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CYCLIC RANDOM MOTIONS IN $\mathbb{R}^d$-SPACE WITH n DIRECTIONS

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Abstract. We study the probability distribution of the location of a particle performing a cyclic random motion in $\mathbb{R}^d$. The particle can take n possible directions with different velocities and the changes of direction occur at random times. The speed-vectors as well as the support of the distribution form a polyhedron (the first one having constant sides and the other expanding with time t). The distribution of the location of the particle is made up of two components: a singular component (corresponding to the beginning of the travel of the particle) and an absolutely continuous component.

We completely describe the singular component and exhibit an integral representation for the absolutely continuous one. The distribution is obtained by using a suitable expression of the location of the particle as well as some probability calculus together with some linear algebra. The particular case of the minimal cyclic motion ($n = d + 1$) with Erlangian switching times is also investigated and the related distribution can be expressed in terms of hyper-Bessel functions with several arguments.

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1. Introduction

Historically, Fok [4], Goldstein [5], Griego and Hersh [6], Kac [7] studied the connection between random evolutions and hyperbolic partial differential equations (see the monograph by Pinsky [20] for more references). In this domain, 1-D telegrapher's process which is related to the 1-D waves equation is certainly the most famous motion.

Many authors attempt to define similar processes in higher dimensions with possibly variable velocities. Numerous models in the literature deal with random motions with few directions in low dimension (2 or 3), constant speed and exponential switching times for the changes of direction. We review several of them (the list is not exhaustive):

- in dimension 2: Di Crescenzo [3], Kolesnik and Orsingher [8], Leorato *et al.* [11], Orsingher [13, 14], Orsingher *et al.* [15–18] studied random motions with constant speed with three and four directions, exponential switching times, the changes of direction obeying various rules (cyclic, reflecting, orthogonal or random deviations);
- in dimension 3: Leorato and Orsingher [10], Orsingher and Sommella [19] considered random motions with four directions changing with uniform law.

On the other hand, Di Crescenzo [2] studied a generalization of telegrapher's process (random motion on the line) getting rid of constraints on the speed and switching times. He introduced alternating velocities and considered changes of direction occurring at Erlang distributed times.

Aside from this, the case of higher dimension arose in very few papers: Lachal *et al.* [9], Samoilenko [21, 22] studied cyclic, minimal random motions in $\mathbb{R}^d$, that is with $d+1$ directions forming a regular hyper-hedron, the directions being taken in a deterministic order.

Such evolutions can adequately describe (in simplified versions):

- particles moving in a turbulent medium, for example in the presence of a vortex (see Orsingher and Ratanov [17]);
- electrons moving randomly in a conductor and changing direction (with damping of velocities) when reaching the boundary of the conductor;
- the microscopic behavior of gas particles (or biological microorganisms) changing of direction when collisions with other particles occur...

More specifically, cyclic random motions can be applied to concrete situations arising in various domains (see [2]):

- in insurance: a company get positive incomes from policyholders and pays indemnities when damages occur. Incomes are regular but damages occur at random times. The profit gained by the company can be modeled by a 1-D telegraph-type process with two velocities and random switching times;
- in reliability: consider a system where machines can break down. The profit gained when a machine works and the cost paid when it is under repair can be modeled by a similar process with two different random switching times;
- in queues: a single-server queueing system produces a positive gain when the server is busy and have a cost to pay when the server is idle. This is also an alternating (cyclic) situation. More than two states (busy-idle) may be considered. We postpone an extension of this example to the application displayed in Section 2.

Now we describe the cyclic motion which is examined in this paper. Consider a particle moving in the d-dimensional space $\mathbb{R}^d$ according to the following rules:

- the particle follows a finite number n of possible directions $D_1, \dots, D_n$;
- for each direction, the particle moves with a constant velocity depending on the direction: along the direction D_j, its speed-vector is given by the vector $\overrightarrow{V_j}$ with a constant norm $V_j \in (0, +\infty)$ depending only on j, $1 \le j \le n$;
- the directions change at random instants $T_1, T_2, \dots$ and the laps of time between two switches, $T_{k+1} - T_k$, $k \ge 1$, are independent random variables;

- the motion is cyclic, that is, the particle moves successively in the direction $D_1, D_2, \ldots, D_n, D_1, D_2, \ldots$ and so on, in this order; actually we shall write $D_1, D_2, \ldots, D_n, D_{n+1}, D_{n+2}, \ldots$ with $D_{n+1} = D_1$, $D_{n+2} = D_2$ and so on.

We suppose that the particle starts at the origin $\mathbf{O}$ at time $T_0 = 0$ with direction D_1 and velocity V_1. This assumption is not restrictive. Indeed, the results related to another initial direction may be easily deduced from the case of D_1 as an initial direction by simply translating the indices of the directions. The initial direction may be chosen randomly as well; see for instance the case of the famous Goldstein-Kac telegraph process [5] and Section 4.1). We also suppose that $n \geq d+1$; the case where $n \leq d$ is evoked at the beginning of Section 3.

We introduce the points A_j, $1 \leq j \leq n$, defined by $\overrightarrow{\mathbf{O}A_j} = \overrightarrow{V_j}$. Let $\mathbf{v}_j = (v_{1j}, \ldots, v_{dj})$ be the coordinates of the point A_j (or, equivalently, the components of the vector $\overrightarrow{V_j}$). We also introduce some cyclic notations: for any $i \geq 0$ and $1 \leq j \leq n$, $D_{in+j} = D_j$, $A_{in+j} = A_j$, $\overrightarrow{V}_{in+j} = \overrightarrow{V_j}$ and $\mathbf{v}_{in+j} = \mathbf{v}_j$. We make the following assumption:

> Any ordered cyclic subset $\{A_j, \ldots, A_{j+d}\}$ of $d+1$ points within the set $\{A_1, \ldots, A_n\}$ form a d-dimensional polyhedron. By "d-dimensional", we mean that this set is not contained inside a hyperplane (*i.e.* an affine subvariety of dimension $d-1$) of $\mathbb{R}^d$. Actually, the polyhedron $A_j \cdots A_{j+d}$ is convex because $d+1$ points in dimension d always generate a convex polyhedron.

We call, for any $j \leq d+1$, $\mathcal{P}_j(t)$ the solid $(j-1)$-dimensional convex polyhedron $(tA_1) \cdots (tA_j)$, and $\mathcal{P}_n(t)$ the solid d-dimensional convex hull of the polyhedron $(tA_1) \cdots (tA_n)$; note that this last polyhedron may be convex or not. The sets $\mathcal{P}_j(t)$ and $\mathcal{P}_n(t)$ are analytically defined as

$$\mathcal{P}_j(t) = \left\{ \sum_{k=1}^{j} t_k A_k \text{ with } t_1, \ldots, t_j \geq 0 \text{ and } \sum_{k=1}^{j} t_k = t \right\}, \tag{1.1}$$

$$\mathcal{P}_n(t) = \left\{ \sum_{k=1}^{n} t_k A_k \text{ with } t_1, \ldots, t_n \geq 0 \text{ and } \sum_{k=1}^{n} t_k = t \right\}. \tag{1.2}$$

Write, for any $i \geq 0$ and $1 \leq j \leq n$,

$$T_i^{(j)} = T_{in+j} - T_{in+(j-1)} \quad \text{and} \quad S_i^{(j)} = \sum_{k=0}^{i} T_k^{(j)}$$

respectively for the time during which the particle evolves in the direction D_j for the $(i+1)$th time (the motion starts in the first cycle, which corresponds to $i = 0$), and for the total duration that the particle takes the direction D_j up to the $(i+1)$th cycle (the last one being included but possibly not completed). The random variables $T_i^{(j)}, i \geq 0$, are identically distributed random variables with a probability distribution depending only on the index j of the direction D_j and which is absolutely continuous with support $[0, +\infty)$.

Let $\overline{F}_{T_0^{(j)}}(t) = \mathbb{P}\{T_i^{(j)} > t\}$ be the survival function of $T_i^{(j)}$ (which does not depend on i) and $f_{S_i^{(j)}}(t) = \mathbb{P}\{S_i^{(j)} \in \mathrm{d}t\}/\mathrm{d}t$ be the probability density function (pdf) of $S_i^{(j)}$.

Let us denote by $X(t)$ and $\overrightarrow{V}(t)$ respectively the location and the speed-vector of the particle at time t. We have

$$\overrightarrow{V}(t) = \sum_{k=1}^{\infty} \mathbb{1}_{\{T_{k-1} \leq t < T_k\}} \overrightarrow{V}_k$$

and then, if $T_{r-1} \leq t < T_r$ (with $X(0) = \mathbf{O}$), the location of the particle at time t writes, noting that $\sum_{k=1}^{r-1}(T_k - T_{k-1}) = T_{r-1}$ (since $T_0 = 0$),

$$X(t) = X(0) + \int_0^t \overrightarrow{V}(s)\,\mathrm{d}s = \sum_{k=1}^{r-1}(T_k - T_{k-1})\mathbf{v}_k + (t - T_{r-1})\mathbf{v}_r = \sum_{k=1}^{r-1}(T_k - T_{k-1})(\mathbf{v}_k - \mathbf{v}_r) + t\mathbf{v}_r.$$

By considering all the possible cycles and putting $r = in + j$ (which corresponds to i complete cycles and j directions), this can be rewritten in terms of the n different directions as follows:

If $i = 0$, for $t \in [T_{j-1}, T_j)$,

$$X(t) = \sum_{k=1}^{j-1} T_0^{(k)}\mathbf{v}_k + (t - T_{j-1})\mathbf{v}_j \tag{1.3}$$

$$= \sum_{k=1}^{j-1} T_0^{(k)}(\mathbf{v}_k - \mathbf{v}_j) + t\mathbf{v}_j. \tag{1.4}$$

If $i \geq 1$, for $t \in [T_{in+j-1}, T_{in+j})$,

$$X(t) = \sum_{k=1}^{j-1} S_i^{(k)}\mathbf{v}_k + \sum_{k=j}^{n} S_{i-1}^{(k)}\mathbf{v}_k + (t - T_{in+j-1})\mathbf{v}_j. \tag{1.5}$$

For $j = 1$, it should be understood in (1.3), (1.4) and (1.5) that $\sum_{k=1}^{j-1} = 0$.

The main purpose of this paper is to describe completely the probability distribution of $X(t)$ by expressing it in terms of those of the durations $T_i^{(1)}, \ldots, T_i^{(n)}$ and $S_i^{(1)}, \ldots, S_i^{(j-1)}$, $S_{i-1}^{(j)}, \ldots, S_{i-1}^{(n)}$. For this, we consider the distribution of $X(t)$ subject to follow the j^{th} direction after having performed i complete cycles:

$$\begin{aligned} p_{ij}(\mathrm{d}\mathbf{x}; t) &= \mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x}, i \text{ complete cycles and } j \text{ directions}\} \\ &= \mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x}, T_{in+j-1} \leq t < T_{in+j}\}. \end{aligned}$$

Actually, the probability distribution of $X(t)$ is made up of a singular component and an absolutely continuous component. Indeed, from (1.2), (1.3) and (1.5), we can see that the particle is located inside the time expanding convex polyhedron $\mathcal{P}_n(t)$ (including its boundary). The particle will produce the singular component at the beginning of its travel, that is, more precisely, during the first cycle ($i = 0$) with the first d directions ($1 \leq j \leq d$). By (1.1) and (1.3), the support of this singular component is the hyper-face $(tA_1)\cdots(tA_d)$; this is the $(d-1)$-dimensional polyhedron $\mathcal{P}_{d-1}(t)$. The rest of the travel—the end of the first cycle ($i = 0$) with the $(n-d)$ last directions ($d+1 \leq j \leq n$), and the other cycles ($i \geq 1$)—produce the absolutely continuous part of the distribution; its support is the d-dimensional polyhedron $(tA_1)\cdots(tA_n)$, that is $\mathcal{P}_n(t)$.

The pdf of the absolutely continuous component of $X(t)$, $p(\mathbf{x}; t) = p(\mathrm{d}\mathbf{x}; t)/\mathrm{d}\mathbf{x}$, may be then derived by summing $p_{ij}(\mathbf{x}; t) = p_{ij}(\mathrm{d}\mathbf{x}; t)/\mathrm{d}\mathbf{x}$ with respect to the indices i and j:

$$p(\mathbf{x}; t) = \sum_{i=1}^{\infty}\sum_{j=1}^{n} p_{ij}(\mathbf{x}; t) + \sum_{j=d+1}^{n} p_{0j}(\mathbf{x}; t)$$

whereas the singular part is given by the sum $\sum_{j=1}^{d} p_{0j}(\mathrm{d}\mathbf{x}; t)$.

Our method relies on relations (1.3), (1.4) and (1.5) and consists of computing the joint distribution of the underlying durations $S_i^{(j)}$. This way has been efficiently used by Di Crescenzo [2,3] in the cases $d = 1$, $n = 2$ and $d = 2$, $n = 3$. Let us finally mention that a very close approach has also been used by Leorato and Orsingher [10], and by Lachal, Leorato and Orsingher [9] in the case of a minimal cyclic motion ($n = d + 1$)

with exponential switching times. In that particular case, the joint distribution of the $S_i^{(j)}$'s can be evaluated by invoking order statistics.

2. Application: a multi-files queueing system

Consider a queueing system constituted of n single parallel queues $Q_1, \ldots, Q_n$ and only one server. The server successively treats the queues $Q_1, Q_2, \ldots, Q_n, Q_1, Q_2, \ldots$ and so on in a cyclic order according to the following rule: for $1 \leq j \leq n-1$, the server treats ν_j customers waiting in queue Q_j, the duration $T^{(j)}$ of the service offered to each customer being exponentially distributed with rate λ_j, and then instantaneously passes to next queue Q_{j+1}. As a byproduct, the duration of the service offered to queue Q_j is distributed as Erlang's law of parameters ν_j and λ_j. After completing the service of queue Q_n, the server return to queue Q_1 and the process goes on similarly.

On the other hand, the authority of the system have to pay numerous charges: running costs, maintenance expenses, refund of loans, supplies, taxes, insurances... The authority can also place its profits into a diversified portfolio. Assume that there are d kinds of charge/profit which have $c_1, \ldots, c_d$ as costs/values per time unit. Consequently, the profit per time unit gained in queue Q_j must be divided into d parts according to different percentages; so, it is convenient to write this profit per time unit as a d-dimensional vector $(v_{1j}, \ldots, v_{dj})$ in $\mathbb{R}^d$. Precisely, component v_{ij} will be the fraction of the profit per time unit won at queue Q_j placed into the asset of value c_i if $v_{ij} > 0$, or devoted to paying charge c_i if $v_{ij} < 0$.

Globally, the quantity $X(t)$ represents the total amounts gained in the whole system intended to supply the portfolio and to refund the various debts at time t.

Other rules may be appended to the foregoing situation:

- the time spent for the server to pass from a queue to the next one may be random and would possibly generate a supplementary cost to the system. This is the so-called switchover time; see *e.g.* [1] for more details about this situation;
- the server may leave a queue only when it is empty and then pass to the next one. In this case, the duration of service $T^{(j)}$ is the busy time of the server in queue Q_j.

Let us point out that this last queueing model may be applied in reliability theory. Indeed, a machine can pass cyclically by several stages from well-working to the failure through different intermediate working states. The last state corresponds to the failure and the machine must be repaired before performing a new working cycle. Each state generates a net profit and/or a maintenance cost which can be modeled by the system depicted above.

3. Evaluation of the measure $p_{ij}(\mathrm{d}\mathbf{x}; t)$

In this section, we evaluate the measure $p_{ij}(\mathrm{d}\mathbf{x}; t)$. This study is essentially divided into two parts: the first one deals with the singular component of the distribution probability of $X(t)$ which is related to the first cycle ($i = 0$) with j directions, $1 \leq j \leq d$. The other part concerns the absolutely continuous component and corresponds to the case of i complete cycles plus j directions with $i \geq 1$ and $1 \leq j \leq n$, or $i = 0$ and $d + 1 \leq j \leq n$.

We suppose $n \geq d+2$. The case $n = d+1$ will be considered in Section 4 and the case $n \leq d$ gives raise only to a singular component which is similar to the case where $n \geq d+2$, $i = 0$ and $1 \leq j \leq d$.

We split this section into four parts: in Section 3.1, we give a preliminary representation for the measure $p_{ij}(\mathbf{x}; t)$; in Section 3.2, we achieve the computations of the singular component of $p_{ij}(\mathbf{x}; t)$ while in Sections 3.3 and 3.4, we achieve those of the absolutely continuous component of $p_{ij}(\mathbf{x}; t)$.

3.1. A preliminary representation for $p_{ij}(\mathrm{d}\mathbf{x};t)$

We begin this study by providing a first integral representation for $p_{ij}(\mathrm{d}\mathbf{x};t)$.

3.1.1. *The case $i=0$ and $1\le j\le n$*

For $i=0$ and $j=1$, we plainly have $X(t)=t\mathbf{v}_1$ for $t\in[0,T_1)$ and then the probability distribution of $X(t)$ is simply

$$p_{01}(\mathrm{d}\mathbf{x};t)=\mathbb{P}\{X(t)\in\mathrm{d}\mathbf{x},t<T_1\}=\mathbb{P}\{t\mathbf{v}_1\in\mathrm{d}\mathbf{x},t<T_1\}=\overline{F}_{T_0^{(1)}}(t)\,\delta(\mathrm{d}\mathbf{x}-t\mathbf{v}_1)$$

where δ denotes the usual Dirac distribution. For $i=0$ and $2\le j\le n$, the probability distribution of $X(t)$ writes, in view of (1.3) and by using the independence of the $T_0^{(k)}$'s, $1\le k\le j$,

$$\begin{aligned}
p_{0j}(\mathrm{d}\mathbf{x};t)&=\mathbb{P}\{X(t)\in\mathrm{d}\mathbf{x},T_{j-1}\le t<T_j\}\\
&=\mathbb{P}\left\{T_{j-1}\le t,\sum_{k=1}^{j-1}T_0^{(k)}(\mathbf{v}_k-\mathbf{v}_j)+t\mathbf{v}_j\in\mathrm{d}\mathbf{x},T_0^{(j)}>t-T_{j-1}\right\}\\
&=\int_0^t\mathbb{P}\left\{T_{j-1}\in\mathrm{d}s,\sum_{k=1}^{j-1}T_0^{(k)}(\mathbf{v}_k-\mathbf{v}_j)\in\mathrm{d}\mathbf{x}-t\mathbf{v}_j,T_0^{(j)}>t-s\right\}\\
&=\int_0^t\mathbb{P}\left\{\sum_{k=1}^{j-1}T_0^{(k)}(\mathbf{v}_k-\mathbf{v}_j)\in\mathrm{d}\mathbf{x}-t\mathbf{v}_j,\sum_{k=1}^{j-1}T_0^{(k)}\in\mathrm{d}s\right\}\mathbb{P}\left\{T_0^{(j)}>t-s\right\}\\
&=\int_0^t\overline{F}_{T_0^{(j)}}(t-s)\,\mathbb{P}\left\{\sum_{k=1}^{j-1}T_0^{(k)}(\mathbf{v}_k-\mathbf{v}_j)\in\mathrm{d}\mathbf{x}-t\mathbf{v}_j,\sum_{k=1}^{j-1}T_0^{(k)}\in\mathrm{d}s\right\}.
\end{aligned}\tag{3.1}$$

Similarly, using (1.4), the probability distribution of $X(t)$ can also be rewritten as

$$\begin{aligned}
p_{0j}(\mathrm{d}\mathbf{x};t)&=\mathbb{P}\left\{T_{j-1}\le t,\sum_{k=1}^{j-1}T_0^{(k)}\mathbf{v}_k+(t-T_{j-1})\mathbf{v}_j\in\mathrm{d}\mathbf{x},T_0^{(j)}>t-T_{j-1}\right\}\\
&=\int_0^t\mathbb{P}\left\{T_{j-1}\in\mathrm{d}s,\sum_{k=1}^{j-1}T_0^{(k)}\mathbf{v}_k\in\mathrm{d}\mathbf{x}-(t-s)\mathbf{v}_j,T_0^{(j)}>t-s\right\}\\
&=\int_0^t\overline{F}_{T_0^{(j)}}(t-s)\,\mathbb{P}\left\{\sum_{k=1}^{j-1}T_0^{(k)}\mathbf{v}_k\in\mathrm{d}\mathbf{x}-(t-s)\mathbf{v}_j,\sum_{k=1}^{j-1}T_0^{(k)}\in\mathrm{d}s\right\}.
\end{aligned}\tag{3.2}$$

In view of (3.1) and (3.2), we are led to introduce the linear maps $\varphi,\phi:\mathbb{R}^{j-1}\longrightarrow\mathbb{R}^{d+1}$ defined by

$$\varphi(t_1,\dots,t_{j-1})=\left(\sum_{k=1}^{j-1}(v_{1k}-v_{1j})t_k,\dots,\sum_{k=1}^{j-1}(v_{dk}-v_{dj})t_k,\sum_{k=1}^{j-1}t_k\right),\tag{3.3}$$

$$\phi(t_1,\dots,t_{j-1})=\left(\sum_{k=1}^{j-1}v_{1k}t_k,\dots,\sum_{k=1}^{j-1}v_{dk}t_k,\sum_{k=1}^{j-1}t_k\right).\tag{3.4}$$

We can rewrite (3.1) and (3.2) as

$$p_{0j}(\mathrm{d}\mathbf{x};t)=\int_0^t\overline{F}_{T_0^{(j)}}(t-s)\,\mathbb{P}\left\{\varphi\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)\in(\mathrm{d}\mathbf{x}-t\mathbf{v}_j)\,\mathrm{d}s\right\}\tag{3.5}$$

$$=\int_0^t\overline{F}_{T_0^{(j)}}(t-s)\,\mathbb{P}\left\{\phi\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)\in(\mathrm{d}\mathbf{x}-(t-s)\mathbf{v}_j)\,\mathrm{d}s\right\}.\tag{3.6}$$

For computing the probabilities lying in (3.5) and (3.6), we shall use an elementary result concerning the probability distribution of the linear image of a random vector; the details of that way are postponed to Appendix A.

