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ERGODICITY OF A CERTAIN CLASS OF NON FELLER MODELS: APPLICATIONS TO *ARCH* AND MARKOV SWITCHING MODELS *, **

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Abstract. We provide an extension of topological methods applied to a certain class of Non Feller Models which we call Quasi-Feller. We give conditions to ensure the existence of a stationary distribution. Finally, we strengthen the conditions to obtain a positive Harris recurrence, which in turn implies the existence of a strong law of large numbers.

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Introduction

It is well-known that to ensure the existence of invariant probability measure for a Feller Markov chain with a Polish state space S, that is whose transition operator maps the span of bounded and continuous function from S to $\mathbb{R}$ into itself, it suffices to establish the tightness of the sequence of probability measures $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x_0, \mathrm{d}y))_{n\in\mathbb{N}^*}$ for a certain x_0 in S (see *e.g.* [2] or [5]). This elementary approach is very well adapted to the problem of Markov chains in $\mathbb{R}^d$. For instance, the existence of at least one invariant probability measure for a nonlinear regression model NAR of the form $X_{t+1} = F(X_t) + \varepsilon_{t+1}$, where X_t takes values in $\mathbb{R}^d$ and the ε_t are i.i.d., can be easily proven providing that $F(x)$ is a continuous function, to ensure that the transition probability is Feller, and that it grows slowly when x goes to infinity, to ensure the tightness of the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x, \mathrm{d}y))_{n\in\mathbb{N}^*}$. For the more general model $ARCH$ of the form $X_{t+1} = F(X_t) + G(X_t)\varepsilon_{t+1}$, it suffices that $F(x)$ and $G(x)$ be continuous functions and have a good behavior when x goes to infinity. Unfortunately, this topological approach doesn't work for non Feller models. For instance, considering a $ARCH$ model, whenever F or G possess a point of discontinuity it is not possible to apply this method.

However, *via* techniques of petite sets (see *e.g.* [3] or [5]), we know that some NAR models, where F is a threshold function, have invariant probability although they are not Feller. Thus, our aim is to extend the topological approach to the study of non Feller models as well. In this context, we introduce the concept of Quasi-Feller models.

The paper is organized as follows. In Section 1 we define Quasi-Feller models. In Section 2 we then give conditions to obtain at least one invariant probability measure and finally, in Section 3 we closely study the

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* *To my Father.*

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problem of the positive Harris recurrence for such models. $ARCH$ models and nonlinear autoregressive models with Markov switching (NAR-MS) illustrate our results throughout the paper.

1. Definition of Quasi-Feller models

1.1. **Definitions**

(S, d) is a separable complete metric space and $\mathcal{B}(S)$ is its Borel σ-field. We denote the class of bounded continuous functions from S to $\mathbb{R}$ by $C_b\,(S, \mathbb{R})$. Let us first define the Quasi-Feller models as follows:

Definition 1.1. A transition kernel $P(x, \mathrm{d}y)$ is Quasi-Feller if there exist a topological space W, a Borel function $H : S \to W$ such that, for all compact subset K, $\overline{H(K)}$ is a compact subset and a family of probability measures $(Q(w, \mathrm{d}y))_{w\in W}$ on $(S, \mathcal{B}(S))$, verifying:

i) the kernel Q defined for any measurable and bounded function $g : S \to \mathbb{R}$ by $Qg(w) := \int_S g(y)Q(w, \mathrm{d}y)$ is Feller that is:
$$g \in C_b\,(S, \mathbb{R}) \Rightarrow Qg \in C_b(W, \mathbb{R});$$
ii) $\forall g$ measurable mapping from S to $\mathbb{R}$, $Pg(x) = Qg(H(x))$;
iii) $\forall w \in \bigcup_{K\in\mathcal{K}} \overline{H(K)}$ we have $Q\,(w, D_H) = 0$, where D_H denotes the set of discontinuity points of H and $\mathcal{K}$ is the span of compact subsets of W.

Remark 1.2.

- Clearly $\bigcup_{K\in\mathcal{K}} \overline{H(K)} \subset \overline{H(S)}$. If H is a continuous function, the above inclusion holds as an equality. On the other hand, as soon as H is not continuous at one point, the inclusion may be strict.
- D_H is a measurable set (see *e.g.* [1]).
- The decomposition of P following Q and H is not unique.
- A Feller transition kernel P is Quasi-Feller with $W = S$, $H = Id_S$, $D_H = \emptyset$ and $Q = P$.
- Q is a transition probability if and only if $W = S$.
- In our examples, W generally will be $\mathbb{R}^p$ or one of its subsets.

