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# POSITIVITY OF THE DENSITY FOR THE STOCHASTIC WAVE EQUATION IN TWO SPATIAL DIMENSIONS *, **

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**Abstract**. We consider the random vector $u(t,\underline{x}) = (u(t,x_1),\dots,u(t,x_d))$, where $t>0$, $x_1,\dots,x_d$ are distinct points of $\mathbb{R}^2$ and $u$ denotes the stochastic process solution to a stochastic wave equation driven by a noise white in time and correlated in space. In a recent paper by Millet and Sanz–Solé [10], sufficient conditions are given ensuring existence and smoothness of density for $u(t,\underline{x})$. We study here the positivity of such density. Using techniques developped in [1] (see also [9]) based on Analysis on an abstract Wiener space, we characterize the set of points $y\in\mathbb{R}^d$ where the density is positive and we prove that, under suitable assumptions, this set is $\mathbb{R}^d$.

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## 1. Introduction

Consider the stochastic wave equation with two-dimensional spatial variable

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}-\Delta\right)\, u(t,x) &= \sigma\big(u(t,x)\big)\ F(\mathrm{d}t,\mathrm{d}x)+b\big(u(t,x)\big),\\ u(0,x) &= u_0(x),\\ \frac{\partial u}{\partial t}(0,x) &= v_0(x)\end{aligned}\tag{1.1}$$

$(t,x)\in[0,\infty[\times\mathbb{R}^2$.

The coefficients $\sigma$ and $b$ are $\mathcal{C}^\infty$ functions with bounded derivatives of any order $i\geq 1$. The noise $F(t,x)$ is supposed to be a martingale measure (in the sense given by Walsh in [16]) obtained as the extension of a centered Gaussian field $\{F(\varphi),\ \varphi\in\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^2)\}$ with covariance

$$E\big(F(\varphi)\ F(\psi)\big)=\int_{\mathbb{R}_+}\mathrm{d}t\int_{\mathbb{R}^2}\mathrm{d}x\int_{\mathbb{R}^2}\mathrm{d}y\ \varphi(t,x)\ f(|x-y|)\ \psi(t,y).\tag{1.2}$$

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Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous on $]0,\infty[$. Moreover, we suppose that the measure $\Gamma(\mathrm{d}x) = f(x)\mathrm{d}x$ on $\mathbb{R}^2$ is a nonnegative tempered distribution. The initial conditions $u_0, v_0$, are real functions satisfying some conditions that will be determined later.

We give a meaning to the formal expression (1.2) using its mild formulation, as follows. Let $\mathcal{H}$ be the completion of the inner-product space of measurable functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\int_{\mathbb{R}^2} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y\ |\varphi(x)|\ f(|x-y|)\ |\varphi(y)| < \infty$, endowed with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^2} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y\ \varphi(x)\ f(|x-y|)\,\psi(y)$. Denote by $\{e_j, j \ge 0\}$ a CONS of functions of $\mathcal{H}$. Then $\{W_j(t) = \int_0^t \int_{\mathbb{R}^2} e_j(x)\ F(\mathrm{d}s, \mathrm{d}x),\ j \ge 0\}$ is a sequence of independent standard Brownian motions.

Let $S(t,x),\ (t,x) \in [0,\infty[\times\mathbb{R}^2$, be the fundamental solution of $\left(\frac{\partial^2}{\partial t^2} - \Delta\right)g = 0$. That is,

$$S(t,x) = \frac{1}{2\pi}\left(t^2 - |x|^2\right)^{-1/2}\ 1_{\{|x|<t\}}.$$

Set

$$X^0(t,x) = \int_{\mathbb{R}^2} S(t,x-y)\ v_0(y)\ \mathrm{d}y + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t,x-y)\ u_0(y)\ \mathrm{d}y\right).$$

By a solution of equation (1.2) we mean a stochastic process $\{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ satisfying

$$\begin{aligned} u(t,x) = X^0(t,x) &+ \sum_{j\ge 0} \int_0^t \big\langle S(t-s,\ x-*)\ \sigma(u(s,*)),\ e_j(*)\big\rangle_{\mathcal{H}}\ W_j(\mathrm{d}s) \\ &+ \int_0^t \int_{\mathbb{R}^2} S(t-s,\ x-y)\ b\big(u(s,y)\big)\ \mathrm{d}s\,\mathrm{d}y. \end{aligned} \tag{1.3}$$

In [10] we studied the existence and uniqueness of solution of (1.3) (see also [4]). We also addressed the smoothness of $u(t,x)$, at a fixed point $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, in the sense of Malliavin Calculus, and the existence and smoothness of density for the probability law on $\mathbb{R}^d$ of

$$u(t,\underline{x}) := \big(u(t,x_1), \dots, u(t,x_d)\big),$$

with $t > 0$ and $x_1, \dots, x_d$ distinct points of $\mathbb{R}^2$.

This last result has been obtained under the following set of assumptions

(i) there exist $a_1 \ge a_2 > 0$ such that $2(1+a_2)(a_1-a_2) < a_2 \le a_1 < 2$, positive constants $C_1$ and $C_2$ such that for $t \in [0,T]$,

$$C_1 t^{a_1} \le \int_0^t y\,f(y)\ \ell n\left(1+\frac{t}{y}\right)\mathrm{d}y \le C_2 t^{a_2};$$

(ii) $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathcal{C}^1$, bounded, with $a_2/2(1+a_2)$-Hölder continuous partial derivatives, $v_0 : \mathbb{R}^2 \to \mathbb{R}$ and there exists $q_0 \in ]4,+\infty]$ such that $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$;

(iii) $\sigma$ and $b$ are $\mathcal{C}^\infty$ with bounded derivatives of any order $i \ge 1$;

(iv) there exists $a > 0$ such that $\big|\sigma(u(t,x_j))\big| \ge a$, for any $j = 1,\dots,d$, a.s.

These conditions are satisfied for instance by the function $f(x) = x^{-\alpha},\ 0 < \alpha < 2$, with $a_1 = a_2 = 2-\alpha$.

In Millet and Morien in [8] there is a slight improvement of the previous result. These authors show that in the above-quoted set of hypothesis, assumptions (i) and (ii) can be replaced by the weaker ones:

(i') there exist $0 < a_2 \le a_1 < 2$ such that $2(a_1 - a_2) < a_2 \wedge 1$, positive constant $C_1$ such that for $t \in [0,T]$,

$$C_1 t^{a_1} \le \int_0^t y f(y)\ \ell n\left(1+\frac{t}{y}\right)\mathrm{d}y \tag{1.4}$$

and

$$\int_{0^+} y^{1-a_2} f(y)\ \mathrm{d}y < \infty; \tag{1.5}$$

(ii') $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathcal{C}^1$, bounded, with $\frac{1}{2}(a_2 \wedge 1)$-Hölder continuous partial derivatives, $\nabla u_0 \in L^{q_1}(\mathbb{R}^2)$ for some $q_1 > 2$; $v_0 : \mathbb{R}^2 \to \mathbb{R}$ belongs to $L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in \left[4 \vee \frac{2}{1-(a_2\wedge 1)}, \infty\right]$.

Recently, in [6] a related problem has been studied. The results apply in particular to equations like (1.2) driven by a centered Gaussian noise with covariance given by

$$E\big(F(\varphi)\ F(\psi)\big) = \int_{\mathbb{R}_+} \mathrm{d}t \int_{\mathbb{R}^2} \Gamma(\mathrm{d}x)\big(\varphi(s,.) * \tilde{\psi}(s,.)\big)(x), \tag{1.6}$$

where $\Gamma$ is a non-negative, non-negative definite tempered measure, the symbol $*$ means the convolution and $\tilde{\psi}(s,x) = \psi(s,-x)$. Notice that if $\Gamma(\mathrm{d}x) = f(|x|)\mathrm{d}x$ then (1.6) reduces to (1.2). It is proved that for $d = 1$ the following set of assumptions yield the existence and smoothness of density for the law of $u(t,x)$, with $t > 0$ and $x \in \mathbb{R}^2$: the preceding condition (iii)

(iv') there exists $C > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq C$;

(v) there exists $\eta \in (0, \frac{3}{4})$ such that

$$\int_{\mathbb{R}^2} \frac{\mu(\mathrm{d}\xi)}{(1+|\xi|^2)^\eta} < \infty, \tag{1.7}$$

where $\mu$ is the spectral measure of $\Gamma$.