We shall use either (3.5) or (3.6) according as $j \le d+1$ or $j \ge d+2$. The reason for this is that when $2 \le j \le d+1$, the map φ is injective whereas for $d+3 \le j \le n$, ϕ is surjective. When $j = d+2$, ϕ is bijective. In order the terminology be clear we recall the definitions of injective, surjective and bijective maps.

Definition 3.1. Let $f : E \longrightarrow F$ be a map. This map is said *injective* (resp. *surjective*, *bijective*) when for every point $y \in F$, there exists *at most* (resp. *at least*, *exactly*) one point $x \in E$ such that $y = f(x)$.

3.1.2. *The case* $i \ge 1$

Suppose now that $i \ge 1$. In order to reduce the sum $\sum_{k=1}^{j-1} S_i^{(k)}\mathbf{v}_k + \sum_{k=j}^{n} S_{i-1}^{(k)}\mathbf{v}_k$ lying in expression (1.5) to only one sum, we introduce the duration $\overline{S}_{ij}^{(k)}$ defined for $1 \le k \le n$ by

$$\overline{S}_{ij}^{(k)} = \begin{cases} S_i^{(k)} & \text{if } 1 \le k \le j-1, \\ S_{i-1}^{(k)} & \text{if } j \le k \le n. \end{cases} \tag{3.7}$$

In these settings, we simply have

$$\sum_{k=1}^{j-1} S_i^{(k)}\mathbf{v}_k + \sum_{k=j}^{n} S_{i-1}^{(k)}\mathbf{v}_k = \sum_{k=1}^{n} \overline{S}_{ij}^{(k)}\mathbf{v}_k.$$

Observe also that

$$T_{in+j-1} = \sum_{k=1}^{j-1} S_i^{(k)} + \sum_{k=j}^{n} S_{i-1}^{(k)} = \sum_{k=1}^{n} \overline{S}_{ij}^{(k)}.$$

Using $T_{in+j} = T_{in+j-1} + T_i^{(j)}$ and the fact that $T_i^{(j)}$ is independent of the $\overline{S}_{ij}^{(k)}$'s, $1 \le k \le n$, we have

$$\begin{aligned} p_{ij}(\mathrm{d}\mathbf{x};t) &= \mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x}, T_{in+j-1} \le t < T_{in+j}\} \\ &= \mathbb{P}\left\{ T_{in+j-1} \le t, \sum_{k=1}^{n} \overline{S}_{ij}^{(k)}\mathbf{v}_k + (t - T_{in+j-1})\mathbf{v}_j \in \mathrm{d}\mathbf{x}, T_i^{(j)} > t - T_{in+j-1} \right\} \\ &= \int_0^t \overline{F}_{T_i^{(j)}}(t-s)\,\mathbb{P}\left\{ \sum_{k=1}^{n} \overline{S}_{ij}^{(k)}\mathbf{v}_k \in \mathrm{d}\mathbf{x} - (t-s)\mathbf{v}_j, \sum_{k=1}^{n} \overline{S}_{ij}^{(k)} \in \mathrm{d}s \right\}. \end{aligned} \tag{3.8}$$

Let us introduce the linear map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{d+1}$ defined by

$$\psi(t_1, \ldots, t_n) = \left(\sum_{k=1}^{n} v_{1k}t_k, \ldots, \sum_{k=1}^{n} v_{dk}t_k, \sum_{k=1}^{n} t_k \right). \tag{3.9}$$

Formula (3.8) can be rewritten as

$$p_{ij}(\mathrm{d}\mathbf{x};t) = \int_0^t \overline{F}_{T_0^{(j)}}(s)\,\mathbb{P}\Big\{ \psi\Big(\overline{S}_{ij}^{(1)}, \ldots, \overline{S}_{ij}^{(n)}\Big) \in (\mathrm{d}\mathbf{x} - s\mathbf{v}_j)\,(t - \mathrm{d}s) \Big\}. \tag{3.10}$$

For computing the probability lying in integral (3.10), we shall use again the results of Appendix A and more specifically those of Appendix A.2 for the map ψ is surjective in this case.

3.2. **Singular component** ($i = 0$, $1 \le j \le d$)

3.2.1. *Some settings*

We decompose the linear map φ defined by (3.3), as described in Appendix A.1, into

$$\varphi(t_1,\dots,t_{j-1}) = (\varphi_1(t_1,\dots,t_{j-1}),\varphi_2(t_1,\dots,t_{j-1})).$$

The linear maps $\varphi_1 : \mathbb{R}^{j-1} \longrightarrow \mathbb{R}^{j-1}$ and $\varphi_2 : \mathbb{R}^{j-1} \longrightarrow \mathbb{R}^{d-j+2}$ (corresponding to the dimensions $p = j-1$ and $q = d+1$ of Appendix A.1) are defined as

$$\varphi_1(t_1,\dots,t_{j-1}) = \left(\sum_{k=1}^{j-1}(v_{1k}-v_{1j})t_k,\dots,\sum_{k=1}^{j-1}(v_{j-1\,k}-v_{j-1\,j})t_k\right),$$

$$\varphi_2(t_1,\dots,t_{j-1}) = \left(\sum_{k=1}^{j-1}(v_{jk}-v_{jj})t_k,\dots,\sum_{k=1}^{j-1}(v_{dk}-v_{dj})t_k,\sum_{k=1}^{j-1}t_k\right).$$

Set also

$$\tilde{\varphi}_2(t_1,\dots,t_{j-1}) = \left(\sum_{k=1}^{j-1}(v_{jk}-v_{jj})t_k,\dots,\sum_{k=1}^{j-1}(v_{dk}-v_{dj})t_k\right).$$

By hypothesis, the solid polyhedron $A_1\cdots A_j$ is not included in an affine subvariety of dimension $(j-1)$, so at least one of its projections on the $(j-1)$-dimensional spaces of coordinates has a non-vanishing $(j-1)$-dimensional volume. Let us suppose that, *e.g.*, this condition is fulfilled for the first projection $p : \mathbb{R}^d \longrightarrow \mathbb{R}^{j-1}$ defined by $p(x_1,\dots,x_d) = (x_1,\dots,x_{j-1})$. By means of a well-known formula, the oriented (positive or negative) volume of the $(j-1)$-polyhedron $p(A_1)\cdots p(A_j)$ is given by

$$\mathbf{V}_{j-1} = \mathrm{Vol}(p(A_1)\cdots p(A_j)) = \frac{1}{(j-1)!}\begin{vmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{vmatrix}.$$

We assume for instance that $\mathbf{V}_{j-1} > 0$. On the other hand, we remark that

$$\det\varphi_1 = \begin{vmatrix} v_{11}-v_{1j} & \dots & v_{1\,j-1}-v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1}-v_{j-1\,j} & \dots & v_{j-1\,j-1}-v_{j-1\,j} \end{vmatrix} = \begin{vmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{vmatrix} = (j-1)!\,\mathbf{V}_{j-1}.$$

So, the map φ_1 is bijective and the map φ is injective. Invoking Lemma A.1, the probability lying in (3.5) is given by

$$\begin{aligned}\mathbb{P}\left\{\varphi\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)\in(\mathrm{d}\mathbf{x}-t\mathbf{v}_j)\,\mathrm{d}s\right\} = {} & \frac{1}{(j-1)!\,\mathbf{V}_{j-1}}\,\delta\big((\mathrm{d}x_j-tv_{jj},\dots,\mathrm{d}x_d-tv_{dj},\mathrm{d}s) \\ & -\big(\varphi_2\circ\varphi_1^{-1}\big)(x_1-tv_{1j},\dots,x_{j-1}-tv_{j-1\,j})\big) \\ & \times f_{\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)}\left(\varphi_1^{-1}(x_1-tv_{1j},\dots,x_{j-1}-tv_{j-1\,j})\right)\mathrm{d}x_1\cdots\mathrm{d}x_{j-1},\end{aligned} \tag{3.11}$$

where $f_{\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)}$ stands for the joint pdf of the random variables $T_0^{(1)},\dots,T_0^{(j-1)}$. By independence, we have

$$f_{\left(T_0^{(1)},\dots,T_0^{(j-1)}\right)} = f_{T_0^{(1)}} \otimes \dots \otimes f_{T_0^{(j-1)}}.$$

The symbol $\otimes$ in the foregoing equality denotes the usual tensorial product of functions which is defined by

$$(f_1 \otimes \dots \otimes f_j)(x_1,\dots,x_j) = \prod_{k=1}^{j} f_k(x_k).$$

In (3.11), the variables $x_1,\dots,x_{j-1},s$ must obey the constraint that all the coordinates of $\varphi_1^{-1}(x_1 - sv_{1j},\dots, x_{j-1} - sv_{j-1\,j})$ are positive.

Let us introduce the coordinates $\tilde{\tau}_k^{(j)}(x_1,\dots,x_{j-1})$, $1 \le k \le j-1$, of the $(j-1)$-uple $\varphi_1^{-1}(x_1,\dots,x_{j-1})$, that is:

$$\varphi_1^{-1}(x_1,\dots,x_{j-1}) = \left(\tilde{\tau}_1^{(j)}(x_1,\dots,x_{j-1}),\dots,\tilde{\tau}_{j-1}^{(j)}(x_1,\dots,x_{j-1})\right). \tag{3.12}$$

We also introduce the function $\vartheta^{(j)}$ defined by

$$\vartheta^{(j)}(x_1,\dots,x_{j-1}) = \sum_{k=1}^{j-1} \tilde{\tau}_k^{(j)}(x_1,\dots,x_{j-1}) \tag{3.13}$$

so that

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)(x_1,\dots,x_{j-1}) = \left(\tilde{\varphi}_2\left(\tilde{\tau}_1^{(j)}(x_1,\dots,x_{j-1}),\dots,\tilde{\tau}_{j-1}^{(j)}(x_1,\dots,x_{j-1})\right),\vartheta^{(j)}(x_1,\dots,x_{j-1})\right). \tag{3.14}$$

Notice that the $\tilde{\tau}_k^{(j)}$'s and $\vartheta^{(j)}$ are temporal variables when the x_k's are spatial variables.

3.2.2. *Deriving the singular component*

With these settings at hand, we can write down the following equalities, in view to clarifying the Dirac measure lying in (3.11):

$$\begin{aligned}
&(x_j - tv_{jj},\dots,x_d - tv_{dj},s) - \left(\varphi_2 \circ \varphi_1^{-1}\right)(x_1 - tv_{1j},\dots,x_{j-1} - tv_{j-1\,j}) \\
&\quad = \left[(x_j,\dots,x_d,s) - \left(\varphi_2 \circ \varphi_1^{-1}\right)(x_1,\dots,x_{j-1})\right] - t\left[(v_{jj},\dots,v_{dj},0) - \left(\varphi_2 \circ \varphi_1^{-1}\right)(v_{1j},\dots,v_{j-1\,j})\right] \\
&\quad = (x_j,\dots,x_d,s) - \varphi_2\left(\tilde{\tau}_1^{(j)}(x_1,\dots,x_{j-1}),\dots,\tilde{\tau}_{j-1}^{(j)}(x_1,\dots,x_{j-1})\right) \\
&\qquad - t\left[(v_{jj},\dots,v_{dj},0) - \varphi_2\left(\tilde{\tau}_1^{(j)}(v_{1j},\dots,v_{j-1\,j}),\dots,\tilde{\tau}_{j-1}^{(j)}(v_{1j},\dots,v_{j-1\,j})\right)\right] \\
&\quad = \left(x_j - \theta_j^{(j)}(x_1,\dots,x_{j-1};t),\dots,x_d - \theta_d^{(j)}(x_1,\dots,x_{j-1};t), s - t + \tau_j^{(j)}(x_1,\dots,x_{j-1};t)\right)
\end{aligned}$$

where we set, in the last displayed equality, for $j \le k \le d$,

$$\begin{aligned}
\left(\theta_j^{(j)}(x_1,\dots,x_{j-1};t),\dots,\theta_d^{(j)}(x_1,\dots,x_{j-1};t)\right) &= \tilde{\varphi}_2\left(\tilde{\tau}_1^{(j)}(x_1,\dots,x_{j-1}),\dots,\tilde{\tau}_{j-1}^{(j)}(x_1,\dots,x_{j-1})\right) \\
&\quad + t\left[(v_{jj},\dots,v_{dj}) - \tilde{\varphi}_2\left(\tilde{\tau}_1^{(j)}(v_{1j},\dots,v_{j-1\,j}),\dots,\tilde{\tau}_{j-1}^{(j)}(v_{1j},\dots,v_{j-1\,j})\right)\right]
\end{aligned} \tag{3.15}$$

and

$$\tau_j^{(j)}(x_1,\dots,x_{j-1};t) = t\left[1 + \vartheta^{(j)}(v_{1j},\dots,v_{j-1\,j})\right] - \vartheta^{(j)}(x_1,\dots,x_{j-1}). \tag{3.16}$$

These computations lead to the following expression for the Dirac measure in (3.11):

$$\delta\big((\mathrm{d}x_j - tv_{jj}, \dots, \mathrm{d}x_d - tv_{dj}, \mathrm{d}s) - \big(\varphi_2 \circ \varphi_1^{-1}\big)(x_1 - tv_{1j}, \dots, x_{j-1} - tv_{j-1\,j})\big) \\ = \delta\Big(\mathrm{d}s - t + \tau_j^{(j)}(x_1, \dots, x_{j-1}; t)\Big) \times \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1, \dots, x_{j-1}; t)\Big). \quad (3.17)$$

On the other hand, for simplifying the arguments of the densities $f_{T_0^{(k)}}$ in (3.11), it is convenient to introduce the following notation: for $1 \le k \le j-1$,

$$\tau_k^{(j)}(x_1, \dots, x_{j-1}; t) = \tilde{\tau}_k^{(j)}(x_1, \dots, x_{j-1}) - t\, \tilde{\tau}_k^{(j)}(v_{1j}, \dots, v_{j-1\,j}). \quad (3.18)$$

Indeed, the argument in $f_{T_0^{(k)}}$ is

$$\varphi_1^{-1}(x_1 - tv_{1j}, \dots, x_{j-1} - tv_{j-1\,j}) = \varphi_1^{-1}(x_1, \dots, x_{j-1}) - t\, \varphi_1^{-1}(v_{1j}, \dots, v_{j-1\,j}) \\ = (\tau_1^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \tau_{j-1}^{(j)}(x_1, \dots, x_{j-1}; t)). \quad (3.19)$$

Remark 3.2. We observe, using successively (3.18), (3.13) and (3.16), that

$$\sum_{k=1}^{j} \tau_k^{(j)}(x_1, \dots, x_{j-1}; t) = \sum_{k=1}^{j-1} \tau_k^{(j)}(x_1, \dots, x_{j-1}; t) + \tau_j^{(j)}(x_1, \dots, x_{j-1}; t) \\ = \tau_j^{(j)}(x_1, \dots, x_{j-1}; t) + \sum_{k=1}^{j-1} \tilde{\tau}_k^{(j)}(x_1, \dots, x_{j-1}) - t \sum_{k=1}^{j-1} \tilde{\tau}_k^{(j)}(v_{1j}, \dots, v_{j-1\,j}) \\ = \tau_j^{(j)}(x_1, \dots, x_{j-1}; t) + \vartheta^{(j)}(x_1, \dots, x_{j-1}) - t\, \vartheta^{(j)}(v_{1j}, \dots, v_{j-1\,j}) \;=\; t.$$

As a byproduct, we have proved the following relation which will be used later:

$$\sum_{k=1}^{j} \tau_k^{(j)}(x_1, \dots, x_{j-1}; t) = t. \quad (3.20)$$

Remark 3.3. The $\tau_k^{(j)}(x_1, \dots, x_{j-1}; t)$'s, $1 \le k \le j$, which are defined successively by (3.16), (3.18), (3.12) and (3.13), and which are related to the inverse of the map φ_1 with matrix

$$\mathbf{A}_1 = \begin{pmatrix} v_{11} - v_{1j} & \dots & v_{1\,j-1} - v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} - v_{j-1\,j} & \dots & v_{j-1\,j-1} - v_{j-1\,j} \end{pmatrix},$$

can be more directly obtained by inverting the map ϕ_1 whose matrix is

$$\begin{pmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{pmatrix}.$$

The linear maps φ_1 and ϕ_1 are linked by the following relation:

$$\begin{aligned}\phi_1(t_1,\dots,t_j) &= \left(\sum_{k=1}^{j} v_{1k}t_k,\dots,\sum_{k=1}^{j} v_{j-1\,k}t_k,\sum_{k=1}^{j} t_k\right)\\ &= \left(\sum_{k=1}^{j-1}(v_{1k}-v_{1j})t_k,\dots,\sum_{k=1}^{j-1}(v_{j-1\,k}-v_{j-1\,j})t_k,0\right)+\left(\sum_{k=1}^{j} t_k\right)(v_{1j},\dots,v_{j-1\,j},1)\\ &= (\varphi_1(t_1,\dots,t_{j-1}),0)+\left(\sum_{k=1}^{j} t_k\right)(v_{1j},\dots,v_{j-1\,j},1).\end{aligned} \tag{3.21}$$

We claim that the $\tau_k^{(j)}(x_1,\dots,x_{j-1};t)$'s, $1\le k\le j$, are also characterized by the simple relation

$$\phi_1^{-1}(x_1,\dots,x_{j-1},t)=\left(\tau_1^{(j)}(x_1,\dots,x_{j-1};t),\dots,\tau_j^{(j)}(x_1,\dots,x_{j-1};t)\right). \tag{3.22}$$

To prove this, we compute, using (3.21),

$$\begin{aligned}\phi_1\left(\tau_1^{(j)}(x_1,\dots,x_{j-1};t),\dots,\tau_j^{(j)}(x_1,\dots,x_{j-1};t)\right) &= \left(\varphi_1\left(\tau_1^{(j)}(x_1,\dots,x_{j-1};t),\dots,\tau_{j-1}^{(j)}(x_1,\dots,x_{j-1};t)\right),0\right)\\ &\quad+\left(\sum_{k=1}^{j}\tau_k^{(j)}(x_1,\dots,x_{j-1};t)\right)(v_{1j},\dots,v_{j-1\,j},1).\end{aligned} \tag{3.23}$$

But we have

$$\begin{aligned}&\varphi_1\left(\tau_1^{(j)}(x_1,\dots,x_{j-1};t),\dots,\tau_{j-1}^{(j)}(x_1,\dots,x_{j-1};t)\right)\\ &\quad=\varphi_1\left(\tilde\tau_1^{(j)}(x_1,\dots,x_{j-1})-t\,\tilde\tau_1^{(j)}(v_{1j},\dots,v_{j-1\,j}),\dots,\ \tilde\tau_{j-1}^{(j)}(x_1,\dots,x_{j-1})-t\,\tilde\tau_{j-1}^{(j)}(v_{1j},\dots,v_{j-1\,j})\right)\\ &\quad=\varphi_1\left(\tilde\tau_1^{(j)}(x_1,\dots,x_{j-1}),\dots,\tilde\tau_{j-1}^{(j)}(x_1,\dots,x_{j-1})\right)-\,t\,\varphi_1\left(\tilde\tau_1^{(j)}(v_{1j},\dots,v_{j-1\,j}),\dots,\tilde\tau_{j-1}^{(j)}(v_{1j},\dots,v_{j-1\,j})\right)\\ &\quad=(x_1,\dots,x_{j-1})-t\,(v_{1j},\dots,v_{j-1\,j}).\end{aligned} \tag{3.24}$$

In the last above equality, we used the reciprocal formula (3.12). Now, putting (3.24) into (3.23) and using (3.20) yield

$$\begin{aligned}\phi_1\left(\tau_1^{(j)}(x_1,\dots,x_{j-1};t),\dots,\tau_j^{(j)}(x_1,\dots,x_{j-1};t)\right)&=(x_1,\dots,x_{j-1},0)-t\,(v_{1j},\dots,v_{j-1\,j},0)+t\,(v_{1j},\dots,v_{j-1\,j},1)\\ &=(x_1,\dots,x_{j-1},t)\end{aligned}$$

which proves (3.22).