1.2. **Examples**

1.2.1. *ARCH model*

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \to S^+(d, \mathbb{R})$ (where $S^+(d, \mathbb{R})$ denotes the set of definite positive matrix) be measurable functions. Let $(\varepsilon_t)_{t\geq 1}$ be a white noise whose distribution μ is defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The $ARCH$ model is then defined by:

$$X_{t+1} = F(X_t) + G(X_t)\varepsilon_{t+1},\ X_0 \in \mathbb{R}^d. \tag{1}$$

Let us define the function:

$$\begin{aligned} G^{(-1)} \quad : \quad & \mathbb{R}^d \to S^+(d, \mathbb{R}), \\ & x \mapsto (G(x))^{-1}, \end{aligned}$$

and the norm of a $d \times d$ matrix:

$$\|A\| := \sup_{|u|=1} \frac{|Au|}{|u|}.$$

The transition probability $P(x, \mathrm{d}y)$ of the $ARCH$ is given by:

$$\forall g \text{ bounded measurable function},\ Pg(x) = \int g\,(F(x) + G(x)y)\,\mu(\mathrm{d}y). \tag{2}$$

Proposition 1.3. *If the set $D_{F,G} := D_F \cup D_G$ satisfies:*

$$\forall \left(w^1, w^2\right) \in \bigcup_{K \in \mathcal{K}} \overline{F(K)} \times \overline{G(K)}, \qquad \mu\left(\left(w^2\right)^{-1}\left(D_{F,G} - w^1\right)\right) = 0, \tag{3}$$

then the ARCH model is Quasi-Feller.

Proof. Let us define, for any bounded measurable function $g : \mathbb{R}^d \to \mathbb{R}$ the function Qg as follows:

$$\forall \left(w^1, w^2\right) \in \mathbb{R}^d \times S^+(d, \mathbb{R}),\ Qg\left(w^1, w^2\right) = \int g\left(w^1 + w^2 y\right) \mu(\mathrm{d}y). \tag{4}$$

We have $Pg(x) = Qg\left(F(x), G(x)\right)$. Q obviously is Feller according to the Definition 1.1 (Q is no longer a transition probability because $E = \mathbb{R}^d \times S^+(d, \mathbb{R})$).

The hypothesis $\forall w \in \bigcup_{K \in \mathcal{K}} \overline{F(K)} \times \overline{G(K)},\ Q(w, D_{F,G}) = 0$ turns into

$$\forall w := \left(w^1, w^2\right) \in \bigcup_{K \in \mathcal{K}} \overline{F(K)} \times \overline{G(K)}, \quad \mu\left(\left(w^2\right)^{-1}\left(D_{F,G} - w^1\right)\right) = 0. \qquad \square$$

Remark 1.4. In this case, we have: $W = \mathbb{R}^d \times S^+(d, \mathbb{R})$, and $\overline{G(K)} \subset S^+(d, \mathbb{R})$.

1.2.2. *Nonlinear autoregressive model with Markov switching (NAR-MS)*

Definition 1.5. $(Y_t)_{t \in \mathbb{N}} \in \left(\mathbb{R}^d\right)^{\mathbb{N}}$ is a *NAR-MS* if:

$$Y_t := f_{X_t}\left(Y_{t-1}\right) + \varepsilon_t,$$

where $(X_t)_{t \geq 0}$ is a positive Harris recurrent chain on $\{1, \cdots, m\}$, for all $k \in \{1, \cdots, m\}$, f_k is a measurable function from $\mathbb{R}^d$ into itself and the ε_t are i.i.d. random variables taking values in $\mathbb{R}^d$.

Example 1.6. Let $(Y_t)_{t \in \mathbb{N}}$ be a *NAR-MS*. Let μ denote the distribution of ε_0 and D_k denote the set of discontinuity points of f_k. If:

$$\forall k \in \{1, \cdots, m\}, \quad \forall z \in \bigcup_{K \in \mathcal{K}} \overline{f_k(K)},\ \mu\left(D_k - z\right) = 0,$$

then the *NAR-MS* is Quasi-Feller.

