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A TWO ARMED BANDIT TYPE PROBLEM REVISITED

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Abstract. In Benaïm and Ben Arous (2003) is solved a multi-armed bandit problem arising in the theory of learning in games. We propose a short and elementary proof of this result based on a variant of the Kronecker lemma.

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In [2] a multi-armed bandit problem is addressed and investigated by Benaïm and Ben Arous. Let $f_0, \dots, f_d$ denote $d+1$ real-valued continuous functions defined on $[0,1]^{d+1}$. Given a sequence $x = (x_n)_{n\geq 1} \in \{0,\dots,d\}^{\mathbb{N}^*}$ (the *strategy*), set for every $n \geq 1$

$$\bar{x}_n := (\bar{x}_n^0, \bar{x}_n^1, \dots, \bar{x}_n^d) \quad \text{with} \quad \bar{x}_n^i := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{x_k = i\}}, \;\; i = 0, \dots, d,$$

and

$$Q(x) = \liminf_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{x_{k+1}}(\bar{x}_k).$$

($\bar{x}_0 := (\bar{x}_0^0, \bar{x}_0^1, \dots, \bar{x}_0^d) \in [0,1]^{d+1}$, $\bar{x}_0^0 + \dots + \bar{x}_0^d = 1$, is a starting distribution). Imagine $d+1$ players enrolled in a cooperative/competitive game with the following simple rules: if player $i \in \{0,\dots,d\}$ plays at time n he is rewarded by $f_i(\bar{x}_n)$, otherwise he gets nothing; only one player can play at any given time. Then the sequence x is a playing strategy adopted by the group of players and $Q(x)$ is the *global* worst cumulative payoff rate of the strategy x for the whole community of players (regardless of the cumulative payoff rate of each player). This interpretation slightly differs from that proposed in [2] where a single player is considered. This player has the choice among $d+1$ "arms" at every time n with a reward $f_i(\bar{x}_n)$ when choosing "arm" i. We adopt the first one in view of our illustration.

In [2] an answer (see Th. 1 below) is provided to the following question

What are the good strategies (for the group)?

The authors rely on some recent tools developed in stochastic approximation theory (see *e.g.* [1]). The aim of this note is to provide an elementary and shorter proof based on a slight improvement of the Kronecker lemma. As an illustration, we emphasize that in such a game a *greedy* strategy is usually not optimal, even for the "individual winner".

Keywords and phrases. Two-armed bandit problem, Kronecker lemma, learning theory, stochastic fictitious play.

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Let $\mathcal{S}_d := \{v = (v^1, \dots, v^d) \in [0,1]^d,\ \sum_{i=1}^d v_i \le 1\}$ and $\mathcal{P}_{d+1} := \{u = (u^0, u^1, \dots, u^d) \in [0,1]^{d+1},\ \sum_{i=1}^{d+1} u_i = 1\}$. Furthermore, for notational convenience, set

$$\begin{aligned}\forall v \in \mathcal{S}_d,\ \tilde{v} &:= \left(1 - \sum_{i=1}^d v^i, v^1, \dots, v^d\right) \in \mathcal{P}_{d+1},\\ \forall u \in \mathcal{P}_{d+1},\ {}^\sigma u &:= (u^1, \dots, u^d) \in \mathcal{S}_d. \end{aligned} \tag{1}$$

The canonical inner product on $\mathbb{R}^d$ will be denoted by $(v|w) = \sum_{i=1}^d v^i w^i$. The interior of a subset A of $\mathbb{R}^d$ will be denoted by $\mathring{A}$. For a sequence $u = (u_n)_{n\ge 0}$, $\Delta u_n := u_n - u_{n-1}$, $n \ge 1$.

The main result is the following theorem (first established in [2]).