In the particular case $\Gamma(\mathrm{d}x) = f(|x|)\mathrm{d}x$ the above condition (1.7) implies property (1.5) of (i') with $a_2 = 2(1-\eta) \in (\frac{1}{2}, 2)$ (see for instance [5,15]).

Denote by $p_{t,\underline{x}}(y)$ the density of $u(t,\underline{x})$ at $y \in \mathbb{R}^d$. The purpose of this paper is to prove the next statement.

**Theorem 1.** *Assume that the conditions (i', ii', iii) and (iv') hold. Suppose in addition that the functional*

$$J(\varphi,\psi) = \int_{\mathbb{R}^2} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y\ \varphi(x)\ f(|x-y|)\ \psi(y),\ \ \varphi,\psi \in \mathcal{D}(\mathbb{R}^2), \tag{1.8}$$

*is positive. Then, for any $y \in \mathbb{R}^d$, $p_{t,\underline{x}}(y)$ is stricly positive.*

In the remaining of this section we give a brief description of the method we have used to approach this problem.

Let $T > 0$ be fixed. The reproducing kernel Hilbert space of the Gaussian process $\{(W_j(t))_{j\geq 0},\ t \in [0,T]\}$ is the set $H_T$ of functions $h : [0,T] \to \mathbb{R}^{\mathbb{N}}$ such that $\sum_{j\geq 0} \int_0^T |h_j(s)|^2\,\mathrm{d}s < \infty$. For any $h \in H_T$, let $\{\Phi^h(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^2\}$ be the solution of

$$\begin{aligned}\Phi^h(t,x) = X^0(t,x) + \sum_{j\geq 0} \int_0^t \big\langle S(t-s, x-*)\ \sigma\big(\Phi^h(s,*)\big),\ e_j(*)\big\rangle_{\mathcal{H}}\ h_j(s)\ \mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^2} S(t-s,\ x-y)\ b\big(\Phi^h(s,y)\big)\ \mathrm{d}s\ \mathrm{d}y.\end{aligned} \tag{1.9}$$

This process is called the skeleton of $\{u(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^2\}$. The function $h \in H_T \to \Phi^h(t,x) \in \mathbb{R}$ is Fréchet differentiable (see the Appendix). Set $\Phi^h(t,\underline{x}) = (\Phi^h(t,x_1),\ldots,\Phi^h(t,x_d))$. Denote by $\gamma_{\Phi^h(t,\underline{x})}$ the $d\times d$ matrix whose entries are $\big\langle \bar{D}\Phi^h(t,x_i),\ \bar{D}\Phi^h(t,x_j)\big\rangle_{H_T}, i,j = 1,\ldots,d$, where $\bar{D}$ means the Fréchet derivative operator. It is called the deterministic Malliavin matrix.

Then we proceed in two steps:

**Step 1.** Assume (i', ii', iii) and (iv). We prove the equivalence between the next two properties on $y \in \mathbb{R}^d$: (a) $p_{t,\underline{x}}(y) > 0$ and (b) there exists $h \in H_T$ such that $\Phi^h(t, \underline{x}) = y$ and $\det \gamma_{\Phi^h(t,\underline{x})} > 0$.

**Step 2.** Suppose (i', ii', iii) and (iv'). Then, if the functional $J(\varphi, \psi)$ defined in (1.8) is positive, any $y \in \mathbb{R}^d$ satisfies (b).

For diffusion processes satisfying some non-degeneracy requirements the equivalence between the analogue of properties (a, b) above has been proved in [3].

A characterization of the points of positive density, analogous to the equivalence between (a) and (b), for functionals defined in an abstract Wiener space has been developed in [1] and then applied to diffusions.

A similar general setting, more close to the ideas of [3], has been presented in [14]. These abstract formulations allow to analyze many interesting examples in the infinite dimensional case, like solutions of stochastic partial differential equations (see, for instance [2, 9]).

We achieve the goal quoted in Step 1 proving first a weaker (localized) version of the criterium given in [14]. Step 2 requires to solve an inverse problem and also requires a careful analysis of the matrix $\gamma_{\Phi^h(t,\underline{x})}$. The positivity of the functional $J(\varphi, \psi)$ is used in the study of the first question, the second one is carried out by exploiting assumption (i').

The programme of the paper is as follows. In Section 2 we precise the tools used in the above-mentioned Step 1. For the proof of (b) $\Rightarrow$ (a) we use Proposition 4.2.2 in [14]. The proof of (a) $\Rightarrow$ (b) needs the weaker version of Proposition 4.2.1 in [14] stated as Proposition 2. Section 3 is devoted to complete Step 1. We give an approximation result on a class of evolution equations which includes (1.3) and (1.9). This ensures the validity of the hypothesis needed to apply the criterium established in Section 2; but it has its own interest. Finally, we devote Section 4 to the proof of Step 2.

## 2. Points of positive density of functionals defined on an abstract Wiener space

We devote this section to set up the method of the proof of the first step of Theorem 1 in Section 1. We follow the approach of [14]; however some modifications are needed.

For the sake of understanding we start by giving some basic notions and facts on Malliavin Calculus and refer the reader to [13, 14] for a complete presentation of this topic.

Let $(\Omega, H, P)$ be an abstract Wiener space. For any $h \in H$ we denote be $W(h)$ the Itô–Wiener integral. Let $\mathcal{S}$ be the class of cylindrical Wiener functionals, that is, the set of random vectors of the form

$$F = f\big(W(h_1), \dots, W(h_n)\big), \tag{2.1}$$

with $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in H$. For $F$ as in (2.1) the Malliavin derivative is the $H$-valued random variable defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \big(W(h_1), \dots, W(h_n)\big)\, h_i.$$

For any integer $k \geq 1$ and any real number $p \in [1, \infty)$ we define $\mathbb{D}^{k,p}$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \left( E\big(|F|^p\big) + \sum_{j=1}^k E\big(\|D^j F\|^p_{H^{\otimes j}}\big) \right)^{1/p}.$$

Suppose that $F = (F^1, \dots, F^d)$ is a random vector whose components belong to $\mathbb{D}^{1,2}$. The Malliavin matrix of $F$ is the $d \times d$ matrix with entries $\big\langle DF^i, DF^j \big\rangle_H$, $i, j = 1, \dots, d$; it is denoted by $\gamma_F$.

Set $\mathbb{D}^\infty = \cap_{k,p}\, \mathbb{D}^{k,p}$. A random vector $F$ is nondegenerate if $F \in \mathbb{D}^\infty(\mathbb{R}^d)$, that means, all the components of $F$ belong to $\mathbb{D}^\infty$, and moreover $\det(\gamma_F)^{-1} > 0$. It is well-known that the law of a nondegenerate random vector has a $\mathcal{C}^\infty$ density with respect to the Lebesgue measure.

Let $\Phi : H \to \mathbb{R}^d$ be Fréchet differentiable. By analogy with the random case we define $\gamma_{\Phi(h)}$ as the matrix with entries $\langle \bar{D}\Phi^i(h),\, \bar{D}\Phi^j(h)\rangle_H,\ \ i,j = 1,\ldots,d$; it is called (after Bismut) the deterministic Malliavin matrix.

For any $y \in \mathbb{R}^d$ we consider the following properties:

(A) the density $p$ of the random vector $F$ is strictly positive at $y$;
(B) there exists $h \in H$ with $\Phi(h) = y$ and $\det\gamma_{\Phi(h)} > 0$.

