ERGODICITY OF A CERTAIN CLASS OF NON FELLER MODELS:
APPLICATIONS TO ARCH AND MARKOV SWITCHING MODELS∗, ∗∗

JEAN-GABRIEL ATTALI

Abstract. We provide an extension of topological methods applied to a certain class of Non Feller Models which we call Quasi-Feller. We give conditions to ensure the existence of a stationary distribution. Finally, we strengthen the conditions to obtain a positive Harris recurrence, which in turn implies the existence of a strong law of large numbers.

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INTRODUCTION

It is well-known that to ensure the existence of invariant probability measure for a Feller Markov chain with a Polish state space $S$, that is whose transition operator maps the span of bounded and continuous function from $S$ to $\mathbb{R}$ into itself, it suffices to establish the tightness of the sequence of probability measures $(\frac{1}{n} \sum_{k=0}^{n-1} P_k(x_0, dy))_{n \in \mathbb{N}^*}$ for a certain $x_0$ in $S$ (see e.g. [2] or [5]). This elementary approach is very well adapted to the problem of Markov chains in $\mathbb{R}^d$. For instance, the existence of at least one invariant probability measure for a nonlinear regression model $NAR$ of the form $X_{t+1} = F(X_t) + \varepsilon_{t+1}$, where $X_t$ takes values in $\mathbb{R}^d$ and the $\varepsilon_t$ are i.i.d., can be easily proven providing that $F(x)$ is a continuous function, to ensure that the transition probability is Feller, and that it grows slowly when $x$ goes to infinity, to ensure the tightness of the sequence $(\frac{1}{n} \sum_{k=0}^{n-1} P_k(x, dy))_{n \in \mathbb{N}^*}$. For the more general model $ARCH$ of the form $X_{t+1} = F(X_t) + G(X_t)\varepsilon_{t+1}$, it suffices that $F(x)$ and $G(x)$ be continuous functions and have a good behavior when $x$ goes to infinity. Unfortunately, this topological approach doesn’t work for non Feller models. For instance, considering a $ARCH$ model, whenever $F$ or $G$ possess a point of discontinuity it is not possible to apply this method.

However, via techniques of petite sets (see e.g. [3] or [5]), we know that some $NAR$ models, where $F$ is a threshold function, have invariant probability although they are not Feller. Thus, our aim is to extend the topological approach to the study of non Feller models as well. In this context, we introduce the concept of Quasi-Feller models.

The paper is organized as follows. In Section 1 we define Quasi-Feller models. In Section 2 we then give conditions to obtain at least one invariant probability measure and finally, in Section 3 we closely study the
problem of the positive Harris recurrence for such models. ARCH models and nonlinear autoregressive models with Markov switching (NAR-MS) illustrate our results throughout the paper.

1. Definition of Quasi-Feller models

1.1. Definitions

(S, d) is a separable complete metric space and \( \mathcal{B}(S) \) is its Borel \( \sigma \)-field. We denote the class of bounded continuous functions from \( S \) to \( \mathbb{R} \) by \( C_b(S, \mathbb{R}) \). Let us first define the Quasi-Feller models as follows:

**Definition 1.1.** A transition kernel \( P(x, dy) \) is Quasi-Feller if there exist a topological space \( W \), a Borel function \( H: S \to W \) such that, for all compact subset \( K \), \( \overline{H(K)} \) is a compact subset and a family of probability measures \( (Q(w, dy))_{w \in W} \) on \( (S, \mathcal{B}(S)) \), verifying:

i) the kernel \( Q \) defined for any measurable and bounded function \( g: S \to \mathbb{R} \) by \( Qg(w) := \int_S g(y)Q(w, dy) \) is Feller that is:

\[ g \in C_b(S, \mathbb{R}) \Rightarrow Qg \in C_b(W, \mathbb{R}); \]

ii) \( \forall g \) measurable mapping from \( S \) to \( \mathbb{R} \), \( Pg(x) = Qg(H(x)) \);

iii) \( \forall w \in \bigcup_{K \in K} \overline{H(K)} \) we have \( Q(w, D_H) = 0 \), where \( D_H \) denotes the set of discontinuity points of \( H \) and \( K \) is the span of compact subsets of \( W \).

**Remark 1.2.**

- Clearly \( \bigcup_{K \in K} \overline{H(K)} \subset \overline{H(S)} \). If \( H \) is a continuous function, the above inclusion holds as an equality.

  On the other hand, as soon as \( H \) is not continuous at one point, the inclusion may be strict.

- \( D_H \) is a measurable set (see e.g. [1]).