Remark 3.4. In the same spirit as in the previous remark, the $\theta_k^{(j)}(x_1,\dots,x_{j-1};t)$'s, $j\le k\le d$, which are defined by (3.15) and (3.14), and which are related to the map $\tilde\varphi_2\circ\varphi_1^{-1}$ with matrix

$$\mathbf{A}_1=\begin{pmatrix}v_{j1}-v_{jj} & \dots & v_{j\,j-1}-v_{jj}\\ \vdots & & \vdots\\ v_{d1}-v_{dj} & \dots & v_{d\,j-1}-v_{dj}\end{pmatrix}\times\begin{pmatrix}v_{11}-v_{1j} & \dots & v_{1\,j-1}-v_{1j}\\ \vdots & & \vdots\\ v_{j-1\,1}-v_{j-1\,j} & \dots & v_{j-1\,j-1}-v_{j-1\,j}\end{pmatrix}^{-1},$$

can be more directly obtained with the aid of the map $\phi_2 \circ \phi_1^{-1}$ whose matrix is

$$\begin{pmatrix} v_{j1} & \dots & v_{jj} \\ \vdots & & \vdots \\ v_{d1} & \dots & v_{dj} \end{pmatrix} \times \begin{pmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{pmatrix}^{-1}.$$

We just have introduced new linear maps $\phi_1 : \mathbb{R}^j \longrightarrow \mathbb{R}^j$ and $\phi_2 : \mathbb{R}^j \longrightarrow \mathbb{R}^{d-j+1}$ defined by

$$\phi_1(t_1, \dots, t_j) = \left(\sum_{k=1}^{j} v_{1k} t_k, \dots, \sum_{k=1}^{j} v_{j-1\,k} t_k, \sum_{k=1}^{j} t_k \right),$$
$$\phi_2(t_1, \dots, t_j) = \left(\sum_{k=1}^{j} v_{jk} t_k, \dots, \sum_{k=1}^{j} v_{dk} t_k \right).$$

The linear maps $\tilde{\varphi}_2$ and ϕ_2 are linked by the following relation:

$$\begin{aligned} \phi_2(t_1, \dots, t_j) &= \left(\sum_{k=1}^{j-1} (v_{jk} - v_{jj}) t_k, \dots, \sum_{k=1}^{j-1} (v_{dk} - v_{dj}) t_k \right) + \left(\sum_{k=1}^{j} t_k \right) (v_{jj}, \dots, v_{dj}) \\ &= \tilde{\varphi}_2(t_1, \dots, t_{j-1}) + \left(\sum_{k=1}^{j} t_k \right) (v_{jj}, \dots, v_{dj}). \end{aligned} \tag{3.25}$$

We claim that the $\theta_k^{(j)}(x_1, \dots, x_{j-1}; t)$'s, $1 \le k \le j$, are also characterized by the simple relation

$$\left(\phi_2 \circ \phi_1^{-1} \right)(x_1, \dots, x_{j-1}, t) = \left(\theta_j^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \theta_d^{(j)}(x_1, \dots, x_{j-1}; t) \right). \tag{3.26}$$

To prove (3.26), we compute, using successively (3.22), (3.25), (3.20), (3.18) and (3.15),

$$\begin{aligned} \left(\phi_2 \circ \phi_1^{-1} \right)(x_1, \dots, x_{j-1}; t) &= \phi_2\Big(\tau_1^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \tau_j^{(j)}(x_1, \dots, x_{j-1}; t) \Big) \\ &= \tilde{\varphi}_2\Big(\tau_1^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \tau_{j-1}^{(j)}(x_1, \dots, x_{j-1}; t) \Big) \\ &\quad + \left(\sum_{k=1}^{j} \tau_k^{(j)}(x_1, \dots, x_{j-1}; t) \right) (v_{jj}, \dots, v_{dj}) \\ &= \tilde{\varphi}_2\Big(\tilde{\tau}_1^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \tilde{\tau}_{j-1}^{(j)}(x_1, \dots, x_{j-1}; t) \Big) \\ &\quad - t\, \tilde{\varphi}_2\Big(\tilde{\tau}_1^{(j)}(v_{1j}, \dots, v_{j-1\,j}; t), \dots, \tilde{\tau}_{j-1}^{(j)}(v_{1j}, \dots, v_{j-1\,j}; t) \Big) + t\,(v_{jj}, \dots, v_{dj}) \\ &= \Big(\theta_j^{(j)}(x_1, \dots, x_{j-1}; t), \dots, \theta_d^{(j)}(x_1, \dots, x_{j-1}; t) \Big). \end{aligned}$$

The $\theta_k^{(j)}(x_1, \dots, x_{j-1}; t)$'s may be explicitly written as (see Appendix A.1.2)

$$\theta_k^{(j)}(x_1, \dots, x_{j-1}; t) = \frac{1}{(j-1)!\, \mathbf{V}_{j-1}} \left(\sum_{l=1}^{j-1} \Delta_{kl} x_l + \Delta_{kj} t \right)$$

with

$$\Delta_{kl} = \begin{vmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{l-1\,1} & \dots & v_{l-1\,j} \\ v_{k1} & \dots & v_{kj} \\ v_{l+1\,1} & \dots & v_{l+1\,j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{vmatrix} \text{ if } 1 \le l \le j-1, \quad \Delta_{kj} = \begin{vmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ v_{k1} & \dots & v_{kj} \end{vmatrix} \text{ if } l = j.$$

Plugging now (3.17) and (3.19) into (3.11), we obtain

$$\mathbb{P}\Big\{\varphi\Big(T_0^{(1)},\dots,T_0^{(j-1)}\Big) \in (\mathrm{d}\mathbf{x} - t\mathbf{v}_j)\,\mathrm{d}s\Big\} = \frac{1}{(j-1)!\,\mathbf{V}_{j-1}} \prod_{k=1}^{j-1} f_{T_0^{(k)}}\Big(\tau_k^{(j)}(x_1,\dots,x_{j-1};t)\Big)\,\mathrm{d}x_k$$
$$\times\, \delta\Big(\mathrm{d}s - t + \tau_j^{(j)}(x_1,\dots,x_{j-1};t)\Big) \times \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1,\dots,x_{j-1};t)\Big). \quad (3.27)$$

Finally, multiplying (3.27) by $\overline{F}_{T_0^{(j)}}(t-s)$ and next integrating with respect to time s on $(0,t)$, we easily derive expression (3.28) for $p_{0j}(\mathrm{d}\mathbf{x};t)$ in Theorem 3.5 below. Equality (3.20) entails that if $\tau_k^{(j)}(x_1,\dots,x_{j-1};t) \ge 0$ for $1 \le k \le j$, then we have $\tau_j^{(j)}(x_1,\dots,x_{j-1};t) \in [0,t]$. This remark justifies the fact that the integration of (3.27) with respect to s on $[0,t]$ provides the term $\overline{F}_{T_0^{(j)}}(\tau_j^{(j)}(x_1,\dots,x_{j-1};t))$ in (3.28).

On the other hand, some constraints on the variables $x_1,\dots,x_{j-1}$ must be added in the measure $p_{0j}(\mathrm{d}\mathbf{x};t)$. They stipulate that the arguments lying in the functions $f_{T_0^{(1)}},\dots,f_{T_0^{(j-1)}}$ must be positive and that the argument lying in $\overline{F}_{T_0^{(j)}}$ must be in $[0,t]$. We specify this point after the statement of the theorem.

Theorem 3.5. *The singular component of the distribution of $X(t)$ is given by the family of measures $(p_{0j}(\mathrm{d}\mathbf{x};t))_{1\le j\le d}$ defined by $p_{01}(\mathrm{d}\mathbf{x};t) = \overline{F}_{T_0^{(1)}}(t)\,\delta(\mathrm{d}\mathbf{x} - t\mathbf{v}_1)$ and for $2 \le j \le d$:*

$$p_{0j}(\mathrm{d}\mathbf{x};t) = \frac{1}{(j-1)!\,\mathbf{V}_{j-1}} \mathbb{1}_{\mathcal{D}_j(t)}(x_1,\dots,x_{j-1})\,\overline{F}_{T_0^{(j)}}\Big(\tau_j^{(j)}(x_1,\dots,x_{j-1};t)\Big)$$
$$\times \prod_{k=1}^{j-1} f_{T_0^{(k)}}\Big(\tau_k^{(j)}(x_1,\dots,x_{j-1};t)\Big)\,\mathrm{d}x_k \times \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1,\dots,x_{j-1};t)\Big) \quad (3.28)$$

where the $\tau_k^{(j)}(x_1,\dots,x_{j-1};t)$'s and the $\theta_k^{(j)}(x_1,\dots,x_{j-1};t)$'s are respectively defined by the matricial relations

$$\begin{pmatrix} \tau_1^{(j)}(x_1,\dots,x_{j-1};t) \\ \vdots \\ \tau_j^{(j)}(x_1,\dots,x_{j-1};t) \end{pmatrix} = \begin{pmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_{j-1} \\ t \end{pmatrix},$$
$$\begin{pmatrix} \theta_j^{(j)}(x_1,\dots,x_{j-1};t) \\ \vdots \\ \theta_d^{(j)}(x_1,\dots,x_{j-1};t) \end{pmatrix} = \begin{pmatrix} v_{j1} & \dots & v_{jj} \\ \vdots & & \vdots \\ v_{d1} & \dots & v_{dj} \end{pmatrix} \begin{pmatrix} \tau_1^{(j)}(x_1,\dots,x_{j-1};t) \\ \vdots \\ \tau_j^{(j)}(x_1,\dots,x_{j-1};t) \end{pmatrix}$$

and

$$\mathbf{V}_{j-1} = \frac{1}{(j-1)!} \begin{vmatrix} v_{11} & \dots & v_{1j} \\ \vdots & & \vdots \\ v_{j-1\,1} & \dots & v_{j-1\,j} \\ 1 & \dots & 1 \end{vmatrix},$$

$$\mathcal{D}_j(t) = \{(x_1, \dots, x_{j-1}) \in \mathbb{R}^{j-1} : \tau_k^{(j)}(x_1, \dots, x_{j-1}; t) \geq 0 \text{ for } 1 \leq k \leq j\}.$$

Remark 3.6. Expression (3.28) is made up of two terms: the factor

$$\frac{1}{(j-1)!\,\mathbf{V}_{j-1}} \overline{F}_{T_0^{(j)}}\Big(\tau_j^{(j)}(x_1, \dots, x_{j-1}; t)\Big) \times \prod_{k=1}^{j-1} f_{T_0^{(k)}}\Big(\tau_k^{(j)}(x_1, \dots, x_{j-1}; t)\Big)\, \mathrm{d}x_k$$

is related to the location of the particle at time t while the factor

$$\mathbb{1}_{\mathcal{D}_j(t)}(x_1, \dots, x_{j-1}) \times \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1, \dots, x_{j-1}; t)\Big)$$

refers to the support of the measure $p_{0j}(\mathrm{d}\mathbf{x}; t)$ as it will be seen in the next subsubsection.

3.2.3. *The support of* $p_{0j}(\mathrm{d}\mathbf{x}; t)$

In view of (3.28), it emerges that the support of the measure $p_{0j}(\mathrm{d}\mathbf{x}; t)$ is the intersection of the set of constraints $\mathcal{D}_j(t)$ and the affine subvariety $\mathcal{H}_j$ of dimension $(j-1)$ defined by the equations

$$x_k = \theta_k^{(j)}(x_1, \dots, x_{j-1}; t), \quad j \leq k \leq d. \tag{3.29}$$

Our aim now is to describe this support in a simpler manner. See also Lemma A.3.

Introducing some parameters $s_1, \dots, s_j$ defined by $(s_1, \dots, s_j) = \phi_1^{-1}(x_1, \dots, x_{j-1}, t)$, the implicit representation (3.29) of $\mathcal{H}_j$ can be rewritten, by (3.26), as

$$\begin{aligned} (x_j, \dots, x_d) &= \big(\phi_2 \circ \phi_1^{-1}\big)(x_1, \dots, x_{j-1}, t) \;=\; \phi_2(s_1, \dots, s_j) \\ &= \left(\sum_{l=1}^{j} v_{jl} s_l, \dots, \sum_{l=1}^{j} v_{dl} s_l\right). \end{aligned} \tag{3.30}$$

Concerning the variables $x_1, \dots, x_{j-1}$, we reciprocally have $(x_1, \dots, x_{j-1}, t) = \phi_1(s_1, \dots, s_j)$, which implies

$$(x_1, \dots, x_{j-1}) = \left(\sum_{l=1}^{j} v_{1l} s_l, \dots, \sum_{l=1}^{j} v_{j-1\,l} s_l\right). \tag{3.31}$$

Putting (3.30) and (3.31) together gives

$$(x_1, \dots, x_d) = \left(\sum_{l=1}^{j} v_{1l} s_l, \dots, \sum_{l=1}^{j} v_{dl} s_l\right).$$

We therefore obtain the concise parametric representation for $\mathcal{H}_j$

$$x_k = \sum_{l=1}^{j} v_{kl} s_l \quad \text{for } 1 \leq k \leq d \text{ with } \sum_{l=1}^{j} s_l = t.$$

In this form, we recognize for $\mathcal{H}_j$ the set of barycentric combinations $\sum_{l=1}^{j} s_l A_l$ with real coefficients $s_1, \dots, s_j$ satisfying $\sum_{l=1}^{j} s_l = t$, so $\mathcal{H}_j$ is the affine subvariety containing the points $(tA_1), \dots, (tA_j)$.

On the other hand, the constraints $\tau_k^{(j)}(x_1, \dots, x_{j-1}; t) \geq 0$ for $1 \leq k \leq j$ are equivalent, because of (3.20), to the constraints for the s_k's: $s_k \geq 0$ for $1 \leq k \leq j$ and $\sum_{k=1}^{j} s_k = t$. Consequently, the support of the measure $p_{0j}(\mathrm{d}\mathbf{x}; t)$, which is $\tilde{\mathcal{D}}_j(t) \cap \mathcal{H}_j$ where

$$\tilde{\mathcal{D}}_j(t) = \{\mathbf{x} \in \mathbb{R}^d : (x_1, \dots, x_{j-1}) \in \mathcal{D}_j(t)\},$$

is also the set of points $\sum_{k=1}^{j} s_k A_k$ with *positive* coefficients $s_1, \dots, s_j$ satisfying the condition $\sum_{k=1}^{j} s_k = t$. It is nothing but the convex hull $\mathcal{P}_j(t)$ of the set $\{tA_1, \dots, tA_j\}$.

Theorem 3.7. *The support of the measure $p_{0j}(\mathrm{d}\mathbf{x}; t)$ is the solid $(j-1)$-dimensional polyhedron $\mathcal{P}_j(t)$.*

Example 3.8. For instance, for $j = 2$, the matrices of ϕ_1, ϕ_2 and $\phi_2 \circ \phi_1^{-1}$ are respectively

$$\begin{pmatrix} v_{11} & v_{12} \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} v_{21} & v_{22} \\ \vdots & \vdots \\ v_{d1} & v_{d2} \end{pmatrix} \quad \text{and} \quad \frac{1}{v_{11} - v_{21}} \begin{pmatrix} v_{21} - v_{22} & v_{11}v_{22} - v_{21}v_{12} \\ \vdots & \vdots \\ v_{d1} - v_{d2} & v_{11}v_{d2} - v_{d1}v_{12} \end{pmatrix}.$$

So, the settings read

$$\mathbf{V}_1 = v_{11} - v_{12}, \quad \tau_1^{(2)}(x_1; t) = \frac{x_1 - v_{12}t}{v_{11} - v_{12}}, \quad \tau_2^{(2)}(x_1; t) = \frac{v_{11}t - x_1}{v_{11} - v_{12}},$$

$$\theta_k^{(2)}(x_1; t) = \frac{(v_{k1} - v_{k2})x_1 + (v_{11}v_{k2} - v_{12}v_{k1})t}{v_{11} - v_{12}} \text{ for } 2 \leq k \leq d,$$

$$\mathcal{D}_2(t) = \left\{x_1 \in \mathbb{R} : \frac{v_{11}t - x_1}{v_{11} - v_{12}} \geq 0, \frac{x_1 - v_{12}t}{v_{11} - v_{12}} \geq 0\right\}.$$

Formula (3.28) writes

$$p_{02}(\mathrm{d}\mathbf{x}; t) = \frac{1}{v_{11} - v_{12}} \mathbb{1}_{\mathcal{D}_2(t)}(x_1) f_{T_0^{(1)}}\left(\frac{x_1 - v_{12}t}{v_{11} - v_{12}}\right) \overline{F}_{T_0^{(2)}}\left(\frac{v_{11}t - x_1}{v_{11} - v_{12}}\right)$$
$$\times \mathrm{d}x_1 \prod_{k=2}^{d} \delta\left(\mathrm{d}x_k - \frac{(v_{k1} - v_{k2})x_1 + (v_{11}v_{k2} - v_{12}v_{k1})t}{v_{11} - v_{12}}\right).$$

The measure $p_{02}(\mathrm{d}\mathbf{x}; t)$ is carried by the segment $(tA_1)(tA_2)$.

Remark 3.9. When times $T_0^{(k)}$ are exponentially distributed with parameter λ, formula (3.28) can be simplified into

$$p_{0j}(\mathrm{d}\mathbf{x}; t) = \frac{\lambda^{j-1}}{(j-1)!\,\mathbf{V}_{j-1}} \mathbb{1}_{\mathcal{D}_j(t)}(x_1, \dots, x_{j-1}) \exp\left[-\lambda\left(\sum_{k=1}^{j} \tau_k^{(j)}(x_1, \dots, x_{j-1}; t)\right)\right]$$
$$\times \prod_{k=1}^{j-1} \mathrm{d}x_k \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1, \dots, x_{j-1}; t)\Big).$$

The sum lying within the above exponential equals t, see (3.20). Hence,

$$p_{0j}(\mathrm{d}\mathbf{x}; t) = \frac{\lambda^{j-1}\,\mathrm{e}^{-\lambda t}}{(j-1)!\,\mathbf{V}_{j-1}} \mathbb{1}_{\mathcal{D}_j(t)}(x_1, \dots, x_{j-1}) \prod_{k=1}^{j-1} \mathrm{d}x_k \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1, \dots, x_{j-1}; t)\Big). \tag{3.32}$$

The total mass of this measure is

$$\int_{\mathcal{P}_j(t)} p_{0j}(\mathrm{d}\mathbf{x};t) = \frac{\lambda^{j-1}\,\mathrm{e}^{-\lambda t}}{(j-1)!\,\mathbf{V}_{j-1}} \int_{\mathcal{D}_j(t)} \mathrm{d}x_1 \cdots \mathrm{d}x_{j-1}.$$

Because of the relation $\mathcal{P}_j(t) = \tilde{\mathcal{D}}_j(t) \cap \mathcal{H}_j$, we see that the integral $\int_{\mathcal{D}_j(t)} \mathrm{d}x_1 \cdots \mathrm{d}x_{j-1}$ is nothing but the $(j-1)$-volume of the projection on $\mathbb{R}^{j-1}$ of the face $(tA_1)\cdots(tA_j)$:

$$\int_{\mathcal{D}_j(t)} \mathrm{d}x_1 \cdots \mathrm{d}x_{j-1} = \mathbf{V}_{j-1}\, t^{j-1}.$$

Therefore,

$$\int_{\mathcal{P}_j(t)} p_{0j}(\mathrm{d}\mathbf{x};t) = \frac{(\lambda t)^{j-1}}{(j-1)!}\,\mathrm{e}^{-\lambda t} = \mathbb{P}\{T_{j-1} \le t < T_j\}, \tag{3.33}$$

the last above equality coming from the fact that the exponential times are related to Poissonian occurrences. This relation is in good agreement with the definition of the density $p_{0j}(\mathbf{x};t)$. Thus, dividing (3.32) by (3.33) provides

$$\mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x} \,|\, T_{j-1} \le t \le T_j\} = \frac{1}{\mathbf{V}_{j-1}\, t^{j-1}}\, 1\!\!1_{\mathcal{D}_j(t)}(x_1,\dots,x_{j-1}) \prod_{k=1}^{j-1} \mathrm{d}x_k \prod_{k=j}^{d} \delta\Big(\mathrm{d}x_k - \theta_k^{(j)}(x_1,\dots,x_{j-1};t)\Big).$$

In words, the location $X(t)$ of the particle conditioned on moving with the speed-vector $\overrightarrow{V_j}$ for the first time $(i=0)$ is uniformly distributed on the $(j-1)$-dimensional polyhedron $\mathcal{P}_j(t)$.

3.3. Absolutely continuous component: the case $i \ge 1$, $n \ge d+2$

In this subsection, we assume that $i \ge 1$. We put, for having homogeneous settings, $v_{d+1\,1} = \cdots = v_{d+1\,n} = 1$.