For elements $h_1,\ldots,h_d \in H,\ z \in \mathbb{R}^d$, set $\underline{h} = (h_1,\ldots,h_d)$ and define

$$\big(T_z\, W\big)(h) = W(h) + \sum_{j=1}^d z_j\, \langle h, h_j\rangle_H, \quad h \in H.$$

Moreover, for $p \in [1,\infty),\ k \geq 0$, set

$$R_{\underline{h},k,p}\, F = \int_{\{|z|\leq 1\}} \big\|(D^k F)\ (T_z W)\big\|^p_{H^{\otimes k}}\, \mathrm{d}z.$$

We next quote Proposition 4.2.2 of [14].

**Proposition 1.** *Let $F$ be a nondegenerate random vector, $\Phi : H \to \mathbb{R}^d$ be a Fréchet continuously differentiable mapping. Fix $h \in H$ and assume that there exists a sequence of measurable, absolutely continuous transformations $T_n^h : \Omega \to \Omega,\ n \geq 0$, such that, for every $\varepsilon > 0,\ k = 0,1,2,3$ and some $p > d$,*

$$\lim_{n\to\infty} P\Big\{|F \circ T_n^h - \Phi(h)| > \varepsilon\Big\} = 0, \tag{2.2}$$

$$\lim_{n\to\infty} P\Big\{\|(DF)\circ T_n^h - (\bar{D}\Phi)(h)\|_H > \varepsilon\Big\} = 0, \tag{2.3}$$

$$\lim_{M\to\infty}\ \sup_n\ P\Big\{\big(R_{\bar{D}\Phi(h),\,k,p}\, F\big)\circ T_n^h > M\Big\} = 0. \tag{2.4}$$

Then (B) implies (A).

In order to set up the conditions ensuring $(A) \Rightarrow (B)$ we need a localized version of Proposition 4.2.1 in [14]. In fact, the Wiener functional $u(t,\underline{x})$ does not satisfy the convergence assumption needed to apply this proposition. A particular localization on $\Omega$ is required. This leads to a convergence in probability on $\mathbb{D}^\infty(\mathbb{R}^d)$.

Let $(H_n)_{n\geq 1}$ be an increasing sequence of finite dimensional subspaces of $H$ such that $\cup_{n\geq 1} H_n$ is dense in $H$. Let $W^n : \Omega \to H_n$ be a sequence of random variables belonging to $\mathbb{D}^\infty$. We introduce a localizing sequence, as follows. Let

$$A_n^\gamma = \Big\{\omega : \|W^n(\omega)\|_H^2 \leq \gamma\, C(n)\Big\}, \quad n \in \mathbb{N}, \quad \gamma \in (0,\infty),$$

where $\{C(n),\, n \geq 1\}$ is an increasing sequence such that

$$\lim_{n\to\infty} P\Big(\big(A_n^{\gamma_0}\big)^c\Big) = 0,$$

for some $\gamma_0 > 0$.

Let $\Psi^\gamma : \mathbb{R}_+ \to [0,1]$ be a $\mathcal{C}^\infty$ function with bounded derivatives of any order, such that

$$\Psi^\gamma(x) = \begin{cases} 1, & \text{if} \quad x < \gamma \\ 0, & \text{if} \quad x > 2\gamma. \end{cases} \tag{2.5}$$

Set $G^{\gamma,n}(\omega) = \Psi^\gamma\big(\frac{\|W^n(\omega)\|_H^2}{C(n)}\big)$. Notice that

$$1\!\!1_{A_n^\gamma} \le G^{\gamma,n} \le 1\!\!1_{A_n^{2\gamma}}. \tag{2.6}$$

Assume that $G^{\gamma,n} \in \mathbb{D}^\infty$ uniformly in $n$. That means, for all integer $k \ge 1$ and $p \in [1,\infty)$,

$$\sup_{n\ge 1} \|G^{\gamma,n}\|_{k,p} < +\infty. \tag{2.7}$$

**Proposition 2.** *Let $F : \Omega \to \mathbb{R}^d$ be a nondegenerate random vector, $\Phi : H \to \mathbb{R}^d$ be infinitely Fréchet differentiable such that*

(H1) $\Phi(W^n) \in \mathbb{D}^\infty(\mathbb{R}^d)$, *for any* $n \ge 1$, *and*

$$\lim_{n\to\infty} E\Big(\|D^k\big(\Phi(W^n) - F\big)\|^p_{H_T^{\otimes k}} 1\!\!1_{A_n^\gamma}\Big) = 0,$$

*for any* $k \ge 1,\ p \in [1,\infty),\ \gamma \ge \gamma_0$.

*Then, for each $y \in \mathbb{R}^d$, (A) implies (B).*

*Proof.* The arguments are similar to those of Proposition 4.2.1 [14] (see also [3] and [1]) with an additional ingredient of localization.

Let $f$ be any continuous positive function with compact support $[a^0, b^0]$ containing $y$; then

$$c := \mathbb{E}\big(f(F)\big) > 0.$$

Fix $\gamma \ge \gamma_0,\ \varepsilon \in ]0,\ c[$; let $n_0 \in \mathbb{N}$ be such that $\mathbb{P}\Big((A_n^\gamma)^c\Big) < \frac{\varepsilon}{\|f\|_\infty}$ for any $n \ge n_0$. Then, $0 < \mathbb{E}\big(f(F)\big) \le \mathbb{E}\Big(f(F)1\!\!1_{A_n^\gamma}\Big) + \varepsilon$ and consequently, $\mathbb{E}\Big(f(F)1\!\!1_{A_n^\gamma}\Big) > 0$, for any $n \ge n_0$. Therefore (2.6) implies that $\mathbb{E}\Big(f(F)\, G^{\gamma,n}\Big) > 0$ for $\gamma > \gamma_0,\ n \ge n_0$.

For every $M \ge 1$, let $\alpha_M \in C_b^\infty(\mathbb{R})$ be such that $0 \le \alpha_M \le 1,\ \alpha_M(x) = 0$ if $|x| \le \frac{1}{M}$ and $\alpha_M(x) = 1$ if $|x| \ge \frac{2}{M}$. Since $F$ is nondegenerate we have that $\lim_{M\to+\infty} \alpha_M(\det \gamma_F) = 1$, a.s. Consequently, $0 < \mathbb{E}\Big(f(F)\, G^{\gamma,n}\Big) = \lim_{M\to+\infty} \mathbb{E}\Big(f(F)\ G^{\gamma,n}\, \alpha_M(\det \gamma_F)\Big)$ and there exists a positive integer $M$ such that $\mathbb{E}\Big(f(F)\ G^{\gamma,n}\alpha_M(\det \gamma_F)) > 0$.

We want to prove that for $\gamma \ge \gamma_0$

$$\lim_{n\to+\infty} \Big|\mathbb{E}\Big(\big(f(F)\, \alpha_M(\det \gamma_F) - f\big(\Phi(W^n)\, \alpha_M(\det \gamma_{\Phi(W^n)})\big)\, G^{\gamma,n}\Big)\Big| = 0. \tag{2.8}$$

This will imply the existence of a positive integer $n_0$ such that

$$\mathbb{E}\Big(f\big(\Phi(W^n)\big)\, G^{\gamma,n}\alpha_M(\det \gamma_{\Phi(W^n)})\Big) > 0, \tag{2.9}$$

for any $n \ge n_0$. Let $\bar{f}(x_1,\dots,x_d) = \int_{a_1^0}^{x_1} \cdots \int_{a_d^0}^{x_d} f(u_1,\dots,u_d)\, du_1 \dots du_d$, $a^0 = (a_1^0,\dots,a_d^0)$.

The integration by parts formula of the Malliavin Calculus implies that

$$\begin{aligned} T_n &:= \mathbb{E}\Big[\big(f(F)\,\alpha_M(\det\,\gamma_F) - f(\Phi(W^n))\,\alpha_M(\det\,\gamma_{\Phi(W^n)})\big)\,G^{\gamma,n}\Big] \\ &= \mathbb{E}\Big[\bar f(F) H_{\{1,\dots,d\}}\big(F, G^{\gamma,n}\alpha_M(\det\,\gamma_F)\big) \\ &\qquad - \bar f\big(\Phi(W^n)\big) H_{\{1,\dots,d\}}\big(\Phi(W^n), G^{\gamma,n}\alpha_M(\det\,\gamma_{\Phi(W^n)})\big)\Big], \end{aligned}$$

where, for any multiindex $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$, $H_\alpha$ are random variables defined recursively as follows. If $U$ is a nondegenerate functional and $V \in \mathbb{D}^\infty$ then $H_{\{i\}}(U,V) = \sum_{j=1}^d \delta\Big(V(\gamma_U^{-1})^{ij} DU^j\Big)$ and $H_\alpha(U,V) = H_{\{\alpha_k\}}\Big(U, H_{\{\alpha_1,\dots,\alpha_{k-1}\}}(U,V)\Big)$, where $\delta$ is the adjoint operator of $D$ which is, as $D$, a local operator. $\delta$ is also called the Skorohod integral.