Proof. X_t being positive Harris, there exist a measurable function $h : \{1, \cdots, m\} \times \mathbb{R} \to \{1, \cdots, m\}$ and a sequence of i.i.d. real valued variables $(\eta_t)_{t \geq 1}$ with distribution ν and which are independent from the process $(\varepsilon_t)_{t \geq 0}$ such that:

$$X_t = h\left(X_{t-1}, \eta_t\right).$$

Then, the process:

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} h\left(X_{t-1}, \eta_t\right) \\ f_{h\left(X_{t-1}, \eta_t\right)}\left(Y_{t-1}\right) + \varepsilon_t \end{pmatrix},$$

is a Markov chain whose transition kernel P verifies:

$$\forall g \in C_b\left(\{1, \cdots, m\} \times \mathbb{R}^d, \mathbb{R}\right), \quad Pg(x, y) = \int_{\mathbb{R}} \nu(\mathrm{d}\eta) \int_{\mathbb{R}^d} g\left(h(x, \eta), f_{h(x,\eta)}(y) + \varepsilon\right) \mu(\mathrm{d}\varepsilon).$$

Now let define, for all $g \in C_b\left(\{1,\cdots,m\}\times\mathbb{R}^d,\mathbb{R}\right)$ the function Qg as follows:

$\forall (x,z_1,\cdots,z_m)\in\{1,\cdots,m\}\times\left(\mathbb{R}^d\right)^m$,

$$Qg(x,z_1,\cdots,z_m)=\int_{\mathbb{R}}\nu(\mathrm{d}\eta)\int_{\mathbb{R}^d} g\left(h(x,\eta),\sum_{k=1}^{m}\mathbf{1}_{\{k\}}\left(h(x,\eta)\right)z_k+\varepsilon\right)\mu(\mathrm{d}\varepsilon). \quad (5)$$

For all x and η, $(z_1,\cdots,z_m)\mapsto g(h(x,\eta),\sum_{k=1}^m \mathbf{1}_{\{k\}}(h(x,\eta))z_k+\varepsilon)$ is a continuous function. So, the function $(z_1,\cdots,z_m)\mapsto Qg(x,z_1,\cdots,z_m)$ is continuous. But x takes values in a finite set, hence the function $(x,z_1,\cdots,z_m)\mapsto Qg(x,z_1,\cdots,z_m)$ is also continuous.

Setting $H(x,y)=(x,f_1(y),\cdots,f_m(y))$, we have:

$$\forall g\in C_b\left(\{1,\cdots,m\}\times\mathbb{R}^d,\mathbb{R}\right),\quad Pg(x,y)=Qg\left(H(x,y)\right).$$

To finish, $D_H\subset\{1,\cdots,m\}\times\bigcup_{k=1}^m D_k$. Hence, the hypothesis:

$$\forall k\in\{1,\cdots,m\},\ \forall z_k\in\bigcup_{K\in\mathcal{K}}\overline{f_k(K)},\quad \mu\left(D_k-z_k\right)=0,$$

implies that:

$$\forall (x,z_1,\cdots,z_m)\in\bigcup_{K\in\mathcal{K}}\overline{H(K)},\quad Q\left(x,z_1,\cdots,z_m,D_H\right)=0,$$

and so the *NAR-MS* is Quasi-Feller. □

2. Existence of a stationary probability measure for Quasi-Feller models

Theorem 2.1. *Let us consider a Quasi-Feller transition probability $P(x,\mathrm{d}y)$ associated to a function H a kernel Q according to Definition 1.1.*
Suppose there exists $x_0\in S$ such that $(\frac{1}{n}\sum_{k=0}^{n-1}P^k(x_0,\mathrm{d}y))_{n\in\mathbb{N}^*}$ is tight. Then

(a) *the model has an invariant probability measure;*
(b) *for all invariant probability measure ν and all measurable set C satisfying:*

$$\forall w\in\bigcup_{K\in\mathcal{K}}\overline{H(K)},\qquad Q\left(w,C\right)=0, \quad (6)$$

then $\nu(C)=0$. In particular, $\nu(D_H)=0$.

Proof. The idea is to show, as for Feller Models, that any limit point ν for weak convergence of the tight sequence $\frac{1}{n}\sum_{k=0}^{n-1}P^k(x_0,\mathrm{d}y)$ is an invariant probability distribution.

The key is to show that $\nu\left(D_H\right)=0$. Let us prove directly point (b). Let C be a measurable subset of S satisfying (6). Let prove that $\nu(C)=0$.