Theorem 1. *Assume there is a continuous function $\Phi : \mathcal{S}_d \to \mathbb{R}$, continuously differentiable on $\mathring{\mathcal{S}}_d$, having a continuous extension of its gradient $\nabla\Phi$ on $\mathcal{S}_d$ and satisfying:*

$$\forall v \in \mathcal{S}_d, \quad \nabla\Phi(v) = (f_i(\tilde{v}) - f_0(\tilde{v}))_{1\le i\le d}\,. \tag{2}$$

Set for every $u \in \mathcal{P}_{d+1}$,

$$q(u) := \sum_{i=0}^{d+1} u^i f_i(u)$$

and $Q^* := \max\{q(u),\ u \in \mathcal{P}_{d+1}\}$. Then, for every strategy $x \in \{0, 1, \dots, d\}^{\mathbb{N}^*}$,

$$Q(x) \le Q^*.$$

Furthermore, for any strategy x such that $\bar{x}_n \to \bar{x}_\infty$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f_{x_{k+1}}(\bar{x}_k) \to q(\bar{x}_\infty) \quad \text{as} \quad n \to \infty \qquad (\text{so that } Q(x) = q(\bar{x}_\infty)).$$

In particular there is no better strategy than choosing the player at random according to an i.i.d. "Bernouilli strategy" with parameter $\bar{x}^* \in \operatorname{argmax} q$.

The key of the proof is the following slight extension of the Kronecker lemma.

Lemma 1 ("à la Kronecker" lemma)**.** *Let $(b_n)_{n\ge1}$ be a nondecreasing sequence of positive real numbers converging to $+\infty$ and let $(a_n)_{n\ge1}$ be a sequence of real numbers. Then*

$$\liminf_{n\to+\infty} \sum_{k=1}^n \frac{a_k}{b_k} \in \mathbb{R} \quad \Longrightarrow \quad \liminf_{n\to+\infty} \frac{1}{b_n}\sum_{k=1}^n a_k \le 0.$$

Proof. Set $C_n = \displaystyle\sum_{k=1}^n \frac{a_k}{b_k}$, $n \ge 1$, and $C_0 = 0$ so that $a_n = b_n \Delta C_n$. As a consequence, an Abel transform yields

$$\begin{aligned}\frac{1}{b_n}\sum_{k=1}^n a_k &= \frac{1}{b_n}\sum_{k=1}^n b_k \Delta C_k = \frac{1}{b_n}\left(b_n C_n - \sum_{k=1}^n C_{k-1}\Delta b_k\right)\\ &= C_n - \frac{1}{b_n}\sum_{k=1}^n C_{k-1}\Delta b_k.\end{aligned}$$

Now, $\liminf_{n\to+\infty} C_n$ being finite, for every $\varepsilon > 0$, there is an integer n_ε such that for every $k \geq n_\varepsilon$, $C_k \geq \liminf_{n\to+\infty} C_n - \varepsilon$. Hence

$$\frac{1}{b_n}\sum_{k=1}^{n} C_{k-1}\Delta b_k \geq \frac{1}{b_n}\sum_{k=1}^{n_\varepsilon} C_{k-1}\Delta b_k + \frac{b_n - b_{n_\varepsilon}}{b_n}\left(\liminf_k C_k - \varepsilon\right).$$

Consequently, $\liminf_{n\to+\infty} C_n$ being finite, one concludes that for every $\varepsilon > 0$,

$$\liminf_{n\to+\infty} \frac{1}{b_n}\sum_{k=1}^{n} a_k \leq \liminf_{n\to+\infty} C_n - 0 - 1 \times \left(\liminf_{k\to+\infty} C_k - \varepsilon\right) = \varepsilon. \qquad \square$$

Proof of Theorem 1. First note that for every $u = (u^0, u^1, \ldots, u^d) \in \mathcal{P}_{d+1}$,

$$q(u) := \sum_{i=0}^{d+1} u^i f_i(u) = f_0(u) + \sum_{i=1}^{d} u^i (f_i(u) - f_0(u))$$

so that

$$Q^* = \sup_{v\in\mathcal{S}_d}\left\{ f_0(\tilde{v}) + \sum_{i=1}^{d} v^i (f_i(\tilde{v}) - f_0(\tilde{v}))\right\} = \sup_{v\in\mathcal{S}_d}\left\{ f_0(\tilde{v}) + (v\,|\nabla\Phi(v))\right\}.$$