3.3.1. *Deriving the pdf*

With the linear map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{d+1}$ defined by (3.9), we associate the maps $\psi_1 : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1}$ and $\psi_2 : \mathbb{R}^{n-d-1} \longrightarrow \mathbb{R}^{d+1}$ defined by, as described in Appendix A.2 (this case is related to the dimensions $p = n$ and $q = d+1$),

$$\psi_1(t_1,\dots,t_{d+1}) = \left(\sum_{k=1}^{d+1} v_{1k} t_k, \dots, \sum_{k=1}^{d+1} v_{d+1\,k} t_k\right),$$

$$\psi_2(t_{d+2},\dots,t_n) = \left(\sum_{k=d+2}^{n} v_{1k} t_k, \dots, \sum_{k=d+2}^{n} v_{d+1\,k} t_k\right).$$

In this part, it is convenient to work with matrices. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the matrices of ψ_1 and ψ_2:

$$\mathbf{B}_1 = \begin{pmatrix} v_{11} & \dots & v_{1\,d+1} \\ \vdots & & \vdots \\ v_{d+1\,1} & \dots & v_{d+1\,d+1} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} v_{1\,d+2} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{d+1\,d+2} & \dots & v_{d+1\,n} \end{pmatrix}.$$

Put $\Delta_d = \det \mathbf{B}_1$. The determinant Δ_d does not vanish (see Remark 3.10 below), thus ψ_1 is bijective and ψ is surjective. Therefore, the conditions for applying Lemma A.4 are fulfilled. Set also

$$w_{pq} = \frac{(-1)^{p+q}}{\Delta_d} \begin{vmatrix} v_{11} & \dots & v_{1\,p-1} & v_{1\,p+1} & \dots & v_{1\,d+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_{q-1\,1} & \dots & v_{q-1\,p-1} & v_{q-1\,p+1} & \dots & v_{q-1\,d+1} \\ v_{q+1\,1} & \dots & v_{q+1\,p-1} & v_{q+1\,p+1} & \dots & v_{q+1\,d+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_{d+1\,1} & \dots & v_{d+1\,p-1} & v_{d+1\,p+1} & \dots & v_{d+1\,d+1} \end{vmatrix}$$

and

$$\Delta_{pq} = \Delta_d \sum_{k=1}^{d+1} w_{pk} v_{kq} \quad \text{for } 1 \le p \le d+1,\ d+2 \le q \le n.$$

The matrices of ψ_1^{-1} and $\psi_1^{-1} \circ \psi_2$ are given by

$$\mathbf{B}_1^{-1} = \begin{pmatrix} w_{11} & \dots & w_{1\,d+1} \\ \vdots & & \vdots \\ w_{d+1\,1} & \dots & w_{d+1\,d+1} \end{pmatrix}, \quad \mathbf{B}_1^{-1}\mathbf{B}_2 = \frac{1}{\Delta_d} \begin{pmatrix} \Delta_{1\,d+2} & \dots & \Delta_{1n} \\ \vdots & & \vdots \\ \Delta_{d+1\,d+2} & \dots & \Delta_{d+1\,n} \end{pmatrix}.$$

Remark 3.10. Actually, the Δ_{pq}'s are defined also for $1 \le q \le d+1$: in this case, we have $\Delta_{pq}/\Delta_d = \delta_{pq}$ since the matrix of the (Δ_{pj}/Δ_d)'s, $1 \le p, q \le d+1$, is nothing but the matrix $\mathbf{B}_1^{-1}\mathbf{B}_1 = I$. The quantities Δ_d and Δ_{pq}'s can be interpreted by means of volumes. Indeed, as already mentioned in Section 3.2.1, it is well-known that, if $\mathbf{V}_d = \mathrm{Vol}(A_1 \cdots A_{d+1})$ denotes the oriented volume of the polyhedron $A_1 \cdots A_{d+1}$, then

$$\begin{vmatrix} v_{11} & \dots & v_{1\,d+1} \\ \vdots & & \vdots \\ v_{d1} & \dots & v_{d\,d+1} \\ 1 & \dots & 1 \end{vmatrix} = d!\,\mathbf{V}_d.$$

Therefore $\Delta_d = d!\,\mathbf{V}_d$. On the other hand, by hypothesis, the polyhedron $A_1 \cdots A_{d+1}$ is not contained in any hyperplane. So, its volume does not vanish. We assume for simplicity that $\mathbf{V}_d > 0$. Similarly, if $\mathbf{V}_{pqd} = \mathrm{Vol}(A_1 \cdots A_{p-1} A_q A_{p+1} \cdots A_{d+1})$ denotes the oriented (positive or negative) volume of the polyhedron deduced from $A_1 \cdots A_{d+1}$ by replacing the point A_p by A_q, we have, by (A.1),

$$\Delta_{pq} = \begin{vmatrix} v_{11} & \dots & v_{1\,p-1} & v_{1q} & v_{1\,p+1} & \dots & v_{1\,d+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{d1} & \dots & v_{d\,p-1} & v_{dq} & v_{d\,p+1} & \dots & v_{d\,d+1} \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{vmatrix} = d!\,\mathbf{V}_{pqd}.$$

Remark 3.11. We can note that the sum $\sum_{p=1}^{d+1} w_{pk}$ vanishes if $k \le d$ and equals 1 if $k = d+1$. Indeed, if $A = (a_{ij})_{1\le i,j\le m}$ is any matrix, $A^T = (a_{ji})_{1\le i,j\le m}$ is its transposed matrix and $A' = (A_{ij})_{1\le i,j\le m}$ is the complementary matrix of A, *i.e.* the matrix of the co-factors of A, one has the well-known relation $AA'^T = (\det A)I$ which writes $\sum_{k=1}^m a_{ik}A_{jk} = (\det A)\delta_{ij}$. If the last row of A is made up of 1, that is $\forall k \in \{1, \dots, m\}$, $a_{mk} = 1$, then $\sum_{k=1}^m A_{jk} = (\det A)\delta_{mj} = \begin{cases} 0 & \text{if } j \le m-1 \\ \det A & \text{if } j = m \end{cases}$. In our study, $(w_{ij})_{1\le i,j\le d+1}$ is the inverse matrix of $\mathbf{B}_1$ which coincides with $\Delta_d^{-1}\mathbf{B}_1'^T$, the last row of $\mathbf{B}_1$ being made up of 1. Hence $\sum_{p=1}^{d+1} w_{pk} = \delta_{d+1\,j}$. We then deduce that

$$\sum_{p=1}^{d+1} \frac{\mathbf{V}_{pkd}}{\mathbf{V}_d} = \sum_{p=1}^{d+1} \left(\sum_{q=1}^{d+1} w_{pq} v_{qk} \right) = \sum_{q=1}^{d+1} \left(\sum_{p=1}^{d+1} w_{pq} \right) v_{qk} = v_{d+1\,k} = 1,$$

or, equivalently,

$$\sum_{p=1}^{d+1} \mathbf{V}_{pkd} = \mathbf{V}_d \quad \text{for } 1 \le k \le d+1. \tag{3.34}$$

We now introduce the coordinates $\tau_k(\mathbf{x};t)$, $1 \le k \le d+1$, of the $(d+1)$-uple $\psi_1^{-1}(\mathbf{x},t)$:

$$\psi_1^{-1}(x_1,\ldots,x_d,t) = \psi_1^{-1}(\mathbf{x};t) = (\tau_1(\mathbf{x};t),\ldots,\tau_{d+1}(\mathbf{x};t))$$

with, referring to the form of $\mathbf{B}_1^{-1}$,

$$\tau_p(\mathbf{x};t) = \sum_{k=1}^{d} w_{pk} x_k + w_{p\,d+1} t \quad \text{for } 1 \le p \le d+1.$$

For evaluating (3.10), we use Lemma A.4. For this, we need to specify the coordinates of the point $\psi_1^{-1}(\mathbf{x} - s\mathbf{v}_j; t-s)$. Observing that

$$\tau_p(\mathbf{v}_j;1) = \sum_{k=1}^{d+1} w_{pk} v_{kj} = \frac{\Delta_{pj}}{\Delta_d} = \frac{\mathbf{V}_{pjd}}{\mathbf{V}_d}, \tag{3.35}$$

the coordinates $\tau_p(\mathbf{x} - s\mathbf{v}_j; t-s)$ are given by

$$\tau_p(\mathbf{x} - s\mathbf{v}_j; t-s) = \tau_p(\mathbf{x};t) - s\,\tau_p(\mathbf{v}_j;1) = \tau_p(\mathbf{x};t) - \frac{\mathbf{V}_{pjd}}{\mathbf{V}_d}\, s.$$

We also need to specify $\psi_1^{-1} \circ \psi_2$: thanks to the form of $\mathbf{B}_1^{-1}\mathbf{B}_2$,

$$\left(\psi_1^{-1} \circ \psi_2\right)(s_{d+2},\ldots,s_n) = \left(\sum_{k=d+2}^{n} \frac{\mathbf{V}_{1kd}}{\mathbf{V}_d}\, s_k, \ldots, \sum_{k=d+2}^{n} \frac{\mathbf{V}_{d+1\,kd}}{\mathbf{V}_d}\, s_k \right).$$

Applying Lemma A.4 to (3.10), we derive for the density $p_{ij}(\mathbf{x};t) = p_{ij}(\mathrm{d}\mathbf{x};t)/\mathrm{d}\mathbf{x}$ the integral representation displayed in Theorem 3.12 below. Referring to Appendix A.2.3, we see that some constraints must be appended to this representation. For taking into account these constraints, we introduce the set

$$\mathcal{D}_j(\mathbf{x};s,t) = \left\{ (s_{d+2},\ldots,s_n) \in (0,+\infty)^{n-d-1} : \sum_{k=d+2}^{n} \frac{\mathbf{V}_{pkd}}{\mathbf{V}_d}\, s_k \le \tau_p(\mathbf{x};t) - \frac{\mathbf{V}_{pjd}}{\mathbf{V}_d}\, s \quad \text{for } 1 \le p \le d+1 \right\}$$

together with the set $\{s \in [0,t] : \mathcal{D}_j(\mathbf{x};s,t) \neq \emptyset\}$ which will be proven later to be an interval $[\varsigma_j(\mathbf{x};t), \sigma_j(\mathbf{x};t)]$.

Theorem 3.12. *If $i \ge 1$ and $1 \le j \le n$, the density $p_{ij}(\mathbf{x};t)$ is given, for $\mathbf{x} \in \mathcal{P}_n(t)$, by*

$$p_{ij}(\mathbf{x};t) = \frac{1}{d!\,\mathbf{V}_d} \int_{\varsigma_j(\mathbf{x};t)}^{\sigma_j(\mathbf{x};t)} \overline{F}_{T_0^{(j)}}(s)\,\mathrm{d}s \int_{\mathcal{D}_j(\mathbf{x};s,t)} \prod_{k=1}^{d+1} f_{\overline{S}_{ij}^{(k)}}\left(\tau_k(\mathbf{x};t) - \frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\, s - \sum_{l=d+2}^{n} \frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\, s_l \right) \times \prod_{k=d+2}^{n} \left(f_{\overline{S}_{ij}^{(k)}}(s_k)\,\mathrm{d}s_k \right) \tag{3.36}$$

where $\mathbf{V}_d = \mathrm{Vol}(A_1 \cdots A_{d+1})$*,* $\mathbf{V}_{kld} = \mathrm{Vol}(A_1 \cdots A_{k-1} A_l A_{k+1} \cdots A_{d+1})$ *and the* $\tau_k(\mathbf{x};t)$*'s are given by the matricial relation*

$$\begin{pmatrix} \tau_1(\mathbf{x};t) \\ \vdots \\ \tau_{d+1}(\mathbf{x};t) \end{pmatrix} = \begin{pmatrix} v_{11} & \dots & v_{1\,d+1} \\ \vdots & & \vdots \\ v_{d1} & \dots & v_{d\,d+1} \\ 1 & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ t \end{pmatrix}.$$

Remark 3.13. By definition, the $\tau_p(\mathbf{x};t)$'s can be viewed as the expansion of a determinant:

$$\tau_p(\mathbf{x};t) = \frac{1}{\Delta_d} \begin{vmatrix} v_{11} & \dots & v_{1\,p-1} & x_1 & v_{1\,p+1} & \dots & v_{1\,d+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{d1} & \dots & v_{d\,p-1} & x_d & v_{d\,p+1} & \dots & v_{d\,d+1} \\ 1 & \dots & 1 & t & 1 & \dots & 1 \end{vmatrix}$$

$$= \frac{1}{\Delta_d} \begin{vmatrix} v_{11} - v_{1\,d+1} & \dots & v_{1\,p-1} - v_{1\,d+1} & x_1 - t v_{1\,d+1} & v_{1\,p+1} - v_{1\,d+1} & \dots & v_{1d} - v_{1\,d+1} & v_{1\,d+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ v_{d1} - v_{d\,d+1} & \dots & v_{d\,p-1} - v_{d\,d+1} & x_d - t v_{d\,d+1} & v_{d\,p+1} - v_{d\,d+1} & \dots & v_{1d} - v_{d\,d+1} & v_{d\,d+1} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{\Delta_d} \det\left(\overrightarrow{A_{d+1}A_1}, \dots, \overrightarrow{A_{d+1}A_{p-1}}, \mathbf{x} - tA_{d+1}, \overrightarrow{A_{d+1}A_{p+1}}, \dots, \overrightarrow{A_{d+1}A_d} \right).$$

The determinant lying in the last displayed equation is $d!$ times the volume $V_{pd}(\mathbf{x};t)$ of the polyhedron $A_1 \cdots A_{p-1}(t^{-1}M)A_{p+1} \cdots A_{d+1}$ where M is the point with coordinates $\mathbf{x}$. Then $\tau_p(\mathbf{x};t) = \mathbf{V}_{pd}(\mathbf{x};t)/\mathbf{V}_d$. On the other hand, the relation $\tau_p(\mathbf{x};t) = 0$ is nothing but the equation of the affine hyperplane containing the points $(tA_1), \dots, (tA_{p-1}), (tA_{p+1}), \dots, (tA_{d+1})$.

Remark 3.14. Let us compute the sum of the $\tau_p(\mathbf{x};t)$'s:

$$\sum_{p=1}^{d+1} \tau_p(\mathbf{x};t) = \sum_{p=1}^{d+1} \left(\sum_{k=1}^{d} w_{pk} x_k + w_{p\,d+1} t \right) = \sum_{k=1}^{d} \left(\sum_{p=1}^{d+1} w_{pk} \right) x_k + \left(\sum_{p=1}^{d+1} w_{p\,d+1} \right) t.$$

By Remark 3.11, the sum $\sum_{p=1}^{d+1} w_{pk}$ vanishes if $k \le d$ and equals 1 if $k = d+1$. Consequently, the following relation which is the analogous of (3.20) holds:

$$\sum_{p=1}^{d+1} \tau_p(\mathbf{x};t) = t. \tag{3.37}$$

Remark 3.15. For the similar probabilities related to a particle subject to taking the r^{th} direction at time 0,

$$p_{ij}^{(r)}(\mathbf{x};t) = \mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x}, i \text{ complete cycles and } j \text{ directions} \,|\, \overrightarrow{V}(0) = \overrightarrow{V_r}\}/\mathrm{d}\mathbf{x},$$

the quantities $\tau_p(\mathbf{x};t)$'s should be replaced by

$$\tau_p^{(r)}(\mathbf{x};t) = \sum_{k=1}^{d} w_{pk}^{(r)} x_k + w_{p\,d+1}^{(r)} t \quad \text{for } r \le p \le r+d,$$

where $(w_{ij}^{(r)})_{1\le i,j\le d+1}$ is the inverse of the matrix of the $(d+1)$-dimensional vectors $\left(\begin{pmatrix}\overrightarrow{V}_r\\1\end{pmatrix},\begin{pmatrix}\overrightarrow{V}_{r+1}\\1\end{pmatrix},\dots,\begin{pmatrix}\overrightarrow{V}_{r+d}\\1\end{pmatrix}\right)$. The relations $\tau_p^{(r)}(\mathbf{x};t)=0$, $r\le p\le r+d$, are the equations of the affine hyperplanes containing the points $A_r,A_{r+1},\dots,A_{p-1},A_{p+1},\dots,A_{r+d}$. Similarly, $\mathbf{V}_d$ and the $\mathbf{V}_{pqd}$'s should be replaced by the volumes $\mathbf{V}_d^{(r)}=\mathrm{Vol}(A_r\cdots A_{r+d})$ and $\mathbf{V}_{pqd}^{(r)}=\mathrm{Vol}(A_r\cdots A_{p-1}A_qA_{p+1}\cdots A_{r+d})$.

3.3.2. *Two families of formulas for the pdf*

Relation (3.36) recovers two families of formulas.

- **Case** $1\le j\le d+1$**:**
  Referring to (3.7) and observing that $\mathbf{V}_{kjd}=0$ if $k\ne j$ (for $1\le j,k\le d+1$) and $\mathbf{V}_{jjd}=\mathbf{V}_d$, formula (3.36) takes the form

$$p_{ij}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\int_{\varsigma_j(\mathbf{x};t)}^{\sigma_j(\mathbf{x};t)}\overline{F}_{T_0^{(j)}}(s)\,\mathrm{d}s\int_{\mathcal{D}_j(\mathbf{x};s,t)}\prod_{k=1}^{j-1}f_{S_i^{(k)}}\left(\tau_k(\mathbf{x};t)-\sum_{l=d+2}^{n}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\right)$$
$$\times\prod_{k=j+1}^{d+1}f_{S_{i-1}^{(k)}}\left(\tau_k(\mathbf{x};t)-\sum_{l=d+2}^{n}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\right)$$
$$\times f_{S_{i-1}^{(j)}}\left(\tau_j(\mathbf{x};t)-s-\sum_{l=d+2}^{n}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\right)\prod_{k=d+2}^{n}\left(f_{S_{i-1}^{(k)}}(s_k)\,\mathrm{d}s_k\right).$$

  In the above formula (and also below), we adopt the convention $\prod_{k=\alpha}^{\beta}=1$ if $\alpha>\beta$. For instance, $\prod_{k=1}^{j-1}=1$ when $j=1$ and $\prod_{k=j+1}^{d+1}=1$ when $j=d+1$.
- **Case** $d+2\le j\le n$**:**
  Formula (3.36) reads in this case

$$p_{ij}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\int_{\varsigma_j(\mathbf{x};t)}^{\sigma_j(\mathbf{x};t)}\overline{F}_{T_0^{(j)}}(s)\,\mathrm{d}s\int_{\mathcal{D}_j(\mathbf{x};s,t)}\prod_{k=1}^{d+1}f_{S_i^{(k)}}\left(\tau_k(\mathbf{x};t)-\frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\,s-\sum_{l=d+2}^{n}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\right)$$
$$\times\left(\prod_{k=d+2}^{j-1}f_{S_i^{(k)}}(s_k)\prod_{k=j}^{n}f_{S_{i-1}^{(k)}}(s_k)\right)\mathrm{d}s_{d+2}\cdots\mathrm{d}s_n.$$

3.3.3. *The set of integration and the support of the density in (3.36)*

Thanks to (3.34) and (3.37), summing the inequalities $\sum_{k=d+2}^{n}\frac{\mathbf{V}_{pkd}}{\mathbf{V}_d}\,s_k\le\tau_p(\mathbf{x};t)-\frac{\mathbf{V}_{pjd}}{\mathbf{V}_d}\,s$ with respect to the index p yields the inequality $\sum_{k=d+2}^{n}s_k\le t-s$. Therefore, the set $\mathcal{D}_j(\mathbf{x};s,t)$ is convex, compact and is included in the simplex

$$\left\{(s_{d+2},\dots,s_n)\in(0,+\infty)^{n-d-1}:\sum_{k=d+2}^{n}s_k\le t-s\right\}.$$

Moreover, the set $\{s\in[0,t]:\mathcal{D}_j(\mathbf{x};s,t)\ne\emptyset\}$ is a convex, compact subset of $[0,t]$, so it is an interval: $[\varsigma_j(\mathbf{x};t),\sigma_j(\mathbf{x};t)]$.

If $j\le d+1$, since $\mathbf{V}_{pjd}=0$ for $1\le p\le d+1$ and $p\ne j$, and $\mathbf{V}_{jjd}=\mathbf{V}_d$, the conditions lying in the set $\mathcal{D}_j(\mathbf{x};s,t)$ write

$$\sum_{k=d+2}^{n}\frac{\mathbf{V}_{pkd}}{\mathbf{V}_d}\,s_k\le\tau_p(\mathbf{x};t)\quad\text{for }1\le p\le d+1,p\ne j,\qquad\sum_{k=d+2}^{n}\frac{\mathbf{V}_{jkd}}{\mathbf{V}_d}\,s_k\le\tau_j(\mathbf{x};t)-s.$$

We then have

$$\mathcal{D}_j(\mathbf{x};0,t) = \left\{(s_{d+2},\dots,s_n)\in(0,+\infty)^{n-d-1} : \sum_{k=d+2}^{n}\frac{\mathbf{V}_{pkd}}{\mathbf{V}_d}\,s_k \le \tau_p(\mathbf{x};t) \quad \text{for } 1\le p\le d+1\right\}$$

and

$$\mathcal{D}_j(\mathbf{x};\tau_j(\mathbf{x};t),t) = \Bigg\{(s_{d+2},\dots,s_n)\in(0,+\infty)^{n-d-1} : \\ \sum_{k=d+2}^{n}\frac{\mathbf{V}_{pkd}}{\mathbf{V}_d}\,s_k \le \tau_p(\mathbf{x};t) \quad \text{for } 1\le p\le d+1 \text{ and } p\neq j,\ \sum_{k=d+2}^{n}\frac{\mathbf{V}_{jkd}}{\mathbf{V}_d}\,s_k\le 0\Bigg\}.$$

For $\mathbf{x}\in\mathcal{P}_{d+1}(t)$, we have $\tau_p(\mathbf{x};t)\ge 0$ when $1\le p\le d+1$. So, the sets $\mathcal{D}_j(\mathbf{x};0,t)$ and $\mathcal{D}_j(\mathbf{x};\tau_j(\mathbf{x};t),t)$ contain the $(n-d-1)$-uple $(0,\dots,0)$, they are non-empty. This entails the inclusion $[0,\tau_j(\mathbf{x};t)]\subset[\varsigma_j(\mathbf{x};t),\sigma_j(\mathbf{x};t)]$, or, equivalently, $\varsigma_j(\mathbf{x};t)=0$ and $\tau_j(\mathbf{x};t)\le\sigma_j(\mathbf{x};t)\le t$. If, in addition, we make the assumption that the polyhedron $A_1\cdots A_n$ is convex (and then coincides with $\mathcal{P}_n(t)$), all the volumes $\mathbf{V}_{jkd}$, $1\le j\le d+1$, $d+1\le k\le n$, are positive (with the reference $\mathbf{V}_d>0$). In this case, if $s>\tau_j(\mathbf{x};t)$, the condition $\sum_{k=d+2}^{n}\frac{\mathbf{V}_{jkd}}{\mathbf{V}_d}\,s_k\le\tau_j(\mathbf{x};t)-s$ is not fulfilled (for $s_{d+1},\dots,s_n>0$), then $\mathcal{D}_j(\mathbf{x};s,t)=\emptyset$. As a result, $\sigma_j(\mathbf{x};t)=\tau_j(\mathbf{x};t)$.