If F be a closed subset of C. then F also satisfies (6), *i.e.*:

$$\forall w\in\bigcup_{K\in\mathcal{K}}\overline{H(K)},\ Q\left(w,F\right)=0.$$

Let $(h_p)_{p\in\mathbb{N}}$ be the sequence of functions from S to $\mathbb{R}$ defined by:

$$\forall p\in\mathbb{N}^*,\ \forall x\in S,\ h_p:=1-\min\left(pd\left(x,F\right),1\right).$$

For all $p \in \mathbb{N}$, h_p are continuous functions (even p-Lipschitz). Moreover, they verify $\|h_p\|_\infty \leq 1$ and $h_p \searrow \mathbf{1}_{\overline{F}} = \mathbf{1}_F$ for F is a closed subset.

Let us consider now $(Qh_p)_{p\in\mathbb{N}}$ the sequence of functions from W to $\mathbb{R}$. For Q is Feller, Qh_p are so continuous functions. On the other hand, $Q(w, \mathrm{d}y)$ is a probability measure for all $w \in W$, hence following dominated convergence Theorem, we have:

$$\forall\ w \in \bigcup_{K\in\mathcal{K}} \overline{H(K)},\ Qh_p(w) \searrow Q(w, F).$$

Let K be a compact subset of S. $Qh_p \searrow 0$ on $\overline{H(K)}$ for $Q(w, F) = 0$ if $w \in \overline{H(K)}$. At last, $\overline{H(K)}$ is, by hypothesis, a compact subset. Then, the Dini's Theorem implies that the above convergence is uniform on $\overline{H(K)}$ *i.e.*

$$\sup_{w\in\overline{H(K)}} Qh_p(w) \to 0 \text{ when } p \to +\infty. \tag{7}$$

Define $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} P^k(x_0, \mathrm{d}y)$ with the usual convention $P^0(x_0, \mathrm{d}y) = \delta_{x_0}(\mathrm{d}y)$. The sequence $(\nu_n)_{n\in\mathbb{N}^*}$ is tight, so we have:

$$\forall\varepsilon > 0,\ \exists K_\varepsilon \text{ compact subset of } S \text{ such that, } \forall n \in \mathbb{N}^*,\ \nu_n({}^cK_\varepsilon) \leq \varepsilon. \tag{8}$$

It follows that, for all n, $p \geq 1$,

$$\nu_n(Ph_p) = \int_{K_\varepsilon} Ph_p \mathrm{d}\nu_n + \int_{{}^cK_\varepsilon} Ph_p \mathrm{d}\nu_n \leq \sup_{y\in K_\varepsilon} Ph_p(y) + \nu_n({}^cK_\varepsilon) \leq \sup_{y\in K_\varepsilon} Ph_p(y) + \varepsilon.$$

But $Ph_p = Qh_p \circ H$ hence, for all n, $p \geq 1$,

$$\nu_n(Ph_p) \leq \sup_{w\in\overline{H(K_\varepsilon)}} Qh_p(w) + \varepsilon.$$

Consider a subsequence $\left(\nu_{\varphi(n)}\right)_{n\in\mathbb{N}^*}$ that converges weakly to a probability measure ν. Because for all bounded function f we have:

$$|\nu_n(Pf) - \nu_n(f)| \leq \frac{2\|f\|_\infty}{n},$$

it follows immediately that:

$$\nu(h_p) = \lim_{n\to\infty} \nu_{\varphi(n)}(h_p) = \lim_{n\to\infty} \nu_{\varphi(n)}(Ph_p) \leq \sup_{w\in\overline{H(K_\varepsilon)}} Qh_p(w) + \varepsilon.$$

Thus, by (7) we have:

$$\forall\ \varepsilon > 0,\ \limsup_{p\to\infty} \nu(h_p) \leq \varepsilon,$$

showing that, $\nu(F) = \lim_{p\to\infty} \nu(h_p) = 0$.

Then, for all closed subset F of C we have $\nu(F) = 0$. By regularity of the probability measure ν, we have:

$$\nu(C) := \sup\{\nu(F),\ F \subset C,\ F \text{ closed subset}\} \quad \text{(see [1])}.$$

Finally $\nu(C) = 0$. In particular, we have $\nu(D_H) = 0$.

