}^p(Q)$ the Lebesgue space of measurable real valued functions $f$ such that $|f|^p$ is $Q$-integrable. For a Borel measure on $\mathbb{N}$, we also denote $Q(n) := Q(\{n\})$ for any $n\in\mathbb{N}$.

## 1.2. The Ornstein-Uhlenbeck process as a fluid limit of M/M/$\infty$ queues

The Ornstein-Uhlenbeck process can be recovered from the M/M/$\infty$ queue as a fluid limit, by using a Kelly scaling. See for instance [24,25] and the books [18,26,31]. Namely, for any $N\in\mathbb{N}$, let $(X_t^N)_{t\geqslant 0}$ be the M/M/$\infty$ queue with input rate $N\lambda$ and service rate $\mu>0$. Define the process $(Y_t^N)_{t\geqslant 0}$ by

$$Y_t^N := \frac{1}{N}X_t^N.$$

For any $x\in\mathbb{R}_+$, let $m:\mathbb{R}_+\to\mathbb{R}$ be defined by $m(t) := \rho + (x-\rho)p(t)$ for any $t\in\mathbb{R}_+$, where $p(t) := \mathrm{e}^{-\mu t}$. Consider a sequence of initial states $(X_0^N)_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} Y_0^N = \lim_{N\to\infty}\frac{1}{N}X_0^N = x.$$

Then, for any $t\in\mathbb{R}_+$, the sequence of random variables $\big(\sup_{0\leqslant s\leqslant t}\big|Y_s^N - m(s)\big|\big)_{N\in\mathbb{N}}$ converges in $\mathrm{L}^1$ towards 0 when $N\to\infty$, see for instance [31], Section 6.5. In particular, for any $\varepsilon>0$,

$$\lim_{N\to\infty}\mathbb{P}\left(\sup_{0\leqslant s\leqslant t}\big|Y_t^N - m(s)\big| > \varepsilon\right) = 0.$$

Moreover, this Law of Large Numbers is complemented by a Central Limit Theorem, see for instance [7] and [31]. Namely, define the process $(Z_t^N)_{t\geqslant 0}$ by

$$Z_t^N := \sqrt{N}\big(Y_t^N - m(t)\big) = \frac{X_t^N - Nm(t)}{\sqrt{N}}\cdot$$

Notice that $m(0) = x$. Let $y \in \mathbb{R}$ and assume that the initial states $(X_0^N)_{N\in\mathbb{N}}$ satisfy additionally that

$$\lim_{N\to\infty} Z_0^N = \lim_{N\to\infty} \sqrt{N}\big(Y_0^N - x\big) = y.$$

A basic example is given by $X_0^N = \lfloor Nx + \sqrt{N}y \rfloor$ where $\lfloor\cdot\rfloor$ denotes the integer part. Then, the sequence of processes $(Z_t^N)_{t\geqslant 0}$ converges in distribution, when $N \to \infty$, towards a process denoted $(Z_t^\infty)_{t\geqslant 0}$, with non-homogeneous independent increments, given by

$$Z_t^\infty := yp(t) + \int_0^t p(t-s)\sqrt{\lambda + x\mu}\,\mathrm{d}B_s,$$

where $(B_s)_{s\geqslant 0}$ is a standard Brownian Motion on the real line. In particular, when $x = \rho$, then $m(s) = \rho$ for any $s \in \mathbb{R}_+$, and $(Z_t^\infty)_{t\geqslant 0}$ is in that case an Ornstein-Uhlenbeck process, solution of the Stochastic Differential Equation $Z_0^\infty = y$ and $dZ_t^\infty = \sqrt{2\lambda}\,dB_t - \mu Z_t^\infty \mathrm{d}t$, where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on the real line. Additionally, for any $t \in \mathbb{R}_+$,

$$\mathcal{L}(Z_t^\infty \,|\, Z_0^\infty = y) = \delta_{yp(t)} * \mathcal{N}\big(0, \big(1 - p(t)^2\big)\rho\big),$$

where $\mathcal{N}(a,b)$ denotes the standard Gaussian law on $\mathbb{R}$ of mean $a$ and variance $b$. This Mehler formula is the continuous space analogue of (3). The Markov infinitesimal generator of $(Z_t^\infty)_{t\geqslant 0}$ is the linear differential operator which maps function $y \mapsto f(y)$ to function

$$y \mapsto \lambda f''(y) - \mu y f'(y).$$

The symmetric invariant measure of $(Z_t^\infty)_{t\geqslant 0}$ is the Gaussian law $\mathcal{N}(0,\rho)$. The $\mu$ parameter appears clearly here as a curvature, whereas the $\lambda$ parameter appears as a diffusive coefficient.

### 1.3. The M/M/$\infty$ queue as a discrete Ornstein-Uhlenbeck process

Let $(X_t)_{t\geqslant 0}$ be an M/M/$\infty$ with rates $\lambda$ and $\mu$. When $\mu$ vanishes, $(X_t)_{t\geqslant 0}$ reduces to a simple Poisson process of intensity $\lambda$, and admits the counting measure on $\mathbb{N}$ as a symmetric measure. A contrario, when $\lambda$ vanishes, $(X_t)_{t\geqslant 0}$ is a pure death process, and admits $\delta_0$ as an invariant probability measure.

The M/M/$\infty$ queue plays for the simple Poisson process the role played by the Ornstein-Uhlenbeck process for standard Brownian Motion. The law of the M/M/$\infty$ queue $(X_t)_{t\geqslant 0}$ is explicitly given for any $n \in \mathbb{N}$ by the following Mehler like formula

$$\mathcal{L}(X_t|X_0 = n) = \mathcal{B}(n, p(t)) * \mathcal{P}(\rho q(t)), \tag{3}$$

where

$$p(t) := \mathrm{e}^{-\mu t} \quad \text{and} \quad q(t) := 1 - p(t).$$

When $\mu = 0$, we set $\rho q(t) = \lambda$, since $\lambda = \lim_{\mu\to 0^+} \rho q(t)$. Here and in the sequel, $\mathcal{B}(n,p)$ stands for the binomial distribution $\mathcal{B}(n,p) := (p\delta_1 + q\delta_0)^{*n}$ of size $n \in \mathbb{N}$ and parameter $p \in [0,1]$, with the convention $\mathcal{B}(n,0) := \delta_0$ and $\mathcal{B}(n,1) := \delta_n$. The notation $\mathcal{P}(\sigma)$ stands for the Poisson measure on $\mathbb{N}$ of intensity $\sigma > 0$, defined by $\mathcal{P}(\sigma) := \mathrm{e}^{-\sigma}\sum_{k=0}^\infty \frac{1}{k!}\sigma^k\delta_k$. When $\mu > 0$, the process $(X_t)_{t\geqslant 0}$ is ergodic and admits $\mathcal{P}(\rho)$ as a reversible invariant measure. In other words, for any $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$,

$$\lim_{n\to\infty} \mathcal{L}(X_t|X_s = n) = \mathcal{P}(\rho).$$

Moreover, $\mathbf{E}_{\mathcal{P}(\rho)}(\mathbf{P}_t f) = \mathbf{E}_{\mathcal{P}(\rho)}(f)$ for any $f \in \mathrm{L}^1(\mathcal{P}(\rho))$ and any $t \in \mathbb{R}_+$, or equivalently $\mathbf{E}_{\mathcal{P}(\rho)}(\mathbf{L}f) = 0$ for any $f \in \mathrm{L}^1(\mathcal{P}(\rho))$. As for the Ornstein-Uhlenbeck process, this convergence is not uniform in $n$ since for any $\alpha > 0$,

$$\lim_{n\to\infty} \mathcal{L}(X_{\mu^{-1}\log(n/\alpha)} \,|\, X_s = n) = \mathcal{P}(\alpha + \rho).$$

The mean and variance of $\mathcal{L}(X_t|X_s = n)$ with $t \geqslant s \geqslant 0$ are given respectively by

$$np(t-s) + \rho q(t-s) \quad \text{and} \quad (np(t-s)+\rho)q(t-s).$$

The semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ of the M/M/$\infty$ queue shares the nice "constant curvature" property with the Ornstein-Uhlenbeck semi-group. Namely, for any $t \in \mathbb{R}_+$, any $n \in \mathbb{N}$, and any bounded $f : \mathbb{N} \to \mathbb{R}$,

$$\mathrm{D}\mathbf{P}_t f = \mathrm{e}^{-\mu t}\mathbf{P}_t \mathrm{D} f. \tag{4}$$

The infinitesimal version writes $[\mathbf{L}, \mathrm{D}] := \mathbf{L}\mathrm{D} - \mathrm{D}\mathbf{L} = \mu \mathrm{D}$. The commutation (4) can be deduced simply from (3). Namely, if $X_1, \ldots, X_{n+1}, Y$ are independent random variables with $X_i \sim \mathcal{B}(1, p(t))$ and $Y \sim \mathcal{P}(\rho q(t))$,

$$\begin{aligned}
\mathbf{P}_t(f)\,(n+1) &= \mathbf{E}(f(X_1 + \cdots + X_{n+1} + Y)) \\
&= \mathbf{E}(\mathbf{E}(f(X_1 + \cdots + X_{n+1} + Y)|X_{n+1})) \\
&= p(t)\mathbf{P}_t(f(1+\cdot))\,(n) + q(t)\mathbf{P}_t(f)\,(n) \\
&= p(t)\mathbf{P}_t(\mathrm{D}f)\,(n) + \mathbf{P}_t(f)\,(n).
\end{aligned}$$

This fact and the properties of the $A - B - C$ transforms introduced in the sequel give rise to various entropic inequalities, by using the semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ as an interpolation flow.

We give now various binomial-Poisson "integration by parts" formulas. Let $H_{n,p}(m) := \binom{n}{m} p^m q^{n-m}$ for any $p \in [0,1]$ and any integers $n$ and $m$ with $0 \leqslant m \leqslant n$. We have then $mH_{n,p}(m) = npH_{n-1,p}(m-1)$ as soon as $0 < m \leqslant n$. As a consequence, for any function $f : \mathbb{N} \to \mathbb{R}$ , any $n \in \mathbb{N}^*$ and any $p \in [0,1]$

$$\mathbf{E}_{\mathcal{B}(n,p)}(hf) = np\mathbf{E}_{\mathcal{B}(n-1,p)}(f(1+\cdot))\,, \tag{5}$$

where $h : \mathbb{N} \to \mathbb{R}$ is defined by $h(k) = k$ for any $k \in \mathbb{N}$. Similarly, $(n-m)H_{n,p}(m) = nqH_{n-1,p}(m)$ as soon as $0 \leqslant m < n$, which gives for any function $f : \mathbb{N} \to \mathbb{R}$ , any $n \in \mathbb{N}^*$ and any $p \in [0,1]$

$$\mathbf{E}_{\mathcal{B}(n,p)}((n-h)f) = nq\mathbf{E}_{\mathcal{B}(n-1,p)}(f)\,. \tag{6}$$

For $\rho > 0$, the binomial approximation of Poisson measure which lets $np$ tend to $\rho$ when $n \to \infty$ gives from (5)

$$\mathbf{E}_{\mathcal{P}(\rho)}(hf) = \rho\mathbf{E}_{\mathcal{P}(\rho)}(f(1+\cdot))\,. \tag{7}$$

Some algebra provides, by conditioning, a mixed binomial-Poisson version

$$\mathbf{E}_{\mathcal{B}(n,p)*\mathcal{P}(\rho)}(hf) = np\mathbf{E}_{\mathcal{B}(n-1,p)*\mathcal{P}(\rho)}(f(1+\cdot)) + \rho\mathbf{E}_{\mathcal{B}(n,p)*\mathcal{P}(\rho)}(f(1+\cdot))\,. \tag{8}$$

In particular, the Mehler like formula (3) gives for any $n \in \mathbb{N}^*$ and $t \in \mathbb{R}_+$,

$$\mu\mathbf{P}_t(hf)\,(n) = \mu np(t)\mathbf{P}_t(f(1+\cdot))\,(n-1) + \lambda q(t)\mathbf{P}_t(f(1+\cdot))\,(n), \tag{9}$$

where $h : \mathbb{N} \to \mathbb{N}$ is defined by $h(n) := n$ for any $n \in \mathbb{N}$. The binomial-Poisson nature of the M/M/$\infty$ queue is related to the fact that the coefficients of its infinitesimal generator (1) are affine functions of $n$. The reader may find an analysis of linear growth birth and death processes in [23] and [34] and references therein.

### 1.4. **Convex functionals**

For any convex domain $\mathcal{D}$ of $\mathbb{R}^n$, let us denote by $\mathcal{C}_\mathcal{D}$ the convex set of smooth convex functions from $\mathcal{D}$ to $\mathbb{R}$. Let $\mathcal{I} \subset \mathbb{R}$ be an open interval of $\mathbb{R}$ and $\Phi \in \mathcal{C}_\mathcal{I}$. We denote by $\mathrm{L}^{1,\Phi}(Q)$ the convex subset of functions $f \in \mathrm{L}^1(Q)$ taking their values in $\mathcal{I}$ and such that $\Phi(f) \in \mathrm{L}^1(Q)$. We define the $\Phi$-entropy $\mathbf{Ent}^\Phi_Q(f)$ of $f \in \mathrm{L}^{1,\Phi}(Q)$ by

$$\mathbf{Ent}^\Phi_Q(f) := \mathbf{E}_Q(\Phi(f)) - \Phi(\mathbf{E}_Q f).$$

Is is also known as "Jensen divergence" since Jensen inequality gives $\mathbf{Ent}_Q^\Phi(f) \geqslant 0$. Moreover, when $\Phi$ is strictly convex, $\mathbf{Ent}_Q^\Phi(f) = 0$ if and only if $\Phi(f)$ is $Q$-a.s. constant. One can distinguish for function $\Phi$ the following three usual special cases.