We now consider the problem of the support of the density (3.36). Fix $\eta>0$ and $M=\sum_{k=1}^{n}s_kA_k$, with $s_1,\dots,s_n\ge 0$ and $\sum_{k=1}^{n}s_k=t$, a point in the interior of $\mathcal{P}_n(t)$. Let us introduce the set $V(M,\eta)$ of the points $P\in\mathcal{P}_n(t)$ for which there exists positive coefficients (not necessarily unique) $t_1,\dots,t_n$ summing to t such that $P=\sum_{k=1}^{n}t_kA_k$ and $|t_k-s_k|\le\eta$ for all $k\neq j$. For any $P\in V(M,\eta)$, since $\sum_{k=1}^{n}(t_k-s_k)=0$, we have that

$$\overrightarrow{MP}=\sum_{k=1}^{n}(t_k-s_k)A_k=\sum_{\substack{1\le k\le n\\ k\neq j}}(t_k-s_k)\overrightarrow{A_jA_k}$$

and then

$$\|\overrightarrow{MP}\|\le\sum_{\substack{1\le k\le n\\ k\neq j}}|t_k-s_k|\times\|\overrightarrow{A_jA_k}\|\le(n-1)\eta\,\mathrm{diam}(\mathcal{P}_n(t)) \tag{3.38}$$

where $\mathrm{diam}(\mathcal{P}_n(t))$ stands for the diameter of $\mathcal{P}_n(t)$. Indeed, for all indices k,l: $\|\overrightarrow{A_kA_l}\|\le\mathrm{diam}(\mathcal{P}_n(t))$.

Fix now $\varepsilon>0$ and choose $\eta=\frac{\varepsilon}{(n-1)\,\mathrm{diam}(\mathcal{P}_n(t))}$. Inequality (3.38) simply reads $\|\overrightarrow{MP}\|\le\varepsilon$ for any $P\in V(M,\eta)$, hence proving the inclusion $V(M,\eta)\subset B(M,\varepsilon)$ where $B(M,\varepsilon)$ is the ball with center M and radius ε. So, we firstly get

$$\mathbb{P}\{X(t)\in V(M,\eta),T_{in+j-1}\le t\le T_{in+j}\}\le\mathbb{P}\{X(t)\in B(M,\varepsilon),T_{in+j-1}\le t\le T_{in+j}\}. \tag{3.39}$$

Secondly, we have by the definition of $V(M,\eta)$ and (1.5)

$$\mathbb{P}\{X(t)\in V(M,\eta),T_{in+j-1}\le t\le T_{in+j}\}\ge\mathbb{P}\Big\{\overline{S}_{ij}^{(1)}\in[s_1-\eta,s_1+\eta],\dots,\overline{S}_{ij}^{(j-1)}\in[s_{i-1}-\eta,s_{i-1}+\eta], \\ \overline{S}_{ij}^{(j+1)}\in[s_{i+1}-\eta,s_{i+1}+\eta],\dots,\overline{S}_{ij}^{(n)}\in[s_n-\eta,s_n+\eta], \\ T_{in+j-1}\le t\le T_{in+j}\Big\}. \tag{3.40}$$

It can be easily checked that the distribution of the vector $\left(\overline{S}_{ij}^{(1)},\dots,\overline{S}_{ij}^{(j-1)},\overline{S}_{ij}^{(j+1)},\dots,\ \overline{S}_{ij}^{(n)}\right)$ admits for pdf

$$\begin{aligned}\mathbb{P}\Big\{\left(\overline{S}_{ij}^{(1)},\dots,\overline{S}_{ij}^{(j-1)},\overline{S}_{ij}^{(j+1)},\dots,\overline{S}_{ij}^{(n)}\right)\in \mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n,\\ T_{in+j-1}\le t\le T_{in+j}\Big\}/\,\mathrm{d}t_1\cdots\mathrm{d}t_{j-1}\mathrm{d}t_{j+1}\cdots\mathrm{d}t_n\\ =\int_{t_j=0}^{t-\sum_{\substack{1\le k\le n\\ k\ne j}}t_k}\overline{F}_{T_0^{(n)}}\left(t-\sum_{k=1}^n t_k\right)\left(\prod_{k=1}^n f_{\overline{S}_{ij}^{(k)}}(t_k)\right)\mathrm{d}t_j>0.\end{aligned}$$

As a result, we deduce by (3.39) and (3.40) that

$$\mathbb{P}\{X(t)\in B(M,\varepsilon),T_{in+j-1}\le t\le T_{in+j}\}>0.$$

In words, the probability of lying in a neighborhood of M for $X(t)$ does not vanish. Therefore each point $M\in\mathcal{P}_n(t)$ is in the support of the distribution of $X(t)$ subject to $T_{in+j-1}\le t\le T_{in+j}$, and then the support of the measure $p_{ij}(\mathrm{d}\mathbf{x};t)$ is the whole polyhedron $\mathcal{P}_n(t)$.

3.4. Absolutely continuous component: the case $i=0$, $d+1\le j\le n$

We now pay attention to the case $i=0$, $d+1\le j\le n$. Recall that the underlying linear maps $\varphi,\phi:\mathbb{R}^{j-1}\longrightarrow\mathbb{R}^{d+1}$ are defined by (3.3) and (3.4):

$$\begin{aligned}\varphi(t_1,\dots,t_{j-1})&=\left(\sum_{k=1}^{j-1}(v_{1k}-v_{1j})t_k,\dots,\sum_{k=1}^{j-1}(v_{dk}-v_{dj})t_k,\sum_{k=1}^{j-1}t_k\right),\\ \phi(t_1,\dots,t_{j-1})&=\left(\sum_{k=1}^{j-1}v_{1k}t_k,\dots,\sum_{k=1}^{j-1}v_{dk}t_k,\sum_{k=1}^{j-1}t_k\right).\end{aligned}$$

We split this study into three parts: the case $d+3\le j\le n$ for which ϕ is surjective, the case $j=d+2$ for which ϕ is bijective and the case $j=d+1$ for which φ is injective.

3.4.1. *The case $d+3\le j\le n$*

The map ϕ_1 associated with ϕ, defined by

$$\phi_1(t_1,\dots,t_{d+1})=\left(\sum_{k=1}^{d+1}v_{1k}t_k,\dots,\sum_{k=1}^{d+1}v_{dk}t_k,\sum_{k=1}^{d+1}t_k\right),$$

is bijective. Introduce the coordinates $\tau_k(\mathbf{x};t)$, $1\le k\le d+1$, of $\phi_1^{-1}(\mathbf{x};t)$, that is: $\phi_1^{-1}(\mathbf{x};t)=(\tau_1(\mathbf{x};t),\dots,\tau_{d+1}(\mathbf{x};t))$. Applying (3.6) and Lemma A.4 yields the expression of the density $p_{0j}(\mathbf{x};t)$ below.

Theorem 3.16. *If $d+3\le j\le n$, we have, for $\mathbf{x}\in\mathcal{P}_j(t)$,*

$$\begin{aligned}p_{0j}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\int_{\varsigma^{(j)}(\mathbf{x};t)}^{\sigma^{(j)}(\mathbf{x};t)}\overline{F}_{T_0^{(j)}}(s)\,\mathrm{d}s\int_{\mathcal{D}^{(j)}(\mathbf{x};s,t)}\prod_{k=1}^{d+1}f_{T_0^{(k)}}\left(\tau_k(\mathbf{x};t)-\frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\,s-\sum_{l=d+2}^{j-1}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\right)\\ \times\left(\prod_{k=d+2}^{j-1}f_{T_0^{(k)}}(s_k)\right)\mathrm{d}s_{d+2}\cdots\mathrm{d}s_{j-1}\end{aligned}\tag{3.41}$$

where

$$\mathcal{D}^{(j)}(\mathbf{x};s,t)=\left\{(s_{d+2},\dots,s_{j-1})\in(0,+\infty)^{j-d-2}:\sum_{l=d+2}^{j-1}\frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\,s_l\le\tau_k(\mathbf{x};t)-\frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\,s\quad\textit{for } 1\le k\le d+1\right\}$$

and

$$\left\{s\in[0,t]:\mathcal{D}^{(j)}(\mathbf{x};s,t)\neq\emptyset\right\}=\left[\varsigma^{(j)}(\mathbf{x};t),\sigma^{(j)}(\mathbf{x};t)\right].$$

Note that the difference between expressions (3.41) and (3.36) lies in the number of variables s_k's and in the last product which ends up at index n when $i\ge 1$ and at index $j-1$ when $i=0$.

3.4.2. *The case* $j=d+2$

For $j=d+2$, the linear map $\phi:\mathbb{R}^{d+1}\longrightarrow\mathbb{R}^{d+1}$, defined by

$$\phi(t_1,\dots,t_{d+1})=\left(\sum_{k=1}^{d+1}v_{1k}t_k,\dots,\sum_{k=1}^{d+1}v_{dk}t_k,\sum_{k=1}^{d+1}t_k\right),$$

is bijective. Put $\phi^{-1}(\mathbf{x};t)=(\tau_1(\mathbf{x};t),\dots,\tau_{d+1}(\mathbf{x};t))$. In these settings, we have

$$\begin{aligned}\phi^{-1}(\mathbf{x}-s\mathbf{v}_{d+2};t-s)&=\phi^{-1}(\mathbf{x};t)-s\phi^{-1}(\mathbf{v}_{d+2};1)\\&=(\tau_1(\mathbf{x};t)-s\,\tau_1(\mathbf{v}_{d+2};1),\dots,\tau_{d+1}(\mathbf{x};t)-s\,\tau_{d+1}(\mathbf{v}_{d+2};1)).\end{aligned}$$

By (3.35), we have

$$\tau_k(\mathbf{v}_{d+2};1)=\frac{\mathbf{V}_{k\,d+2\,d}}{\mathbf{V}_d}\cdot$$

The following expression for $p_{0\,d+2}(\mathbf{x};t)$ emerges from this together with a straightforward change of variables in (3.6).

Theorem 3.17. *If $j=d+2$, we have, for $\mathbf{x}\in\mathcal{P}_{d+2}(t)$,*

$$p_{0\,d+2}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\int_{\varsigma^{(d+2)}(\mathbf{x};t)}^{\sigma^{(d+2)}(\mathbf{x};t)}\overline{F}_{T_0^{(d+2)}}(s)\,\mathrm{d}s\prod_{k=1}^{d+1}f_{T_0^{(k)}}\left(\tau_k(\mathbf{x};t)-\frac{\mathbf{V}_{k\,d+2\,d}}{\mathbf{V}_d}\,s\right).\tag{3.42}$$

In (3.42), the integration must be carried out on the s's for which one has $\tau_k(\mathbf{x};t)-\frac{\mathbf{V}_{k\,d+2\,d}}{\mathbf{V}_d}\,s\ge 0$ for $1\le k\le d+1$. Introducing the subsets I^+ and I^-, say, of indices k for which respectively $\mathbf{V}_{k\,d+2\,d}>0$ and $\mathbf{V}_{k\,d+2\,d}<0$ and using the notations of Remark 3.13, we see that

$$\varsigma^{(d+2)}(\mathbf{x};t)=\max\left(\max_{k\in I^-}\frac{\mathbf{V}_{kd}(\mathbf{x};t)}{\mathbf{V}_{k\,d+2\,d}},0\right)\quad\text{and}\quad\sigma^{(d+2)}(\mathbf{x};t)=\min_{k\in I^+}\frac{\mathbf{V}_{kd}(\mathbf{x};t)}{\mathbf{V}_{k\,d+2\,d}}\cdot$$

3.4.3. *The case* $j=d+1$

For $j=d+1$, the computations are quite similar to those of Section 3.2.2. Theorem 3.5 may apply in this case with very slight modifications: the sole difference is that the measure $p_{0\,d+1}(\mathrm{d}\mathbf{x};t)$ is absolutely continuous. Set

$$\begin{pmatrix}\tau_1^{(d+1)}(\mathbf{x};t)\\\vdots\\\tau_{d+1}^{(d+1)}(\mathbf{x};t)\end{pmatrix}=\begin{pmatrix}v_{11}&\dots&v_{1\,d+1}\\\vdots&&\vdots\\v_{d1}&\dots&v_{d\,d+1}\\1&\dots&1\end{pmatrix}^{-1}\begin{pmatrix}x_1\\\vdots\\x_d\\t\end{pmatrix}.$$

The set of constraints $\mathcal{D}_{d+1}(t) = \{\mathbf{x} \in \mathbb{R}^d : \tau_k^{(d+1)}(\mathbf{x};t) \geq 0 \text{ for } 1 \leq k \leq d\}$ is nothing but the hyper-hedron $(tA_1)\cdots(tA_{d+1})$: $\mathcal{D}_{d+1}(t) = \mathcal{P}_{d+1}(t)$.

Theorem 3.18. *If $j = d+1$, the density $p_{0\,d+1}(\mathbf{x};t)$ writes, for $x \in \mathcal{P}_{d+1}(t)$,*

$$p_{0\,d+1}(\mathbf{x};t) = \frac{1}{d!\,\mathbf{V}_d}\,\overline{F}_{T_0^{(d+1)}}\Big(\tau_{d+1}^{(d+1)}(\mathbf{x};t)\Big)\prod_{k=1}^{d} f_{T_0^{(k)}}\Big(\tau_k^{(d+1)}(\mathbf{x};t)\Big). \tag{3.43}$$

3.4.4. *Exponential switching times with equal parameters*

We assume here that $i = 0$, $d+1 \leq j \leq n$ and that times $T_0^{(k)}$ are exponentially distributed with parameter λ. We have

$$\overline{F}_{T_0^{(k)}}(t) = \mathrm{e}^{-\lambda t}, \quad f_{T_0^{(k)}}(t) = \lambda\,\mathrm{e}^{-\lambda t}.$$

In this case, expression (3.43) simplifies into a remarkable formula.

- If $j = d+1$, we have for (3.43)

$$p_{0\,d+1}(\mathbf{x};t) = \frac{\lambda^d}{d!\,\mathbf{V}_d}\,\exp\left[-\lambda\left(\sum_{k=1}^{d+1}\tau_k(\mathbf{x};t)\right)\right].$$

By (3.37), the sum lying within the above exponential equals t. As a result,

$$p_{0\,d+1}(\mathbf{x};t) = \frac{\lambda^d}{d!\,\mathbf{V}_d}\,\mathrm{e}^{-\lambda t} \quad \text{for } x \in \mathcal{P}_{d+1}(t).$$

Let us compute the total mass of this density:

$$\int_{\mathcal{P}_{d+1}(t)} p_{0\,d+1}(\mathbf{x};t)\,\mathrm{d}\mathbf{x} = \frac{\lambda^d}{d!\,\mathbf{V}_d}\,\mathrm{e}^{-\lambda t}\,\mathrm{Vol}(\mathcal{P}_{d+1}(t)) = \frac{(\lambda t)^d}{d!}\,\mathrm{e}^{-\lambda t} = \mathbb{P}\{T_d \leq t < T_{d+1}\}.$$

We deduce that

$$\mathbb{P}\{X(t) \in \mathrm{d}\mathbf{x} \,|\, T_d \leq t < T_{d+1}\} = \frac{1}{\mathbf{V}_d\,t^d}\,\mathbb{1}_{\mathcal{P}_{d+1}(t)}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

which means that the position $X(t)$ of the particle conditioned on having $\overrightarrow{V_d}$ as speed-vector for the first time ($i = 0$) is uniformly distributed over the polyhedron $\mathcal{P}_{d+1}(t)$.

- If $j = d+2$, we have for (3.42)

$$p_{0\,d+2}(\mathbf{x};t) = \frac{\lambda^{d+1}}{d!\,\mathbf{V}_d}\int_{\varsigma^{(d+2)}(\mathbf{x};t)}^{\sigma^{(d+2)}(\mathbf{x};t)} \exp\left[-\lambda\left(s + \sum_{k=1}^{d+1}(\tau_k(\mathbf{x};t) - s\,\tau_k(\mathbf{v}_{d+2};1))\right)\right]\mathrm{d}s.$$

By (3.37), the sum lying within the above exponential simply equals $t-s$ and then, for $\mathbf{x} \in \mathcal{P}_{d+2}(t)$,

$$p_{0\,d+2}(\mathbf{x};t) = \frac{\lambda^{d+1}}{d!\,\mathbf{V}_d}\,\mathrm{e}^{-\lambda t}\left[\sigma^{(d+2)}(\mathbf{x};t) - \varsigma^{(d+2)}(\mathbf{x};t)\right].$$

• If $d+3 \le j \le n$, formula (3.41) becomes

$$p_{0j}(\mathbf{x};t) = \frac{\lambda^n}{d!\,\mathbf{V}_d} \int_{\varsigma^{(j)}(\mathbf{x};t)}^{\sigma^{(j)}(\mathbf{x};t)} \mathrm{d}s \int_{\mathcal{D}^{(j)}(\mathbf{x};s,t)} \exp\left[-\lambda\left(s + \sum_{k=1}^{d+1}\left(\tau_k(\mathbf{x};t) - \frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\, s - \sum_{l=d+2}^{j-1} \frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\, s_l\right) + \sum_{l=d+2}^{j-1} s_l\right)\right] \mathrm{d}s_{d+2}\cdots\mathrm{d}s_{j-1}. \quad (3.44)$$

With the aid of (3.34) and (3.37), we see that the sums lying in (3.44) simplify into

$$s + \sum_{k=1}^{d+1}\left(\tau_k(\mathbf{x};t) - \frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\, s - \sum_{l=d+2}^{j-1} \frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\, s_l\right) + \sum_{l=d+2}^{j-1} s_l$$
$$= \left(1 - \sum_{k=1}^{d+1} \frac{\mathbf{V}_{kjd}}{\mathbf{V}_d}\right) s + \sum_{k=1}^{d+1} \tau_k(\mathbf{x};t) - \sum_{l=d+2}^{j-1}\left(1 - \sum_{k=1}^{d+1} \frac{\mathbf{V}_{kld}}{\mathbf{V}_d}\right) s_l \;=\; t.$$

Hence, (3.44) writes

$$p_{0j}(\mathbf{x};t) = \frac{\lambda^{j-1}}{d!\,\mathbf{V}_d} \int_{\varsigma^{(j)}(\mathbf{x};t)}^{\sigma^{(j)}(\mathbf{x};t)} \mathrm{d}s \int_{\mathcal{D}^{(j)}(\mathbf{x};s,t)} \mathrm{d}s_{d+2}\cdots\mathrm{d}s_{j-1}.$$

The integration seems to be difficult to carry out because of the complexity of the domain $\mathcal{D}^{(j)}(\mathbf{x};s,t)$ and of the bounds $\varsigma^{(j)}(\mathbf{x};t)$ and $\sigma^{(j)}(\mathbf{x};t)$.

4. Particular case: the minimal cyclic motion $(n = d+1)$

The minimal cyclic motion corresponds to the case where $n = d+1$. The variables $s_{d+2},\dots,s_n$ in (3.36) disappear; in fact the function ψ defined by

$$\psi(t_1,\dots,t_{d+1}) = \left(\sum_{k=1}^{d+1} v_{1k}t_k, \dots, \sum_{k=1}^{d+1} v_{dk}t_k, \sum_{k=1}^{d+1} t_k\right),$$

is bijective and the change of variable is quite easy to perform in this case. Let $\tau_k(\mathbf{x};t)$, $1 \le k \le d+1$, be the coordinates of $\psi^{-1}(\mathbf{x};t)$: $\psi^{-1}(\mathbf{x};t) = (\tau_1(\mathbf{x};t),\dots,\tau_{d+1}(\mathbf{x};t))$. Observing that

$$\psi(0,\dots,0,1,0,\dots,0) = (v_{1j},\dots,v_{dj},1) = (\mathbf{v}_j;1),$$

we get

$$\psi^{-1}(\mathbf{x} - s\mathbf{v}_j; t-s) = \psi^{-1}(\mathbf{x};t) - s\psi^{-1}(\mathbf{v}_j;1) = \psi^{-1}(\mathbf{x};t) - (0,\dots,0,s,0,\dots,0).$$

The term s in the above $(d+1)$-uple lies at the j^{th} position.