- The decomposition of \( P \) following \( Q \) and \( H \) is not unique.

- A Feller transition kernel \( P \) is Quasi-Feller with \( W = S, H = Id_S, D_H = \emptyset \) and \( Q = P \).

- In our examples, \( W \) generally will be \( \mathbb{R}^p \) or one of its subsets.

1.2. Examples

1.2.1. ARCH model

Let \( F: \mathbb{R}^d \to \mathbb{R}^d \) and \( G: \mathbb{R}^d \to S^+(d, \mathbb{R}) \) (where \( S^+(d, \mathbb{R}) \) denotes the set of definite positive matrix) be measurable functions. Let \( (\varepsilon_t)_{t \geq 1} \) be a white noise whose distribution \( \mu \) is defined on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). The ARCH model is then defined by:

\[
X_{t+1} = F(X_t) + G(X_t)\varepsilon_{t+1}, \quad X_0 \in \mathbb{R}^d.
\] (1)

Let us define the function:

\[
G^{-1} : \mathbb{R}^d \to S^+(d, \mathbb{R}),
\]

\[
x \mapsto (G(x))^{-1},
\]

and the norm of a \( d \times d \) matrix:

\[
\|A\| := \sup_{\|u\|=1} \frac{|Au|}{\|u\|}.
\]

The transition probability \( P(x, dy) \) of the ARCH is given by:

\[
\forall g \text{ bounded measurable function, } Pg(x) = \int g(F(x) + G(x)y) \mu(dy).
\] (2)
Proposition 1.3. If the set \( D_{F,G} := D_F \cup D_G \) satisfies:

\[
\forall (w^1, w^2) \in \bigcup_{K \in \mathcal{K}} F(K) \times G(K), \quad \mu \left( (w^2)^{-1} (D_{F,G} - w^1) \right) = 0,
\]

then the ARCH model is Quasi-Feller.

Proof. Let us define, for any bounded measurable function \( g : \mathbb{R}^d \to \mathbb{R} \) the function \( Qg \) as follows:

\[
\forall (w^1, w^2) \in \mathbb{R}^d \times S^+(d, \mathbb{R}), \quad Qg (w^1, w^2) = \int g (w^1 + w^2 y) \mu(dy).
\]

We have \( P g(x) = Qg(F(x), G(x)) \). \( Q \) obviously is Feller according to the Definition 1.1 (\( Q \) is no longer a transition probability because \( E = \mathbb{R}^d \times S^+(d, \mathbb{R}) \)).

The hypothesis \( \forall w \in \bigcup_{K \in \mathcal{K}} F(K) \times G(K), \ Q(w, D_{F,G}) = 0 \) turns into

\[
\forall w := (w^1, w^2) \in \bigcup_{K \in \mathcal{K}} F(K) \times G(K), \quad \mu \left( (w^2)^{-1} (D_{F,G} - w^1) \right) = 0. \]

\[\square\]

Remark 1.4. In this case, we have: \( W = \mathbb{R}^d \times S^+(d, \mathbb{R}) \), and \( \overline{G(K)} \subset S^+(d, \mathbb{R}) \).

1.2.2. Nonlinear autoregressive model with Markov switching (NAR-MS)

Definition 1.5. \((Y_t)_{t \in \mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}\) is a NAR-MS if:

\[
Y_t := f_{X_t} (Y_{t-1}) + \varepsilon_t,
\]

where \((X_t)_{t \geq 0}\) is a positive Harris recurrent chain on \(\{1, \cdots, m\}\), for all \(k \in \{1, \cdots, m\}\), \(f_k\) is a measurable function from \(\mathbb{R}^d\) into itself and the \(\varepsilon_t\) are i.i.d. random variables taking values in \(\mathbb{R}^d\).

Example 1.6. Let \((Y_t)_{t \in \mathbb{N}}\) be a NAR-MS. Let \(\mu\) denote the distribution of \(\varepsilon_0\) and \(D_k\) denote the set of discontinuity points of \(f_k\). If:

\[
\forall k \in \{1, \cdots, m\}, \quad \forall z \in \bigcup_{K \in \mathcal{K}} \overline{f_k(K)}, \quad \mu(D_k - z) = 0,
\]

then the NAR-MS is Quasi-Feller.