**(P1)** $\Phi(u) = u\log(u)$ on $\mathcal{I} = \mathbb{R}_+^*$, and $\mathbf{Ent}_Q^\Phi(f) = \mathbf{Ent}_Q(f) := \mathbf{E}_Q(f\log(f/\mathbf{E}_Q f))$;
**(P2)** $\Phi(u) = u^2$ on $\mathcal{I} = \mathbb{R}$, and $\mathbf{Ent}_Q^\Phi(f) = \mathbf{Var}_Q(f) := \mathbf{E}_Q\big((f - \mathbf{E}_Q f)^2\big)$;
**(P3)** $\Phi(u) = u^\alpha$ on $\mathcal{I} = \mathbb{R}_+^*$ with $\alpha \in (1,2)$.

The $\mathbf{Ent}_Q^\Phi$ functional is linear in $\Phi$ and vanishes when $\Phi$ is affine. Let us define from the interval $\mathcal{I} \subset \mathbb{R}$ the convex subsets $\mathcal{T}_\mathcal{I} \subset \mathcal{T}'_\mathcal{I} \subset \mathbb{R}^2$ by

$$\mathcal{T}_\mathcal{I} := \big\{(u,v) \in \mathbb{R}^2;\ (u, u+v) \in \mathcal{I}\times\mathcal{I}\big\} \quad \text{and} \quad \mathcal{T}'_\mathcal{I} := \big\{(u,v) \in \mathbb{R}^2;\ u \in \mathcal{I}, (v+\mathcal{I}) \cap \mathcal{I} \neq \emptyset\big\}.$$

The $A-B-C$ transforms of $\Phi$ are the functions $A^\Phi, B^\Phi : \mathcal{T}_\mathcal{I} \to \mathbb{R}$ and $C^\Phi : \mathcal{T}'_\mathcal{I} \to \mathbb{R}$ defined by

$$\begin{aligned} A^\Phi(u,v) &:= \Phi(u+v) - \Phi(u) - \Phi'(u)v; \\ B^\Phi(u,v) &:= (\Phi'(u+v) - \Phi'(u))v; \\ C^\Phi(u,v) &:= \Phi''(u)v^2. \end{aligned}$$

These three transforms are linear in $\Phi$, and their kernel contains any affine function. Various additional properties of these three transforms are collected in Section 4. In particular, the convexity of $\Phi$ on $\mathcal{I}$ is equivalent to the non negativity of its $A-B-C$ transforms on $\mathcal{T}_\mathcal{I}$. In particular, the following statements hold.

- $A^\Phi$, $B^\Phi$, $C^\Phi$, $\mathbf{Ent}_Q^\Phi$ are non negative and convex for **(P1-P2-P3)**;
- $2A^\Phi = B^\Phi = C^\Phi$ for **(P2)**, $A^\Phi \leqslant C^\Phi$ for **(P1)**, and $A^\Phi \leqslant B^\Phi$ for **(P1-P2-P3)**.

On the two point space $\{0,1\}$, the $\Phi$-entropy gives rise naturally to the $A$-transform of $\Phi$. Namely, for any $f : \{0,1\} \to \mathcal{I}$ with $(a,b) := (f(0), f(1))$ and $(u,v) := (a, b-a)$,

$$\mathbf{Ent}_{\mathcal{B}(1,p)}^\Phi(f) = q\Phi(a) + p\Phi(b) - \Phi(qa+pb) = pA^\Phi(u,v) - A^\Phi(u,pv). \tag{10}$$

The $A-B-C$ transforms are the germs of discrete Dirichlet forms *via* the identities

$$\begin{aligned} A^\Phi(f, \mathrm{D}f) &= \mathrm{D}\Phi(f) - \Phi'(f)\mathrm{D}f; \\ B^\Phi(f, \mathrm{D}f) &= \mathrm{D}(\Phi'(f))\mathrm{D}f; \\ C^\Phi(f, \mathrm{D}f) &= \Phi''(f)|\mathrm{D}f|^2. \end{aligned}$$

The reader may find explicit examples in table 1. We used above the following identity, valid for any functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{Z} \to \mathbb{R}$,

$$\mathrm{D}(\varphi(f)) = \varphi(f(1+\cdot)) - \varphi(f) = \varphi(f + \mathrm{D}f) - \varphi(f),$$

where $f(1+\cdot)$ stands for $\mathbb{Z} \ni n \mapsto f(1+n)$. In particular, $f(1+\cdot) = f + \mathrm{D}f$. The usage of the $A-B-C$ transforms allows, as presented in the sequel, to derive several entropic inequalities in the same time, including Poincaré inequalities and various modified logarithmic Sobolev inequalities. They reduce most of the proofs to convexity, and they provide various comparisons for discrete Dirichlet forms.

For any open interval $\mathcal{I} \subset \mathbb{R}$ and any probability space $(E, \mathcal{E}, Q)$, we denote in the sequel by $\mathcal{K}(E,\mathcal{I})$ the convex set of measurable functions $E \to \mathcal{I}$ with a relatively compact image in $\mathcal{I}$. These functions are bounded. Notice that $\mathcal{K}(E,\mathcal{I})$ is a convex subset of $\mathrm{L}^{1,\Phi}(Q)$. The introduction of $\mathcal{K}(E,\mathcal{I})$ permits to avoid integrability obstructions at the boundary of $\mathcal{I}$ when dealing with the derivatives of $\Phi$. Any element of $\mathrm{L}^{1,\Phi}(Q)$ is a pointwise limit of elements of $\mathcal{K}(E,\mathcal{I})$.

TABLE 1. Examples of $A-B-C$ transforms. For **(P3)**, $\alpha\in(1,2)$.

| **Function $\Phi$** | $\mathcal{I}$ | $A^\Phi$ | $A^\Phi(f,\mathrm{D}f)$ |
|---|---|---|---|
| **(P1)** $u\log(u)$ | $\mathbb{R}_+^*$ | $(u+v)(\log(u+v)-\log(u))-v$ | $(f+\mathrm{D}f)\mathrm{D}(\log f)-\mathrm{D}f$ |
| **(P2)** $u^2$ | $\mathbb{R}$ | $v^2$ | $\lvert\mathrm{D}f\rvert^2$ |
| **(P3)** $u^\alpha$ | $\mathbb{R}_+^*$ | $(u+v)^\alpha-u^\alpha-\alpha u^{\alpha-1}v$ | $\mathrm{D}(f^\alpha)-\alpha f^{\alpha-1}\mathrm{D}f$ |
| | | $B^\Phi$ | $B^\Phi(f,\mathrm{D}f)$ |
| **(P1)** $u\log(u)$ | $\mathbb{R}_+^*$ | $v(\log(u+v)-\log(u))$ | $\mathrm{D}(f)\mathrm{D}(\log f)$ |
| **(P2)** $u^2$ | $\mathbb{R}$ | $2v^2$ | $2\lvert\mathrm{D}f\rvert^2$ |
| **(P3)** $u^\alpha$ | $\mathbb{R}_+^*$ | $\alpha v((u+v)^{\alpha-1}-u^{\alpha-1})$ | $\alpha\mathrm{D}(f)\mathrm{D}(f^{\alpha-1})$ |
| | | $C^\Phi$ | $C^\Phi(f,\mathrm{D}f)$ |
| **(P1)** $u\log(u)$ | $\mathbb{R}_+^*$ | $v^2u^{-1}$ | $\lvert\mathrm{D}f\rvert^2f^{-1}$ |
| **(P2)** $u^2$ | $\mathbb{R}$ | $2v^2$ | $2\lvert\mathrm{D}f\rvert^2$ |
| **(P3)** $u^\alpha$ | $\mathbb{R}_+^*$ | $\alpha(\alpha-1)v^2u^{\alpha-2}$ | $\alpha(\alpha-1)\lvert\mathrm{D}f\rvert^2f^{\alpha-2}$ |

## 2. From two point space to binomial-Poisson inequalities

Let $p\in[0,1]$ and let $\mathcal{B}(1,p)$ be the Bernoulli measure $q\delta_0+p\delta_1$ on $\{0,1\}$, where $q:=1-p$. We identify the two point space $\{0,1\}$ with the "circle" $\mathbb{Z}/2\mathbb{Z}$, for which $1+1=0$. In particular, the the "+" sign in the definition (2) of D is taken modulo 2. Then, for any $f:\{0,1\}\to\mathcal{I}$, the following identity holds.

$$pq\mathbf{E}_{\mathcal{B}(1,p)}\big(B^\Phi(f,\mathrm{D}f)\big)-\mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f)=A^\Phi(\sigma_p(a,b-a))+A^\Phi(\sigma_q(b,a-b)),$$

where $(a,b):=(f(0),f(1))$ and where $\sigma_p$ is as in (38). Now, Lemma 4.1 gives that $A^\Phi$ is non-negative as soon as $\Phi$ is convex. Consequently, when $\Phi$ is convex, we obtain the following entropic inequality for $\mathcal{B}(1,p)$.

$$\mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f)\leqslant pq\mathbf{E}_{\mathcal{B}(1,p)}\big(B^\Phi(f,\mathrm{D}f)\big)\,. \tag{11}$$

Unfortunately, the inequality (11) is not optimal for **(P2)** since in that case,

$$\mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f)=\mathbf{Var}_{\mathcal{B}(1,p)}(f)=pq(f(1)-f(0))^2\quad\text{whereas}\quad\mathbf{E}_{\mathcal{B}(1,p)}\big(B^\Phi(f,\mathrm{D}f)\big)=2(f(1)-f(0))^2.$$

This is due to the fact that $B^\Phi(f,\mathrm{D}f)=2|\mathrm{D}f|^2$ for **(P2)**. We derive in the sequel the $A$ transform version, which is stronger and optimal for **(P2)** since $A^\Phi(f,\mathrm{D}f)=|\mathrm{D}f|^2$ in that case. All the inequalities obtained in this section involve the $A$ transform in their right hand side. They hold for example in the cases **(P1)**, **(P2)**, **(P3)**. The $A$ transform can be bounded by the $B$ or the $C$ transforms, by using the elementary bounds given by Lemma 4.2. We start with an entropic inequality for the Bernoulli law $\mathcal{B}(1,p)$. By convolution, we derive from this two point space inequality a new entropic inequality for the binomial law $\mathcal{B}(n,p)=\mathcal{B}(1,p)^{*n}$. An inequality for the Poisson law $\mathcal{P}(\rho)$ is then obtained by binomial approximation. The binomial-Poisson case is derived by tensorisation. The following calculus lemma is a $\Phi$ version of [6], Lemma 2, by Bobkov and Ledoux.

**Lemma 2.1** (two point lemma). *Let $\Phi\in\mathcal{C}_\mathcal{I}$ such that $\Phi''\in\mathcal{C}_\mathcal{I}$. Let $U:[0,1]\to\mathbb{R}$ be defined by*

$$U(p):=\mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f)-pq\mathbf{E}_{\mathcal{B}(1,p)}(g)\,,$$

*where $f,g:\{0,1\}\to\mathcal{I}$. Then, $U\leqslant 0$ on $[0,1]$ if and only if*

$$U'(0)\leqslant 0\leqslant U'(1). \tag{12}$$

*Proof.* We denote $(a,b) := (f(0), f(1))$ and $(\alpha,\beta) := (g(0), g(1))$. We get then

$$U(p) = q\Phi(a) + p\Phi(b) - \Phi(qa+pb) - pq(q\alpha + p\beta). \tag{13}$$

The last term is a polynomial in $p$ of degree three. Taking the fourth derivative in $p$ gives

$$U''''(p) = -(b-a)^4\Phi''''(qa+pb).$$

Since $\Phi''$ is convex, we have $U'''' \leqslant 0$ on $(0,1)$ and thus $U''$ is concave. Consequently, there exists $0 \leqslant p_0 \leqslant p_1 \leqslant 1$ such that $U'' \leqslant 0$ on $[0,p_0] \cup [p_1,1]$ and $U'' \geqslant 0$ on $[p_0,p_1]$. We have that $U$ is concave on $[0,p_0]$. But $U(0)=0$ and by assumption $U'(0) \leqslant 0$, thus $U \leqslant 0$ on $[0,p_0]$ by concavity. A consequence is that $U(p_0) \leqslant 0$. Similarly by symmetry we have that $U \leqslant 0$ on $[p_1,1]$ and $U(p_1) \leqslant 0$. Now since $U$ is convex on $[p_0,p_1]$ and non-positive on the boundaries, it is non-positive on the whole interval $[p_0,p_1]$. Therefore (12) implies $U \leqslant 0$ on $[0,1]$. $\square$

One can show by similar arguments that if additionally $f(0) \geqslant f(1)$ and $g(0) \geqslant g(1)$ (respectively $f(0) \leqslant f(1)$ and $g(0) \leqslant g(1)$), then the condition (12) may be weakened into $U'(0) \leqslant 0$ (respectively $U'(1) \geqslant 0$). Notice that in terms of $A$ transform,

$$U'(0) = A^\Phi(a, b-a) - \alpha \quad \text{and} \quad U'(1) = -A^\Phi(b, a-b) + \beta. \tag{14}$$

The following lemma provides the $A$ transform version of (11).