Consider now $h \in C_b$. One clearly has $Ph = Qh \circ H$ is a continuous function on the continuity set of H. So, Ph is ν-*a.s.* continuous. Then:

$$\nu_{\varphi(n)}(Ph) \overset{n\to+\infty}{\longrightarrow} \nu(Ph) \text{ for } \nu_{\varphi(n)} \Rightarrow \nu.$$

But

$$|\nu_n(Ph) - \nu_n(h)| \overset{n\to+\infty}{\longrightarrow} 0 \text{ and } \nu_{\varphi(n)}(h) \overset{n\to+\infty}{\longrightarrow} \nu(h),$$

so we can deduce that $\nu(Ph) = \nu(h)$ and to finish $\nu P = \nu$ because h is an arbitrary function in C_b. □

To ensure the tightness of $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x, \mathrm{d}y))_{n\in\mathbb{N}^*}$ we usually use a Lyapunov function V *i.e.* verifying for $a \in S$:

$$\lim_{d(x,a)\to\infty} V(x) = +\infty.$$

Let recall the following result:

Proposition 2.2. *Let $P(x, \mathrm{d}y)$ be a transition probability on S and V be a continuous Lyapunov function. Pakes' lemma* [6] *suppose that:*

i) $\forall x \in S, \quad PV(x) \leq V(x) + \beta$;
ii) for $a \in S$, $\lim_{d(x,a)\to\infty}(PV - V)(x) = -\infty$.

Then, for all x in S the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x, \mathrm{d}y))_{n\in\mathbb{N}^*}$ is tight.

3. Irreducibility and positive Harris recurrence

The aim of this section is to extend the criterion of positive Harris recurrence from the Feller case as presented in [5] to our Quasi-Feller models. Any ψ-irreducible Feller chain on a polish space S, where the support of ψ has nonempty interior, is such that every compact subset of S is petite. If we add the hypothesis of tightness, for all x in S, of the sequence of probability measures $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x, \mathrm{d}y))_{n\in\mathbb{N}}$ it is easy to show the chain is positive Harris recurrent (see *e.g.* [5]). Our aim is to extend this result to Quasi-Feller chain.

3.1. **Definitions**

Let recall first the notion of ϕ-irreducibility as defined in [4]:

Definition 3.1. Consider a Markov chain with transition kernel $P(x, \mathrm{d}y)$ on S and ϕ a nonnegative measure. The chain is said to be ϕ-irreducible if,

$$\forall x \in S,\ \phi(\mathrm{d}y) \ll \sum_{n=1}^{+\infty} a_n P^n(x, \mathrm{d}y),$$

where the a_n are nonnegative real numbers such that $\sum_{n=1}^{+\infty} a_n = 1$.

Definition 3.2. We will say an homogeneous Markov chain $(X_t)_{t\in\mathbb{N}}$ is positive Harris recurrent if there exists a probability measure ν such that:

$$\forall g : S \to \mathbb{R} \text{ bounded continuous function },\ \forall x \in S,\quad \frac{1}{n}\sum_{k=0}^{n-1} g(X_k) \overset{\mathbb{P}_x\text{-}a.s.}{\longrightarrow} \nu(g).$$

This definition clearly implies that ν is the unique invariant probability measure of the chain. Moreover, if a chain is positive Harris recurrent chain with ν for unique stationary distribution, then it is ν-irreducible (see [5]).

In the following, ψ will denote a maximal measure of irreducibility, *i.e.* a measure of irreducibility that dominates all measures of irreducibility. The existence of such a measure for a ϕ-irreducible Markov chain is proved in [5].

Definition 3.3. Let $P(x, \mathrm{d}y)$ be the transition kernel of a ψ-irreducible Markov chain. A nonempty set $A \in \mathcal{B}(S)$ is called petite if it verifies:

$$\inf_{x \in A} K_a(x, \mathrm{d}y) \geq \delta\psi(\mathrm{d}y),$$

where $K_a(x, \mathrm{d}y) := \sum_{k=1}^{+\infty} a_k P^k(x, \mathrm{d}y)$ and $\sum_{k=1}^{+\infty} a_k = 1,\ \forall k \geq 1,\ a_k > 0,\ \delta > 0.$

Remark 3.4.

- $K_a K_b(x, \mathrm{d}y) := \int K_a(x, \mathrm{d}z) K_a(z, \mathrm{d}y) = K_{a*b}(x, \mathrm{d}y)$.
- The property of petiteness depends only on $P(x, \mathrm{d}y)$ but not on the chosen sequence $(a_n)_{n \in \mathbb{N}^*}$.
- Every subset of a petite set is petite.