Proof. \(X_t\) being positive Harris, there exist a measurable function \(h : \{1, \cdots, m\} \times \mathbb{R} \to \{1, \cdots, m\}\) and a sequence of i.i.d. real valued variables \((\eta_t)_{t \geq 1}\) with distribution \(\nu\) and which are independent from the process \((\varepsilon_t)_{t \geq 0}\) such that:

\[
X_t = h(X_{t-1}, \eta_t).
\]

Then, the process:

\[
Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} h(X_{t-1}, \eta_t) \\ f_{h(X_{t-1}, \eta_t)} (Y_{t-1}) + \varepsilon_t \end{pmatrix},
\]

is a Markov chain whose transition kernel \(P\) verifies:

\[
\forall g \in C_b \left( \{1, \cdots, m\} \times \mathbb{R}^d, \mathbb{R} \right), \quad Pg(x, y) = \int_{\mathbb{R}} \nu(d\eta) \int_{\mathbb{R}^d} g \left( h(x, \eta), f_{h(x, \eta)}(y) + \varepsilon \right) \mu(d\varepsilon).
\]
Now let define, for all \( g \in C_b \left( \{1, \cdots, m\} \times \mathbb{R}^d, \mathbb{R} \right) \) the function \( Qg \) as follows:

\[
\forall (x, z_1, \cdots, z_m) \in \{1, \cdots, m\} \times (\mathbb{R}^d)^m,
Qg(x, z_1, \cdots, z_m) = \int_{\mathbb{R}^d} \nu(d \eta) \int_{\mathbb{R}^d} g \left( h(x, \eta), \sum_{k=1}^{m} 1_{\{k\}} (h(x, \eta))z_k + \varepsilon \right) \mu(d \varepsilon).
\]

(5)

For all \( x \) and \( \eta, (z_1, \cdots, z_m) \mapsto g(h(x, \eta), \sum_{k=1}^{m} 1_{\{k\}} (h(x, \eta))z_k + \varepsilon) \) is a continuous function. So, the function \( (x, z_1, \cdots, z_m) \mapsto Qg(x, z_1, \cdots, z_m) \) is continuous. But \( x \) takes values in a finite set, hence the function \( (x, z_1, \cdots, z_m) \mapsto Qg(x, z_1, \cdots, z_m) \) is also continuous.

Setting \( H(x, y) = (x, f_1(y), \cdots, f_m(y)) \), we have:

\[
\forall g \in C_b \left( \{1, \cdots, m\} \times \mathbb{R}^d, \mathbb{R} \right), \quad Pg(x, y) = Qg(H(x, y)).
\]

To finish, \( D_H \subset \{1, \cdots, m\} \times \bigcup_{k=1}^{m} D_k \). Hence, the hypothesis:

\[
\forall k \in \{1, \cdots, m\}, \forall z_k \in \bigcup_{K \in \mathcal{K}} f_k(K), \quad \mu(D_k - z_k) = 0,
\]

implies that:

\[
\forall (x, z_1, \cdots, z_m) \in \bigcup_{K \in \mathcal{K}} \overline{H(K)}, \quad Q(x, z_1, \cdots, z_m, D_H) = 0,
\]

and so the NAR-MS is Quasi-Feller.

\[\square\]

2. Existence of a Stationary Probability Measure for Quasi-Feller Models

**Theorem 2.1.** Let us consider a Quasi-Feller transition probability \( P(x, dy) \) associated to a function \( H \) a kernel \( Q \) according to Definition 1.1.

Suppose there exists \( x_0 \in S \) such that \( \left( \frac{1}{n} \sum_{k=0}^{n-1} P^k(x_0, dy) \right)_{n \in \mathbb{N}^*} \) is tight.

(a) the model has an invariant probability measure;

(b) for all invariant probability measure \( \nu \) and all measurable set \( C \) satisfying:

\[
\forall w \in \bigcup_{K \in \mathcal{K}} \overline{H(K)}, \quad Q(w, C) = 0,
\]

(6)

then \( \nu(C) = 0 \). In particular, \( \nu(D_H) = 0 \).

**Proof.** The idea is to show, as for Feller Models, that any limit point \( \nu \) for weak convergence of the tight sequence \( \frac{1}{n} \sum_{k=1}^{n-1} P^k(x_0, dy) \) is an invariant probability distribution.

The key is to show that \( \nu(D_H) = 0 \). Let us prove directly point (b). Let \( C \) be a measurable subset of \( S \) satisfying (6). Let prove that \( \nu(C) = 0 \).