**Lemma 2.2** (two point entropic inequalities)**.** *Let $\Phi \in \mathcal{C}_\mathcal{I}$ such that $\Phi'' \in \mathcal{C}_\mathcal{I}$. Then, for any $f : \{0,1\} \to \mathcal{I}$,*

$$\mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f) \leqslant pq\,\mathbf{E}_{\mathcal{B}(1,p)}\left(A^\Phi(f, \mathrm{D}f)\right), \tag{15}$$

*where the "+" in* (2) *of* D *is taken modulo* 2*. Moreover, the inequality becomes an equality for **(P2)**.*

*Proof.* Let $U$ be as in (13) with $g = A^\Phi(f, \mathrm{D}f)$. From (14) we get

$$U'(0) = A^\Phi(a,b-a) - A^\Phi(a,b-a) = 0 \quad \text{and} \quad U'(1) = -A^\Phi(b,a-b) + A^\Phi(b,a-b) = 0,$$

where $(a,b) := (f(0), f(1))$. Therefore (15) follows by virtue of Lemma 2.1. Notice that since $+1 = -1$ in $\mathbb{Z}/2\mathbb{Z}$, we have $\mathrm{D} = \mathrm{D}^*$. In particular, for any $f : \{0,1\} \to \mathcal{I}$, the function $B^\Phi(f, \mathrm{D}f)$ is constant, and $A^\Phi(f, \mathrm{D}f) = A^\Phi(f, \mathrm{D}^*f)$. $\square$

Notice that (15) can be rewritten as (40). Entropic inequalities like (15) belong to the so called family of $\Phi$-Sobolev inequalities, which are known to be stable by convolution, *cf.* [11], Corollary 3.1, page 342. This observation leads to Theorem 2.3 below, by using the tensorisation property (34) of Theorem 4.4.

**Theorem 2.3** (Bernoulli entropic inequalities)**.** *Let $M := \mathcal{B}(1,p_1) * \cdots * \mathcal{B}(1,p_n)$ and $C_M := \max\{p_1q_1, \ldots, p_nq_n\}$ where $p_1, \ldots, p_n \in [0,1]$. Let $\Phi \in \mathcal{C}_\mathcal{I}$ such that $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$. Then, for any $f : \mathbb{N} \to \mathcal{I}$,*

$$\mathbf{Ent}^\Phi_M(f) \leqslant C_M \mathbf{E}_M\left((n-h)A^\Phi(f, \mathrm{D}f) + hA^\Phi(f, \mathrm{D}^*f)\right), \tag{16}$$

*where $h : \mathbb{N} \to \mathbb{R}$ is defined by $h(k) = k$ for any $k \in \mathbb{N}$. In particular,*

$$\mathbf{Ent}^\Phi_{\mathcal{B}(n,p)}(f) \leqslant pq\mathbf{E}_{\mathcal{B}(n,p)}\left((n-h)A^\Phi(f, \mathrm{D}f) + hA^\Phi(f, \mathrm{D}^*f)\right), \tag{17}$$

*for any $n \in \mathbb{N}^*$, any $p \in [0,1]$, and any $f : \mathbb{N} \to \mathcal{I}$. Moreover, if $\tau$ is as in* (32)*,*

$$\mathbf{Ent}^\Phi_{\mathcal{B}(n,p)}(f) \leqslant npq\mathbf{E}_{\mathcal{B}(n-1,p)}\left(qA^\Phi(f, \mathrm{D}f) + pA^\Phi(\tau(f, \mathrm{D}f))\right). \tag{18}$$

The optimality of these inequalities in the case **(P2)** can be checked for a linear function $f$.

*Proof.* First of all, by virtue of Theorem 4.4, the convexity of $A^\Phi$ on $\mathcal{T}_\mathcal{I}$ implies the convexity of $\Phi''$ on $\mathcal{I}$. Let $(E_i, Q_i) = (\{0,1\}, \mathcal{B}(1,p_i))$ for any $i \in \{1,\ldots,n\}$. Let $f : \mathbb{N} \to \mathcal{I}$ and consider the symmetric function $g : E_1 \times \cdots \times E_n \to \mathcal{I}$ defined by $g(x_1,\ldots,x_n) := f(x_1 + \cdots + x_n)$. The tensorisation formula (34) together with the two point entropic inequality (15) of Lemma 2.2 gives

$$\mathbf{Ent}^\Phi_{Q_1\otimes\cdots\otimes Q_n}(g) \leqslant C_Q \mathbf{E}_{Q_1\otimes\cdots\otimes Q_n}\left(\sum_{i=1}^n A^\Phi(g, \mathrm{D}^{(i)}g)\right),$$

where $\mathrm{D}^{(i)}$ denotes the operator D acting on the $i^{\text{th}}$ coordinate with modulo 2 as in Lemma 2.2. At this step, we notice by denoting $s_n := x_1 + \cdots + x_n$ that for any $x \in \{0,1\}^n$,

$$\sum_{i=1}^n A^\Phi(g, \mathrm{D}^{(i)}g)(x) = (n - s_n)A^\Phi(f, \mathrm{D}f)(s_n) + s_n A^\Phi(f, \mathrm{D}^* f)(s_n).$$

Outside $\{0,\ldots,n\}$, the function $f$ takes values which come with a null coefficient in the right hand side. The desired result follows since $M$ is the law of $s_n$ under $Q_1 \otimes \cdots \otimes Q_n$. Inequality (16) reduces to (17) when $p_1 = \cdots = p_n = p$. It remains to establish (18). By virtue of (5) and (6), the right hand side of (17) is equal to

$$npq\mathbf{E}_{\mathcal{B}(n-1,p)}\left(qA^\Phi(f,\mathrm{D}f) + pA^\Phi(f,\mathrm{D}^*f)(1+\cdot)\right).$$

Inequality (18) follows then from the simple identity

$$(f, \mathrm{D}^* f)(1+\cdot) = \tau(f, \mathrm{D}f). \tag{19}$$

When $n = 1$, then $M = \mathcal{B}(1,p)$, and (16) reduces to (15). Beware that D in (15) is taken modulo 2. □

**Corollary 2.4** (Poisson entropic inequality)**.** *Let $\Phi \in \mathcal{C}_\mathcal{I}$ be such that $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$. Let $\rho > 0$ and $\mathcal{P}(\rho)$ be the Poisson measure of mean $\rho$. Then, for any $\rho \in \mathbb{R}_+$ and any $f \in \mathrm{L}^{1,\Phi}(\mathcal{P}(\rho))$,*

$$\mathbf{Ent}^\Phi_{\mathcal{P}(\rho)}(f) \leqslant \rho\, \mathbf{E}_{\mathcal{P}(\rho)}\left(A^\Phi(f, \mathrm{D}f)\right). \tag{20}$$

*Proof.* Notice that the right hand side of (20) takes its values in $[0,+\infty]$. By approximation, we can assume that $f \in \mathcal{K}(\mathbb{N},\mathcal{I})$. Consider now (18). Let $p$ depend on $n$ is such a way that $\lim_{n\to\infty} np_n = \rho$. Since $\lim_{n\to\infty} p_n = 0$, we have $q_n \to 1$. Moreover, $\mathcal{B}(n,p_n) \to \mathcal{P}(\rho)$ and $\mathcal{B}(n-1,p_n) \to \mathcal{P}(\rho)$. □

To the author's knowledge, inequality (20) appeared for the first time in [35] for **(P1)**, in [6], p. 357, for **(P2)**, and in [11] for the general case. See also [5]. The $B$ and $C$ transforms versions of (20), which are weaker, appeared in particular in [1,6].

**Corollary 2.5** (Binomial-Poisson entropic inequality)**.** *Let $\Phi \in \mathcal{C}_\mathcal{I}$ be such that $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$. Let $M_n$ be the probability measure $M_n = \mathcal{B}(n,p) * \mathcal{P}(\rho)$ where $p \in [0,1]$, $\rho \in \mathbb{R}_+$, and $n \in \mathbb{N}$. Then, for any $f \in \mathrm{L}^{1,\Phi}(\mathcal{P}(\rho))$,*

$$\mathbf{Ent}^\Phi_{M_n}(f) \leqslant \rho\mathbf{E}_{M_n}\left(A^\Phi(f,\mathrm{D}f)\right) + npq\mathbf{E}_{M_{n-1}}\left(qA^\Phi(f,\mathrm{D}f) + pA^\Phi(\tau(f,\mathrm{D}f))\right). \tag{21}$$

*Proof.* By approximation, we can assume that $f \in \mathcal{K}(\mathbb{N},\mathcal{I})$. If $n = 0$, then (21) reduces to (20). Let us assume now that $n > 0$. Let $(E_1, Q_1) = (\mathbb{N}, \mathcal{B}(n,p))$ and $(E_2, Q_2) = (\mathbb{N}, \mathcal{P}(\rho))$. Let $g : E_1 \times E_2 \to \mathcal{I}$ be defined by $g(x_1, x_2) = f(x_1 + x_2)$. Let us denote by $\mathrm{D}^{(1)}$ and $\mathrm{D}^{(2)}$ the D operator which acts on $x_1$ and $x_2$ respectively. The inequalities (34), (18), (20) yield that $\mathbf{Ent}^\Phi_{Q_1\otimes Q_2}(g)$ is bounded above by

$$npq\mathbf{E}_{Q_2}\Big(\mathbf{E}_{Q_0}\Big(qA^\Phi(g,\mathrm{D}^{(1)}g) + pA^\Phi(\tau(g,\mathrm{D}^{(1)}g))\Big)\Big) + \rho\mathbf{E}_{Q_1}\Big(\mathbf{E}_{Q_2}\Big(A^\Phi(g,\mathrm{D}^{(2)}g)\Big)\Big),$$

where $Q_0 := \mathcal{B}(n-1,p)$. Since D commutes with translations, we get for $i = 1, 2$,

$$(g, \mathrm{D}^{(i)} g)(x_1, x_2) = (f, \mathrm{D} f)(x_1 + x_2).$$

Inequality (21) follows since $M_n$, respectively $M_{n-1}$, is the law of $x_1+x_2$ under $Q_1 \otimes Q_2$, respectively $Q_0 \otimes Q_2$. $\square$

The expectation with respect to $M_{n-1}$ in the right hand side of (21) may be rewritten as an expectation with respect to $M_n$ by using (8).

## 3. Entropies along the M/M/∞ queue

We start with the decay of the Φ-entropy functional along the queue.

**Theorem 3.1** (Φ-entropies dissipation)**.** *Let $\Phi \in \mathcal{C}_{\mathcal{I}}$. Let $(\mathbf{P}_t)_{t \geqslant 0}$ be the M/M/∞ semi-group with input rate $\lambda > 0$ and service rate $\mu > 0$. Then for any $f \in \mathcal{K}(\mathbb{N}, \mathcal{I})$, the real function $t \in \mathbb{R}_+ \mapsto \mathbf{Ent}^{\Phi}_{\mathcal{P}(\rho)}(\mathbf{P}_t f)$ is non-increasing. Moreover, when $A^{\Phi} \in \mathcal{C}_{\mathcal{T}_{\mathcal{I}}}$,*

$$\mathbf{Ent}^{\Phi}_{\mathcal{P}(\rho)}(\mathbf{P}_t f) \leqslant \mathrm{e}^{-c\mu t} \mathbf{Ent}^{\Phi}_{\mathcal{P}(\rho)}(f)\,, \tag{22}$$

*where $c$ is the best (i.e. biggest) constant in the inequality*

$$\forall f \in \mathrm{L}^{1,\Phi}(\mathcal{P}(\rho)), \quad c\mu \mathbf{Ent}^{\Phi}_{\mathcal{P}(\rho)}(f) \leqslant \lambda \mathbf{E}_{\mathcal{P}(\rho)}\big(B^{\Phi}(f, \mathrm{D} f)\big)\,.$$

*It holds with $c = 1$ in general, and with $c = 2$ for **(P2)**.*

*Proof.* Let us denote $Q = \mathcal{P}(\rho)$. Since $\mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f) = \mathbf{E}_Q(\mathbf{P}_t \Phi(f)) - \Phi(\mathbf{E}_Q(f))$, the invariance of $Q$ gives,

$$\partial_t \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f) = \mathbf{E}_Q(\Phi'(\mathbf{P}_t f) \mathbf{L} \mathbf{P}_t f)\,.$$