Since $G^{\gamma,n} = 0$ on $(A_n^{2\gamma})^c$,

$$H_{\{1,\dots,d\}}\Big(F, G^{\gamma,n}\alpha_M(\det\gamma_F)\Big) = 1\!\!1_{A_n^{2\gamma}} H_{\{1,\dots,d\}}\Big(F, G^{\gamma,n}\alpha_M(\det\,\gamma_F)\Big)$$

and

$$H_{\{1,\dots,d\}}\Big(\Phi(W^n), G^{\gamma,n}\alpha_M(\det\,\gamma_{\Phi(W^n)})\Big) = 1\!\!1_{A_n^{2\gamma}} H_{\{1,\dots,d\}}\Big(\Phi(W^n), G^{\gamma,n}\alpha_M(\det\,\gamma_{\Phi(W^n)})\Big).$$

Consequently $|T_n| \le T_n^1 + T_n^2$ with

$$T_n^1 = \Big|\mathbb{E}\Big(\big(\bar f(F) - \bar f(\Phi(W^n))\big)\,1\!\!1_{A_n^{2\gamma}} H_{\{1,\dots,d\}}\big(F, G^{\gamma,n}\alpha_M(\det\gamma_F)\big)\Big)\Big|$$

and

$$\begin{aligned} T_n^2 = \Big|\mathbb{E}\Big(\bar f\big(\Phi(W^n)\big)\Big)\,1\!\!1_{A_n^{2\gamma}} \Big[&H_{\{1,\dots,d\}}\big(F, G^{\gamma,n}\alpha_M(\det\,\gamma_F)\big) \\ &- H_{\{1,\dots,d\}}\Big(\Phi(W^n), G^{\gamma,n}\alpha_M(\det\,\gamma_{\Phi(W^n)})\Big)\Big]\Big|. \end{aligned}$$

Let us first check that $\lim_{n\to+\infty} T_n^1 = 0$. Indeed, $T_n^1 \le\ T_n^{1,1} \times T_n^{1,2}$ where

$$\begin{aligned} T_n^{1,1} &= \Big(\mathbb{E}\big(|\bar f(F) - \bar f(\Phi(W^n))|^p\,1\!\!1_{A_n^{2\gamma}}\big)\Big)^{1/p} \\ T_n^{1,2} &= \Big\|H_{\{1,\dots,d\}}\big(F, G^{\gamma,n}\,\alpha_M(\det\,\gamma_F)\big)\Big\|_{L^q(\Omega)} \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Assumption (H1) implies that

$$\overline{\lim_n}\, T_n^{1,1} \le \|f\|_\infty \lim_{n\to+\infty} \Big(\mathbb{E}\big(|F - \Phi(W^n)|^p\,1\!\!1_{A_n^{2\gamma}}\big)\Big)^{1/p} = 0.$$

Moreover, $\sup_n T_n^{1,2} < +\infty$. Indeed, Proposition 3.2.2 in [14] yields:

$$T_n^{1,2} \le C(q,d)\Big(|\gamma_F^{-1}|_k^a\ \|F\|_{b,c}^{a'}\ \|G^{\gamma,n}\alpha_M(\det\,\gamma_F)\|_{b',c'}^{a''}\Big),$$

for some positive real numbers $k, a, c, a', c', a'' > 1$ and positive integers $b, b'$.

The properties on $G^{\gamma,n}$ imply that $\sup_n T_n^{1,2} < +\infty$. We now prove that $\lim_{n\to+\infty} T_n^2 = 0$. For any $p > 1$,

$$T_n^2 \le \|\bar f\|_\infty \, \mathbb{E}\, \Big| 1\!\!1_{A_n^{2\gamma}} \Big[ H_{\{1,\dots,d\}}\big(F, G^{\gamma,n}\alpha_M(\det \gamma_F)\big) - H_{\{1,\dots,d\}}\big(\Phi(W^n), G^{\gamma,n}\alpha_M(\det \gamma_{\Phi(W^n)})\big)\Big]\Big|.$$

The $L^p$-inequalities for the Skorohod integral yield

$$T_n^2 \le C(d)\|\bar f\|_\infty \, \mathbb{E}\Big| 1\!\!1_{A_n^{2\gamma}}\Big(\big\|F - \Phi(W^n)\big\|_{e,r} + \big\|G^{\gamma,n}(\alpha_M(\det \gamma_F)) - \alpha_M(\det \gamma_{\Phi(W^n)})\big\|_{e',r'}\Big)\Big|,$$

for some $r, r' > 1,\ e, e' \in \mathbb{N}$ (see [12] for the one dimensional case).

Thus $\lim_{n\to\infty} T_n^2 = 0$. This finishes the proof of (2.8) and therefore that of (2.9).

Next we consider a function $\beta_K : \mathbb{R} \to [0,1]$, $\mathcal{C}^\infty$, such that $\beta_K(x) = 1$ if $|x| \le K$ and $\beta_K(x) = 0$ if $|x| \le K+1$. Since $\|W^n\|_H^2$ is finite a.s., it is clear that for any fixed $n \ge 1$, $\beta_K(\|W^n\|_H^2)$ converges to 1 a.s. as $K \to \infty$.

Notice that since the functions $f$ and $\alpha_M$ are positive and the random variable $G^{\gamma,n}$ is positive and bounded by 1, the inequality (2.9) implies that for any $n \ge n_0$,

$$\mathbb{E}\Big(f\big(\Phi(W^n)\big)\,\alpha_M(\det \gamma_{\Phi(W^n)})\Big) > 0. \tag{2.10}$$

Thus, for any fixed $n \ge n_0$ there exists $K_0(n)$ such that for any $K \ge K_0(n)$,

$$\mathbb{E}\Big(f\big(\Phi(W^n)\big)\,\alpha_M(\det \gamma_{\Phi(W^n)})\beta_K(\|W^n\|_H^2)\Big) > 0. \tag{2.11}$$

This yields, for any $K \ge K_0(n)$ and any $\varepsilon > 0$

$$\mathbb{P}\Big(\big|\Phi(W^n) - y\big| < \varepsilon,\ \big|\det \gamma_{\Phi(W^n)}\big| \ge \frac{1}{M}, \|W^n\|_H^2 \le K+1\Big) > 0. \tag{2.12}$$

Indeed, otherwise, if for some $K \ge K_0(n)$ and some $\varepsilon > 0$

$$\mathbb{P}\Big(\big|\Phi(W^n) - y\big| < \varepsilon,\ \big|\det \gamma_{\Phi(W^n)}\big| \ge \frac{1}{M}, \|W^n\|_H^2 \le K+1\Big) = 0, \tag{2.13}$$

one could find a function $f$ bounded, positive and continuous such that $y \in \operatorname{supp} f$ and

$$\mathbb{E}\Big(f\big(\Phi(W^n)\big)\alpha_M\big(\det \gamma_{\Phi(W^n)}\big)\beta_K(\|W^n\|_H^2)\Big) = 0, \tag{2.14}$$

because $1\!\!1_{\{|\det \gamma_{\Phi(W^n)}| \ge \frac{1}{M}\}} \ge \alpha_M\big(\det \gamma_{\Phi(W^n)}\big)$ and $1\!\!1_{\{\|W^n\|_H^2 \le K+1\}} \ge \beta_K(\|W^n\|_H^2)$. This contradicts (2.11). Let $\bar k = K_0(n) + 1$. Then from (2.12) we can find a sequence of elements $h_m \in H_n$ such that for any $m \ge 1$, $\big|\Phi(h_m) - y\big| < \frac{1}{m}$, $\|h_m\|_H^2 \le \bar k$ and $|\det \gamma_{\Phi(h_m)}| \ge \frac{1}{M}$.