Now, for every $k \geq 0$,

$$\begin{aligned} f_{x_{k+1}}(\bar{x}_k) - q(\bar{x}_k) &= \sum_{i=0}^{d}(f_i(\bar{x}_k)\,\mathbf{1}_{\{x_{k+1}=i\}} - \bar{x}_k^i f_i(\bar{x}_k)) = \sum_{i=0}^{d} f_i(\bar{x}_k)(\,\mathbf{1}_{\{x_{k+1}=i\}} - \bar{x}_k^i) \\ &= \sum_{i=0}^{d} f_i(\bar{x}_k)(k+1)\Delta\bar{x}_{k+1}^i \\ &= (k+1)\sum_{i=1}^{d}(f_i(\bar{x}_k) - f_0(\bar{x}_k))\Delta\bar{x}_{k+1}^i. \end{aligned}$$

The last equality reads using Assumption (2) and notation (1),

$$f_{x_{k+1}}(\bar{x}_k) - q(\bar{x}_k) = (k+1)(\nabla\Phi({}^\sigma\bar{x}_k)\,|\,\Delta\,{}^\sigma\bar{x}_{k+1}).$$

Consequently, by the fundamental formula of calculus applied to Φ on $({}^\sigma\bar{x}_k,\ {}^\sigma\bar{x}_{k+1}) \subset \mathring{\mathcal{S}}_d$,

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} f_{x_{k+1}}(\bar{x}_k) - q(\bar{x}_k) &= \frac{1}{n}\sum_{k=0}^{n-1}(k+1)\left(\Phi({}^\sigma\bar{x}_{k+1}) - \Phi({}^\sigma\bar{x}_k)\right) - R_n \\ \text{with} \qquad R_n &:= \frac{1}{n}\sum_{k=0}^{n-1}\left(\nabla\Phi(\xi_k) - \nabla\Phi({}^\sigma\bar{x}_k)\,|\,(k+1)\Delta\,{}^\sigma\bar{x}_{k+1}\right) \end{aligned}$$

and $\xi_k \in ({}^\sigma\bar{x}_k, {}^\sigma\,\bar{x}_{k+1}),\ k = 0, \ldots, n-1$. The fact that $|(k+1)\Delta\,{}^\sigma\bar{x}_{k+1}| \leq 1$ implies

$$|R_n| \leq \frac{1}{n}\sum_{k=0}^{n-1} w(\nabla\Phi, |\Delta\,{}^\sigma\bar{x}_{k+1}|)$$

where $w(g,\delta)$ denotes the uniform continuity δ-modulus of a function g. One derives from the uniform continuity of $\nabla\Phi$ on the compact set $\mathcal{S}_d$ that

$$R_n \to 0 \quad \text{as} \quad n \to +\infty.$$

Finally, the continuous function Φ being bounded on the compact set $\mathcal{S}_d$, the partial sums

$$\sum_{k=0}^{n-1} \Phi({}^\sigma\bar{x}_{k+1}) - \Phi({}^\sigma\bar{x}_k) = \Phi({}^\sigma\bar{x}_{n+1}) - \Phi({}^\sigma\bar{x}_0)$$

remain bounded as n goes to infinity. Lemma 1 then implies that

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1}(k+1)\left(\Phi({}^\sigma\bar{x}_{k+1}) - \Phi({}^\sigma\bar{x}_k)\right) \le 0.$$

One concludes by noting that on one hand

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} q(\bar{x}_k) \le Q^* = \sup_{\mathcal{P}_{d+1}} q$$

and that, on the other hand, the function q being continuous,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} q(\bar{x}_k) = q(x^*) \quad \text{as soon as} \quad \bar{x}_n \to x^*. \qquad \square$$

Corollary 1. *When $d+1=2$ (two players), Assumption (2) is satisfied as soon as f_0 and f_1 are continuous on $\mathcal{P}_2$ and then the conclusions of Theorem 1 hold true.*