Jensen inequality yields $\mathbf{P}_t(\Phi(f)) \geqslant \Phi(\mathbf{P}_t(f))$ as soon as $\Phi$ is convex. In particular $\mathbf{L}\Phi(f) \geqslant \Phi'(f)\mathbf{L} f$, which gives $\mathbf{E}_Q(\Phi'(f)\mathbf{L} f) \leqslant 0$ as soon as $\Phi(f)$ is $Q$-integrable. Hence, $t \mapsto \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f)$ is non-increasing, and we used only the convexity of $\Phi$, the Markovian nature of $(\mathbf{P}_t)_{t \geqslant 0}$, and the invariance of $Q$. The Poisson integration by parts (7) – which is this time specific to our settings – yields for any $g$

$$\mathbf{E}_Q(\Phi'(g)\mathbf{L} g) = -\lambda \mathbf{E}_Q(\mathrm{D}(g)\mathrm{D}(\Phi'(g))) = -\lambda \mathbf{E}_Q\big(B^{\Phi}(g, \mathrm{D} g)\big)\,. \tag{23}$$

In particular, for $g = \mathbf{P}_t(f)$, we get,

$$\partial_t \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f) = -\lambda \mathbf{E}_Q\big(B^{\Phi}(\mathbf{P}_t f, \mathrm{D}\mathbf{P}_t f)\big)\,.$$

Notice that since $\Phi$ is convex, we have $B^{\Phi} \geqslant 0$ by virtue of Lemma 4.1. In the other hand, when $A^{\Phi}$ is convex, the Poisson entropic inequality (20) together with the bound $A^{\Phi} \leqslant B^{\Phi}$ of Lemma 4.2 gives

$$-\lambda \mathbf{E}_Q\big(B^{\Phi}(\mathbf{P}_t f, \mathrm{D}\mathbf{P}_t f)\big) \leqslant -\mu \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f)\,.$$

Putting all together yields $\partial_t \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f) \leqslant -\mu \mathbf{Ent}^{\Phi}_Q(\mathbf{P}_t f)$, which gives (22). Finally, the correct constant for **(P2)** comes from the fact that $2A^{\Phi} = B^{\Phi}$ in that case. $\square$

For any probability measure $\gamma$ on $\mathbb{N}$, and any $t \in \mathbb{R}_+$, we denote by $\gamma \mathbf{P}_t$ the probability measure on $\mathbb{N}$ defined for any bounded function $g : \mathbb{N} \to \mathbb{R}$ by $\mathbf{E}_{\gamma \mathbf{P}_t}(g) := \mathbf{E}_\gamma(\mathbf{P}_t(g))$. In particular, when $\gamma = \delta_n$ for some fixed $n \in \mathbb{N}$, we get $\delta_n \mathbf{P}_t = \mathbf{P}_t(\cdot)\,(n)$. We have $\gamma \ll \mathcal{P}(\rho)$ for any probability measure $\gamma$ on $\mathbb{N}$, as soon as $\rho > 0$. Let us define $f_\gamma := \mathrm{d}\gamma/\mathrm{d}\mathcal{P}(\rho)$. Since $\mathcal{P}(\rho)$ is symmetric for $\mathbf{L}$, we have that $\mathbf{L}$ and $\mathbf{P}_t(\cdot)$ are self-adjoint in $\mathrm{L}^2(\mathcal{P}(\rho))$. Therefore, one can write for any $g \in \mathrm{L}^2(\mathcal{P}(\rho))$

$$\int_{\mathbb{N}} \mathbf{P}_t(f_\gamma)\, g \,\mathrm{d}\mathcal{P}(\rho) = \int_{\mathbb{N}} \mathbf{P}_t(g)\, f_\gamma \,\mathrm{d}\mathcal{P}(\rho) = \int_{\mathbb{N}} \mathbf{P}_t(g)\, \mathrm{d}\gamma = \int_{\mathbb{N}} g \,\mathrm{d}(\gamma \mathbf{P}_t).$$

Recall that the total variation norm $\|\alpha\|_{\mathrm{TV}}$ of a Borel measure $\alpha$ on an at most countable set $\mathcal{S}$ is defined by $\|\alpha\|_{\mathrm{TV}} = \frac{1}{2}\|\alpha\|_1 = \frac{1}{2}\sum_{x\in\mathcal{S}} |\alpha(x)|$. If $\alpha$ and $\beta$ are two probability measures on $\mathcal{S}$, the distance $\|\alpha-\beta\|_{\mathrm{TV}}$ is

$$\|\alpha-\beta\|_{\mathrm{TV}} = \sup_{A\subset\mathcal{S}} |\alpha(A)-\beta(B)| = \frac{1}{2}\sup_{\|f\|_\infty \leqslant 1}\left|\int f\,\mathrm{d}\alpha - \int f\,\mathrm{d}\beta\right|.$$

Recall the well known bound for any $a,b\in\mathbb{R}_+$, $\|\mathcal{P}(a)-\mathcal{P}(b)\|_{\mathrm{TV}} \leqslant 1-\mathrm{e}^{-(b-a)}$, *cf.* [31], Proposition 6.1, p. 143, which gives from (3) for any $t\in\mathbb{R}_+$

$$\|\mathbf{P}_t(\cdot)\,(0) - \mathcal{P}(\rho)\|_{\mathrm{TV}} \leqslant 1 - \mathrm{e}^{-\rho \mathrm{e}^{-\mu t}}.$$

Theorem 3.1 for **(P1)** produces in particular a bound for $\|\mathbf{P}_t(\cdot)\,(n) - \mathcal{P}(\rho)\|_{\mathrm{TV}}$, as stated in Corollary 3.2.

**Corollary 3.2.** *Let $(\mathbf{P}_t)_{t\geqslant 0}$ be the semi-group of the M/M/$\infty$ queue with input rate $\lambda>0$ and service rate $\mu>0$. For any $n\in\mathbb{N}$ and any $t\in\mathbb{R}_+$,*

$$2\|\mathbf{P}_t(\cdot)\,(n) - \mathcal{P}(\rho)\|_{\mathrm{TV}}^2 \leqslant \mathrm{e}^{-\mu t}\log(\mathrm{e}^\rho \rho^{-n} n!).$$

The proof follows the lines of a method due to Diaconis and Saloff-Coste, *cf.* for example [17, 33].

*Proof of Corollary 3.2.* Since $Q := \mathcal{P}(\rho)$ is symmetric for $\mathbf{L}$,

$$\frac{\mathrm{d}(\gamma\mathbf{P}_t)}{\mathrm{d}Q} = \mathbf{P}_t(f_\gamma)\,, \tag{24}$$

where $f_\gamma := \mathrm{d}\gamma/\mathrm{d}Q$. The Pinsker-Csiszár-Kullback inequality states that for any couple $(\alpha,\beta)$ of probability measures on the same measured space $2\|\alpha-\beta\|_{\mathrm{TV}}^2 \leqslant \mathbf{Ent}(\alpha\,|\,\beta) = \mathbf{Ent}_\beta(\mathrm{d}\alpha/\mathrm{d}\beta)$, where $\mathbf{Ent}_\beta$ is the $\Phi$-entropy in the case **(P1)**. Let $t\in\mathbb{R}_+$ and $n\in\mathbb{N}$. For $(\alpha,\beta) = (\mathbf{P}_t(\cdot)\,(n), Q)$, we can write by (24) and (22)

$$2\|\gamma\mathbf{P}_t - Q\|_{\mathrm{TV}}^2 \leqslant \mathbf{Ent}_Q(\mathbf{P}_t(f_\gamma)) \leqslant \mathrm{e}^{-\mu t}\mathbf{Ent}_Q(f_\gamma)\,.$$

For $\gamma=\delta_n$ for some fixed $n\in\mathbb{N}$, we get $\gamma\mathbf{P}_t = \mathbf{P}_t(\cdot)\,(n)$ and $f_{\delta_n} = \mathrm{I}_{\{n\}}/Q(n)$. As a consequence, we obtain as expected $\mathbf{Ent}_Q(f_{\delta_n}) = -\log Q(n) = \log(\mathrm{e}^\rho\rho^{-n}n!)$. □

## 3.1. Local inequalities and semi-group interpolation

Standard Brownian motion on $\mathbb{R}$ starting from $x$ interpolates on the time interval $[0,t]$ between the Dirac measure $\delta_x$ and the Gaussian measure $\mathcal{N}(x,t)$. It is known that this interpolation provides the optimal Gaussian logarithmic Sobolev inequality. Similarly, the simple Poisson process of intensity $\lambda$ starting from $n$ interpolates on the time interval $[0,t]$ between the Dirac measure $\delta_n$ and the translated Poisson measure $\delta_n * \mathcal{P}(\lambda t)$. By analogy, let us give a proof of the Poisson entropic inequality (20) by using the simple Poisson process, which corresponds to an M/M/$\infty$ queue with $\mu=0$. In that case, $\mathbf{L} = \lambda\mathrm{D}$ and $\mathbf{P}_t(\cdot)\,(0) = \mathcal{P}(\lambda t)$. Let $\Phi\in\mathcal{C}_{\mathcal{I}}$ such that $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$, *cf.* Theorem 4.4. One can write by abridging $\mathbf{P}_t(\cdot)\,(0)$ in $\mathbf{P}_t(\cdot)$ and denoting $F = \mathbf{P}_{t-s}(f)$,

$$\begin{aligned}\mathbf{Ent}^\Phi_{\mathcal{P}(\lambda)}(f) &= \mathbf{P}_t(\Phi(f)) - \Phi(\mathbf{P}_t(f)) \\ &= \int_0^t \partial_s \mathbf{P}_s(\Phi(\mathbf{P}_{t-s}f))\,\mathrm{d}s \\ &= \int_0^t \mathbf{P}_s(\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F)\,\mathrm{d}s.\end{aligned}$$

Now, $\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = A^\Phi(F, \mathrm{D}F)$, and (4) with $\mu = 0$ gives $\mathrm{D}F = \mathrm{D}\mathbf{P}_{t-s}(f) = \mathbf{P}_{t-s}(\mathrm{D}f)$. Thus, we get,

$$\begin{aligned}\mathbf{Ent}^\Phi_{\mathcal{P}(\lambda)}(f) &= \lambda \int_0^t \mathbf{P}_s\big(A^\Phi(F, \mathrm{D}F)\big)\,\mathrm{d}s \\ &= \lambda \int_0^t \mathbf{P}_s\big(A^\Phi(\mathbf{P}_{t-s}f, \mathbf{P}_{t-s}\mathrm{D}f)\big)\,\mathrm{d}s.\end{aligned}$$

Finally, Jensen inequality for convex function $A^\Phi$ gives then the desired result,

$$\mathbf{Ent}^\Phi_{\mathcal{P}(\lambda)}(f) \leqslant \lambda \int_0^t \mathbf{P}_s\big(\mathbf{P}_{t-s}\big(A^\Phi(f, \mathrm{D}f)\big)\big)\,\mathrm{d}s = \lambda t \mathbf{P}_t\big(A^\Phi(f, \mathrm{D}f)\big)\,.$$

*M/M/$\infty$ semi-group interpolation on the time interval* $[0, +\infty]$

The standard Ornstein-Uhlenbeck process on $\mathbb{R}$ starting from $x$ interpolates on the time interval $[0, +\infty]$ between the Dirac measure $\delta_x$ and the standard Gaussian measure $\mathcal{N}(0,1)$. It is known that this interpolation provides the optimal Gaussian logarithmic Sobolev inequality. Similarly, the M/M/$\infty$ queue with intensities $(\lambda, \mu)$ starting from $n$ interpolates on the time interval $[0, +\infty]$ between the Dirac measure $\delta_n$ and the Poisson measure $\mathcal{P}(\rho)$ where $\rho = \lambda/\mu$. Notice that when $\lambda = 0$, this interpolation holds between $\delta_n$ and $\delta_0$ (pure death process). By analogy, let us give a proof of the Poisson entropic inequality (20) by using the M/M/$\infty$ queue. Let $(\mathbf{P}_t)_{t\geqslant 0}$ be the M/M/$\infty$ queue semi-group with input rate $\lambda$ and service rate $\mu$. Let $\Phi \in \mathcal{C}_\mathcal{I}$ such that $B^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$, *cf.* Theorem 4.4. We denote by $Q$ the Poisson measure $\mathcal{P}(\rho)$. For any $f \in \mathcal{K}(\mathbb{N}, \mathcal{I})$, we write

$$\begin{aligned}\mathbf{Ent}^\Phi_Q(f) &= +\int_\mathbb{N} (\Phi(\mathbf{P}_0 f) - \Phi(\mathbf{P}_\infty f))\,\mathrm{d}Q \\ &= -\int_\mathbb{N}\int_0^\infty \partial_t \Phi(\mathbf{P}_t f)\,\mathrm{d}t\,\mathrm{d}Q \\ &= -\int_0^\infty \int_\mathbb{N} \Phi'(\mathbf{P}_t f)\mathbf{L}\mathbf{P}_t f\,\mathrm{d}Q\,\mathrm{d}t \\ &= \lambda \int_0^\infty \int_\mathbb{N} B^\Phi(\mathbf{P}_t f, \mathrm{D}\mathbf{P}_t f)\,\mathrm{d}Q\,\mathrm{d}t,\end{aligned}$$

where we used (23) for the last equality. Now, the commutation (4) yields

$$\mathbf{Ent}^\Phi_Q(f) = \lambda \int_0^\infty \int_\mathbb{N} B^\Phi(\mathbf{P}_t f, \mathrm{e}^{-\mu t}\mathbf{P}_t \mathrm{D}f)\,\mathrm{d}Q\,\mathrm{d}t.$$

Jensen inequality for $B^\Phi$ and $\mathbf{P}_t(\cdot)$ followed by the invariance of $Q$ give

$$\mathbf{Ent}^\Phi_Q(f) \leqslant \lambda \int_0^\infty \int_\mathbb{N} B^\Phi(f, \mathrm{e}^{-\mu t}\mathrm{D}f)\,\mathrm{d}Q\,\mathrm{d}t.$$

But by Lemma 4.10, $B^\Phi(u, e^{-\mu t} v) \leqslant e^{-\mu t} B^\Phi(u,v)$ for any $(u,v) \in \mathcal{T}_\mathcal{I}$, and thus

$$\mathbf{Ent}^\Phi_Q(f) \leqslant \lambda \int_0^\infty e^{-\mu t}\, dt \int_\mathbb{N} B^\Phi(f, Df)\, dQ = \rho \mathbf{E}_Q\big(B^\Phi(f, Df)\big),$$

which is exactly the $B$ transform version of the Poisson entropic inequality (20).