The compactness of bounded and closed sets in $H_n$ implies that we can select a subsequence converging to some element $h \in H$ which verifies $\Phi(h) = y$ and $|\det \gamma_{\Phi(h)}| > 0$. That means (B) holds. □

In this article we shall apply the preceding results to the following particular case. Fix any $T > 0$. Consider a sequence $\{W_j(t),\ t \in [0,T]\}$, $j \ge 0$, of standard Wiener processes and let $(\Omega, H_T, P)$ be the associated Wiener space. That means $\Omega = \mathcal{C}([0,T];\ \mathbb{R}^{\mathbb{N}})$, $H_T$ is the Hilbert space $L^2([0,T];\mathbb{R}^{\mathbb{N}})$ and $P$ is the Wiener measure on $\Omega$. The random vector $F$ will be $u(t,\underline{x}) = (u(t,x_1),\dots,u(t,x_d))$, the solution to the wave equation (1.3) at different points $(t,x_1),\dots,(t,x_d)$ of $[0,T]\times\mathbb{R}^2$. The functional $\Phi : H \to \mathbb{R}^d$ will be the skeleton $\Phi^h(t,\underline{x}) = (\Phi^h(t,x_1),\dots,\Phi^h(t,x_d))$ defined in (1.9).

Let us precise which are the sequences $(T_n^h)_{n\ge0}$ and $(W^n)_{n\ge0}$ in the above Propositions 1 and 2.

For any fixed $n \in \mathbb{N}$ we denote by $\Delta_i$ the interval $\big[\frac{iT}{2^n}, \frac{(i+1)T)}{2^n}\big)$ and write $W_j(\Delta_i)$ for the increment $W_j\big(\frac{(i+1)T}{2^n}\big) - W_j\big(\frac{iT}{2^n}\big)$.

Let $W^n = \big(W_j^n = \int_0^\cdot \dot{W}_j^n(s)\,\mathrm{d}s,\ j \in \mathbb{N}\big)$ be as in [11], that is, $\dot{W}_j^n = 0$ if $j > n$ and, for $0 \le j \le n$,

$$\dot{W}_j^n(t) = \begin{cases} \sum_{i=1}^{2^n} 2^n\, T^{-1}\, W_j(\Delta_{i-1})\ 1_{\Delta_i}(t), & \text{if } \ t \in [2^{-n}T,\, T], \\ 0, & \text{if } \ t \in [0, 2^{-n}T]. \end{cases}$$

Notice that for each $n \in \mathbb{N}$, $\big(\dot{W}_j^n,\, 0 \le j \le n\big)$ belongs to the finite dimensional subspace of $H_T$ generated by $\big\{2^n T^{-1}\ 1_{\Delta_i},\ i = 1, \ldots, 2^n - 1\big\}$. We identify $\big(\dot{W}_j^n,\, 0 \le j \le n\big)$ with an element of $H_T$ by putting $\dot{W}_j^n \equiv 0$ for $j > n$.

For any $h \in H_T$ define

$$T_n^h(\omega) = \left(W_j - W_j^n + \int_0^\cdot h_j(s)\,\mathrm{d}s,\ j \ge 0\right). \tag{2.15}$$

Girsanov's theorem yields that $P \circ \big(T_n^h\big)^{-1} \ll P$.

Finally, the localizing sequence $A_n^\gamma$, $n \ge 1$ of Proposition 2 is defined as follows. Fix $\gamma > 2T\,\ell n\, 2$, $t \in [0, T]$; then

$$A_n^\gamma(t) = \left\{\|W^n 1\!\!1_{[0,t]}\|_{H_T}^2 \le \gamma n^2\, 2^n T^{-1}\right\}. \tag{2.16}$$

Clearly the sets $A_n^\gamma(t)$ decrease in $t$ and increase in $\gamma$. Moreover,

$$\lim_{n\to\infty} P\left(A_n^\gamma(T)^c\right) = 0. \tag{2.17}$$

Indeed,

$$\begin{aligned} P\big(A_n^\gamma(T)^c\big) &\le \sum_{j=1}^{n} \sum_{i=0}^{2^n-1} P\big(|W_j(\Delta_i)|^2 \ge \gamma n 2^{-n}\big) \\ &\le n 2^n P\big(|Z| \ge \sqrt{\gamma} n^{\frac{1}{2}} T^{-\frac{1}{2}}\big) \le \frac{n 2^n T^{\frac{1}{2}}}{\sqrt{\gamma} n^{\frac{1}{2}}} \exp\left(-\frac{\gamma n}{2T}\right) \\ &\le \gamma^{-\frac{1}{2}} n^{\frac{1}{2}} T^{\frac{1}{2}} \exp\left(-n\left(\frac{\gamma}{2T} - \log 2\right)\right), \end{aligned}$$

where $Z$ is a standard Gaussian random variable. Thus (2.17) holds true.

A simple computation shows that for any $0 \le t < t' \le T$, $p \in [2, \infty)$ and $[t, t'] \subset \Delta_i$ for some $i$,

$$E\big(\|W^n 1\!\!1_{[t,t']}\|_{H_T}^p \le n^{\frac{p}{2}}. \tag{2.18}$$

In fact,

$$E\big(\|W^n\ 1_{[t,t']}\|^p_{H_T}\big) = E\left(\left(\sum_{j=1}^n 2^{2n}T^{-2}(W_j(\Delta_{i-1}))^2(t-t')\right)^{\frac{p}{2}}\right)$$
$$\le 2^{np}T^{-p}(t-t')^{\frac{p}{2}}E\left(\sum_{j=1}^n (W_j(\Delta_{i-1}))^2\right)^{\frac{p}{2}}$$
$$\le n^{\frac{p}{2}}.$$

Let

$$G^{\gamma,n}(\omega) = \Psi^\gamma\left(\frac{\|W^n\|^2_{H_T}}{n^2\,2^nT^{-1}}\right) = \Psi^\gamma\left(n^{-2}\sum_{j=1}^n\sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2\right), \tag{2.19}$$

$n \ge 1$, where $\Psi^\gamma$ is given by (2.5). This sequence satisfies property (2.7). Indeed, this can be proved by direct computations, as follows.

Let $k \in \mathbb{N}$ and fix a set $B_k = \{\alpha_i = (r_i, j_i) \in \mathbb{R}_+ \times \mathbb{N},\ i = 1, \dots k\}$. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and denote by $P_m$ the set of partitions of $B_k$ consisting of $m$ disjoint subsets $p_1, \dots, p_m,\ m = 1, \dots, k$; set $|p_i| = \text{card}\ p_i$. Let $Y$ be any random variable in $\mathbb{D}^{k,2},\ k \ge 1,\ g$ be a real $\mathcal{C}^k$ function with bounded derivatives up to order $k$. Leibniz's rule for Malliavin's derivatives yields

$$D^k_\alpha\big(g(Y)\big) = \sum_{m=1}^k \sum_{P_m} c_m\ g^{(m)}(Y) \prod_{i=1}^m D^{|p_i|}_{p_i} Y, \tag{2.20}$$

with some positive coefficients $c_m,\ m = 1, \dots, k,\ c_1 = 1$. We want to apply this formula to $g = \Psi^\gamma$ and $Y = n^{-2}\sum_{j=1}^n\sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2$, $n \ge 1$. Notice that these random variables have null components on the $n$-th Wiener chaos for $n \ge 3$. Hence, it suffices to prove (2.7) for $k = 0, 1, 2$ and $p \in [1, \infty)$. For these values of $k$, the order of the derivatives in the right hand-side of (2.20) are clearly less or equal to 2.