If \( F \) be a closed subset of \( C \). then \( F \) also satisfies (6), i.e.:

\[
\forall w \in \bigcup_{K \in \mathcal{K}} \overline{H(K)}, \quad Q(w, F) = 0.
\]

Let \( (h_p)_{p \in \mathbb{N}} \) be the sequence of functions from \( S \) to \( \mathbb{R} \) defined by:

\[
\forall p \in \mathbb{N}^*, \forall x \in S, \quad h_p := 1 - \min(pd(x, F), 1).
\]
For all \( p \in \mathbb{N} \), \( h_p \) are continuous functions (even \( p \)-Lipschitz). Moreover, they verify \( \|h_p\|_{\infty} \leq 1 \) and \( h_p \downarrow 1_F \) for \( F \) is a closed subset.

Let us consider now \( (Qh_p)_{p \in \mathbb{N}} \) the sequence of functions from \( W \) to \( \mathbb{R} \). For \( Q \) is Feller, \( Qh_p \) are so continuous functions. On the other hand, \( Q(w, dy) \) is a probability measure for all \( w \in W \), hence following dominated convergence Theorem, we have:

\[
\forall w \in \bigcup_{K \in \mathcal{K}} \overline{H(K)}, \ Qh_p(w) \downarrow Q(w, F).
\]

Let \( K \) be a compact subset of \( S \). \( Qh_p \downarrow 0 \) on \( \overline{H(K)} \) for \( Q(w, F) = 0 \) if \( w \in \overline{H(K)} \). At last, \( \overline{H(K)} \) is, by hypothesis, a compact subset. Then, the Dini’s Theorem implies that the above convergence is uniform on \( \overline{H(K)} \), i.e.

\[
\sup_{w \in \overline{H(K)}} Qh_p(w) \to 0 \text{ when } p \to +\infty.
\]

(7)

Define \( \nu_n := \frac{1}{n} \sum_{k=0}^{n-1} P^k(x_0, dy) \) with the usual convention \( P^0(x_0, dy) = \delta_{x_0}(dy) \). The sequence \( (\nu_n)_{n \in \mathbb{N}^*} \) is tight, so we have:

\[
\forall \varepsilon > 0, \exists K_\varepsilon \text{ compact subset of } S \text{ such that, } \forall n \in \mathbb{N}^*, \ \nu_n(C_{K_\varepsilon}) \leq \varepsilon.
\]

(8)

It follows that, for all \( n, p \geq 1 \),

\[
\nu_n(Ph_p) = \int_{K_\varepsilon} Ph_p d\nu_n + \int_{K_\varepsilon} Ph_p d\nu_n \leq \sup_{y \in K_\varepsilon} Ph_p(y) + \nu_n(C_{K_\varepsilon}) \leq \sup_{y \in K_\varepsilon} Ph_p(y) + \varepsilon.
\]

But \( Ph_p = Qh_p \circ H \) hence, for all \( n, p \geq 1 \),

\[
\nu_n(Ph_p) \leq \sup_{w \in \overline{H(K_\varepsilon)}} Qh_p(w) + \varepsilon.
\]

Consider a subsequence \( (\nu_{\varphi(n)})_{n \in \mathbb{N}^*} \) that converges weakly to a probability measure \( \nu \). Because for all bounded function \( f \) we have:

\[
|\nu_n(Pf) - \nu_n(f)| \leq \frac{2\|f\|_{\infty}}{n},
\]

it follows immediately that:

\[
\nu(h_p) = \lim_{n \to \infty} \nu_{\varphi(n)}(h_p) = \lim_{n \to \infty} \nu_{\varphi(n)}(Ph_p) \leq \sup_{w \in \overline{H(K_\varepsilon)}} Qh_p(w) + \varepsilon.
\]

Thus, by (7) we have:

\[
\forall \varepsilon > 0, \lim_{p \to \infty} \sup \nu(h_p) \leq \varepsilon,
\]

showing that, \( \nu(F) = \lim_{p \to \infty} \nu(h_p) = 0 \).

Then, for all closed subset \( F \) of \( C \) we have \( \nu(F) = 0 \). By regularity of the probability measure \( \nu \), we have:

\[
\nu(C) := \sup \{\nu(F), \ F \subset C, \ F \text{ closed subset} \} \quad (\text{see [1]})
\]

Finally \( \nu(C) = 0 \). In particular, we have \( \nu(D_H) = 0 \).

Consider now \( h \in C_b \). One clearly has \( Ph = Qh \circ H \) is a continuous function on the continuity set of \( H \). So, \( Ph \) is \( \nu \)-a.s. continuous. Then:

\[
\nu_{\varphi(n)}(Ph) \overset{n \to +\infty}{\longrightarrow} \nu(Ph) \quad \text{for} \quad \nu_{\varphi(n)} \Rightarrow \nu.
\]
But

\[ |\nu_n(Ph) - \nu_n(h)| \xrightarrow{n \to \infty} 0 \quad \text{and} \quad \nu_{\phi(n)}(h) \xrightarrow{n \to \infty} \nu(h), \]

so we can deduce that \( \nu(Ph) = \nu(h) \) and to finish \( \nu P = \nu \) because \( h \) is an arbitrary function in \( C_0 \).