On the other hand, the set $\mathcal{D}(t) = \{\mathbf{x} \in \mathbb{R}^d : \tau_k(\mathbf{x};t) \ge 0 \text{ for } 1 \le k \le d+1\}$ coincides with $\mathcal{P}_{d+1}(t)$. Indeed, since the map ψ is bijective, for any point $\mathbf{x} \in \mathbb{R}^d$, there exists a unique $(d+1)$-uple $(t_1,\dots,t_{d+1})$ such that $(\mathbf{x};t) = \psi(t_1,\dots,t_{d+1})$: the t_k's are given by $t_k = \tau_k(\mathbf{x};t)$. So, we have $x_k = \sum_{l=1}^{d+1} v_{kl}t_l$ for $1 \le k \le d$ and $t = \sum_{l=1}^{d+1} t_l$, or, equivalently, $\mathbf{x} = \sum_{l=1}^{d+1} t_l\mathbf{v}_l$ and $\sum_{l=1}^{d+1} t_l = t$. These considerations show that $\mathcal{D}(t)$ is the set of points $\mathbf{x} \in \mathbb{R}^d$ of the form $\sum_{l=1}^{d+1} t_l\mathbf{v}_l$ with *positive* parameters $t_1,\dots,t_{d+1}$ such that $\sum_{l=1}^{d+1} t_l = t$. Then $\mathcal{D}(t) = \mathcal{P}_{d+1}(t)$. Therefore, the main result states as follows.

Theorem 4.1. *For $i \geq 1$ and $1 \leq j \leq d+1$, then density $p_{ij}(\mathbf{x};t)$ is given, for $\mathbf{x} \in \mathcal{P}_{d+1}(t)$, by*

$$p_{ij}(\mathbf{x};t) = \frac{1}{d!\,\mathbf{V}_d} \prod_{k=1}^{j-1} f_{S_i^{(k)}}(\tau_k(\mathbf{x};t)) \prod_{k=j+1}^{d+1} f_{S_{i-1}^{(k)}}(\tau_k(\mathbf{x};t)) \int_0^{\tau_j(\mathbf{x};t)} \overline{F}_{T_0^{(j)}}(s)\, f_{S_{i-1}^{(j)}}(\tau_j(\mathbf{x};t)-s)\,\mathrm{d}s. \tag{4.1}$$

Let us now turn our attention to some examples.

4.1. The telegraph process ($d = 1$, $n = 2$)

This situation corresponds to the case where $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ have opposite directions and $V_1 = c$, $V_2 = v$. The matrices of ψ and ψ^{-1} read here

$$\mathbf{B} = \begin{pmatrix} c & -v \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{B}^{-1} = \frac{1}{v+c}\begin{pmatrix} c & -v \\ 1 & 1 \end{pmatrix}.$$

So,

$$\tau_1(x;t) = \frac{vt+x}{v+c}, \quad \tau_2(x;t) = \frac{ct-x}{v+c}, \quad \mathbf{V}_1 = v+c.$$

We retrieve the results of Di Crescenzo [2]. The notations of [2] are the following: $D_1 = U$ is the upward direction, $D_2 = D$ is the downward direction and the related densities are $f_{S_i^{(1)}} = f_U^{(i+1)}$, $f_{S_i^{(2)}} = f_D^{(i+1)}$. Moreover $\tau_1(x;t) = \tau^*$ and $\tau_2(x;t) = t - \tau^*$. For $i \geq 1$,

$$\begin{aligned} p_{i1}(x;t) &= \frac{1}{v+c} f_{S_{i-1}^{(2)}}(\tau_2(x;t)) \int_0^{\tau_1(x;t)} \overline{F}_{T_0^{(1)}}(s)\, f_{S_{i-1}^{(1)}}(\tau_1(x;t)-s)\,\mathrm{d}s \\ &= \frac{1}{v+c} f_D^{(i)}(t-\tau^*) \int_0^{\tau^*} \overline{F}_U(s)\, f_U^{(i)}(\tau^*-s)\,\mathrm{d}s \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} p_{i2}(x;t) &= \frac{1}{v+c} f_{S_i^{(1)}}(\tau_1(x;t)) \int_0^{\tau_2(x;t)} \overline{F}_{T_0^{(2)}}(s)\, f_{S_{i-1}^{(2)}}(\tau_2(x;t)-s)\,\mathrm{d}s \\ &= \frac{1}{v+c} f_U^{(i+1)}(\tau^*) \int_{\tau*}^{t} \overline{F}_D(t-s)\, f_D^{(i)}(s-\tau^*)\,\mathrm{d}s. \end{aligned} \tag{4.3}$$

Our result related to the second case slightly differs from that of [2] because the particle starts downwards instead of upwards.

Let us focus on the particular case of the historical Goldstein-Kac telegraph process which corresponds to the situation where $v = c$ and the switching times $T_i^{(j)}$ are exponentially distributed with parameter λ. This process is classically represented by

$$X(t) = V(0)\int_0^t (-1)^{N(s)}\,\mathrm{d}s$$

where $V(0)$ is a Bernoulli random variable taking the two values c and $-c$ with probability $1/2$ and $(N(t))_{t\geq 0}$ is a Poisson process of parameter λ independent of $V(0)$. The quantity $V(0)$ is the initial velocity and $N(t)$ describes the number of changes of directions within the laps of time $[0,t]$. The underlying velocity process is simply $V(t) = V(0)(-1)^{N(t)}$. Our aim is to retrieve the well-known distribution of $X(t)$ by using the results of this part.

First, notice that if $N(t)=0$ (this event occurs with probability $\mathrm{e}^{-\lambda t}$), we then have $X(t)=V(0)t\in\{-ct,ct\}$ and this case is related to the singular part of the distribution of $X(t)$:

$$\mathbb{P}\{X(t)=ct\}=\mathbb{P}\{X(t)=-ct\}=\frac{1}{2}\,\mathrm{e}^{-\lambda t}$$

that is, for $x\in\{-ct,ct\}$,

$$\mathbb{P}\{X(t)\in \mathrm{d}x\}=\frac{1}{2}\,\mathrm{e}^{-\lambda t}\,[\delta(\mathrm{d}x+ct)+\delta(\mathrm{d}x-ct)].$$

Second, let us turn our attention to the absolutely continuous part. For this, we introduce the following conditional probabilities. For $x\in(-ct,ct)$ (in this case, $N(t)\geq 1$),

$$\begin{aligned}
p_1^+(x;t)&=\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=c\,|\,V(0)=c\}/\mathrm{d}x\\
&=\mathbb{P}\{X(t)\in \mathrm{d}x, N(t)\text{ is even}\geq 2\,|\,V(0)=c\}/\mathrm{d}x,\\
p_2^+(x;t)&=\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=-c\,|\,V(0)=c\}/\mathrm{d}x\\
&=\mathbb{P}\{X(t)\in \mathrm{d}x, N(t)\text{ is odd}\,|\,V(0)=c\}/\mathrm{d}x,\\
p_1^-(x;t)&=\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=-c\,|\,V(0)=-c\}/\mathrm{d}x\\
&=\mathbb{P}\{X(t)\in \mathrm{d}x, N(t)\text{ is even}\geq 2\,|\,V(0)=-c\}/\mathrm{d}x,\\
p_2^-(x;t)&=\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=c\,|\,V(0)=-c\}/\mathrm{d}x\\
&=\mathbb{P}\{X(t)\in \mathrm{d}x, N(t)\text{ is odd}\,|\,V(0)=-c\}/\mathrm{d}x.
\end{aligned}$$

It is easily seen that $p_j^-(x;t)=p_j^+(-x;t)$ for $j=1,2$. Next, we derive the density of $X(t)$, for $x\in(-ct,ct)$, according as

$$\begin{aligned}
\mathbb{P}\{X(t)\in \mathrm{d}x\}/\mathrm{d}x&=\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=c\}/\mathrm{d}x+\mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=-c\}/\mathrm{d}x\\
&=\mathbb{P}\{V(0)=c\}\big(p_1^+(x;t)+p_2^+(x;t)\big)+\,\mathbb{P}\{V(0)=-c\}\big(p_1^-(x;t)+p_2^-(x;t)\big)\\
&=\frac{1}{2}\left(p_1^+(x;t)+p_1^-(x;t)+p_2^+(x;t)+p_2^-(x;t)\right) &(4.4)\\
&=\frac{1}{2}\left(p_1^+(x;t)+p_1^+(-x;t)+p_2^+(x;t)+p_2^+(-x;t)\right). &(4.5)
\end{aligned}$$

To make the connection with our work, we observe that the densities p_1^+ and p_2^+ are nothing but p_{i1} and p_{i2} respectively given by (4.2) and (4.3). Indeed, on the event $\{N(t)\text{ is even}\geq 2\}$ the particle performs a certain number of cycles plus one direction while on the event $\{N(t)\text{ is odd}\}$ the particle performs a certain number of cycles (possibly zero) plus two directions, that is it performs a certain number – at least one – of complete cycles.

Now, we evaluate the densities p_1^+ and p_2^+. In the settings of the beginning of this part, we have $\overline{F}_U(t)=\overline{F}_D(t)=\mathrm{e}^{-\lambda t}$ and the random variable $S_i^{(j)}=\sum_{k=0}^{i}T_k^{(j)}$ is distributed according to the Erlang law $E(i+1,\lambda)$:

$$f_U^{(i)}(t)=f_D^{(i)}(t)=\mathrm{e}^{-\lambda t}\,\frac{\lambda^i t^{i-1}}{(i-1)!}\cdot$$

The expression (4.2) of p_{i1} becomes, for $i\geq 1$,

$$\begin{aligned}
p_{i1}(x;t)&=\frac{\lambda^{2i}\tau_2(x;t)^{i-1}}{2c\,(i-1)!}\,\mathrm{e}^{-\lambda(\tau_1(x;t)+\tau_2(x;t))}\int_0^{\tau_1(x;t)}\frac{(\tau_1(x;t)-s)^{i-1}}{(i-1)!}\,\mathrm{d}s\\
&=\mathrm{e}^{-\lambda t}\frac{\lambda^{2i}\tau_1(x;t)^i\tau_2(x;t)^{i-1}}{2c\,i!(i-1)!}
\end{aligned}$$

that is, explicitly,

$$p_{i1}(x;t) = \mathrm{e}^{-\lambda t}\left(\frac{\lambda}{2c}\right)^{2i}\frac{(ct+x)^i(ct-x)^{i-1}}{i!(i-1)!}$$

and similarly, for $i \geq 0$,

$$p_{i2}(x;t) = \mathrm{e}^{-\lambda t}\left(\frac{\lambda}{2c}\right)^{2i+1}\frac{(c^2t^2-x^2)^i}{i!^2}.$$

Reminding that

$$p_{i1}(x;t) = \mathbb{P}\{X(t)\in \mathrm{d}x, i \text{ complete cycles and 1 direction}\}/\mathrm{d}x,$$

(with the assumption that the first direction is $\overrightarrow{V_1}$, that is here $V(0)=c$) the density $p_1^+(x;t)$ we are looking for is obtained by adding the $p_{i1}(x;t)$ with respect to $i \geq 1$:

$$\begin{aligned}
p_1^+(x;t) &= \mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=c \,|\, V(0)=c\}/\mathrm{d}x \;=\; \sum_{i=1}^{\infty} p_{i1}(x;t)\\
&= \mathrm{e}^{-\lambda t}\sum_{i=1}^{\infty}\left(\frac{\lambda}{2c}\right)^{2i}\frac{(ct+x)^i(ct-x)^{i-1}}{i!(i-1)!}\\
&= \frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c}\sqrt{\frac{ct+x}{ct-x}}\sum_{i=0}^{\infty}\frac{1}{i!(i+1)!}\left(\frac{\lambda}{2c}\sqrt{c^2t^2-x^2}\right)^{2i+1}\\
&= \frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c}\sqrt{\frac{ct+x}{ct-x}}\,I_1\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \;=\; \frac{(ct+x)\mathrm{e}^{-\lambda t}}{2c^2t}\frac{\partial}{\partial t}I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right).
\end{aligned}$$

In the last equality, the functions I_0 and I_1 are the usual modified Bessel functions and we have used the relation $I_0' = I_1$. Similarly, the density $p_2^+(x;t)$ is given by

$$\begin{aligned}
p_2^+(x;t) &= \mathbb{P}\{X(t)\in \mathrm{d}x, V(t)=-c \,|\, V(0)=c\}/\mathrm{d}x \;=\; \sum_{i=0}^{\infty} p_{i2}(x;t)\\
&= \frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c}\sum_{i=0}^{\infty}\frac{1}{i!^2}\left(\frac{\lambda}{2c}\sqrt{c^2t^2-x^2}\right)^{2i} \;=\; \frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c}I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right).
\end{aligned}$$

Therefore, we have

$$p_1^+(x;t)+p_1^+(-x;t) = \frac{\mathrm{e}^{-\lambda t}}{c}\frac{\partial}{\partial t}I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)$$

and

$$p_2^+(x;t)+p_2^+(-x;t) = \frac{\lambda\,\mathrm{e}^{-\lambda t}}{c}I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right).$$

Finally, putting these two equalities into (4.5), the density of the Goldstein-Kac process emerges:

$$\mathbb{P}\{X(t)\in \mathrm{d}x\}/\mathrm{d}x = \frac{\mathrm{e}^{-\lambda t}}{2c}\left[\lambda\,I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{\partial}{\partial t}I_0\!\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right].$$

Hence, we retrieve formula (13) displayed in Orsingher [12] (see also Pinsky [20], p. 9).

4.2. The case $d = 2$, $n = 3$

Di Crescenzo [3] considered a uniform cyclic motion on the plane with three directions forming an equilateral triangle with vertices $A_1 = c(\frac{\sqrt{3}}{2}, \frac{1}{2})$, $A_2 = c(-\frac{\sqrt{3}}{2}, \frac{1}{2})$, $A_3 = c(0, -1)$. (Actually, the coordinates of A_1 and A_2 in [3] have been incidentally inverted, this provided non correct equations for the edges of the triangle $A_1A_2A_3$; the correct equations are written down below). The velocity of the motion is constant all the time: $V_1 = V_2 = V_3 = c$.

• Suppose first that $i \geq 1$. The matrices of ψ and ψ^{-1} read here

$$\mathbf{B} = \begin{pmatrix} \frac{c\sqrt{3}}{2} & -\frac{c\sqrt{3}}{2} & 0 \\ \frac{c}{2} & \frac{c}{2} & -c \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{B}^{-1} = \frac{1}{3c} \begin{pmatrix} \sqrt{3} & 1 & c \\ -\sqrt{3} & 1 & c \\ 0 & -2 & c \end{pmatrix}.$$

So, $\mathbf{V}_2 = \frac{3c^2\sqrt{3}}{4}$ and

$$\tau_1(x, y; t) = \frac{1}{3c}(\sqrt{3}\,x + y + ct), \quad \tau_2(x, y; t) = \frac{1}{3c}(-\sqrt{3}\,x + y + ct), \quad \tau_3(x, y; t) = \frac{1}{3c}(-2y + ct).$$

In [3], the directions are denoted by $D_1 = U$, $D_2 = V$, $D_3 = W$ and the related densities by $f_{S_i^{(1)}} = f_U^{(i+1)}$, $f_{S_i^{(2)}} = f_V^{(i+1)}$, $f_{S_i^{(3)}} = f_W^{(i+1)}$. Furthermore, the survival functions are denoted by $\overline{F}_{T_0^{(1)}} = \overline{F}_U$, $\overline{F}_{T_0^{(2)}} = \overline{F}_V$, $\overline{F}_{T_0^{(3)}} = \overline{F}_W$ and also $\tau_1(x, y; t) = \tau_1^*$, $\tau_2(x, y; t) = \tau_2^*$, $\tau_3(x, y; t) = \tau_3^* = t - \tau_1^* - \tau_2^*$.

We retrieve the results of Di Crescenzo [3]: for $i \geq 1$,

$$\begin{aligned}
p_{i1}(x, y; t) &= \frac{1}{2\mathbf{V}_2} f_{S_{i-1}^{(2)}}(\tau_2(x, y; t))\, f_{S_{i-1}^{(3)}}(\tau_3(x, y; t)) \int_0^{\tau_1(x,y;t)} \overline{F}_{T_0^{(1)}}(s)\, f_{S_{i-1}^{(1)}}(\tau_1(x, y; t) - s)\,\mathrm{d}s \\
&= \frac{1}{2\mathbf{V}_2} f_V^{(i)}(\tau_2^*)\, f_W^{(i)}(\tau_3^*) \int_0^{\tau_1^*} f_U^{(i)}(\tau_1^* - s)\, \overline{F}_U(s)\,\mathrm{d}s, \\
p_{i2}(x, y; t) &= \frac{1}{2\mathbf{V}_2} f_{S_i^{(1)}}(\tau_1(x, y; t))\, f_{S_{i-1}^{(3)}}(\tau_3(x, y; t)) \int_0^{\tau_2(x,y;t)} \overline{F}_{T_0^{(2)}}(s)\, f_{S_{i-1}^{(2)}}(\tau_2(x, y; t) - s)\,\mathrm{d}s \\
&= \frac{1}{2\mathbf{V}_2} f_U^{(i+1)}(\tau_1^*)\, f_W^{(i)}(\tau_3^*) \int_0^{\tau_2^*} f_V^{(i)}(\tau_2^* - s)\, \overline{F}_V(s)\,\mathrm{d}s, \\
p_{i3}(x, y; t) &= \frac{1}{2\mathbf{V}_2} f_{S_i^{(1)}}(\tau_1(x, y; t))\, f_{S_i^{(2)}}(\tau_2(x, y; t)) \int_0^{\tau_3(x,y;t)} \overline{F}_{T_0^{(3)}}(s)\, f_{S_{i-1}^{(3)}}(\tau_3(x, y; t) - s)\,\mathrm{d}s \\
&= \frac{1}{2\mathbf{V}_2} f_U^{(i+1)}(\tau_1^*)\, f_V^{(i+1)}(\tau_2^*) \int_0^{\tau_3^*} f_W^{(i)}(\tau_3^* - s)\, \overline{F}_W(s)\,\mathrm{d}s.
\end{aligned}$$

• Suppose now that $i = 0$. We describe the measures $p_{0j}(\mathrm{d}x\,\mathrm{d}y; t)$ for $1 \leq j \leq 3$.