For $k = 0$, the result is obvious, since $\Psi^\gamma$ is bounded by 1. Set

$$F_n = n^{-2}\sum_{j=1}^n\sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2,$$

then, $D_{r,j}F_n = 0$, if $j > n$ and for $j \le n$

$$D_{r,j}F_n = n^{-2}\sum_{i=1}^{2^n} 2W_j(\Delta_{i-1})\mathbb{1}_{\Delta_{i-1}}(r).$$

Furthermore, $D^2_{(r_1,j_1)(r_2,j_2)}F_n = 0$, if $j_1 > n$ or $j_1 \le n$ but $j_1 \ne j_2$ and, for $j_1 = j_2 \le n$,

$$D^2_{(r_1,j_1)(r_2,j_2)}F_n = n^{-2}\sum_{i=1}^{2^n} 2\mathbb{1}_{\Delta_{i-1}}(r_1)\mathbb{1}_{\Delta_{i-1}}(r_2).$$

Applying Hölder's inequality it is easy to check that

$$E\left(\sum_{j\in\mathbb{N}}\int_0^T \mathrm{d}r|D_{r,j}F_n|^2\right)^{\frac{p}{2}} \le Cn^{-\frac{3}{2}p}2^{p(-\frac{n}{2}+1)}.$$

Moreover, a direct computation shows that

$$E\left(\sum_{j_1,j_2\in\mathbb{N}}\int_0^T \mathrm{d}r_1\int_0^T \mathrm{d}r_2|D^2_{(r_1,j_1)(r_2,j_2)}F_n|^2\right)^{\frac{p}{2}} \le Cn^{-\frac{3}{2}p}2^{p(-\frac{n}{2}+1)}.$$

Thus, for any $p\in[1,\infty)$ and $k=0,1,2$, we have that $\sup_n\|F_n\|_{k,p}<\infty$. Then, by means of (2.20) we complete the proof of (2.7) for the sequence of random variables defined in (2.19).

## 3. Approximation in probability in $\mathbb{D}^\infty$. Characterization of points of positive density

In this section we present an approximation result for a class of evolution equations which include as particular cases (1.3) and (1.9). This general setting allows to check the validity of the assumptions of Propositions 1 and 2 for the Wiener functional $F=u(t,\underline{x})$.

Let $A,B,G,b:\mathbb{R}\to\mathbb{R},\ h\in H_T$. Consider the evolution equations

$$\begin{aligned} X_n(t,x) &= X^0(t,x)+\sum_{j\ge 0}\int_0^t\Big\{\langle S(t-s,x-*)\,A\big(X_n(s,*)\big),e_j(*)\rangle_{\mathcal{H}}\ W_j(\mathrm{d}s)\\ &\quad+\langle S(t-s,x-*)\,B\big(X_n(s,*)\big),e_j(*)\rangle_{\mathcal{H}}\dot W_j^n(s)\mathrm{d}s\\ &\quad+\langle S(t-s,x-*)\,G\big(X_n(s,*)\big),e_j(*)\rangle_{\mathcal{H}}h_j(s)\mathrm{d}s\Big\}\\ &\quad+\int_0^t\int_{\mathbb{R}^2}S(t-s,x-y)\,b\big(X_n(s,y)\big)\mathrm{d}s\,\mathrm{d}y, \end{aligned} \tag{3.1}$$

$$\begin{aligned} X(t,x) &= X^0(t,x)+\sum_{j\ge 0}\int_0^t\Big\{\langle S(t-s,x-*)\,(A+B)\big(X(s,*)\big),e_j(*)\rangle_{\mathcal{H}}\ W_j(\mathrm{d}s)\\ &\quad+\langle S(t-s,x-*)\,G\big(X(s,*)\big),e_j(*)\rangle_{\mathcal{H}}\ h_j(s)\mathrm{d}s\Big\}\\ &\quad+\int_0^t\int_{\mathbb{R}^2}S(t-s,x-y)\,b\big(X(s,y)\big)\mathrm{d}s\,\mathrm{d}y. \end{aligned} \tag{3.2}$$

The existence and uniqueness of solution for equations (3.1) and (3.2) have been addressed in [11]. Notice that the processes $X_n$ and $X$ depend on $h\in H_T$.

We introduce the following set of assumptions (see (1.5) and (ii') in Sect. 1).

There exists $\beta_0\in]0,2[$ such that

(C1) $\int_{0^+} r^{1-\beta_0}\,f(r)\,\mathrm{d}r<+\infty$;

(C2) $u_0:\mathbb{R}^2\to\mathbb{R}$ is of class $\mathcal{C}^1$, bounded, with $\frac12\,(\beta_0\wedge 1)$-Hölder continuous partial derivatives, $\nabla u_0\in L^{q_1}(\mathbb{R}^2)$ for some $q_1>2$; $v_0:\mathbb{R}^2\to\mathbb{R}$ belongs to $L^{q_0}(\mathbb{R}^2)$ for some $q_0\in]4\vee\frac{2}{1-(\beta_0\wedge 1)},\infty]$;

(C3) the coefficients $A,B,G,b$ of equation (3.1) (and (3.2)) are $\mathcal{C}^\infty$ functions with bounded derivatives of any order $k\ge 1$.

Conditions (C1–C3) ensure that the trajectories of the solution of (1.3) are $\beta$-Hölder continuous in $(t,x)$ for $\beta < \frac{1}{2}(\beta_0 \wedge 1)$ (see Th. 2.2 in [8] and Prop. 1.4 in [10]).

For any separable Hilbert space $\mathbb{E}$ we denote by $H_T(\mathbb{E})$ the Hilbert space of functions $g : [0,T] \to \mathbb{E}^{\mathbb{N}}$ such that $\sum_{j\geq 0} \int_0^T \|g_j(s)\|_{\mathbb{E}}^2 \,\mathrm{d}s < \infty$. Notice that $H_T(\mathbb{R}) = H_T$. In this section we will deal with $\mathbb{E} = \mathbb{R}^d$ and $\mathbb{E} = \mathcal{H}$, where $\mathcal{H}$ has been defined in the introduction.

We now state the main technical result of this section. We shall see later in Theorem 3 that by a particular choice of the coefficients $A$, $B$, $G$ of equations (3.1) and (3.2), the next Theorem 2 allows to complete the first step of our programme.

**Theorem 2.** *Assume (C1–C3). Then, for any* $p \in (1,\infty)$, $k \in \mathbb{Z}_+$, $\gamma > 2T\,\ell n\,2$ *and every compact set* $K \subset \mathbb{R}^2$,

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \sup_{\underline{x}\in K^d} E\Big(\|D^k\big(X_n(t,\underline{x}) - X(t,\underline{x})\big)\|^p_{H_T(\mathbb{R}^d)^{\otimes k}}\, 1_{A_n^\gamma(t)}\Big) = 0, \tag{3.3}$$

*where, for* $k=0$, $D^0 = Id$ *and* $\|\cdot\|_{H_T(\mathbb{R}^d)^{\otimes 0}}$ *is the Euclidean norm in* $\mathbb{R}^d$*. Moreover, the convergence in (3.3) is uniform in* $h$ *on bounded sets of* $H_T$*.*

This theorem provides an extension of Proposition 2.3 in [11], where the convergence stated in (3.3) has been proved in the $L^p$ norm. Actually, in this proposition the localizing sequence is given by $\bar{A}_n^\gamma(t) = \Big\{\sup_{0\leq j\leq n} \ \sup_{0\leq i\leq([2^n tT^{-1}]-1)^+} |W_j(\Delta_i)| \leq \sqrt{\gamma}2^{-n/2}\,n^{1/2}\Big\}$, which is included in $A_n^\gamma(t)$ and also satisfies $\lim_{n\to\infty} P\big((\bar{A}_n^\gamma(T))^c\big) = 0$ (see Lem. 2.1 in [11]). However looking carefully at the proof we realize that only two facts concerning the localization are needed: (a) $E\big(\|W^n\|^p_{H_T}1\!\!1_{\bar{A}_n^\gamma(t)}\big) \leq Cn^p 2^{\frac{np}{2}}$ and (b) $E\big(\|W^n 1\!\!1_{[t_n,t]}\|^p_{H_T}1\!\!1_{\bar{A}_n^\gamma(t)}\big) \leq Cn^p$. Here we have used the following notation: for any $n\geq 1$, $t\in[0,T]$, we set $\underline{t}_n = \max\{k2^{-n}T;\ k = 1,\dots,2^n-1 : k2^{-n}T \leq t\}$, $t_n = (\underline{t}_n - 2^{-n}T)\vee 0$. Property (a) is clearly true with $A_n^\gamma(t)$ instead of $\bar{A}_n^\gamma(t)$, by the very definition of $A_n^\gamma(t)$. Property (b) is a trivial consequence of (2.18).