To ensure the tightness of \( \left( \frac{1}{n} \sum_{k=0}^{n-1} P^k(x,dy) \right)_{n \in \mathbb{N}} \) we usually use a Lyapunov function \( V \) i.e. verifying for \( a \in S \):

\[ \lim_{d(x,a) \to \infty} V(x) = +\infty. \]

Let recall the following result:

**Proposition 2.2.** Let \( P(x,dy) \) be a transition probability on \( S \) and \( V \) be a continuous Lyapunov function. Pakes’ lemma [6] suppose that:

i) \( \forall x \in S, \ PV(x) \leq V(x) + \beta; \)

ii) for \( a \in S \), \( \lim_{d(x,a) \to \infty} (PV - V)(x) = -\infty. \)

Then, for all \( x \) in \( S \) the sequence \( \left( \frac{1}{n} \sum_{k=0}^{n-1} P^k(x,dy) \right)_{n \in \mathbb{N}} \) is tight.

3. **Irreducibility and positive Harris recurrence**

The aim of this section is to extend the criterion of positive Harris recurrence from the Feller case as presented in [5] to our Quasi-Feller models. Any \( \psi \)-irreducible Feller chain on a polish space \( S \), where the support of \( \psi \) has nonempty interior, is such that every compact subset of \( S \) is petite. If we add the hypothesis of tightness, for all \( x \) in \( S \), of the sequence of probability measures \( \left( \frac{1}{n} \sum_{k=0}^{n-1} P^k(x,dy) \right)_{n \in \mathbb{N}} \) it is easy to show the chain is positive Harris recurrent (see e.g. [5]). Our aim is to extend this result to Quasi-Feller chain.

3.1. **Definitions**

Let recall first the notion of \( \phi \)-irreducibility as defined in [4]:

**Definition 3.1.** Consider a Markov chain with transition kernel \( P(x,dy) \) on \( S \) and \( \phi \) a nonnegative measure. The chain is said to be \( \phi \)-irreducible if,

\[ \forall x \in S, \ \phi(dy) \ll \sum_{n=1}^{+\infty} a_n P^n(x,dy), \]

where the \( a_n \) are nonnegative real numbers such that \( \sum_{n=1}^{+\infty} a_n = 1. \)

**Definition 3.2.** We will say an homogeneous Markov chain \( (X_t)_{t \in \mathbb{N}} \) is positive Harris recurrent if there exists a probability measure \( \nu \) such that:

\[ \forall g : S \to \mathbb{R} \text{ bounded continuous } \quad \forall x \in S, \quad \frac{1}{n} \sum_{k=0}^{n-1} g(X_k) \xrightarrow{\mathbb{P}, a.s.} \nu(g). \]

This definition clearly implies that \( \nu \) is the unique invariant probability measure of the chain. Moreover, if a chain is positive Harris recurrent chain with \( \nu \) for unique stationary distribution, then it is \( \nu \)-irreducible (see [5]).

In the following, \( \psi \) will denote a maximal measure of irreducibility, i.e. a measure of irreducibility that dominates all measures of irreducibility. The existence of such a measure for a \( \phi \)-irreducible Markov chain is proved in [5].
Proof. Let us show that if \( P(x, dy) \) be the transition kernel of a \( \psi \)-irreducible Markov chain. A nonempty set \( A \in \mathcal{B}(S) \) is called petite if it verifies:

\[
\inf_{x \in A} K_a(x, dy) \geq \delta \psi(dy),
\]

where \( K_a(x, dy) := \sum_{k=0}^{+\infty} a_k P^k(x, dy) \) and \( \sum_{k=0}^{+\infty} a_k = 1, \forall k \geq 1, a_k > 0, \delta > 0. \)

Remark 3.4.

- \( K_a K_b(x, dy) := \int K_a(x, dz) K_b(z, dy) = K_{a \cdot b}(x, dy). \)
- The property of petiteness depends only on \( P(x, dy) \) but not on the chosen sequence \((a_n)_{n \in \mathbb{N}^*}. \)
- Every subset of a petite set is petite.

3.2. Positive Harris recurrence of Quasi-Feller models

To obtain positive Harris recurrence for our models, the key is to prove the following result:

Proposition 3.5. Let \( P(x, dy) = Q(H(x), dy) \) be a \( \psi \)-irreducible Quasi-Feller transition probability. If \( \text{supp} \{ \psi \} \) has nonempty interior, then all compact subsets of \( S \) are petite.