**Remark 3.3** ($A - B - C$ transforms and discrete space)**.** The interpolation on $[0,t]$ gives rise to the $A^\Phi$ transform whereas the interpolation on $[0,+\infty]$ leads to the $B^\Phi$ transform. In continuous space settings, the diffusion property permits to write $\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = \Phi''(F)\boldsymbol{\Gamma}(F,F)$ which is close to $C^\Phi$ and not to $A^\Phi$ in that case.

*Local inequality and semi-group interpolation on the time interval* $[0,t]$

Consider the semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ of the M/M/$\infty$ queue with input rate $\lambda$ and service rate $\mu$. The family $(\mathbf{P}_s(\cdot)(n))_{0\leqslant s\leqslant t}$ interpolates between $\delta_n$ and $\mathcal{B}(n, e^{-\mu t}) * \mathcal{P}(\rho(1-e^{-\mu t}))$. Let $\Phi \in \mathcal{C}_\mathcal{I}$ such that $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$, *cf.* Theorem 4.4. The inequalities (21) and (3) give for any $n \in \mathbb{N}$, any $t \in \mathbb{R}_+$, and any $f \in \mathcal{K}(\mathbb{N}, \mathcal{I})$,

$$\mathbf{Ent}^\Phi_{\mathbf{P}_t(\cdot)(n)}(f) \leqslant \rho q(t) \mathbf{P}_t\big(A^\Phi(f, Df)\big)(n) + np(t)q(t)\mathbf{P}_t\big(q(t)A^\Phi(f,Df) + p(t)A^\Phi(\tau(f,Df))\big)(n-1). \tag{25}$$

Let us try to recover (25) by semi-group interpolation. We write as for the pure Poisson process case,

$$\begin{aligned}\mathbf{Ent}^\Phi_{\mathbf{P}_t(\cdot)(n)}(f) &= \mathbf{P}_t(\Phi(f))(n) - \Phi(\mathbf{P}_t(f)(n)) \\ &= \int_0^t \partial_s \mathbf{P}_s(\Phi(\mathbf{P}_{t-s} f))(n)\, ds \\ &= \int_0^t \mathbf{P}_s(\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F)(n)\, ds\end{aligned}$$

where $F := \mathbf{P}_{t-s}(f)$. At this step, we notice that

$$\mathbf{L}\Phi(F) - \Phi'(F)\mathbf{L}F = \lambda A^\Phi(F, DF) + \mu h A^\Phi(F, D^* F),$$

where $h : \mathbb{N} \to \mathbb{N}$ is defined by $h(k) = k$ for any $k \in \mathbb{N}$. Thus, we get

$$\mathbf{Ent}^\Phi_{\mathbf{P}_t(\cdot)(n)}(f) = \lambda \int_0^t \mathbf{P}_s\big(A^\Phi(F, DF)\big)(n)\, ds + \mu \int_0^t \mathbf{P}_s\big(hA^\Phi(F, D^*F)\big)(n)\, ds. \tag{26}$$

By virtue of (4), (41), Jensen inequality for the convex functions $A^\Phi$ and $C^\Phi$, and the semi-group property, the first term of the right hand side of (26) is bounded above by

$$\left(\frac{1}{2}\lambda \int_0^t p(t-s)^2 q(t-s)\, ds\right) \mathbf{P}_t\big(C^\Phi(f, Df)\big)(n) + \left(\lambda \int_0^t p(t-s)^3\, ds\right) \mathbf{P}_t\big(A^\Phi(f, Df)\big)(n).$$

For the second term of the right hand side of (26), we first write by virtue of (9) and (19),

$$\mu \mathbf{P}_s\big(hA^\Phi(F, D^*F)\big)(n) = \mu n p(s) \mathbf{P}_s\big(A^\Phi(\tau(F, DF))\big)(n-1) + \lambda q(s) \mathbf{P}_s\big(A^\Phi(\tau(F, DF))\big)(n).$$

Now, by (4), (42), Jensen inequality for the convex functions $A^\Phi(\tau)$ and $C^\Phi$, and the semi-group property,

$$\mathbf{P}_s\big(A^\Phi(\tau(F, DF))\big)(k) \leqslant \frac{1}{2} p(t-s)^2 q(t-s) \mathbf{P}_t\big(C^\Phi(f, Df)\big)(k) + p(t-s)^3 \mathbf{P}_t\big(A^\Phi(\tau(f, Df))\big)(k)$$

for any $k \in \{n-1, n\}$. Thus, the second term of the right hand side of (26) is bounded above by

$$\left(\mu n \int_0^t p(s)p(t-s)^3 \,\mathrm{d}s\right)\mathbf{P}_t\big(A^\Phi(\tau(f,\mathrm{D}f))\big)(n-1) + \left(\frac{1}{2}\mu n \int_0^t p(s)p(t-s)^2 q(t-s)\,\mathrm{d}s\right)\mathbf{P}_t\big(C^\Phi(f,\mathrm{D}f)\big)(n-1)$$
$$+ \left(\lambda \int_0^t q(s)p(t-s)^3 \,\mathrm{d}s\right)\mathbf{P}_t\big(A^\Phi(\tau(f,\mathrm{D}f))\big)(n) + \left(\frac{1}{2}\lambda \int_0^t q(s)p(t-s)^2 q(t-s)\,\mathrm{d}s\right)\mathbf{P}_t\big(C^\Phi(f,\mathrm{D}f)\big)(n).$$

Putting all together, we obtain finally the following local inequality.

$$\mathbf{Ent}^\Phi_{\mathbf{P}_t(\cdot)(n)}(f) \leqslant \rho\mathbf{P}_t\left(\frac{1}{3}(1-p(t)^3)A^\Phi(f,\mathrm{D}f) + \frac{1}{6}q(t)^2(2+p(t))\left[A^\Phi(\tau(f,\mathrm{D}f)) + \frac{1}{2}C^\Phi(f,\mathrm{D}f)\right]\right)(n)$$
$$+\frac{1}{2}np(t)\mathbf{P}_t\left((1-p(t)^2)A^\Phi(\tau(f,\mathrm{D}f)) + \frac{1}{2}q(t)^2C^\Phi(f,\mathrm{D}f)\right)(n-1), \qquad (27)$$

which is not (25). Actually, (27) is in a way stronger than (25) for small $t$, as we will see in the sequel with the fluid limit approximation of the Ornstein-Uhlenbeck process. When $t \to \infty$, we have $p(t) \to 0$, $q(t) \to 1$, and $A^\Phi + A^\Phi(\tau) = B^\Phi$, and in that case, (27) provides the following Poissonian inequality.

$$\mathbf{Ent}^\Phi_{\mathcal{P}(\rho)}(f) \leqslant \frac{1}{2}\rho\mathbf{E}_{\mathcal{P}(\rho)}\left(\frac{2}{3}B^\Phi(f,\mathrm{D}f) + \frac{1}{3}C^\Phi(f,\mathrm{D}f)\right)(n),$$

which is not (20). The proof of (27) given above suggests to use (30) instead of its consequences (41) and (42) for the derivation of local inequalities *via* semi-group interpolation. The investigation of this approach is left to the reader. Notice that (40) is not strong enough. Let us focus on the **(P2)** case, for which we have the simple identity $2A^\Phi(f,\mathrm{D}f) = 2A^\Phi(\tau(f,\mathrm{D}f)) = B^\Phi(f,\mathrm{D}f) = C^\Phi(f,\mathrm{D}f) = 2|\mathrm{D}f|^2$. In that case, (27) is the optimal local Poincaré inequality, *e.g.*

$$\mathbf{Var}_{\mathbf{P}_t(\cdot)(n)}(f) \leqslant \rho q(t)\mathbf{P}_t\left(|\mathrm{D}f|^2\right)(n) + np(t)q(t)\mathbf{P}_t\left(|\mathrm{D}f|^2\right)(n-1).$$

## 3.2. Scaling limit of the entropic inequalities

Let us consider the Poisson distribution $\mathcal{P}(\rho)$ with parameter $\rho > 0$. For any $N \in \mathbb{N}^*$, let $\kappa_N : \mathbb{N} \to \mathbb{R}$ be the function defined by $\kappa_N(n) := N^{-1/2}(n - \rho N)$ for any $n \in \mathbb{N}$. By virtue of the Central Limit Theorem, the image measure of $\mathcal{P}(N\rho) = \mathcal{P}(\rho)^{*N}$ by $\kappa_N$ converges weakly towards the Gaussian measure $\mathcal{N}(0,\rho)$ when $N \to \infty$. Let $g \in \mathcal{K}(\mathbb{R},\mathcal{I})$ be smooth with bounded derivatives, and set $f_N := g \circ \kappa_N$. In one hand, we have

$$\lim_{N\to\infty} \mathbf{Ent}^\Phi_{\mathcal{P}(N\rho)}(f_N) = \mathbf{Ent}^\Phi_{\mathcal{N}(0,\rho)}(g)\,.$$

In the other hand, by a Taylor formula, $D(f_N) = \mathrm{D}(g\circ\kappa_N) = N^{-1/2}(g'\circ\kappa_N) + O(N)$, and by a Taylor formula for $\Phi$ this time, $A^\Phi(f_N,\mathrm{D}f_N) = (2N)^{-1}(g'\circ\kappa_N)^2\Phi''(f_N) + o(N)$. This yields that

$$\lim_{N\to\infty} \rho N\mathbf{E}_{\mathcal{P}(\rho)}\big(A^\Phi(f_N,\mathrm{D}f_N)\big) = \frac{1}{2}\rho\mathbf{E}_{\mathcal{N}(0,\rho)}\big(C^\Phi(g,g')\big)\,.$$

Now, the $A$-transform based Poisson entropic inequality (20) for $\mathcal{P}(N\rho)$ and $f_N$ gives finally that

$$\mathbf{Ent}^\Phi_{\mathcal{N}(0,\rho)}(g) \leqslant \frac{1}{2}\rho\mathbf{E}_{\mathcal{N}(0,1)}\big(C^\Phi(g,g')\big)\,. \qquad (28)$$

Recall that the Poincaré inequality corresponds to **(P2)**. In that case,

$$\mathbf{Ent}^\Phi_{\mathcal{N}(0,\rho)}(g) = \mathbf{Var}_{\mathcal{N}(0,\rho)}(g) \quad \text{and} \quad C^\Phi(g,g') = 2|g'|^2.$$

The logarithmic Sobolev inequality corresponds to **(P1)**. In that case,

$$\mathbf{Ent}^{\Phi}_{\mathcal{N}(0,\rho)}(g) = \mathbf{Ent}_{\mathcal{N}(0,\rho)}(g) \quad \text{and} \quad C^{\Phi}(g,g') = \frac{|g'|^2}{g}\cdot$$

The constant $\rho$ in (28) is known to be optimal. It gives in particular the optimal Poincaré inequality for the Gaussian measure in the case **(P2)**, and the optimal logarithmic Sobolev inequality for the Gaussian measure in the case **(P1)**. The method was used in the case **(P1)** in [35], Remark 1.6. In some sense, the $A$ transform is the right Dirichlet form to consider since it allows the derivation of optimal Gaussian entropic inequalities from their $A$-transform based Poisson versions. In contrast, it is shown in [6], pages 356–357, that the optimal $B$ transform version for the Poisson measure does not lead to the optimal constant in the logarithmic Sobolev inequality for the Gaussian measure (lack of a multiplicative factor 2). The deep reason for this difference between $A$ and $B$ transforms consequences is the fact that the comparison $A^{\Phi} \leqslant B^{\Phi}$ improves by a factor 2 when $v$ goes to 0, as stated in Remark 4.3. This phenomenon does not hold for the Poincaré inequality, since $2A^{\Phi} = B^{\Phi}$ for **(P2)**.

As presented in Section 1.2, the M/M/$\infty$ queueing process gives rise to an Ornstein-Uhlenbeck process *via* a fluid limit procedure. It is quite natural to ask about the behaviour of the binomial-Poisson entropic inequalities under this scaling limit.