3.2. **Positive Harris recurrence of Quasi-Feller models**

To obtain positive Harris recurrence for our models, the key is to prove the following result:

Proposition 3.5. *Let $P(x, \mathrm{d}y) = Q(H(x), \mathrm{d}y)$ be a ψ-irreducible Quasi-Feller transition probability. If* $\operatorname{supp}\{\psi\}$ *has nonempty interior, then all compact subsets of S are petite.*

Proof. The proof relies on the following two Lemmas:

Lemma 3.6. *Let $P(x, \mathrm{d}y) = Q(H(x), \mathrm{d}y)$ be a Quasi-Feller. Then:*

(a) *for any bounded function $f : S \to \mathbb{R}$ which is continuous on $S \backslash D_H$, and for any $k \in \mathbb{N}^*$, $P^k f$ is continuous on $S \backslash D_H$;*
(b) *$\forall F$ closed set, $\forall k \in \mathbb{N}^*,\ x \mapsto P^k(x, F)$ is upper semicontinuous on $S \backslash D_H$;*
(c) *$\forall O$ open set, $\forall k \in \mathbb{N}^*,\ x \mapsto P^k(x, O)$ is lower semicontinuous on $S \backslash D_H$.*

Consequently $K_a(x, \mathrm{d}y)$ is a Quasi-Feller transition probability. More precisely, there exists a Feller transition kernel $\tilde{K}_a(x, \mathrm{d}y)$ such that $K_a(x, \mathrm{d}y) = \tilde{K}_a(H(x), \mathrm{d}y)$.

Proof. Let us show that if f is a lower semicontinuous function on $S \backslash D_H$, so is Pf.

Let f be a lower semicontinuous function on $S \backslash D_H$. Then f is the limit on $S \backslash D_H$ of a growing sequence $(g_p)_{p \in \mathbb{N}}$ of bounded Lipschitz functions on $S \backslash D_H$.

During the proof of Theorem 2.1, we showed if $g : S \to \mathbb{R}$ is a bounded continuous function then Pg is continuous on $S \backslash D_H$.

So we have:

$$Pg(x_n) \to Pg(x) \quad \text{when} \quad x_n \to x.$$

Thus, we can write:

$$\forall x \in S \backslash D_H, P(x_n, \mathrm{d}y) \Rightarrow P(x, \mathrm{d}y) \text{ when } x_n \to x.$$

Then for any $P(x, \mathrm{d}y)$-*a.s.* continuous function g:

$$Pg(x_n) \to Pg(x) \text{ si } x_n \to x. \tag{9}$$

But $P(x, D_H) = 0$ by hypothesis. Consequently Pf is $P(x, \mathrm{d}y)$-*a.s.* continuous whenever f is a bounded and continuous function on $S \backslash D_H$.

It shows that for all $p \in \mathbb{N}$, Pg_p is continuous on $S \backslash D_H$. Then, $x \mapsto Pf(x)$ is a lower semicontinuous function on $S \backslash D_H$ as a growing limit of continuous functions on $S \backslash D_H$. So, Point (*a*) is true. Points (b) and (c) follow easily.

To see the second part of the Lemma, let $g : S \to \mathbb{R}$ be a bounded continuous function. We have:

$$\begin{aligned} K_a g(x) &= \sum_{k=1}^{+\infty} a_k P^k g(x) \\ &= \int P(x, \mathrm{d}y) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y) \\ &= \int Q(H(x), \mathrm{d}y) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y). \end{aligned}$$

By point (a), we know that $\sum_{k=1}^{+\infty} a_k P^{k-1} g(y)$ is continuous on $S \backslash D_H$ and so it is $Q(z, \mathrm{d}y)$-*a.s.* continuous. Q being Feller, the function $f(z) = \int Q(z, \mathrm{d}y) \sum_{k=1}^{+\infty} a_k P^{k-1} g(y)$ is a bounded continuous function from W to $\mathbb{R}$. It implies that $\tilde{K}_a(z, .) = \int Q(z, \mathrm{d}y) \sum_{k=1}^{+\infty} a_k P^{k-1}(y, .)$ is Feller which proves the announced result for $\tilde{K}_a(H(x), .) = K_a(x, .)$. □

The following lemma and the end of the proof of Proposition 3.5 are extensions of Lemma 6.2.7 and Proposition 6.2.8 from [5].

Lemma 3.7. *Let $P(x, \mathrm{d}y) = Q(H(x), \mathrm{d}y)$ be a Quasi-Feller transition probability and let A be a Borel set such that $A \cap (S \backslash D_H)$ is a petite set for P. Then $\overline{A} \cap (S \backslash D_H)$ is a petite set for P.*

Proof. By hypothesis, we have $\inf_{x \in A \cap (S \backslash D_H} K_a(x, \mathrm{d}y) \geq \delta \psi(\mathrm{d}y)$, with $\delta > 0$.