The proof in the $\mathbb{D}^\infty$ convergence consists in a quite long and tricky exercise with almost no new ideas. For this reason we do not give a detailed proof. Instead, we draw an outline and also state the technical lemmas which are needed. With these ingredients we provide the readers interested in a complete proof with the main guidelines to check by themselves this extension.

Let us introduce some additional notation.

$$\begin{aligned} X_n^-(t,x) = X^0(t,x) &+ \sum_{j\geq 0}\int_0^{t_n} \Big\{\langle S(t-s,x-*)\,A\big(X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\,W_j(\mathrm{d}s) \\ &+ \langle S(t-s,x-*)\,B\big(X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\,\dot{W}_j^n(s)\mathrm{d}s \\ &+ \langle S(t-s,x-*)\,G\big(X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\ h_j(s)\mathrm{d}s\Big\} \\ &+ \int_0^{t_n}\int_{\mathbb{R}^2} S(t-s,x-y)\,b\big(X_n(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y, \end{aligned} \tag{3.4}$$

$$\begin{aligned} X^-(t,x) = X^0(t,x) &+ \sum_{j\geq 0}\int_0^{t_n} \Big\{\langle S(t-s,x-*)(A+B)\big(X(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\,W_j(\mathrm{d}s) \\ &+ \langle S(t-s,x-*)\,G\big(X(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\ h_j(s)\mathrm{d}s\Big\} \\ &+ \int_0^{t_n}\int_{\mathbb{R}^2} S(t-s,x-y)\,b\big(X(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned} \tag{3.5}$$

Notice that, although it is not explicit in the notation, $X^-$ depends on $n$.

In Section 2 of [10] we have proved that, if $G = 0$, $X(t,x) \in \mathbb{D}^\infty$, for any $t \in [0,T]$, $x \in \mathbb{R}^2$. A slight modification of the proof allows to establish the same result for $G$ satisfying the assumptions of Theorem 2 as well as for $X_n(t,x)$, $X_n^-(t,x)$ and $X^-(t,x)$, respectively. Moreover, for any $k \in \mathbb{Z}_+$ and $p \in [1,\infty)$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \|D^k X(t,x)\|_{L^p(\Omega;H_T^{\otimes k})} < \infty. \tag{3.6}$$

We also need the explicit form of the evolution equations satisfied by the Malliavin derivatives of these processes. To this end we consider the chain rule given in (2.20). Let $\Delta_\alpha(g,Y) = D_\alpha^k\big(g(Y)\big) - g'(Y)\, D_\alpha^k Y$ and $\Delta(g,Y)$ be the stochastic process with components $\Delta_\alpha(g,Y)$. For any $r_1,\cdots,r_k \in \mathbb{R}$ we define $\bigvee_i r_i = \max(r_1,\cdots,r_k)$ and for $\alpha_i = (r_i,j_i) \in \mathbb{R}_+ \times \mathbb{N}$, $\hat{\alpha}_i = (\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_k)$.

Then, following the proof of Theorem 2.2 in [10], we obtain

$$\begin{aligned}
D_\alpha^k X_n(t,x) \quad = \quad & \sum_{i=1}^k \langle S(t-r_i,x-*)\; D_{\hat{\alpha}_i}^{k-1}\big(A(X_n(r_i,*))\big), e_{j_i}(*)\rangle_{\mathcal{H}} \\
& + \sum_{j\ge 0} \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, \Delta_\alpha\big(A, X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\, W_j(\mathrm{d}s) \\
& + \sum_{i=1}^k \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, D_{\hat{\alpha}_i}^{k-1}\big(B(X_n(s,*))\big), e_{j_i}(*)\rangle_{\mathcal{H}} \\
& \times \sum_{i=1}^{2^n} 2^n\, T^{-1} \times 1_{\Delta_{i-1}}(r_i) \times 1_{\Delta_i}(s)\,\mathrm{d}s \\
& + \sum_{j=0}^n \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, \Delta_\alpha\big(B, X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\, \dot{W}_j^n(s)\,\mathrm{d}s \\
& + \sum_{j\ge 0} \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, \Delta_\alpha\big(G, X_n(s,*)\big), e_j(*)\rangle_{\mathcal{H}}\; h_j(s)\,\mathrm{d}s \\
& + \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t-s,x-y)\, \Delta_\alpha\big(b, X_n(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y \\
& + \sum_{j\ge 0} \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, A'\big(X_n(s,*)\big)\, D_\alpha^k X_n(s,*), e_j(*)\rangle_{\mathcal{H}}\, W_j(\mathrm{d}s) \\
& + \sum_{j=0}^n \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, B'\big(X_n(s,*)\big)\, D_\alpha^k X_n(s,*), e_j(*)\rangle_{\mathcal{H}}\, \dot{W}_j^n(s)\,\mathrm{d}s \\
& + \sum_{j\ge 0} \int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\, G'\big(X_n(s,*)\big)\, D_\alpha^k X_n(s,*), e_j(*)\rangle_{\mathcal{H}}\, h_j(s)\,\mathrm{d}s \\
& + \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t-s,x-y)\, b'\big(X_n(s,y)\big)\, D_\alpha^k\, X_n(s,y)\,\mathrm{d}s\,\mathrm{d}y
\end{aligned} \tag{3.7}$$

and similarly,

$$\begin{aligned}
D_\alpha^k X(t,x) \quad = \quad & \sum_{i=1}^k \langle S(t-r_i, x-*)\, D_{\hat\alpha_i}^{k-1}\big((A+B)(X(r_i,*))\big),\, e_{j_i}(*)\rangle_{\mathcal H} \\
& + \sum_{j\geq 0} \int_{\bigvee_i r_i}^t \langle S(t-s, x-*)\, \Delta_\alpha\big(A+B,\, X(s,*)\big),\, e_j(*)\rangle_{\mathcal H}\, W_j(\mathrm{d}s) \\
& + \sum_{j\geq 0} \int_{\bigvee_i r_i}^t \langle S(t-s, x-*)\, \Delta_\alpha\big(G,\, X(s,*)\big),\, e_j(*)\rangle_{\mathcal H}\; h_j(s)\,\mathrm{d}s \\
& + \int_{\bigvee_i r_i}^t \int_{\mathbb R^2} S(t-s, x-y)\, \Delta_\alpha\big(b, X(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y \\
& + \sum_{j\geq 0} \int_{\bigvee_i r_i}^t \langle S(t-s, x-*)(A+B)'\big(X(s,*)\big)\, D_\alpha^k\, X(s,*),\, e_j(*)\rangle_{\mathcal H}\, W_j(\mathrm{d}s) \\
& + \sum_{j\geq 0} \int_{\bigvee_i r_i}^t \langle S(t-s, x-*)\, G'\big(X(s,*)\big)\, D_\alpha^k\, X(s,*),\, e_j(*)\rangle_{\mathcal H}\; h_j(s)\,\mathrm{d}s \\
& + \int_{\bigvee_i r_i}^t \int_{\mathbb R^2} S(t-s, x-y)\, b'\big(X(s,y)\big)\, D_\alpha^k\, X(s,y)\,\mathrm{d}s\,\mathrm{d}y, \qquad (3.8)
\end{aligned}$$

in the case where $\bigvee_i r_i \leq t$. Otherwise, $D_\alpha^k X_n(t,x) = D_\alpha^k X(t,x) = 0$.