∗ For $j = 1$, we simply have

$$p_{01}(\mathrm{d}x\,\mathrm{d}y; t) = \overline{F}_{T_0^{(1)}}(t)\, \delta(\mathrm{d}x\,\mathrm{d}y - t\mathbf{v}_1) = \overline{F}_{T_0^{(1)}}(t)\, \delta\Big(\mathrm{d}x - \frac{\sqrt{3}}{2}\, ct\Big)\, \delta\Big(\mathrm{d}y - \frac{1}{2}\, ct\Big).$$

∗ For $j = 2$, referring to Example 3.8,

$$p_{02}(\mathrm{d}x\,\mathrm{d}y; t) = \frac{1}{\mathbf{V}_1} \mathbb{1}_{\mathcal{D}_2(t)}(x)\, f_{T_0^{(1)}}\Big(\tau_1^{(2)}(x; t)\Big)\, \overline{F}_{T_0^{(2)}}\Big(\tau_2^{(2)}(x; t)\Big)\,\mathrm{d}x\, \delta\Big(\mathrm{d}y - \theta_2^{(2)}(x; t)\Big)$$

with

$$\mathbf{V}_1=\sqrt{3}\,c,\quad \tau_1^{(2)}(x;t)=\frac{\sqrt{3}\,ct+2x}{2\sqrt{3}\,c},\quad \tau_2^{(2)}(x;t)=\frac{\sqrt{3}\,ct-2x}{2\sqrt{3}\,c},\quad \theta_2^{(2)}(x;t)=\frac{ct}{2}$$

and

$$\mathcal{D}_2(t)=\Big\{x\in\mathbb{R}:\sqrt{3}\,ct-2x\geq 0,\sqrt{3}\,ct+2x\geq 0\Big\}=\Big[-\frac{\sqrt{3}}{2}\,ct,\frac{\sqrt{3}}{2}\,ct\Big].$$

Hence,

$$p_{02}(\mathrm{d}x\,\mathrm{d}y;t)\;=\;\frac{1}{\sqrt{3}\,c}\,\mathbb{1}_{\left[-\frac{\sqrt{3}}{2}\,ct,\frac{\sqrt{3}}{2}\,ct\right]}(x)\,f_{T_0^{(1)}}\left(\frac{\sqrt{3}\,ct+2x}{2\sqrt{3}\,c}\right)\overline{F}_{T_0^{(2)}}\left(\frac{\sqrt{3}\,ct-2x}{2\sqrt{3}\,c}\right)\mathrm{d}x\,\delta\Big(\mathrm{d}y-\frac{ct}{2}\Big).$$

* For $j=3$, by (3.43),

$$p_{03}(x,y;t)\;=\;\frac{1}{2\,\mathbf{V}_2}\,\mathbb{1}_{\mathcal{P}_3(t)}(x,y)\,f_{T_0^{(1)}}\Big(\tau_1^{(3)}(x,y;t)\Big)\,f_{T_0^{(2)}}\Big(\tau_2^{(3)}(x,y;t)\Big)\,\overline{F}_{T_0^{(3)}}\Big(\tau_3^{(3)}(x,y;t)\Big).$$

The settings are exactly identical to those of the case $i\geq 1$: $\mathbf{V}_2=\frac{3c^2\sqrt{3}}{4}$ and

$$\tau_1^{(3)}(x,y;t)=\frac{1}{3c}\,(\sqrt{3}\,x+y+ct),\quad \tau_2^{(3)}(x,y;t)=\frac{1}{3c}\,(-\sqrt{3}\,x+y+ct),\quad \tau_3^{(3)}(x,y;t)=\frac{1}{3c}\,(ct-2y).$$

Consequently,

$$p_{03}(x,y;t)\;=\;\frac{2}{3\sqrt{3}\,c^2}\,\overline{F}_{T_0^{(3)}}\left(\frac{1}{3c}\,(ct-2y)\right)\,f_{T_0^{(1)}}\left(\frac{1}{3c}\,(\sqrt{3}\,x+y+ct)\right)\,f_{T_0^{(2)}}\left(\frac{1}{3c}\,(-\sqrt{3}\,x+y+ct)\right).$$

Remark 4.2. Adding an extra point $A_4=c(\alpha,\beta)$ to the situation we are dealing with in this part (then $d=2$ and $n=4$) is of interest: for $i=0$ and $j=4=d+2$, *e.g.*, we have

$$p_{04}(x,y;t)=\frac{1}{2\,\mathbf{V}_2}\int_{\varsigma^{(4)}(x,y;t)}^{\sigma^{(4)}(x,y;t)}\overline{F}_{T_0^{(4)}}(s)\,\mathrm{d}s\prod_{k=1}^{3}f_{T_0^{(k)}}(\tau_k(x,y;t)-s\,\tau_k(c\alpha,c\beta;1))$$

with

$$\begin{aligned}\tau_1(x,y;t)-s\,\tau_1(c\alpha,c\beta;1)\;&=\;\frac{1}{3c}\,(\sqrt{3}\,x+y+ct)-\frac{s}{3}\,(\sqrt{3}\,\alpha+\beta+1),\\ \tau_2(x,y;t)-s\,\tau_2(c\alpha,c\beta;1)\;&=\;\frac{1}{3c}\,(-\sqrt{3}\,x+y+ct)-\frac{s}{3}\,(-\sqrt{3}\,\alpha+\beta+1),\\ \tau_3(x,y;t)-s\,\tau_3(c\alpha,c\beta;1)\;&=\;\frac{1}{3c}\,(-2y+ct)-\frac{s}{3}\,(-2\beta+1).\end{aligned}$$

If, *e.g.*, the solid quadrilateral $A_1A_2A_3A_4$ is convex, the reals numbers α,β verify the conditions

$$\sqrt{3}\,\alpha+\beta+1\geq 0,\quad -\sqrt{3}\,\alpha+\beta+1\leq 0,\quad -2\beta+1\geq 0.$$

Then, the constraints for s, $\tau_k(x,y;t)-s\,\tau_k(c\alpha,c\beta;1)\geq 0$ for $1\leq k\leq 3$, read

$$\frac{1}{c}\,\max\left(\frac{\sqrt{3}\,x-y-ct}{\sqrt{3}\,\alpha-\beta-1},0\right)\leq s\leq\frac{1}{c}\,\min\left(\frac{\sqrt{3}\,x+y+ct}{\sqrt{3}\,\alpha+\beta+1},\frac{ct-2y}{1-2\beta}\right)$$

which produce the following bounds for s:

$$\varsigma^{(4)}(x,y;t)=\frac{1}{c}\,\max\left(\frac{\sqrt{3}\,x-y-ct}{\sqrt{3}\,\alpha-\beta-1},0\right),\quad \sigma^{(4)}(x,y;t)=\frac{1}{c}\,\min\left(\frac{\sqrt{3}\,x+y+ct}{\sqrt{3}\,\alpha+\beta+1},\frac{ct-2y}{1-2\beta}\right).$$

4.3. Erlangian switching times

4.3.1. *Erlang distribution and Bessel functions with several variables*

Consider the case where the switching times are distributed according to an Erlang law: we suppose that the distributions of random times $T_i^{(j)}$ are Erlang law $E(\nu_j,\lambda_j)$, that is

$$\overline{F}_{T_0^{(j)}}(t)=\mathbb{P}\Big(T_i^{(j)}>t\Big)=\mathrm{e}^{-\lambda_j t}\sum_{l=0}^{\nu_j-1}\frac{(\lambda_j t)^l}{l!}\cdot$$

Then, the random variable $S_i^{(j)}=\sum_{k=0}^{i}T_k^{(j)}$ is distributed according to the Erlang law $E((i+1)\nu_j,\lambda_j)$:

$$f_{S_i^{(j)}}(t)=\mathrm{e}^{-\lambda_j t}\,\frac{\lambda_j^{(i+1)\nu_j}t^{(i+1)\nu_j-1}}{((i+1)\nu_j-1)!}\cdot$$

In this case, formula (4.1) becomes

$$p_{ij}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\prod_{k=1}^{j-1}\left(\mathrm{e}^{-\lambda_k\tau_k(\mathbf{x};t)}\,\frac{\lambda_k^{(i+1)\nu_k}\tau_k(\mathbf{x};t)^{(i+1)\nu_k-1}}{((i+1)\nu_k-1)!}\right)\prod_{k=j+1}^{d+1}\left(\mathrm{e}^{-\lambda_k\tau_k(\mathbf{x};t)}\,\frac{\lambda_k^{i\nu_k}\tau_k(\mathbf{x};t)^{i\nu_k-1}}{(i\nu_k-1)!}\right)$$
$$\times\int_0^{\tau_j(\mathbf{x};t)}\mathrm{e}^{-\lambda_j s}\sum_{l=0}^{\nu_j-1}\frac{(\lambda_j s)^l}{l!}\,\mathrm{e}^{-\lambda_j(\tau_j(\mathbf{x};t)-s)}\,\frac{\lambda_j^{i\nu_j}(\tau_j(\mathbf{x};t)-s)^{i\nu_j-1}}{(i\nu_j-1)!}\,\mathrm{d}s.$$

The above integral can be easily evaluated:

$$\int_0^{\tau_j(\mathbf{x};t)}\mathrm{e}^{-\lambda_j s}\sum_{l=0}^{\nu_j-1}\frac{(\lambda_j s)^l}{l!}\,\mathrm{e}^{-\lambda_j(\tau_j(\mathbf{x};t)-s)}\,\frac{\lambda_j^{i\nu_j}\,(\tau_j(\mathbf{x};t)-s)^{i\nu_j-1}}{(i\nu_j-1)!}\,\mathrm{d}s$$
$$=\mathrm{e}^{-\lambda_j\tau_j(\mathbf{x};t)}\sum_{l=0}^{\nu_j-1}\lambda_j^{i\nu_j+l}\int_0^{\tau_j(\mathbf{x};t)}\frac{s^l}{l!}\,\frac{(\tau_j(\mathbf{x};t)-s)^{i\nu_j-1}}{(i\nu_j-1)!}\,\mathrm{d}s$$
$$=\mathrm{e}^{-\lambda_j\tau_j(\mathbf{x};t)}\sum_{l=0}^{\nu_j-1}\frac{(\lambda_j\tau_j(\mathbf{x};t))^{i\nu_j+l}}{(i\nu_j+l)!}\cdot$$

We then obtain

$$p_{ij}(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\left(\prod_{\substack{1\le k\le d+1\\ k\neq j}}\lambda_k\right)\mathrm{e}^{-\sum_{k=1}^{d+1}\lambda_k\tau_k(\mathbf{x};t)}\prod_{k=1}^{j-1}\frac{(\lambda_k\tau_k(\mathbf{x};t))^{(i+1)\nu_k-1}}{((i+1)\nu_k-1)!}$$
$$\times\sum_{l=0}^{\nu_j-1}\frac{(\lambda_j\tau_j(\mathbf{x};t))^{i\nu_j+l}}{(i\nu_j+l)!}\prod_{k=j+1}^{d+1}\frac{(\lambda_k\tau_k(\mathbf{x};t))^{i\nu_k-1}}{(i\nu_k-1)!}\cdot$$

When summing this expression over all indices i, we get the probability

$$p_j(\mathbf{x};t)=\sum_{i=1}^{\infty}p_{ij}(\mathbf{x};t)=\mathbb{P}\{X(t)\in \mathrm{d}\mathbf{x}, \text{at least a cycle and } j \text{ directions}\}/\mathrm{d}\mathbf{x}.$$

We explicitly have

$$p_j(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\Bigg(\prod_{\substack{1\le k\le d+1\\ k\neq j}}\lambda_k\Bigg)\mathrm{e}^{-\sum_{k=1}^{d+1}\lambda_k\tau_k(\mathbf{x};t)}\sum_{l=0}^{\nu_j-1}\sum_{i=0}^{\infty}\prod_{k=1}^{j-1}\frac{(\lambda_k\tau_k(\mathbf{x};t))^{i\nu_k+2\nu_k-1}}{(i\nu_k+2\nu_k-1)!}$$
$$\times\frac{(\lambda_j\tau_j(\mathbf{x};t))^{i\nu_j+\nu_j+l}}{(i\nu_j+\nu_j+l)!}\prod_{k=j+1}^{d+1}\frac{(\lambda_k\tau_k(\mathbf{x};t))^{i\nu_k+\nu_k-1}}{(i\nu_k+\nu_k-1)!}.$$

This probability can be expressed in terms of generalized Bessel with several arguments. Set

$$I_{\alpha_1,\dots,\alpha_n}^{\beta_1,\dots,\beta_n}(x_1,\dots,x_n)=\sum_{k=0}^{\infty}\frac{x_1^{\alpha_1+k\beta_1}\cdots x_n^{\alpha_n+k\beta_n}}{(\alpha_1+k\beta_1)!\cdots(\alpha_n+k\beta_n)!}.$$

Theorem 4.3. *The density of $X(t)$ subject to having performed at least one complete cycle and to following the jth direction can be expressed as*

$$p_j(\mathbf{x};t)=\frac{1}{d!\,\mathbf{V}_d}\Bigg(\prod_{\substack{1\le k\le d+1\\ k\neq j}}\lambda_l\Bigg)\mathrm{e}^{-\sum_{k=1}^{d+1}\lambda_k\tau_k(\mathbf{x};t)}\sum_{l=0}^{\nu_j-1}I_{\mu_{1l},\dots,\mu_{d+1\,l}}^{\nu_1,\dots,\nu_{d+1}}(\lambda_1\tau_1(\mathbf{x};t),\dots,\lambda_{d+1}\tau_{d+1}(\mathbf{x};t))$$

where

$$\mu_{kl}=\begin{cases}2\nu_k-1 & \text{if } 1\le k\le j-1,\\ \nu_j+l & \text{if } k=j,\\ \nu_k-1 & \text{if } j+1\le k\le d+1.\end{cases}$$

Remark 4.4. The Bessel function can be related to hypergeometric function as follows: writing

$$(\alpha+k\beta)!=\alpha!\prod_{l=1}^{k\beta}(\alpha+l)\;=\;\alpha!\prod_{i=0}^{k-1}\prod_{j=1}^{\beta}(\alpha+i\beta+j)$$
$$=\alpha!\beta^{k\beta}\prod_{i=0}^{k-1}\prod_{j=1}^{\beta}\left(\frac{\alpha+j}{\beta}+i\right)\;=\;\alpha!\beta^{k\beta}\prod_{j=1}^{\beta}\left(\frac{\alpha+j}{\beta}\right)_k$$

with $a_k=a(a+1)\cdots(a+k-1)$, we have

$$I_{\alpha_1,\dots,\alpha_n}^{\beta_1,\dots,\beta_n}(x_1,\dots,x_n)=\frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!}\sum_{k=0}^{\infty}\frac{\left(\frac{x_1}{\beta_1}\right)^{k\beta_1}\cdots\left(\frac{x_n}{\beta_n}\right)^{k\beta_n}}{\prod_{i=1}^{\beta_1}\left(\frac{\alpha_1+i}{\beta_1}\right)_k\cdots\prod_{i=1}^{\beta_n}\left(\frac{\alpha_n+i}{\beta_n}\right)_k}.$$

So, invoking the hypergeometric function

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{z^k}{k!}$$

and putting $\gamma_{ij} = \frac{\alpha_i+j}{\beta_i}$ for $1 \le i \le n$ and $1 \le j \le \beta_i$, we extract the relationship

$$I^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(x_1,\dots,x_n) = \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!}\,{}_1F_n\!\left(1;\gamma_{11},\dots,\gamma_{n\beta_n};\left(\frac{x_1}{\beta_1}\right)^{\beta_1}\cdots\left(\frac{x_n}{\beta_n}\right)^{\beta_n}\right).$$

4.3.2. *Erlang distribution with equal parameters λ_k*

When moreover all parameters λ_k are identical to λ, we have

$$\sum_{k=1}^{d+1}\lambda_k\tau_k(\mathbf{x};t) = \lambda\sum_{k=1}^{d+1}\tau_k(\mathbf{x};t) = \lambda t$$

and then

$$p_j(\mathbf{x};t) = \frac{\lambda^d}{d!\,\mathbf{V}_d}\,\mathrm{e}^{-\lambda t}\sum_{l=0}^{\nu_j-1} I^{\nu_1,\dots,\nu_{d+1}}_{\mu_{1l},\dots,\mu_{d+1\,l}}(\lambda\tau_1(\mathbf{x};t),\dots,\lambda\tau_{d+1}(\mathbf{x};t)).$$

4.4. **Exponential switching times with equal parameters**

If all parameters λ_k are identical to λ and all parameters ν_k are identical to 1 in the Erlang distribution $E(\nu_k,\lambda_k)$, times $T_i^{(j)}$ are exponentially distributed with parameter λ and the Bessel function simplifies into (in the lower indices below, there are j indices equal to 1 and $n-j$ equal to 0)

$$\begin{aligned} I^{1,\dots,1}_{1,\dots,1,0,\dots,0}(x_1,\dots,x_n) &= \sum_{k=0}^{\infty}\frac{x_1^{k+1}\cdots x_j^{k+1}x_{j+1}^k\cdots x_n^k}{(k+1)!\cdots(k+1)!k!\cdots k!} = (x_1\cdots x_j)\sum_{k=0}^{\infty}\frac{(x_1\cdots x_n)^k}{(k+1)!^j k!^{n-j}}\\ &= (x_1\cdots x_j)\mathcal{I}_{j,n}(n\sqrt[n]{x_1\cdots x_n}) \end{aligned}$$

where

$$\mathcal{I}_{j,n}(z) = \sum_{k=0}^{\infty}\frac{(z/n)^{nk}}{k!^{n-j}(k+1)!^j}.$$

In these settings, we obtain

$$p_j(\mathbf{x};t) = \frac{\lambda^{d+j}}{d!\,\mathbf{V}_d}\,\mathrm{e}^{-\lambda t}\left(\prod_{k=1}^{j}\tau_k(\mathbf{x};t)\right)\mathcal{I}_{j,d+1}\!\left(\lambda(d+1)\sqrt[d+1]{\prod_{k=1}^{d+1}\tau_k(\mathbf{x};t)}\right).$$

4.4.1. *Motion with constant velocity*

We suppose in addition that all the speeds V_k are identical to c. In that case, $\mathbf{V}_d = c^d\,\tilde{\mathbf{V}}_d$, $\tilde{\mathbf{V}}_d$ being the volume of the normalized $(d+1)$-hedron $A'_1\cdots A'_{d+1}$ where $A'_k = \frac{1}{c}A_k$. We have

$$\tau_k(\mathbf{x};t) = \frac{w_{k\,d+1}}{c}\,h_k(\mathbf{x};t)\quad\text{with}\quad h_k(\mathbf{x};t) = ct + \sum_{l=1}^{d}\frac{cw_{kl}}{w_{k\,d+1}}\,x_l$$

and, furthermore,

$$\prod_{k=1}^{j}\tau_k(\mathbf{x};t) = \frac{1}{c^j}\prod_{k=1}^{j}w_{k\,d+1}\prod_{k=1}^{j}h_k(\mathbf{x};t) = \frac{\alpha_j}{(d+1)^j c^j}\prod_{k=1}^{j}h_k(\mathbf{x};t)$$

with

$$\alpha_j = \prod_{k=1}^{j}[(d+1)w_{k\,d+1}].$$

As a result, we obtain:

$$p_j(\mathbf{x};t)=\frac{\alpha_j\,\mathrm{e}^{-\lambda t}}{(d+1)^j\,d!\,\tilde{\mathbf{V}}_d}\left(\frac{\lambda}{c}\right)^{d+j}\left(\prod_{k=1}^{j}h_k(\mathbf{x};t)\right)\mathcal{I}_{j,d+1}\!\left(\frac{\lambda}{c}\sqrt[d+1]{\alpha_{d+1}\prod_{k=1}^{d+1}h_k(\mathbf{x};t)}\right).$$

4.4.2. *Motion with directions forming a regular* $(d+1)$*-hedron*

If moreover the $(d+1)$-hedron of the directions is regular, the inner products of the speed-vectors are constant and their values are given by (see [9])

$$\overrightarrow{V_i}.\overrightarrow{V_j}=\begin{cases}-c^2/d & \text{if } i\neq j,\\ c^2 & \text{if } i=j.\end{cases}$$

This yields

$$\frac{d}{d+1}\left(\sum_{k=1}^{d}\frac{v_{ki}}{c^2}\times v_{kj}+\frac{1}{d}\right)=\frac{d}{d+1}\left(\frac{1}{c^2}\,\overrightarrow{V_i}.\overrightarrow{V_j}+\frac{1}{d}\right)=\delta_{ij}$$

and then, the matrix of ψ^{-1} writes

$$\mathbf{B}^{-1}=\begin{pmatrix} w_{11} & \dots & w_{1\,d+1}\\ \vdots & & \vdots\\ w_{d+1\,1} & \dots & w_{d+1\,d+1}\end{pmatrix}=\frac{d}{(d+1)c^2}\begin{pmatrix} v_{11} & \dots & v_{d1} & \frac{c^2}{d}\\ \vdots & & \vdots & \vdots\\ v_{1\,d+1} & \dots & v_{d\,d+1} & \frac{c^2}{d}\end{pmatrix}.$$

So, $w_{kl}=\frac{dv_{lk}}{(d+1)c^2}$ for $l\leq d$, $w_{k\,d+1}=\frac{1}{d+1}$, $\alpha_j=1$ and

$$h_k(\mathbf{x};t)=ct+d\sum_{l=1}^{d}\frac{v_{lk}}{c}\,x_l.$$

In that case, we retrieve a formula by Lachal, Leorato and Orsingher [9]:

$$p_j(\mathbf{x};t)=\frac{\mathrm{e}^{-\lambda t}}{(d+1)^j\,d!\,\tilde{\mathbf{V}}_d}\left(\frac{\lambda}{c}\right)^{d+j}\left(\prod_{k=1}^{j}h_k(\mathbf{x};t)\right)\mathcal{I}_{j,d+1}\!\left(\frac{\lambda}{c}\sqrt[d+1]{\prod_{k=1}^{d+1}h_k(\mathbf{x};t)}\right).$$

A. Appendix: the linear image of a random vector

Let $\mathbf{X}=(X_1,\dots,X_p)$ be a random vector with pdf $f_{\mathbf{X}}$, $\Phi:\mathbb{R}^p\longrightarrow\mathbb{R}^q$ be a linear map, and $\mathbf{Y}=\Phi(\mathbf{X})=(Y_1,\dots,Y_q)$. In this section, we express the probability distribution of $\mathbf{Y}$ in terms of $f_{\mathbf{X}}$. We distinguish the cases where Φ is injective with $p<q$ (we shall call it φ), and Φ is surjective with $p>q$ (we shall call it ψ).

A.1. **Injective case**

Suppose φ injective and $p<q$. Let us introduce the matrix of the linear map φ:

$$\mathbf{A}=\begin{pmatrix} a_{11} & \dots & a_{1p}\\ \vdots & & \vdots\\ a_{p1} & \dots & a_{pp}\\ a_{p+1\,1} & \dots & a_{p+1\,p}\\ \vdots & & \vdots\\ a_{q1} & \dots & a_{qp}\end{pmatrix}.$$

Since φ is injective, the rank of the matrix $\mathbf{A}$ is p. So, we can extract from $\mathbf{A}$ an invertible matrix of type $p\times p$. Upon exchanging some rows of $\mathbf{A}$, we can suppose that this extracted matrix is obtained by picking the p first rows of $\mathbf{A}$. This observation suggests us to split $\mathbf{A}$ into the two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ defined by

$$\mathbf{A}_1=\begin{pmatrix} a_{11} & \dots & a_{1p}\\ \vdots & & \vdots\\ a_{p1} & \dots & a_{pp}\end{pmatrix} \quad\text{and}\quad \mathbf{A}_2=\begin{pmatrix} a_{p+1\,1} & \dots & a_{p+1\,p}\\ \vdots & & \vdots\\ a_{q1} & \dots & a_{qp}\end{pmatrix}$$

where $\mathbf{A}_1$ is the aforementioned invertible matrix. We finally introduce the linear maps $\varphi_1:\mathbb{R}^p\longrightarrow\mathbb{R}^p$ and $\varphi_2:\mathbb{R}^p\longrightarrow\mathbb{R}^{q-p}$ whose matrices are respectively $\mathbf{A}_1$ and $\mathbf{A}_2$. In these settings, we clearly have

$$\varphi(x_1,\dots,x_p)=(\varphi_1(x_1,\dots,x_p),\varphi_2(x_1,\dots,x_p)).$$

The above discussion entails that, in the foregoing decomposition, the map φ_1 is bijective.