Let $B \in \mathcal{B}(S)$ and let $F \subset B$ be a closed set. Lemma 3.6 permits us to ensure that the function $x \mapsto K_a(x, F)$ is a uniform limit of functions that are upper semicontinuous on $S \backslash D_H$. So it is a function that is also upper semicontinuous on $S \backslash D_H$. Thus:

$$\inf_{x \in \overline{A} \cap (S \backslash D_H)} K_a(x, F) = \inf_{x \in A \cap (S \backslash D_H)} K_a(x, F).$$

F being a subset of B, we can deduce:

$$\inf_{x \in \overline{A} \cap (S \backslash D_H)} K_a(x, B) \geq \delta \psi(F).$$

For ψ is a regular measure,

$$\inf_{x \in \overline{A} \cap (S \backslash D_H)} K_a(x, B) \geq \delta \psi(B),$$

which proves that $\overline{A} \cap (S \backslash D_H)$ is a petite set. □

End of the proof of Proposition 3.5: The chain being ψ-irreducible, there exists a petite set A of strictly positive ψ-measure (see [5]).

Let us define for all $k \in \mathbb{N}^*$, the following sets:

$$A_k := \left\{ x \in S / \ K_a(x, A) > \frac{1}{k} \right\}.$$

It is clear that:

$$\bigcup_{k \in \mathbb{N}^*} A_k = \{x \in S / \ K_a(x, A) > 0\} = S \quad \text{for} \quad \psi(A) > 0.$$

So, there exists $k_0 \in \mathbb{N}^*$ such that for all $k \geq k_0$, $A_k \neq \emptyset$. For $k \geq k_0$ and $x \in A_k$, we have:

$$\begin{aligned} K_{a*a}(x, dy) &= \int K_a(x, \mathrm{d}z) K_a(z, \mathrm{d}y) \\ &\geq \int_A K_a(x, \mathrm{d}z) K_a(z, \mathrm{d}y) \\ &\geq \delta\psi(\mathrm{d}y) \times K_a(x, A) \\ &\geq \frac{\delta}{k}\psi(\mathrm{d}y). \end{aligned}$$

Thus, for $k \geq k_0$, A_k is a petite set of strictly positive ψ-measure.

Let $O := \overset{\circ}{\widehat{\mathrm{supp}\{\psi\}}}$. O is a nonempty set and is of strictly positive ψ-measure. Thus, for all $k \geq k_0$, $A_k \cap O$ is of strictly positive ψ-measure. On the other hand, $A_k \cap O$ is a petite set because it is a subset of the petite set A_k. So $A_k \cap O$ is a petite set of strictly positive ψ-measure. By Proposition 2.1, we know that $\nu(D_H) = 0$, so $\psi(D_H) = 0$ for $\psi \ll \nu$. Then, for all $k \geq k_0$, $A_k \cap O \cap (S \backslash D_H)$ is a petite set of strictly positive ψ-measure.

Lemma 3.7 implies that for all $k \geq k_0$, the sets:

$$B_k := \overline{A_k \cap O} \cap (S \backslash D_H),$$

are petite set of strictly positive ψ-measure.

Let us define, for all $k \geq k_0$, the sets $F_k := \overline{A_k \cap O}$. F_k are closed sets and verify $\bigcup_{k \geq k_0} F_k = \overline{O}$. Hence, at least one of the F_k is nonempty. By Baire's Theorem, we know then that at least one of the F_k has a nonempty interior.

Let F_{k_1} $(k_1 \geq k_0)$ be one of the F_k with a nonempty interior and let O_{k_1} be its interior. $O_{k_1} \cap (S \backslash D_H)$ has strictly positive ψ-measure because $O_{k_1} \subset \mathrm{supp}\{\psi\}$ and is a petite set as subset of B_{k_1}. Hence there exists $\delta_1 > 0$ such that:

$$\inf_{x \in O_{k_1} \cap (S \backslash D_H)} K_a(x, \mathrm{d}y) \geq \delta_1 \psi(\mathrm{d}y).$$

Let $x \in S$ and $B \in \mathcal{B}(S)$. Let F be a closed subset of B. We have:

$$\begin{aligned} K_{a*a}(x, F) &= \int K_a(x, \mathrm{d}y) K_a(y, F) \\ &\geq \int_{O_{k_1} \cap (S \backslash D_H)} K_a(x, \mathrm{d}y) K_a(y, F) \\ &\geq \delta_1 \psi(F) K_a\left(x, O_{k_1} \cap (S \backslash D_H)\right). \end{aligned}$$