Proof. This follows from the obvious fact that the continuous function $u^1 \mapsto f_1(1-u^1,u^1) - f_0(1-u^1,u^1)$ on $[0,1]$ has an antiderivative. $\square$

Further comments:

• If one considers a slightly more general game in which some *weighted strategies* are allowed, the final result is not modified in any way provided the weight sequence satisfies a very light assumption. Namely, assume that at time n the reward is

$$\Delta_{n+1} f_{x_{n+1}}(\bar{x}_n) \qquad \text{instead of} \qquad f_{x_{n+1}}(\bar{x}_n)$$

where the weight sequence $\Delta = (\Delta_n)_{n\ge 1}$ satisfies

$$\Delta_n \ge 0,\ n\ge 1, \quad S_n = \sum_{k=1}^{n} \Delta_k \to +\infty, \quad \frac{\Delta_n}{S_n} \to 0 \text{ as } n\to\infty$$

then the quantities $\bar{x}_0^\Delta \in \mathcal{P}_{d+1}$, $\bar{x}_n^\Delta := (\bar{x}_n^{\Delta,0},\dots,\bar{x}_n^{\Delta,d})$ with $\bar{x}_n^{\Delta,i} = \frac{1}{S_n}\sum_{k=1}^n \Delta_k \mathbf{1}_{\{x_k=i\}}$, $i=0,\dots,d$, $n\ge 1$, and $Q^\Delta(x) = \liminf_{n\to+\infty} \dfrac{1}{S_n}\sum_{k=0}^{n-1} \Delta_{k+1} f_{x_{k+1}}(\bar{x}_k^\Delta)$ satisfy all the conclusions of Theorem 1 *mutatis mutandis*.

• Several applications of Theorem 1 to the theory of learning in games and to stochastic fictitious play are extensively investigated in [2] which we refer to for all these aspects. As far as we are concerned we will simply make a remark about some "natural" strategies which illustrates the theorem in an elementary way.

In the reward function at time k, *i.e.* $f_{x_k}(\bar{x}_{k-1})$, x_k represents the competitive term ("who will play?") and $\bar{x}_{k-1}$ represents a cooperative term (everybody's past behaviour has influence on everybody's reward).

This cooperative/competitive antagonism induces that in such a game a *greedy* competitive strategy is usually not optimal (when the players do not play a symmetric role). Let us be more specific. Assume for the sake of simplicity that $d+1=2$ (two players). Then one may consider without loss of generality that $\bar{x}_n = {}^{\sigma}\bar{x}_n$ *i.e.* that $\bar{x}_n$ is a $[0,1]$-valued real number. A *greedy competitive* strategy is defined by

$$\text{player 1 plays at time } n \ \ (i.e.\ x_n = 1) \quad \text{iff} \quad f_1(\bar{x}_{n-1}) \geq f_0(\bar{x}_{n-1}) \tag{3}$$

i.e. the player with the highest reward is nominated to play. Then, for every $n \geq 1$,

$$f_{x_n}(\bar{x}_{n-1}) = \max(f_0(\bar{x}_{n-1}), f_1(\bar{x}_{n-1}))$$

and it is clear that

$$f_{x_n}(\bar{x}_{n-1}) - q(\bar{x}_{n-1}) = \max(f_0(\bar{x}_{n-1}), f_1(\bar{x}_{n-1})) - q(\bar{x}_{n-1}) =: \varphi(\bar{x}_{n-1}) \geq 0.$$

On the other hand, the proof of Theorem 1 implies that

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(\bar{x}_k) \leq 0.$$

Hence, there is at least one weak limiting distribution $\bar{\mu}_\infty$ of the sequence of empirical measures $\bar{\mu}_n := \frac{1}{n}\sum_{0\leq k\leq n-1} \delta_{\bar{x}_k}$ on the compact interval $[0,1]$ which is supported by the closed set $\{\varphi = 0\} \subset \{0,1\} \cup \{f_0 = f_1\}$; on the other hand $\mathrm{supp}(\mu_\infty)$ is contained in the set $\bar{\mathcal{X}}_\infty$ of the limiting values of the sequence $(\bar{x}_n)$ itself (in fact $\bar{\mathcal{X}}_\infty$ is an interval since $(\bar{x}_n)_n$ is bounded and $\bar{x}_{n+1} - \bar{x}_n \to 0$). Hence $\bar{\mathcal{X}}_\infty \cap (\{0,1\} \cup \{f_0 = f_1\}) \neq \emptyset$.