Proof. The proof relies on the following two Lemmas:

Lemma 3.6. Let \( P(x, dy) = Q(H(x), dy) \) be a Quasi-Feller. Then:

(a) for any bounded function \( f : S \to \mathbb{R} \) which is continuous on \( S \setminus D_H \), and for any \( k \in \mathbb{N}^* \), \( P^k f \) is continuous on \( S \setminus D_H \);
(b) \( \forall F \) closed set, \( \forall k \in \mathbb{N}^* \), \( x \mapsto P^k(x,F) \) is upper semicontinuous on \( S \setminus D_H \);
(c) \( \forall O \) open set, \( \forall k \in \mathbb{N}^* \), \( x \mapsto P^k(x,O) \) is lower semicontinuous on \( S \setminus D_H \).

Consequently \( K_a(x, dy) \) is a Quasi-Feller transition probability. More precisely, there exists a Feller transition kernel \( \tilde{K}_a(x, dy) \) such that \( K_a(x, dy) = \tilde{K}_a(H(x), dy) \).

Proof. Let us show that if \( f \) is a lower semicontinuous function on \( S \setminus D_H \), so is \( P^k f \).

Let \( f \) be a lower semicontinuous function on \( S \setminus D_H \). Then \( f \) is the limit on \( S \setminus D_H \) of a growing sequence \((g_p)_{p \in \mathbb{N}} \) of bounded Lipschitz functions on \( S \setminus D_H \).

During the proof of Theorem 2.1, we showed if \( g : S \to \mathbb{R} \) is a bounded continuous function then \( Pg \) is continuous on \( S \setminus D_H \).

So we have:

\[
P g(x_n) \to P g(x) \quad \text{when} \quad x_n \to x.
\]

Thus, we can write:

\[
\forall x \in S \setminus D_H, P(x_n, dy) \Rightarrow P(x, dy) \quad \text{when} \quad x_n \to x.
\]

Then for any \( P(x, dy) \)-a.s. continuous function \( g \):

\[
P g(x_n) \to P g(x) \quad \text{si} \quad x_n \to x.
\]

But \( P(x, D_H) = 0 \) by hypothesis. Consequently \( Pf \) is \( P(x, dy) \)-a.s. continuous whenever \( f \) is a bounded and continuous function on \( S \setminus D_H \).

It shows that for all \( p \in \mathbb{N} \), \( P g_p \) is continuous on \( S \setminus D_H \). Then, \( x \mapsto Pf(x) \) is a lower semicontinuous function on \( S \setminus D_H \) as a growing limit of continuous functions on \( S \setminus D_H \). So, Point (a) is true. Points (b) and (c) follow easily.
To see the second part of the Lemma, let \( g : S \to \mathbb{R} \) be a bounded continuous function. We have:

\[
K_a g(x) = \sum_{k=1}^{+\infty} a_k P^k g(x)
\]

\[
= \int P(x, dy) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y)
\]

\[
= \int Q(H(x), dy) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y).
\]

By point (a), we know that \( \sum_{k=1}^{+\infty} a_k P^{k-1} g(y) \) is continuous on \( S \setminus D_H \) and so it is \( Q(z, dy) \)-a.s. continuous. \( Q \) being Feller, the function \( f(z) = \int Q(z, dy) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y) \) is a bounded continuous function from \( W \) to \( \mathbb{R} \). It implies that \( \tilde{K}_a(z, \cdot) = \int Q(z, dy) \sum_{k=1}^{+\infty} a_k P^{k-1} (y, \cdot) \) is Feller which proves the announced result for \( \tilde{K}_a(H(x), \cdot) = K_a(x, \cdot) \).

The following lemma and the end of the proof of Proposition 3.5 are extensions of Lemma 6.2.7 and Proposition 6.2.8 from [5].

**Lemma 3.7.** Let \( P(x, dy) = Q(H(x), dy) \) be a Quasi-Feller transition probability and let \( A \) be a Borel set such that \( A \cap (S \setminus D_H) \) is a petite set for \( P \). Then \( \overline{A} \cap (S \setminus D_H) \) is a petite set for \( P \).

**Proof.** By hypothesis, we have \( \inf_{x \in A \cap (S \setminus D_H)} K_a(x, dy) \geq \delta \psi(dy) \), with \( \delta > 0 \).