But $K_a(x, D_H) = 0$ so $K_{a*a}(x, F) \geq \delta_1 \psi(F) K_a\left(x, O_{k_1}\right)$. By taking the supremum over the closed sets included in B, we obtain:

$$K_{a*a}(x, B) \geq \delta_1 \psi(B) K_a\left(x, O_{k_1}\right). \tag{10}$$

If we choose $a < 1$ and $a_k = (1-a)a^{k-1}$ for all $k \geq 1$, we have

$$\begin{aligned} K_a(x, O_{k_1}) &= \sum_{k=1}^{+\infty}(1-a)a^{k-1}P^k(x, O_{k_1}) \\ &= (1-a)P(x, O_{k_1}) + a\sum_{k=1}^{+\infty}(1-a)a^{k-1}\int P(x, \mathrm{d}y)P^k(y, O_{k_1}) \\ &= (1-a)P(x, O_{k_1}) + a\int P(x, \mathrm{d}y)K_a(y, O_{k_1}) \\ &= (1-a)P(x, O_{k_1}) + a\int Q(H(x), \mathrm{d}y)\,K_a(y, O_{k_1}) \\ &\geq a\int Q(H(x), \mathrm{d}y)\,K_a(y, O_{k_1}). \end{aligned}$$

We already know that $K_a(y, O_{k_1})$ is lower semicontinuous on $S\backslash D_H$. Hence, $\int Q(w, \mathrm{d}y)K_a(y, O_{k_1})$ is a positive lower semicontinuous on W which implies that its infimum on any compact subset of W is also positive. By hypothesis, for all K compact subset of S, $\overline{H(K)}$ is a compact subset of W. So we have:

$$\inf_{x\in K} K_a(y, O_{k_1}) \geq a \inf_{w\in\overline{H(K)}}\int Q(w, \mathrm{d}y)\,K_a(y, O_{k_1}) = \alpha > 0,$$

what replaced in (10) gives:

$$\inf_{x\in K} K_{a*a}(x, \mathrm{d}y) \geq \alpha\delta_1\psi(\mathrm{d}y),$$

which proves that every compact subset K of S is petite. □

Theorem 3.8. *Let $P(x, \mathrm{d}y) = Q(H(x), \mathrm{d}y)$ be the transition kernel of a Quasi-Feller chain. Suppose that:*

i) $\forall x \in S$, $(\frac{1}{n}\sum_{k=0}^{n-1}P^k(x, \mathrm{d}y))_{n\in\mathbb{N}}$ *is tight;*
ii) *the chain is ψ-irreducible and* supp$\{\psi\}$ *has a nonempty interior.*

Then the chain is positive Harris recurrent.

Proof. Proposition 3.5 proves that we are under assumptions of Theorem 6.2.5 of [5]. Then, the result follows from Theorem 9.2.2 of [5]. □

3.3. Applications

Example 3.9. Consider a *ARCH* model. Recall that μ denotes the distribution of ε_0. Let suppose:

i) $\mu \sim \lambda_d$;
ii) F and G are Riemann integrables;
iii) F, G et G^* are bounded on compact sets;
iv) $\forall x \in \mathbb{R}^d$, $\left(\frac{1}{n}\sum_{k=0}^{n-1}P^k(x, \mathrm{d}y)\right)_{n\in\mathbb{N}}$ is tight.

Then the model is positive Harris recurrent.

Proof. For F and G are Riemann integrable, we have $\lambda_d(D_{F,G}) = 0$, so the model is Quasi-Feller. The result follows then from Theorem 3.8 for i) implies that the transition probability is λ_d-irreducible. □

Example 3.10. Consider $(Y_t)_{t\in\mathbb{N}}$ a NAR-MS. Recall that μ denotes the distribution of ε_0 and suppose that:

i) $\mu \sim \lambda_d$;

ii) the f_k are Riemann integrables;

iii) $\forall x \in \mathbb{R}^d$, $(\frac{1}{n}\sum_{k=0}^{n-1} P^k(x,\mathrm{d}y))_{n\in\mathbb{N}}$ is tight.

Then $(Z_t)_{t\geq 0} := (X_t, Y_t')'_{y\geq 0}$ is positive Harris.

Proof. For the f_k are Riemann integrable, we have, for all $k \in \{1,\cdots,m\}$, $\lambda_d(D_k) = 0$, what proves that the model is Quasi-Feller. Again, the result follows then from Theorem 3.8 for the transition probability of $(Z_t)_{t\geq 0}$ is a $\alpha\otimes\lambda_d$-irreducible chain, where α is the measure on $\{1,\cdots,m\}$ such that, for all k, $\alpha\{k\} = 1$. □

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