A.1.1. *The distribution of* $\mathbf{Y}$

Lemma A.1. *The random vector* $\mathbf{Y}$ *admits for probability distribution the singular measure*

$$\begin{aligned} f_{\mathbf{Y}}(\mathrm{d}y_1\cdots\mathrm{d}y_q) &= \frac{1}{|\det\varphi_1|}\,\delta\big((y_{p+1},\dots,y_q)-\big(\varphi_2\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\big)\,\big(f_{\mathbf{X}}\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\,\mathrm{d}y_1\cdots\mathrm{d}y_q\\ &= \frac{1}{|\det\varphi_1|}\,\big(f_{\mathbf{X}}\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\,\mathrm{d}y_1\cdots\mathrm{d}y_p\delta_{(\varphi_2\circ\varphi_1^{-1})(y_1,\dots,y_p)}\,(\mathrm{d}y_{p+1}\cdots\mathrm{d}y_q).\end{aligned}$$

The symbol $\circ$ *in the foregoing relations denotes the usual composition of functions.*

Remark A.2. We emphasize that $f_{\mathbf{Y}}(\mathrm{d}y_1\cdots\mathrm{d}y_q)$ is not a density (we did not write it as $f_{\mathbf{Y}}(y_1,\dots,y_q)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$) because of the presence of the generalized function or the Dirac measure δ. In the expression $\delta(y-a)$, δ stands for the generalized function such that $\int\delta(y-a)\psi(y)\,\mathrm{d}y=\psi(a)$ while in $\delta_a(\mathrm{d}y)$, δ_a stands for the Dirac measure with an atom lying at a: $\delta_a(A)=\int_A\delta_a(\mathrm{d}y)=\mathbb{1}_A(a)$.

Proof. By applying the change of variables defined by the bijective map φ_1, we plainly derive for any Borel sets E of $\mathbb{R}^p$ and F of $\mathbb{R}^{q-p}$:

$$\begin{aligned}\mathbb{P}\{Y\in E\times F\} &= \mathbb{P}\{\varphi_1(X)\in E,\varphi_2(X)\in F\} = \mathbb{P}\Big\{X\in\varphi_1^{-1}(E)\cap\varphi_2^{-1}(F)\Big\}\\ &= \int_{\varphi_1^{-1}(E)}\mathbb{1}_{\varphi_2^{-1}(F)}(\mathbf{x})\,f_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\varphi_1^{-1}(E)}\mathbb{1}_F(\varphi_2(\mathbf{x}))\,f_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= \frac{1}{|\det\varphi_1|}\int_E\mathbb{1}_F\left(\big(\varphi_2\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\right)\big(f_{\mathbf{X}}\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\,\mathrm{d}y_1\dots\mathrm{d}y_p.\end{aligned}$$

Let us mention that in the second above equality, the notations $\varphi_1^{-1}(E)$ and $\varphi_2^{-1}(F)$ (although φ_2 is not bijective) refer to the inverse images of the sets E and F by φ_1 and φ_2 respectively. We have

$$\begin{aligned}\mathbb{1}_F\left(\big(\varphi_2\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\right) &= \delta_{(\varphi_2\circ\varphi_1^{-1})(y_1,\dots,y_p)}(F) = \int_F\delta_{(\varphi_2\circ\varphi_1^{-1})(y_1,\dots,y_p)}(\mathrm{d}y_{p+1}\dots\mathrm{d}y_q)\\ &= \int_F\delta\big((y_{p+1},\dots,y_q)-\big(\varphi_2\circ\varphi_1^{-1}\big)(y_1,\dots,y_p)\big)\,\mathrm{d}y_{p+1}\dots\mathrm{d}y_q.\end{aligned}$$

Then, putting $\mathbf{y} = (y_1, \ldots, y_p, y_{p+1}, \ldots, y_q)$,

$$\mathbb{P}\{Y \in E \times F\} = \frac{1}{|\det \varphi_1|} \int_{E\times F} \delta\big((y_{p+1}, \ldots, y_q) - \big(\varphi_2 \circ \varphi_1^{-1}\big)(y_1, \ldots, y_p)\big)\, \big(f_{\mathbf{X}} \circ \varphi_1^{-1}\big)(y_1, \ldots, y_p)\, \mathrm{d}\mathbf{y}$$

which proves the result. □

Lemma A.3. *The support of the measure $f_{\mathbf{Y}}(\mathrm{d}y_1 \cdots \mathrm{d}y_q)$ is the range of φ.*

Proof. The support of the measure $f_{\mathbf{Y}}(\mathrm{d}y_1 \cdots \mathrm{d}y_q)$ is the set characterized by the equations

$$(y_{p+1}, \ldots, y_q) = \big(\varphi_2 \circ \varphi_1^{-1}\big)(y_1, \ldots, y_p).$$

Introducing the parameters $u_1, \ldots, u_p$ defined by $(u_1, \ldots, u_p) = \varphi_1^{-1}(y_1, \ldots, y_p)$, we extract the relations

$$(y_1, \ldots, y_p) = \varphi_1(u_1, \ldots, u_p) \quad \text{and} \quad (y_{p+1}, \ldots, y_q) = \varphi_2(u_1, \ldots, u_p)$$

which write, in a more concise form,

$$(y_1, \ldots, y_q) = \varphi(u_1, \ldots, u_p), \quad u_1, \ldots, u_p \in \mathbb{R}.$$

This is a parametrical representation of the range of φ which is also the vectorial space spanned by the vectors $\sum_{i=1}^{q} a_{ij}\, \vec{e}_i$, $1 \le j \le p$, where $\{\vec{e}_1, \ldots, \vec{e}_q\}$ is the canonical basis of $\mathbb{R}^q$. □

A.1.2. *Computation of $\varphi_2 \circ \varphi_1^{-1}$*

We specify the explicit form of $\varphi_2 \circ \varphi_1^{-1}$. For this, we use the matrices $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$ of the linear maps φ, φ_1, φ_2. Set $\Delta = \det \mathbf{A}_1$ and introduce the co-factors A_{ij}, $1 \le i, j \le q$, of $\mathbf{A}_1$ defined by

$$A_{ij} = (-1)^{i+j} \det(a_{kl})_{\substack{1\le k,l\le p\\ k\ne i, l\ne j}} = \begin{vmatrix} a_{11} & \ldots & a_{1\,j-1} & a_{1\,j+1} & \ldots & a_{1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1\,1} & \ldots & a_{i-1\,j-1} & a_{i-1\,j+1} & \ldots & a_{i-1\,p} \\ a_{i+1\,1} & \ldots & a_{i+1\,j-1} & a_{i+1\,j+1} & \ldots & a_{i+1\,p} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{p1} & \ldots & a_{p\,j-1} & a_{p\,j+1} & \ldots & a_{pp} \end{vmatrix}.$$

The matrix of the map φ_1^{-1} is then given by

$$\mathbf{A}_1^{-1} = \frac{1}{\Delta} \begin{pmatrix} A_{11} & \ldots & A_{p1} \\ \vdots & & \vdots \\ A_{1p} & \ldots & A_{pp} \end{pmatrix}.$$

Let us perform some intermediate computations:

$$\sum_{k=1}^{p} A_{jk} x_k = \sum_{k=1}^{p} (-1)^{j+k} x_k \begin{vmatrix} a_{11} & \ldots & a_{1\,k-1} & a_{1\,k+1} & \ldots & a_{1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{j-1\,1} & \ldots & a_{j-1\,k-1} & a_{j-1\,k+1} & \ldots & a_{j-1\,p} \\ a_{j+1\,1} & \ldots & a_{j+1\,k-1} & a_{j+1\,k+1} & \ldots & a_{j+1\,p} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{p1} & \ldots & a_{p\,k-1} & a_{p\,k+1} & \ldots & a_{pp} \end{vmatrix} = \begin{vmatrix} a_{11} & \ldots & a_{1p} \\ \vdots & & \vdots \\ a_{j-1\,1} & \ldots & a_{j-1\,p} \\ x_1 & \ldots & x_p \\ a_{j+1\,1} & \ldots & a_{j+1\,p} \\ \vdots & & \vdots \\ a_{p1} & \ldots & a_{pp} \end{vmatrix}.$$

In the case where $x_k = a_{ik}$, $p+1 \le i \le q$, we get the matrix of the map $\varphi_2 \circ \varphi_1^{-1}$:

$$\mathbf{A}_2\mathbf{A}_1^{-1} = \frac{1}{\Delta}\begin{pmatrix} \Delta_{p+1\,1} & \dots & \Delta_{p+1\,p} \\ \vdots & & \vdots \\ \Delta_{q1} & \dots & \Delta_{qp} \end{pmatrix}$$

where

$$\Delta_{ij} = \sum_{k=1}^{p} a_{ik}A_{jk} = \begin{vmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{j-1\,1} & \dots & a_{j-1\,p} \\ a_{i1} & \dots & a_{ip} \\ a_{j+1\,1} & \dots & a_{j+1\,p} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pp} \end{vmatrix} \quad \text{for } p+1 \le i \le q,\ 1 \le j \le p.$$

A.1.3. *Some constraints*

In the case where all the random variables $X_1, \dots, X_p$ are positive, the support of the measure $f_{\mathbf{Y}}(\mathrm{d}y_1 \cdots \mathrm{d}y_q)$ may be a little more specified. The coordinates of $\varphi_1^{-1}(y_1, \dots, y_p)$ must be positive. Assuming for instance $\Delta > 0$, this generates the constraints that $\sum_{k=1}^{p} A_{jk}y_k \ge 0$ for $1 \le j \le p$. Hence the support is the intersection of the range of φ and the subset $\mathcal{D}$ of $\mathbb{R}^q$ given by

$$\mathcal{D} = \left\{(y_1, \dots, y_q) \in \mathbb{R}^q : \sum_{k=1}^{p} A_{jk}y_k \ge 0 \text{ for } 1 \le j \le p\right\}.$$

A.2. **Surjective case**

Suppose ψ surjective and $p > q$. Let us assume that the vectors $\mathbf{X}_1 = (X_1, \dots, X_q)$ and $\mathbf{X}_2 = (X_{q+1}, \dots, X_p)$ are independent. We introduce the matrix of the linear map ψ:

$$\mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1q} & b_{1\,q+1} & \dots & b_{1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{q1} & \dots & b_{qq} & b_{q\,q+1} & \dots & b_{qp} \end{pmatrix}.$$

Since ψ is surjective, the rank of the matrix $\mathbf{B}$ is q. So, we can extract from $\mathbf{B}$ an invertible matrix of type $q \times q$. Upon exchanging some columns of $\mathbf{B}$, we can suppose that this extracted matrix is obtained by picking the q first columns of $\mathbf{B}$. This observation suggests us to split $\mathbf{B}$ into the two following matrices $\mathbf{B}_1$ and $\mathbf{B}_2$:

$$\mathbf{B}_1 = \begin{pmatrix} b_{11} & \dots & b_{1q} \\ \vdots & & \vdots \\ b_{q1} & \dots & b_{qq} \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \begin{pmatrix} b_{1\,q+1} & \dots & b_{1p} \\ \vdots & & \vdots \\ b_{q\,q+1} & \dots & b_{qp} \end{pmatrix}$$

where $\mathbf{B}_1$ is the invertible aforementioned matrix. We also introduce the linear maps $\psi_1 : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ and $\psi_2 : \mathbb{R}^{p-q} \longrightarrow \mathbb{R}^q$ whose matrices are respectively $\mathbf{B}_1$ and $\mathbf{B}_2$. Actually, the maps ψ_1 and ψ_2 are nothing but the partial maps of ψ defined by

$$\psi_1(x_1, \dots, x_q) = \psi(x_1, \dots, x_q, 0, \dots, 0) \quad \text{and} \quad \psi_2(x_{q+1}, \dots, x_p) = \psi(0, \dots, 0, x_{q+1}, \dots, x_p).$$

From the above discussion the map ψ_1 is evidently bijective.

A.2.1. *The distribution of* $\mathbf{Y}$

Lemma A.4. *The random vector* $\mathbf{Y}$ *admits for pdf*

$$f_{\mathbf{Y}}(y_1,\dots,y_q)=\frac{1}{|\det\psi_1|}\int_{\mathbb{R}^{p-q}} f_{\mathbf{X}_1}\left(\psi_1^{-1}(y_1,\dots,y_q)-(\psi_1^{-1}\circ\psi_2)(x_{q+1},\dots,x_p)\right)\times f_{\mathbf{X}_2}(x_{q+1},\dots,x_p)\,\mathrm{d}x_{q+1}\dots\mathrm{d}x_p.$$

Proof. First, observe that $\psi(x_1,\dots,x_p)=\psi_1(x_1,\dots,x_q)+\psi_2(x_{q+1},\dots,x_p)$. Now, applying the change of variables defined by the bijective map ψ_1, we plainly derive

$$\begin{aligned}\mathbb{P}\{Y\in E\}&=\mathbb{P}\big\{\mathbf{X}_1\in\psi_1^{-1}(E)-(\psi_1^{-1}\circ\psi_2)(\mathbf{X}_2)\big\}\\&=\int_{\mathbb{R}^{p-q}}\mathbb{P}\big\{\mathbf{X}_1\in\psi_1^{-1}(E)-(\psi_1^{-1}\circ\psi_2)(x_{q+1},\dots,x_p)\big\}\,f_{\mathbf{X}_2}(x_{q+1},\dots,x_p)\,\mathrm{d}x_{q+1}\dots\mathrm{d}x_p\\&=\int_{\psi_1^{-1}(E)}\mathrm{d}x_1\dots\mathrm{d}x_q\int_{\mathbb{R}^{p-q}}f_{\mathbf{X}_1}\left((x_1,\dots,x_q)-(\psi_1^{-1}\circ\psi_2)(x_{q+1},\dots,x_p)\right)\\&\quad\times f_{\mathbf{X}_2}(x_{q+1},\dots,x_p)\,\mathrm{d}x_{q+1}\dots\mathrm{d}x_p\\&=\frac{1}{|\det\psi_1|}\int_E\mathrm{d}y_1\dots\mathrm{d}y_q\int_{\mathbb{R}^{p-q}}f_{\mathbf{X}_1}\left(\psi_1^{-1}(y_1,\dots,y_q)-(\psi_1^{-1}\circ\psi_2)(x_{q+1},\dots,x_p)\right)\\&\quad\times f_{\mathbf{X}_2}(x_{q+1},\dots,x_p)\,\mathrm{d}x_{q+1}\dots\mathrm{d}x_p\end{aligned}$$

which proves the result. □

A.2.2. *Computation of* $\psi_1^{-1}\circ\psi_2$

We specify the explicit form of $\psi_1^{-1}\circ\psi_2$. For this, we use the matrices $\mathbf{B}$, $\mathbf{B}_1$, $\mathbf{B}_2$ of the linear maps ψ, ψ_1, ψ_2. Put $\Delta=\det\mathbf{B}_1$ and introduce the co-factors B_{ij}, $1\le i,j\le q$, of $\mathbf{B}_1$:

$$B_{ij}=(-1)^{i+j}\det(b_{kl})_{\substack{1\le k,l\le q\\ k\ne i,l\ne j}}=(-1)^{i+j}\begin{vmatrix}b_{11}&\dots&b_{1\,j-1}&b_{1\,j+1}&\dots&b_{1q}\\ \vdots&&\vdots&\vdots&&\vdots\\ b_{i-1\,1}&\dots&b_{i-1\,j-1}&b_{i-1\,j+1}&\dots&b_{i-1\,q}\\ b_{i+1\,1}&\dots&b_{i+1\,j-1}&b_{i+1\,j+1}&\dots&b_{i+1\,q}\\ \vdots&&\vdots&\vdots&&\vdots\\ b_{q1}&\dots&b_{q\,j-1}&b_{q\,j+1}&\dots&b_{qq}\end{vmatrix}.$$

We then have

$$\mathbf{B}_1^{-1}=\frac{1}{\Delta}\begin{pmatrix}B_{11}&\dots&B_{q1}\\ \vdots&&\vdots\\ B_{1q}&\dots&B_{qq}\end{pmatrix}.$$

Let us perform some intermediate computations:

$$\sum_{k=1}^{q} B_{ki}x_k = \sum_{k=1}^{q}(-1)^{i+k}x_k \begin{vmatrix} b_{11} & \dots & b_{1\,i-1} & b_{1\,i+1} & \dots & b_{1q} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{k-1\,1} & \dots & b_{k-1\,i-1} & b_{k-1\,i+1} & \dots & b_{k-1\,q} \\ sb_{k+1\,1} & \dots & b_{k+1\,i-1} & b_{k+1\,i+1} & \dots & b_{k+1\,q} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{q1} & \dots & b_{q\,i-1} & b_{q\,i+1} & \dots & b_{qq} \end{vmatrix}$$

$$= \begin{vmatrix} b_{11} & \dots & b_{1\,i-1} & x_1 & b_{1\,i+1} & \dots & b_{1q} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{q1} & \dots & b_{q\,i-1} & x_q & b_{q\,i+1} & \dots & b_{qq} \end{vmatrix} = \det(\vec{v}_1,\dots,\vec{v}_{i-1},\vec{u},\vec{v}_{i+1},\dots,\vec{v}_q)$$

where we set in the last displayed equality

$$\vec{u} = \sum_{j=1}^{q} x_j\vec{e}_j \quad \text{and} \quad \vec{v}_k = \sum_{j=1}^{q} b_{jk}\vec{e}_j \quad \text{for } 1 \le k \le q.$$

Observe that the relation $\sum_{k=1}^{q} B_{ki}x_k = 0$ is the equation of the vectorial space spanned by the vectors $\vec{v}_1,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_q$. In the case where $x_k = b_{jk}$, $q+1 \le j \le p$, we get the matrix of the map $\psi_1^{-1} \circ \psi_2$:

$$\mathbf{B}_1^{-1}\mathbf{B}_2 = \frac{1}{\Delta}\begin{pmatrix} \Delta_{1\,q+1} & \dots & \Delta_{1p} \\ \vdots & & \vdots \\ \Delta_{q\,q+1} & \dots & \Delta_{qp} \end{pmatrix}.$$

where, for $1 \le i \le q$, $q+1 \le j \le p$,

$$\Delta_{ij} = \sum_{k=1}^{q} B_{ki}b_{kj} = \begin{vmatrix} b_{11} & \dots & b_{1\,i-1} & b_{1j} & b_{1\,i+1} & \dots & b_{1q} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{q1} & \dots & b_{q\,i-1} & b_{qj} & b_{q\,i+1} & \dots & b_{qq} \end{vmatrix}$$

$$= \det(\vec{v}_1,\dots,\vec{v}_{i-1},\vec{v}_j,\vec{v}_{i+1},\dots,\vec{v}_q). \tag{A.1}$$

We also have

$$\Delta = \det(\vec{v}_1,\dots,\vec{v}_q).$$

A.2.3. *Some constraints*

In the case where all the random variables $X_1,\dots,X_p$ are positive, we specify a little more the domain over which the integration in Lemma A.4 should be carried. For this, we see that we must have $(x_{q+1},\dots,x_p) \in (0,+\infty)^{p-q}$ together with $\psi_1^{-1}(y_1,\dots,y_q) - (\psi_1^{-1} \circ \psi_2)(x_{q+1},\dots,x_p) \in (0,+\infty)^q$. Assume for instance $\Delta > 0$. The domain of integration then writes

$$\mathcal{D} = \left\{(x_{q+1},\dots,x_p) \in (0,+\infty)^{p-q} : \sum_{k=1}^{q} B_{ki}y_k - \sum_{j=q+1}^{p} \Delta_{ij}x_j \ge 0 \quad \text{for } 1 \le i \le q\right\}$$

and the variables $y_1, \dots, y_q$ are subject to the condition that this set is non-empty. Notice that

$$\sum_{j=q+1}^{p} \Delta_{ij} x_j = \sum_{j=q+1}^{p} \left(\sum_{k=1}^{q} B_{ki} b_{kj} \right) x_j = \sum_{k=1}^{q} B_{ki} \left(\sum_{j=q+1}^{p} b_{kj} x_j \right) = \sum_{k=1}^{q} B_{ki} x'_k$$

where, in the last above equality, the x'_k's are the components with respect to the canonical basis of $\mathbb{R}^q$ of the vector

$$\sum_{k=1}^{q} x'_k \vec{e}_k = \sum_{k=1}^{q} \left(\sum_{j=q+1}^{p} b_{kj} x_j \right) \vec{e}_k = \sum_{j=q+1}^{p} x_j \left(\sum_{k=1}^{q} b_{kj} \vec{e}_k \right) = \sum_{j=q+1}^{p} x_j \vec{v}_j .$$

Thus

$$\sum_{k=1}^{q} B_{ki} y_k - \sum_{j=q+1}^{p} \Delta_{ij} x_j = \det\left(\vec{v}_1, \dots, \vec{v}_{i-1}, \sum_{k=1}^{q} y_k \vec{e}_k - \sum_{j=q+1}^{p} x_j \vec{v}_j, \vec{v}_{i+1}, \dots, \vec{v}_p \right).$$

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