Let \( B \in \mathcal{B}(S) \) and let \( F \subset B \) be a closed set. Lemma 3.6 permits us to ensure that the function \( x \mapsto K_a(x, F) \) is a uniform limit of functions that are upper semicontinuous on \( S \setminus D_H \). So it is a function that is also upper semicontinuous on \( S \setminus D_H \). Thus:

\[
\inf_{x \in \overline{A} \cap (S \setminus D_H)} K_a(x, F) = \inf_{x \in A \cap (S \setminus D_H)} K_a(x, F).
\]

\( F \) being a subset of \( B \), we can deduce:

\[
\inf_{x \in \overline{A} \setminus (S \setminus D_H)} K_a(x, B) \geq \delta \psi(F).
\]

For \( \psi \) is a regular measure,

\[
\inf_{x \in \overline{A} \setminus (S \setminus D_H)} K_a(x, B) \geq \delta \psi(B),
\]

which proves that \( \overline{A} \cap (S \setminus D_H) \) is a petite set.

**End of the proof of Proposition 3.5:** The chain being \( \psi \)-irreducible, there exists a petite set \( A \) of strictly positive \( \psi \)-measure (see [5]).

Let us define for all \( k \in \mathbb{N}^* \), the following sets:

\[
A_k := \left\{ x \in S / K_a(x, A) > \frac{1}{k} \right\}.
\]

It is clear that:

\[
\bigcup_{k \in \mathbb{N}^*} A_k = \{ x \in S / K_a(x, A) > 0 \} = S \quad \text{for} \quad \psi(A) > 0.
\]
So, there exists \( k_0 \in \mathbb{N}^* \) such that for all \( k \geq k_0 \), \( A_k \neq \emptyset \). For \( k \geq k_0 \) and \( x \in A_k \), we have:

\[
K_{\ast a}(x, dy) = \int K_a(x, dz)K_a(z, dy) \\
\geq \int_A K_a(x, dz)K_a(z, dy) \\
\geq \delta \psi(dy) \times K_a(x, A) \\
\geq \frac{\delta}{k} \psi(dy).
\]

Thus, for \( k \geq k_0 \), \( A_k \) is a petite set of strictly positive \( \psi \)-measure.

Let \( O := \text{supp} \{ \psi \} \). \( O \) is a nonempty set and is of strictly positive \( \psi \)-measure. Thus, for all \( k \geq k_0 \), \( A_k \cap O \) is of strictly positive \( \psi \)-measure. On the other hand, \( A_k \cap O \) is a petite set because it is a subset of the petite set \( A_k \). So \( A_k \cap O \) is a petite set of strictly positive \( \psi \)-measure. By Proposition 2.1, we know that \( \nu(D_H) = 0 \), so \( \psi(D_H) = 0 \) for \( \psi \ll \nu \). Then, for all \( k \geq k_0 \), \( A_k \cap O \cap (S \setminus D_H) \) is a petite set of strictly positive \( \psi \)-measure.

Lemma 3.7 implies that for all \( k \geq k_0 \), the sets:

\[ B_k := A_k \cap O \cap (S \setminus D_H), \]

are petite set of strictly positive \( \psi \)-measure.

Let us define, for all \( k \geq k_0 \), the sets \( F_k := A_k \cap O \). \( F_k \) are closed sets and verify \( \bigcup_{k \geq k_0} F_k = O \). Hence, at least one of the \( F_k \) is nonempty. By Baire’s Theorem, we know then that at least one of the \( F_k \) has a nonempty interior.

Let \( F_{k_1} \) \((k_1 \geq k_0)\) be one of the \( F_k \) with a nonempty interior and let \( O_{k_1} \) be its interior. \( O_{k_1} \cap (S \setminus D_H) \) has strictly positive \( \psi \)-measure because \( O_{k_1} \subset \text{supp} \{ \psi \} \) and is a petite set as subset of \( B_{k_1} \). Hence there exists \( \delta_1 > 0 \) such that:

\[
\inf_{x \in O_{k_1} \cap (S \setminus D_H)} K_a(x, dy) \geq \delta_1 \psi(dy).
\]

Let \( x \in S \) and \( B \in \mathcal{B}(S) \). Let \( F \) be a closed subset of \( B \). We have:

\[
K_{\ast a}(x, F) = \int K_a(x, dy)K_a(y, F) \\
\geq \int_{O_{k_1} \cap (S \setminus D_H)} K_a(x, dy)K_a(y, F) \\
\geq \delta_1 \psi(F)K_a(x, O_{k_1} \cap (S \setminus D_H)).
\]