Let $(X_t^N)_{t\geqslant 0}$ be an M/M/$\infty$ queueing process with input rate $N\lambda > 0$ and service rate $\mu > 0$, where $N \in \mathbb{N}^*$. Let $(U_t)_{t\geqslant 0}$ be an Ornstein-Uhlenbeck process, solution of the Stochastic Differential Equation $\mathrm{d}U_t = \lambda \mathrm{d}B_t - \mu U_t\,\mathrm{d}t$. Let $g \in \mathcal{K}(\mathbb{R},\mathcal{I})$ be smooth with bounded derivatives. For any $y \in \mathbb{R}$, we define $z_N := \lfloor N\rho + N^{1/2}y\rfloor$ where $\lfloor\cdot\rfloor$ denotes the integer part. According to Section 1.2, the image measure of $\mathcal{L}(X_t^N \,|\, X_0 = z_N)$ by function $\kappa_N$ converges weakly towards $\mathcal{L}(U_t \,|\, U_0 = y)$ when $N$ goes to $\infty$. Notice that

$$\mathcal{L}(X_t^N \,|\, X_0 = z_N) = \mathcal{B}(z_N, p(t)) * \mathcal{P}(N\rho q(t))$$

and that $\mathcal{L}(U_t \,|\, U_0 = y) = \mathcal{N}(yp(t), \rho(1-p(t)^2))$. In particular, if $f_N := g \circ \kappa_N$, then

$$\lim_{N\to\infty} \mathbf{Ent}^{\Phi}_{\mathcal{L}(X_t^N \,|\, X_0^N = z_N)}(f_N) = \mathbf{Ent}^{\Phi}_{\mathcal{L}(U_t \,|\, U_0=y)}(g)\,.$$

In the other hand, as for the pure Poisson measure case, we have

$$\lim_{N\to\infty} N\mathbf{E}_{\mathcal{L}(X_t^N \,|\, X_0^N = z_N)}\big(A^{\Phi}(f_N, \mathrm{D}f_N)\big) = \frac{1}{2}\mathbf{E}_{\mathcal{L}(U_t \,|\, U_0=y)}\big(C^{\Phi}(g,g')\big)\,.$$

A similarly identity holds for $A^{\Phi}(\tau(f_N, \mathrm{D}f_N))$. Putting all together, we deduce from (25) that

$$\mathbf{Ent}^{\Phi}_{\mathcal{L}(U_t \,|\, U_0=y)}(g) \leqslant K(t)\mathbf{E}_{\mathcal{L}(U_t \,|\, U_0=y)}\big(C^{\Phi}(g,g')\big)\,, \tag{29}$$

where $K(t) := \frac{1}{2}\rho q(t)(1+2p(t))$. It is known that the best constant in (29) is $K^*(t) := \frac{1}{2}\rho q(t)(1+p(t))$. Let us consider now the function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ defined for any $t \in (0,\infty)$ by

$$\theta(t) := \frac{K(t)}{K^*(t)} = 1 + \frac{1}{1+\frac{1}{p(t)}}\cdot$$

This function is non-increasing, with $\theta(0) = \frac{3}{2}$ and $\lim_{t\to+\infty}\theta(t) = 1$. Consequently, the constant $K(t)$ in the inequality (29) improves when $t$ increases. It is asymptotically optimal, when $t$ goes to $+\infty$. Surprisingly, it turns out that the usage of (27) instead of (25) provides (29) with constant $K^*(t)$ instead of constant $K(t)$. As a consequence, (27) is in a way stronger than (25), at least in terms of their fluid limit.

**Remark 3.4** (the M/M/1 case)**.** The M/M/1 queue with input rate $\lambda$ and service rate $\mu$ is the birth and death process on $\mathbb{N}$ with generator $\mathbf{L} = \mu \mathrm{D}^* + \lambda \mathrm{D}$. We have $[\mathbf{L}, \mathrm{D}] = 0$ and the "curvature" is identically zero. When $\lambda > 0$ and $\mu > 0$, the symmetric invariant measure $Q$ is given by $Q(n) = \rho^n$ for any $n \in \mathbb{N}$, with $\rho := \lambda/\mu$. The associated Markov semi-group $(\mathbf{P}_t)_{t\geqslant 0}$ satisfies to the exact commutation formula $\mathrm{D}\mathbf{P}_t = \mathbf{P}_t(\mathrm{D})$. Measure $Q$ is finite if and only if $\rho \leqslant 1$, and $Q$ is in that case the geometric measure $\mathcal{G}(1-\rho)$ of mean $\lambda/(\mu-\lambda)$. This leads to Poisson like entropic inequalities for $\mathbf{P}_t(\cdot)$. The M/M/1 is a discrete space analog of the continuous process $(E_t)_{t\geqslant 0}$ solution of the Stochastic Differential Equation $dE_t = dB_t - \mathrm{sign}(E_t)\mathrm{d}t$.

**Remark 3.5** (spectrum)**.** A function $f : \mathbb{N} \to \mathbb{R}$ is an eigenvector associated to the eigenvalue $\alpha \in \mathbb{R}$ for the M/M/∞ infinitesimal generator $\mathbf{L}$ defined by (1) if and only if $\lambda f(n+1) = (\lambda + \alpha + n\mu) f(n) - n\mu f(n-1)$ for any $n \in \mathbb{N}$. Obviously, for any $\alpha \in \mathbb{R}$ and any starting value $f(0) \neq 0$, the equation above has a unique non null solution. As a consequence, the spectrum of $\mathbf{L}$ is $\mathbb{R}$. We will denote by $f_\alpha$ the unique solution such that $f_\alpha(0) = 1$. By the equation $\mathbf{L} f_\alpha = \alpha f$ and the invariance of $Q := \mathcal{P}(\rho)$ we get that $\mathbf{E}_Q(f_\alpha) = 0$ as soon as $f_\alpha \in \mathrm{L}^1(Q)$. Suppose that $f_\alpha \in \mathrm{L}^2(Q)$, then $0 \leqslant \mathbf{E}_Q(\mathbf{\Gamma}(f,f)) = -\mathbf{E}_Q(f\mathbf{L}f) = -\alpha \mathbf{E}_Q\big(f^2\big)$ and thus $\alpha \leqslant 0$. Moreover, Theorem 3.1 for **(P2)** gives $\inf\big\{-\alpha \in \mathbb{R},\ f_\alpha \in \mathrm{L}^2(Q)\big\} = \mu^{-1}$, *cf.* [3], Proposition 2.3.

**Remark 3.6** (Bakry $\mathbf{\Gamma_2}$ calculus)**.** Let us define the Markovian functional quadratic forms $\mathbf{\Gamma}$ and $\mathbf{\Gamma_2}$ by $2\mathbf{\Gamma}(f,f) := \mathbf{L}(f^2) - 2f\mathbf{L}f$ and $2\mathbf{\Gamma_2}(f,f) := \mathbf{L}\mathbf{\Gamma}(f,f) - 2\mathbf{\Gamma}(f,\mathbf{L}f)$. After some algebra based on (1), we get

$$2\mathbf{\Gamma}(f,f)(n) = n\mu |\mathrm{D}^* f|^2(n) + \lambda |\mathrm{D} f|^2(n)$$

for any $f : \mathbb{N} \to \mathbb{R}$ and any $n \in \mathbb{N}$, and

$$2\mathbf{\Gamma_2}(f,f)(n) = \frac{3}{2}\lambda\mu |\mathrm{D} f|^2(n) + \frac{n}{2}\mu^2 |\mathrm{D}^* f|^2(n) + \mathbf{R}(f,f)(n),$$

where $2\mathbf{R}(f,f)(n) := n(n-1)\mu^2 |\mathrm{D}^*\mathrm{D}^* f|^2 + 2n\lambda\mu |\mathrm{D}\mathrm{D}^* f|^2 + \lambda^2 |\mathrm{D}\mathrm{D} f|^2$. Notice that for the linear function $f(n) = n$, we get

$$2\mathbf{\Gamma}(f,f)(n) = \lambda + n\mu \quad \text{and} \quad 4\mathbf{\Gamma_2}(f,f)(n) = 3\lambda\mu + n\mu^2.$$

Since $\mathbf{R}(f,f) \geqslant 0$ for any $f$, we obtain immediately the bound $\mathbf{\Gamma_2} \geqslant \mu\frac{1}{2}\mathbf{\Gamma}$, which is the infinitesimal version of the commutation $\mathbf{\Gamma}\mathbf{P}_t \leqslant \exp(-t\frac{\mu}{2})\mathbf{P}_t\mathbf{\Gamma}$. Moreover, an integration by parts similar to (7) gives the integrated bound $\mathbf{E}_Q(\mathbf{\Gamma_2} f) \geqslant \mu \mathbf{E}_Q(\mathbf{\Gamma} f)$, where $Q := \mathcal{P}(\rho)$. Such a bound gives, *via* integration by parts, the Poincaré inequality $\mathbf{Var}_Q(f) \leqslant \rho \mathbf{E}_Q(|\mathrm{D} f|^2)$, which is exactly (20) for **(P2)**. However, the $\mathbf{\Gamma_2}$ bound above suggests that $\mathbf{\Gamma_2}$ is not the right tool in order to derive $\Phi$-entropic inequalities beyond the **(P2)** case. Bakry-Émery type approaches are designed for diffusion. In discrete space settings, the lack of chain rule reduces their strength for the derivation of entropic inequalities beyond the **(P2)** case.

## 4. Convexity and $\Phi$-calculus on $A - B - C$ transforms

We give in the sequel various convexity properties, which extend in particular many aspects of [11]. Let $\Phi : \mathcal{I} \to \mathbb{R}$ be a smooth function defined on an open interval $\mathcal{I} \subset \mathbb{R}$. The usage of suitable Taylor formulas provide for any $(u,v) \in \mathcal{T}_{\mathcal{I}}$,

$$A^\Phi(u,v) = \int_0^1 (1-p) C^\Phi(u+pv, v)\,\mathrm{d}p \quad \text{and} \quad B^\Phi(u,v) = \int_0^1 C^\Phi(u+pv, v)\,\mathrm{d}p, \tag{30}$$

and for any $(u,v) \in \mathcal{T}'_{\mathcal{I}}$ and small enough $\varepsilon$,

$$A^\Phi(u,\varepsilon v) = \frac{1}{2} C^\Phi(u,v)\varepsilon^2 + o(\varepsilon^2) \quad \text{and} \quad B^\Phi(u,\varepsilon v) = C^\Phi(u,v)\varepsilon^2 + o(\varepsilon^2). \tag{31}$$

We denote by $\tau : \mathcal{T}_\mathcal{I} \to \mathcal{T}_\mathcal{I}$ the bijective linear map defined for any $(u,v) \in \mathcal{T}_\mathcal{I}$ by

$$\tau(u,v) := (u+v,-v). \tag{32}$$

Notice that $\tau$ is well defined since $(u,v) \in \mathcal{T}_\mathcal{I}$ implies that $(u+v,u+v-v) \in \mathcal{I}\times\mathcal{I}$ and thus $(u+v,-v) \in \mathcal{T}_\mathcal{I}$. Writing $(a,b) := (u,u+v)$ shows that the map $\tau$ transposes $a$ and $b$, and $\tau^2$ is the identity map. Moreover,

$$\begin{aligned} A^\Phi(u,v) &= \Phi(b) - \Phi(a) - \Phi'(a)(b-a),\\ B^\Phi(u,v) &= (b-a)(\Phi'(b) - \Phi'(a)),\\ C^\Phi(u,v) &= \Phi''(a)(b-a)^2. \end{aligned}$$

**Lemma 4.1.** *Let $\Phi : \mathcal{I} \to \mathbb{R}$ be smooth on an open interval $\mathcal{I} \subset \mathbb{R}$. Then the following statements hold.*
(1) $A^\Phi + A^\Phi(\tau) = B^\Phi$ *and* $B^\Phi(\tau) = B^\Phi$;
(2) *Each of $A^\Phi$, $B^\Phi$, $C^\Phi$ is non-negative if and only if $\Phi \in \mathcal{C}_\mathcal{I}$.*

*Proof.* The first statement of the Lemma is immediate. For the second statement, we observe first that $C^\Phi$ is non-negative if and only if $\Phi''$ is non-negative, *e.g.* if and only if $\Phi \in \mathcal{C}_\mathcal{I}$. The same holds then for $A^\Phi$ and $B^\Phi$ by using (31) and (30). □

Lemma 4.1 tells that the $A$–$B$–$C$ transforms map the set of convex functions on $\mathcal{I}$ into the set of non-negative functions on $\mathcal{T}_\mathcal{I}$. Moreover, their null space contains any real valued affine functions on $\mathcal{I}$.