If the greedy strategy $(\bar{x}_n)_n$ is optimal then $\mathrm{dist}(\bar{x}_n, \mathrm{argmax}\, q) \to 0$ as $n \to \infty$ *i.e.* $\bar{\mathcal{X}}_\infty \subset \mathrm{argmax}\, q$. Consequently if

$$\mathrm{argmax}\, q \cap (\{0,1\} \cup \{f_0 = f_1\}) = \emptyset \tag{4}$$

then *the purely competitive strategy is never optimal* for the group of two players.

Let us be more specific on the following example: set for two positive parameters $a \neq b$

$$f_0(x) := a\,x \qquad \text{and} \qquad f_1(x) := b\,(1-x), \qquad x \in [0,1].$$

Then one checks that

$$\mathrm{argmax}\, q = \{1/2\} \qquad \text{and} \qquad f_0(1/2) \neq f_1(1/2).$$

One first shows that the greedy strategy $x = (x_n)_{n\geq 1}$ defined by (3) satisfies

$$\bar{x}_n \to \frac{b}{a+b} \qquad \text{and} \qquad Q(x) = \frac{ab}{a+b} \qquad \text{as} \qquad n \to \infty.$$

On the other hand, any optimal (cooperative) strategy (like the *i.i.d.* Bernoulli(1/2) one) yields an asymptotic (relative) global payoff rate

$$Q^* = \max_{[0,1]} q = \frac{a+b}{4}\cdot$$

Note that $Q^* > \frac{ab}{a+b}$ since $a \neq b$. (When $a = b$ the greedy strategy becomes optimal.)

Now, if one looks at the *individual* performances (*i.e.* $\lim_n \frac{1}{n}\sum_{0\leq k\leq n-1} f_i(\bar{x}_k)\mathbf{1}_{\{x_{k+1}=i\}}$, $i = 0,1$) of both players when the greedy strategy is played, one checks that:

– the "winner" of the game is player 1 if $b > a$ and player 0 if $a > b$,

– the asymptotic (relative) payoff rate of the winner is equal to $\frac{ab\max(a,b)}{(a+b)^2}$ (and $\frac{ab\min(a,b)}{(a+b)^2}$ for the "looser").

If an optimal cooperative strategy is adopted by the players the "winner" remains the same but with an asymptotic payoff rate equal to $\frac{\max(a,b)}{4}$ (the "looser" gets $\frac{\min(a,b)}{4}$). Consequently (when $a \neq b$), *an optimal cooperative strategy always yields to the winner a strictly higher asymptotic payoff rate than the greedy one.* This is also true for the looser.

• A more abstract version of Theorem 1 can be established using the same approach. The finite set $\{0, 1, \ldots, d\}$ is replaced by a compact metric set K, $\mathcal{P}_{d+1}$ is replaced by the convex set $\mathcal{P}_K$ of probability distributions on K equipped with the weak topology and the continuous function $f : K \times \mathcal{P}_K \to \mathbb{R}$ is still supposed to derive from a potential function in some sense.

References

[1] M. Benaïm, Dynamics of stochastic algorithms, in *Séminaire de probabilités XXXIII*, J. Azéma *et al.* Eds., Springer-Verlag, Berlin. *Lect. Notes Math.* **1708** (1999) 1–68.

[2] M. Benaïm and G. Ben Arous, A two armed bandit type problem. *Game Theory* **32** (2003) 3–16.