But \( K_a(x, D_H) = 0 \) so \( K_{\ast a}(x, F) \geq \delta_1 \psi(F)K_a(x, O_{k_1}) \). By taking the supremum over the closed sets included in \( B \), we obtain:

\[
K_{\ast a}(x, B) \geq \delta_1 \psi(B)K_a(x, O_{k_1}). \quad (10)
\]
If we choose $a < 1$ and $a_k = (1 - a)a^{k-1}$ for all $k \geq 1$, we have

$$K_a(x, O_{k_1}) = \sum_{k=1}^{+\infty} (1 - a)a^{k-1}P^k(x, O_{k_1})$$

$$= (1 - a)P(x, O_{k_1}) + a \sum_{k=1}^{+\infty} (1 - a)a^{k-1} \int P(x, dy)P^k(y, O_{k_1})$$

$$= (1 - a)P(x, O_{k_1}) + a \int P(x, dy)K_a(y, O_{k_1})$$

$$= (1 - a)P(x, O_{k_1}) + a \int Q(H(x), dy)K_a(y, O_{k_1})$$

$$\geq a \int Q(H(x), dy)K_a(y, O_{k_1}).$$

We already know that $K_a(y, O_{k_1})$ is lower semicontinuous on $S \setminus D_H$. Hence, $\int Q(w, dy)K_a(y, O_{k_1})$ is a positive lower semicontinuous on $W$ which implies that its infimum on any compact subset of $W$ is also positive. By hypothesis, for all $K$ compact subset of $S$, $H(K)$ is a compact subset of $W$. So we have:

$$\inf_{x \in K} K_a(y, O_{k_1}) \geq a \inf_{w \in H(K)} \int Q(w, dy)K_a(y, O_{k_1}) = \alpha > 0,$$

what replaced in (10) gives:

$$\inf_{x \in K} K_{a+a}(x, dy) \geq \alpha \delta_1 \psi(dy),$$

which proves that every compact subset $K$ of $S$ is petite. □

**Theorem 3.8.** Let $P(x, dy) = Q(H(x), dy)$ be the transition kernel of a Quasi-Feller chain. Suppose that:

i) \(\forall x \in S, \left(\frac{1}{n} \sum_{k=0}^{n-1} P^k(x, dy)\right)_{n \in \mathbb{N}}\) is tight;

ii) the chain is $\psi$-irreducible and $\text{supp}\{\psi\}$ has a nonempty interior.

Then the chain is positive Harris recurrent.

**Proof.** Proposition 3.5 proves that we are under assumptions of Theorem 6.2.5 of [5]. Then, the result follows from Theorem 9.2.2 of [5]. □

### 3.3. Applications

**Example 3.9.** Consider a ARCH model. Recall that $\mu$ denotes the distribution of $\varepsilon_0$. Let suppose:

i) $\mu \sim \lambda_d$;

ii) $F$ and $G$ are Riemann integrable;

iii) $F$, $G$, and $G^*$ are bounded on compact sets;

iv) $\forall x \in \mathbb{R}^d, \left(\frac{1}{n} \sum_{k=0}^{n-1} P^k(x, dy)\right)_{n \in \mathbb{N}}$ is tight.

Then the model is positive Harris recurrent.

**Proof.** For $F$ and $G$ are Riemann integrable, we have $\lambda_d(D_{F,G}) = 0$, so the model is Quasi-Feller. The result follows then from Theorem 3.8 for i) implies that the transition probability is $\lambda_d$-irreducible. □
Example 3.10. Consider \((Y_t)_{t \in \mathbb{N}}\) a NAR-MS. Recall that \(\mu\) denotes the distribution of \(\varepsilon_0\) and suppose that:

i) \(\mu \sim \lambda_d\);

ii) the \(f_k\) are Riemann integrables;

iii) \(\forall x \in \mathbb{R}^d, \left(\frac{1}{n} \sum_{k=0}^{n-1} P^k(x, dy)\right)_{n \in \mathbb{N}}\) is tight.

Then \((Z_t)_{t \geq 0} := (X_t, Y'_t)_{t \geq 0}\) is positive Harris.

Proof. For the \(f_k\) are Riemann integrable, we have, for all \(k \in \{1, \cdots, m\}\), \(\lambda_d(D_k) = 0\), what proves that the model is Quasi-Feller. Again, the result follows then from Theorem 3.8 for the transition probability of \((Z_t)_{t \geq 0}\) is a \(\alpha \otimes \lambda_d\)-irreducible chain, where \(\alpha\) is the measure on \(\{1, \cdots, m\}\) such that, for all \(k\), \(\alpha(k) = 1\). \(\square\)

References