**Lemma 4.2.** *Let $\Phi : \mathcal{I} \to \mathbb{R}$ be smooth on an open interval $\mathcal{I} \subset \mathbb{R}$. The following statements hold.*
(1) *for **(P1-P2-P3)**, we have $\Phi'' > 0$ on $\mathcal{I}$ and $\Phi$, $-\Phi'$, $\Phi''$, $-1/\Phi''$ belong to $\mathcal{C}_\mathcal{I}$;*
(2) $2A^\Phi = B^\Phi = C^\Phi$ *for **(P2)** and* $A^\Phi \leqslant C^\Phi$ *for **(P1)**;*
(3) *if $\Phi \in \mathcal{C}_\mathcal{I}$ then $A^\Phi \leqslant B^\Phi$;*
(4) *if $\Phi'' \in \mathcal{C}_\mathcal{I}$ then $C^\Phi(u+v/3,v) \leqslant 2A^\Phi(u,v)$ and $C^\Phi(u+v/2,v) \leqslant B^\Phi(u,v)$ for any $(u,v) \in \mathcal{T}_\mathcal{I}$.*

*Proof.* Statement 1 and the first part of statement 2 are immediate. Notice that $1/\Phi''$ is affine for **(P1)** and **(P2)**. The second part of statement 2 follows from the first part of (30). For statement 3, we notice that by Lemma 4.1, $B^\Phi = A^\Phi + A^\Phi(\tau)$, where $A^\Phi(\tau) \geqslant 0$ when $\Phi \in \mathcal{C}_\mathcal{I}$. Statement 4 follows by using (30), the definition of $C^\Phi$, and Jensen inequality with respect to the integral over $[0,1]$ for the convex function $p \in [0,1] \mapsto \Phi''(u+pv)$. □

**Remark 4.3** (optimality of *A*-*B*-*C* comparisons)**.** The bound $A^\Phi \leqslant B^\Phi$ is optimal in the sense that for **(P1)**, we have $B^\Phi(u,v) \sim A^\Phi(u,v)$ at $v = +\infty$ for any $u \in \mathcal{I}$. However, $B^\Phi = 2A^\Phi = C^\Phi$ for **(P2)**; and in general

$$\lim_{v\to 0} v^{-2} 2A^\Phi(u,v) = \lim_{v\to 0} v^{-2} B^\Phi(u,v) = v^{-2} C^\Phi(u,v) = \Phi''(u).$$

Theorem 4.4 below states that the convexity of the $A-B-C$ transforms of $\Phi$ are deeply related to the convexity of the $\Phi$-entropy functional. It provides in particular a synthesis of some results by Latała and Oleszkiewicz in [27], by the author in [11], and by Massart in his Saint-Flour course [28] (see also the article [4]). We say that a collection $\mathcal{P}$ of probability spaces is a *covering collection* if $\{Q(T); T \in \mathcal{E}, (E,\mathcal{E},Q) \in \mathcal{P}\} = [0,1]$. An example is given for instance by the family of Bernoulli probability measures on the two point space $\{0,1\}$, or by any collection containing a probability measure on $\mathbb{R}$ with a continuous cumulative distribution function.

**Theorem 4.4.** *For any smooth $\Phi : \mathcal{I} \to \mathbb{R}$ on an open interval $\mathcal{I} \subset \mathbb{R}$, the following statements are equivalent.*
(1) $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$;
(2) $B^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$;
(3) $C^\Phi \in \mathcal{C}_{\mathcal{T}'_\mathcal{I}}$;
(4) *either $\Phi$ is affine on $\mathcal{I}$, or $\Phi'' > 0$ on $\mathcal{I}$ with $-1/\Phi'' \in \mathcal{C}_\mathcal{I}$;*
(5) $(a,b) \in \mathcal{I}\times\mathcal{I} \mapsto t\Phi(a) + (1-t)\Phi(b) - \Phi(ta+(1-t)b)$ *belongs to $\mathcal{C}_{\mathcal{I}\times\mathcal{I}}$, for any $t \in [0,1]$;*

(6) *for any probability space* $(E,\mathcal{E},Q)$, $\mathbf{Ent}_Q^\Phi \in \mathcal{C}_{\mathcal{K}(E,\mathcal{I})}$;

(7) *there exists a covering collection* $\mathcal{P}$ *such that* $\mathbf{Ent}_Q^\Phi \in \mathcal{C}_{\mathcal{K}(E,\mathcal{I})}$ *for any* $(E,\mathcal{E},Q)$ *in* $\mathcal{P}$;

(8) *for any probability space* $(E,\mathcal{E},Q)$ *and any* $f \in \mathcal{K}(E,\mathcal{I})$,

$$\mathbf{Ent}_Q^\Phi(f) = \sup_{g\in\mathcal{K}(E,\mathcal{I})} \{\mathbf{E}_Q((\Phi'(g) - \Phi'(\mathbf{E}_Q g))(f-g)) + \mathbf{Ent}_Q^\Phi(g)\}; \tag{33}$$

(9) *there exists a covering collection* $\mathcal{P}$ *such that* (33) *holds for any* $(E,\mathcal{E},Q)\in\mathcal{P}$ *and any* $f\in\mathcal{K}(E,\mathcal{I})$;

(10) *for any product probability space* $(E,\mathcal{E},Q) := (E_1\times\cdots\times E_n, \mathcal{E}_1\otimes\cdots\otimes\mathcal{E}_n, Q_1\otimes\cdots\otimes Q_n)$, *and for any* $f\in\mathcal{K}(E,\mathcal{I})$,

$$\mathbf{Ent}_Q^\Phi(f) \leqslant \mathbf{E}_Q\Big(\mathbf{Ent}_{Q_1}^\Phi(f)\Big) + \cdots + \mathbf{E}_Q\Big(\mathbf{Ent}_{Q_n}^\Phi(f)\Big)\,, \tag{34}$$

*where the expectation with respect to* $Q_i$ *in* $\mathbf{Ent}_{Q_i}^\Phi(f)$ *concerns only the* $i^{th}$ *coordinate;*

(11) *there exists a covering collection* $\mathcal{P}$ *such that* (34) *holds for* $n=2$, *any* $Q_1\in\{\mathcal{B}(1,p); p\in[0,1]\}$, *any* $Q_2\in\mathcal{P}$, *and any* $f\in\mathcal{K}(E_1\times E_2,\mathcal{I})$.

*Moreover, if these statements hold, then* $\Phi$ *and* $\Phi''$ *belong to* $\mathcal{C}_\mathcal{I}$.

**Remark 4.5** (functional spaces)**.** By approximation, the convex set $\mathcal{K}(E,\mathcal{I})$ can be replaced by the convex set $\mathrm{L}^{1,\Phi}(Q)$ in statements 6, 7, 8, 9, 10, 11 of Theorem 4.4. More precisely, statement 4 implies the convexity of $\Phi$, which implies in turn that $\Phi'(g)(f-g)+\Phi(g)\leqslant\Phi(f)$ for any $f,g\in\mathrm{L}^{1,\Phi}(Q)$. This yields that $\mathbf{E}_Q((\Phi'(g)-\mathbf{E}_Q g)(f-g))$ is well defined in $[-\infty,+\infty)$, as noticed in the proof of [28], Lemma 2.26.

**Remark 4.6** (meaning of the variational formula)**.** Despite its functional expression, the variational formula (33) is actually a unidimensional statement, taken in all directions. Namely, for any probability space $(E,\mathcal{E},Q)$ and any $f,g\in\mathrm{L}^{1,\Phi}(Q)$, let us denote by $\alpha_{f,g}:[0,1]\to\mathbb{R}$ the function defined for any $\lambda\in[0,1]$ by

$$\alpha_{f,g}(\lambda) := \mathbf{Ent}_Q^\Phi(\lambda f + (1-\lambda)g)\,. \tag{35}$$

Notice that $\alpha_{f,g}(0)=\mathbf{Ent}_Q^\Phi(g)$ and $\alpha_{f,g}(1)=\mathbf{Ent}_Q^\Phi(f)$. The consideration of convex combinations reveals that the convexity of the $\Phi$-entropy functional on $\mathrm{L}^{1,\Phi}(Q)$ is equivalent to the convexity of $\alpha_{f,g}$ on $[0,1]$ for any $f$ and $g$. Assume now that $\alpha_{f,g}$ is convex on $[0,1]$ for any $f$ and $g$ in $\mathrm{L}^{1,\Phi}(Q)$. Assume for the moment that $f,g\in\mathcal{K}(E,\mathcal{I})$. In that case, there are no boundary effects, and $\alpha_{f,g}$ is smooth. Recall that a real convex function is the envelope of its tangents, *cf.* [32]. In particular, $\alpha_{f,g}(1)\geqslant\alpha_{f,g}(0)+\alpha'_{f,g}(0)$. Moreover, equality is achieved for $f=g$. As a consequence, we get

$$\mathbf{Ent}_Q^\Phi(f) = \sup_{g\in\mathcal{K}(E,\mathcal{I})}\big\{\alpha'_{f,g}(0)+\alpha_{f,g}(0)\big\}. \tag{36}$$

It turns out that $\alpha'_{f,g}(0)=\mathbf{E}_Q((\Phi'(g)-\Phi'(\mathbf{E}_Q g))(f-g))$. We thus recover exactly (33). By virtue of Remark 4.5, the formula above for $\alpha'_{f,g}(0)$ still makes sense in $[-\infty,\infty)$ when $f,g$ are in $\mathrm{L}^{1,\Phi}(Q)$, and consequently, the variational formula (36) remains true when $f,g\in\mathrm{L}^{1,\Phi}(Q)$. Notice that $\alpha_{f,g}(\lambda)=\mathbf{Ent}_Q^\Phi(g+\lambda(f-g))$, and hence $\alpha'_{f,g}(0)$ is the directional derivative of the $\Phi$-entropy functional at point $g$ in the direction $f-g$.

*Proof of Theorem 4.4. 1⇒2.* Follows from the identity $B^\Phi = A^\Phi + A^\Phi(\tau)$ where $\tau$ is linear, given by Lemma 4.1.

*3⇒1* and *3⇒2.* Follow from (30) used on a convex combination.

*1⇒3* and *3⇒2.* Follow from (31) used on a convex combination.

*1⇒4.* The Hessian matrix of $A^\Phi$ writes for any $(u,v)\in\mathcal{T}_\mathcal{I}$,

$$\nabla^2 A^\Phi(u,v) = \begin{pmatrix} A^{\Phi''}(u,v) & \Phi''(u+v)-\Phi''(u) \\ \Phi''(u+v)-\Phi''(u) & \Phi''(u+v)\end{pmatrix}.$$

Since $A^\Phi$ is convex, the diagonal elements of $\nabla^2 A^\Phi$ are non-negative, and thus $\Phi'' \geqslant 0$ on $\mathcal{I}$. Moreover, the convexity of $A^\Phi$ yields that $\det(\nabla^2 A^\Phi)$ is non-negative. Suppose now that $(u,v) \in \mathcal{T}_\mathcal{I}$ is such that $\Phi''(u+v)=0$. Then $\det(\nabla^2 A^\Phi(u,v)) = -\Phi''(u)^2$, and thus $\Phi''(u)=0$. Consequently, the set $\{w \in \mathcal{I}; \Phi''(w)=0\}$ is either empty of equal to $\mathcal{I}$, as required. When $\Phi''>0$ on $\mathcal{I}$, we get $\det(\nabla^2 A^\Phi(u,v)) = \Phi''(u+v)\Phi''^2(u)A^{-1/\Phi''}(u,v)$, which is non-negative since $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$. But $\Phi''>0$, and thus $A^{-1/\Phi''} \geqslant 0$. Lemma 4.1 gives then $-1/\Phi'' \in \mathcal{C}_\mathcal{I}$.

*4⇒1.* If $\Phi$ is affine on $\mathcal{I}$, then $A^\Phi$ is identically zero, and thus belongs to $\mathcal{C}_{\mathcal{T}_\mathcal{I}}$. Let us consider the second case. Assume that $\Phi''>0$ on $\mathcal{I}$ with $-1/\Phi'' \in \mathcal{C}_\mathcal{I}$. It turns out that $(-1/\Phi'')'' = (\Phi''''\Phi'' - 2\Phi'''^2)/\Phi''^3$. Hence, $\Phi''''\Phi'' \geqslant 2\Phi'''^2$ on $\mathcal{I}$, and thus $\Phi'''' \geqslant 0$ on $\mathcal{I}$. In other words, $\Phi'' \in \mathcal{C}_\mathcal{I}$. By Lemma 4.1, it follows that $A^{\Phi''}$ is non-negative. Therefore, the diagonal elements of $\nabla^2 A^\Phi$ are non-negative on $\mathcal{C}_\mathcal{I}$. In the other hand, for any $(u,v) \in \mathcal{T}_\mathcal{I}$, $\det(\nabla^2 A^\Phi(u,v)) = \Phi''(u+v)\Phi''^2(u)A^{-1/\Phi''}(u,v)$, which is non-negative again by virtue of Lemma 4.1. Putting all together, the two dimensional matrix $\nabla^2 A^\Phi(u,v)$ has a non-negative trace and determinant for any $(u,v) \in \mathcal{C}_\mathcal{I}$, as expected.

*4⇒5* and *5⇒6*. Follow from the definitions. See for instance [27].

*6⇒7* and *8⇒9* and *10⇒11* are immediate.

*1⇒8.* Let $f$ and $g$ be in $\mathcal{K}(E,\mathcal{I})$. Since $A^\Phi \in \mathcal{C}_{\mathcal{T}_\mathcal{I}}$, the following sort of "$A^\Phi$-entropy"

$$J(f,g) := \mathbf{E}_Q\big(A^\Phi(g, f-g)\big) - A^\Phi(\mathbf{E}_Q g, \mathbf{E}_Q(f-g)) \tag{37}$$

is non negative by Jensen inequality. Moreover, it vanishes when $f=g$. The desired result follows from the identity $\mathbf{Ent}_Q^\Phi(f) = J(f,g) + \mathbf{E}_Q((\Phi'(g) - \Phi'(\mathbf{E}_Q g))(f-g)) + \mathbf{Ent}_Q^\Phi(g)$.

*9⇒1.* For any $(E,\mathcal{E},Q) \in \mathcal{P}$ and $f,g \in \mathcal{K}(E,\mathcal{I})$, the identity (33) implies that the quantity $J(f,g)$ defined by (37) is non-negative. By approximation, $J(f,g)$ is non-negative for any $f,g \in \mathrm{L}^{1,\Phi}(Q)$. Now, let $(u,v)$ and $(u',v')$ be in $\mathcal{T}_\mathcal{I}$, and let $\lambda \in [0,1]$. Since $\mathcal{P}$ is a covering collection, there exists $(E,\mathcal{E},Q) \in \mathcal{P}$ and $T \in \mathcal{E}$ such that $Q(T)=\lambda$. Let $g := u\mathrm{I}_T + u'\mathrm{I}_{T^c}$ and $f := g + v\mathrm{I}_T + v'\mathrm{I}_{T^c}$. The desired result follows from the identity $J(f,g) = \lambda A^\Phi(u,v) + (1-\lambda)A^\Phi(u',v') - A^\Phi(\lambda(u,v) + (1-\lambda)(u',v'))$.

*3⇒6* and *3⇒8.* Let $f,g \in \mathcal{K}(E,\mathcal{I})$, and let $\alpha_{f,g}$ be as in (35). It turns out that

$$\alpha''(t) = \mathbf{E}_Q\big(C^\Phi(h_t, f-g)\big) - C^\Phi(\mathbf{E}_Q h_t, \mathbf{E}_Q(f-g)).$$

Notice that $(h_t, f-g)$ takes its values in $\mathcal{T}'_\mathcal{I}$. Since $C^\Phi \in \mathcal{C}_{\mathcal{T}'_\mathcal{I}}$, we get $\alpha''(t) \geqslant 0$. In other words, $\alpha \in \mathcal{C}_{[0,1]}$. In particular $\alpha(\lambda) \leqslant \lambda\alpha(1) + (1-\lambda)\alpha(0)$ for $\lambda \in [0,1]$ writes $\mathbf{Ent}_Q^\Phi(\lambda f + (1-\lambda)g) \leqslant \lambda\mathbf{Ent}_Q^\Phi(f) + (1-\lambda)\mathbf{Ent}_Q^\Phi(g)$, which is nothing else but the expression of the convexity of $\mathbf{Ent}_Q^\Phi$. Additionally, since every convex function on an interval is the envelope of its tangents, see [32], one gets $\mathbf{Ent}_Q^\Phi(f) = \alpha(1) = \sup_{t\in[0,1]}\{\alpha(t) + \alpha'(t)(1-t)\}$. In particular, $\mathbf{Ent}_Q^\Phi(f) \geqslant \alpha(0) + \alpha'(0)$, with equality when $f=g$. Taking the supremum with respect to $g$ leads to (33).

*7⇒9* and *6⇒8.* Let $f,g \in \mathcal{K}(E,\mathcal{I})$, and let $\alpha_{f,g}$ be as in (35). Then, for any $s,t \in [0,1]$ and any $\lambda \in [0,1]$, $\alpha(\lambda s + (1-\lambda)t) = \mathbf{Ent}_Q^\Phi(\lambda(tf + (1-t)g) + (1-\lambda)(sf + (1-s)g))$. Since $\mathbf{Ent}_Q^\Phi \in \mathcal{C}_{\mathcal{K}(E,\mathcal{I})}$, we get $\alpha(\lambda s + (1-\lambda)t) \leqslant \lambda\alpha(s) + (1-\lambda)\alpha(t)$, and thus $\alpha \in \mathcal{C}_{[0,1]}$. Since every convex function on an interval is the envelope of its tangents, see [32], we obtain $\mathbf{Ent}_Q^\Phi(f) = \alpha(1) = \sup_{t\in[0,1]}\{\alpha(t) + \alpha'(t)(1-t)\}$. In particular, $\mathbf{Ent}_Q^\Phi(f) \geqslant \alpha(0) + \alpha'(0)$, with equality when $f=g$. Taking the supremum with respect to $g$ leads to (33).

*9⇒7* and *8⇒6.* Use $\lambda f_1 + (1-\lambda)f_2 - g = \lambda(f_1 - g) + (1-\lambda)(f_2 - g)$ and $\mathbf{Ent}_Q^\Phi(g) = \lambda\mathbf{Ent}_Q^\Phi(g) + (1-\lambda)\mathbf{Ent}_Q^\Phi(g)$ in the expression inside the supremum in (33), then use the fact that the supremum of the sum is less than or equal to the sum of the suprema.

*11⇒7.* The proof can be found in [28], introduction of section 2.5. Namely, let $g_1, g_2 \in \mathcal{K}(E_2,\mathcal{I})$, and consider $f : \{0,1\} \times E_2 \to \mathbb{R}$ defined by $f(x,y) := g_1(y)$ if $x=0$ and $f(x,y) := g_2(y)$ if $x=1$. The tensorisation formula (34) expressed for $f$ rewrites $\mathbf{Ent}_{Q_2}^\Phi((1-p)g_1 + pg_2) \leqslant (1-p)\mathbf{Ent}_{Q_2}^\Phi(g_1) + p\mathbf{Ent}_{Q_2}^\Phi(g_2)$, as expected.

*8⇒10.* The proof can be found in the Saint-Flour course [28], Theorem 2.27. Roughly speaking, it consists in the usage of the variational formula (33) on each entropy in the right hand side of (34), which gives rise, *via* a telescopic sum, to the variational formula for the left hand side of (34).

Finally, if the statements hold, then $\Phi \in \mathcal{C}_\mathcal{I}$ by statement 4, and the proof of *4⇒1* given above provides in particular that $\Phi'' \in \mathcal{C}_\mathcal{I}$. □

**Example 4.7.** For **(P1-P2-P3)**, both $\Phi$, $-\Phi'$, and $\Phi''$ are convex on $\mathcal{I}$. Moreover, $\Phi'' > 0$ on $\mathcal{I}$ and $-1/\Phi''$ is convex on $\mathcal{I}$. Actually, $-1/\Phi''$ is affine for **(P1)** and **(P2)**. Consider the case where $\Phi(u) := -u\log(-u)$ on $\mathcal{I} = (-\infty, 0)$. Then $\Phi'' > 0$ on $\mathcal{I}$ , and $-1/\Phi'' > 0$ on $\mathcal{I}$. However, $-\Phi'$ is concave and not convex on $\mathcal{I}$. Consider now the case where $\Phi(u) = -\log(u)$ on $\mathcal{I} = (0, +\infty)$. Then $\Phi$, $-\Phi'$, $\Phi''$ are convex on $\mathcal{I}$, and $\Phi'' > 0$ on $\mathcal{I}$. However, $-1/\Phi''$ is concave and not convex. These examples rely on the stability by symmetry of the convexity of $-1/\Phi''$, and the absence of such a stability for $-\Phi'$.

**Example 4.8.** Following [11], the convexities of the $A$-$B$-$C$ transforms of $\Phi$ and of the $\Phi$-entropy functional are stable by any linear combination on $\Phi$ with non-negative coefficients. Theorem 4.4 shows in particular that this stability still holds for the convexity of $-1/\Phi''$, for which it is less apparent. The consideration of continuous linear combinations on $\Phi$ by mean of an integral with respect to a positive Borel measure provides several interesting examples. For instance, $\Phi(u) := \int_1^2 u^p\,\mathrm{d}p = u(u-1)/\log(u)$ on $\mathcal{I} = \mathbb{R}_+^*$ is obtained from **(P3)**, and satisfies to the required convexities of Theorem 4.4.

**Example 4.9.** A curious example is given by $\Phi(u) = -\mathbf{I}(u)$ on $\mathcal{I} = (0,1)$, where $\mathbf{I}$ is the Gaussian isoperimetric function defined by $\mathbf{I} := F' \circ F^{-1}$ where $F$ is the cumulative distribution function of $\mathcal{N}(0,1)$. Function $\mathbf{I}$ is positive and concave on $\mathcal{I}$, and satisfies to the identity $\mathbf{I}\mathbf{I}'' = -1$. Consequently, $\Phi$ and $-1/\Phi'' = \Phi$ are convex. In particular, Theorem 4.4 shows that the $\mathbf{I}$-entropy is concave, and provides a reversed tensorisation formula.

For any $p \in [0,1]$, let $\sigma_p : \mathcal{T}_\mathcal{I} \to \mathcal{T}_\mathcal{I}$ be the linear map defined for any $(u,v) \in \mathcal{T}_\mathcal{I}$ by

$$\sigma_p(u,v) := (u, pv). \tag{38}$$

The map $\sigma_p$ is well defined since for any $(u,v) \in \mathcal{T}_\mathcal{I}$ and any $p \in [0,1]$, we have $u + pv \in [u, u+v] \subset \mathcal{I}$ by convexity of $\mathcal{I}$, and thus $(u, u+pv) \in \mathcal{T}_\mathcal{I}$. Notice that $C^\Phi(\sigma_p) = p^2 C^\Phi$.

**Lemma 4.10.** *Let $\Phi \in \mathcal{C}_\mathcal{I}$ and $p \in [0,1]$. Let $\sigma_p$ be as in and* (38). *The following inequalities hold on $\mathcal{T}_\mathcal{I}$,*

$$A^\Phi(\sigma_p) \leqslant pA^\Phi \quad \text{and} \quad B^\Phi(\sigma_p) \leqslant pB^\Phi, \tag{39}$$

*where $q := 1-p$. Moreover, $A^\Phi(\sigma_p) = p^2 A^\Phi$ and $B^\Phi(\sigma_p) = p^2 B^\Phi$ for **(P2)**. Let $\tau$ be as in* (32). *Assume in addition that $\Phi'' \in \mathcal{C}_\mathcal{I}$, then the following inequalities hold on $\mathcal{T}_\mathcal{I}$,*

$$\begin{aligned} pA^\Phi - A^\Phi(\sigma_p) &\leqslant pq\big(pA^\Phi(\tau) + qA^\Phi\big); && (40)\\ A^\Phi(\sigma_p) &\leqslant \frac{1}{2}p^2 q\, C^\Phi + p^3 A^\Phi; && (41)\\ A^\Phi(\tau(\sigma_p))) &\leqslant \frac{1}{2}p^2 q\, C^\Phi + p^3 A^\Phi(\tau); && (42)\\ B^\Phi(\sigma_p) &\leqslant p^2 q\, C^\Phi + p^3 B. && (43)\end{aligned}$$

*Proof.* The $A^\Phi$ part of (39) is a rewriting of (10). For the $B^\Phi$ part of (39), we notice that $\Phi'$ is non-decreasing since $\Phi$ is convex, and thus $pv\Phi'(u+pv) \leqslant pv\Phi'(u+v)$ regardless of the sign of $v$, which gives the desired result. The inequality (40) is a rewriting of (15). Namely, $pA^\Phi(u,v) - A^\Phi(\sigma_p(u,v)) = \mathbf{Ent}^\Phi_{\mathcal{B}(1,p)}(f)$ for any $(u,v) \in \mathcal{T}_\mathcal{I}$, where $(f(0), f(1)) := (u, u+v)$, whereas $\mathbf{E}_{\mathcal{B}(1,p)}\big(A^\Phi(f, \mathrm{D}f)\big) = pA^\Phi(\tau(u,v)) + qA^\Phi(u,v)$. The inequalities (41), (42), and (43) follow from (30) by using the definition of $C^\Phi$ and a suitable Jensen inequality for $\Phi''$. □

**Remark 4.11.** The bounds (41) and (42) become equalities for **(P2)**. However, (43) is not sharp for **(P2)**. The bound $B^\Phi(\sigma_p) \leqslant pB^\Phi$ is optimal in the sense that for **(P1)**, we have $B^\Phi(u,pv) \sim pB^\Phi(u,v)$ at $v = +\infty$ for any $(p,u) \in (0,1) \times \mathcal{I}$. However, $B^\Phi(\sigma_p) = p^2 B^\Phi$ for **(P2)**; and in general

$$\lim_{v\to 0} v^{-2} B^\Phi(u,pv) = p^2 \lim_{v\to 0} B^\Phi(u,v) = p^2 \Phi''(u).$$

The same remark holds for $A^\Phi$ (up to a factor 2 in the case **(P2)**).

Some of the results of this section correct mistakes discovered by the author in [11] after publication. In contrary to what appears in [11], p. 330, $\mathcal{H}$**(2')** does not imply $\mathcal{H}$**(2)**. Actually, $\mathcal{H}$**(1)** and $\mathcal{H}$**(2)** are equivalent and $\mathcal{H}$**(2')** should be removed from [11]. In particular, [11], Remarks 8 and 11, pages 354–356, should be replaced by Lemma (4.1). These corrections are minor and simplifying and do not impact the results of [11] at all.

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