The equations satisfied by $D_\alpha^k X_n^-(t,x)$ (resp. $D_\alpha^k X^-(t,x)$) are obtained substituting in (3.7) (resp. in (3.8)) the upper bound in the integral by $t_n$ and multiplying the first term of the right hand-side of (3.7) (resp. of (3.8)) by $1_{[0,t_n]}(r_i)$.

*Outline of the proof of Theorem 2.* We shall apply induction on $k$ and assume that $d=1$. The proof for $d>1$ is analogous. For $k=0$, the convergence (3.3) has been proved in Proposition 2.3 of [11].

Set

$$D_\alpha^k\big(X_n(t,x) - X(t,x)\big) = \sum_{i=1}^8 V_{i,\alpha}^n(t,x),$$

where

$$\begin{aligned}
V_{1,\alpha}^n(t,x) = & \sum_{i=1}^k \big\langle S(t-r_i, x-*)\big[D_{\hat\alpha_i}^{k-1}(A(X_n(r_i,*))) - D_{\hat\alpha_i}^{k-1}(A(X(r_i,*)))\big], \\
& e_{j_i}(*)\big\rangle_{\mathcal H} \\
V_{2,\alpha}^n(t,x) = & \sum_{i=1}^k \int_0^t \big\langle S(t-s, x-*) D_{\hat\alpha_i}^{k-1}(B(X_n(s,*))),\; e_{j_i}(*)\big\rangle_{\mathcal H} \\
& \times \sum_{\ell=1}^{2^n} 2^n\, T^{-1}\; 1_{\Delta_{\ell-1}}(r_i)\; 1_{\Delta_\ell}(s)\, ds - \big\langle S(t-r_i, x-*) D_{\hat\alpha_i}^{k-1}(B(X(r_i,*))), \\
& e_{j_i}(*)\big\rangle_{\mathcal H},
\end{aligned}$$

$$V^n_{3,\alpha}(t,x) = \sum_{j\geq 0}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)[\Delta_\alpha(G,X_n(s,*)) - \Delta_\alpha(G,X(s,*))],\, e_j(*)\big\rangle_{\mathcal{H}} \times h_j(s)\,\mathrm{d}s,$$

$$V^n_{4,\alpha}(t,x) = \int_{\bigvee r_i}^t \int_{\mathbb{R}^2} S(t-s,x-y)\big(\Delta_\alpha(b,X_n(s,y)) - \Delta_\alpha(b,X(s,y))\big) \times \mathrm{d}s\,\mathrm{d}y,$$

$$V^n_{5,\alpha}(t,x) = \sum_{j\geq 0}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)[G'(X_n(s,*))D^k_\alpha X_n(s,*) - G'(X(s,x)) \times D^k_\alpha X(s,*)],\, e_j(*)\big\rangle_{\mathcal{H}} h_j(s)\,\mathrm{d}s,$$

$$V^n_{6,\alpha}(t,x) = \int_{\bigvee r_i}^t \int_{\mathbb{R}^2} S(t-s,x-y)\left[b'(X_n(s,y))D^k_\alpha X_n(s,y) - b'(X(s,y))D^k_\alpha X(s,y)\right] \times \mathrm{d}s\,\mathrm{d}y,$$

$$\begin{aligned} V^n_{7,\alpha}(t,x) = {} & \sum_{j\geq 0}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)\Delta_\alpha(A,X_n(s,*)),\, e_j(*)\big\rangle_{\mathcal{H}} W_j(\mathrm{d}s) \\ & + \sum_{j=0}^{n}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)\Delta_\alpha(B,X_n(s,*)),\, e_j(*)\big\rangle_{\mathcal{H}} \dot{W}^n_j(s)\,\mathrm{d}s \\ & - \sum_{j\geq 0}^{n}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)\Delta_\alpha(A+B,X(s,*)),\, e_j(*)\big\rangle_{\mathcal{H}} W_j(\mathrm{d}s), \end{aligned}$$

$$\begin{aligned} V^n_{8,\alpha}(t,x) = {} & \sum_{j\geq 0}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)A'(X_n(s,*))\,D^k_\alpha X_n(s,*),\, e_j(*)\big\rangle_{\mathcal{H}} W_j(\mathrm{d}s) \\ & + \sum_{j=0}^{n}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)B'(X_n(s,*))\,D^k_\alpha X_n(s,*),\, e_j(*)\big\rangle_{\mathcal{H}} \dot{W}^n_j(s)\,\mathrm{d}s \\ & - \sum_{j\geq 0}^{n}\int_{\bigvee r_i}^t \big\langle S(t-s,x-*)(A+B)'(X(s,*))\,D^k_\alpha X(s,*),\, e_j(*)\big\rangle_{\mathcal{H}} W_j(\mathrm{d}s). \end{aligned}$$

Let $U^n_i(t,x) = E\Big(\|V^n_i(t,x)\|^p_{H_T^{\otimes k}}\ 1_{A^\gamma_n(t)}\Big)$, $i = 1,\ldots,8$.

We have to prove that

$$\lim_{n\to\infty}\ \sup_{(t,x)\in K^T_m} U^n_\rho(t,x) = 0, \tag{3.9}$$

for any $\rho = 1,2,3,4,7$, and

$$U^n_\rho(t,x) \leq C\int_{\bigvee r_i}^t \sup_{(u,x)\in K^s_{m+T}} E\Big(\big\|D^k(X_n(u,x) - X(u,x))\big\|^p_{H_T^{\otimes k}}\ 1_{A^\gamma_n(u)}\Big)\,\mathrm{d}s, \tag{3.10}$$

for $\rho = 5,6,8$, where $K^t_m = [0,t]\times\{x\in\mathbb{R}^2 : \|x\|\leq m\}$, $t\in[0,T]$ and $m\in\mathbb{N}$, and we are assuming that (3.3) holds up to the order of derivation $k-1$. This can be done using the same ideas as in the proof of the above mentioned Proposition 2.3 in [11] taking into account the results given in the next Lemmas 1 to 3. Basically, the proof of (3.9) for $\rho = 1,2,3,4$ follows easily from the induction hypothesis, the proof for $\rho = 2$ uses Lemma 3 and induction; the proof of (3.10) for $\rho = 5,6$ uses the boundedness and Lipschitz properties of the coefficients.

Notice that the structure of $U_\rho^n(t,x)$ for $\rho = 7,8$ is similar. These terms are the most difficult to study. However the ideas used in Proposition 2.3 in [11] can be adapted using the additional ingredient of induction for the term corresponding to $\rho = 7$. Once (3.9) and (3.10) have been established, the proof of the theorem is completed by means of a Gronwall-type argument.

The next statements correspond to the appropriate extensions of Lemmas 2.2–2.4 in [11], they are proved by induction on the derivative order $k$. The third one is related with the result stated in Proposition 2.2 in [11], we give a complete proof of it; as a by-product of this lemma we obtain Hölder continuity in time of the Malliavin derivative process $D^k X(t,x), t \geq 0$.

**Lemma 1.** *We assume the hypothesis of Theorem 2. Then, for any $c \geq 0$, $k \in \mathbb{Z}_+$, $p \in [1,\infty)$, every integer $n \geq 1$ and $0 < \beta < \beta_0 \wedge 1$,*

$$\sup_{\|h\|_{H_T} \leq c} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \|D^k\big(X(t,x) - X^-(t,x)\big)\|_{L^p(\Omega;H_T^{\otimes k})} \leq C\, 2^{-n\,\frac{\beta+1}{2}}.$$

**Lemma 2.** *We suppose that the assumptions of Theorem 2 are satisfied. For any $k \in \mathbb{Z}_+, p \in [1,\infty), \gamma > 2T\,\ell n\, 2$, $c \geq 0$, every integer $n \geq 1$ and $0 < \beta < \beta_0 \wedge 1$,*

$$\begin{aligned}
&\sup_{\|h\|_{H_T} \leq c} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \big\|D^k(X_n(t,x) - X_n^-(t,x))\; 1_{A_n^\gamma(t)}\big\|_{L^p(\Omega;H_T^{\otimes k})} \\
&\leq C\, n\, 2^{-n\frac{1+\beta}{2}} \left(1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \big\|D^k\big(X_n(t,x)\big)\; 1_{A_n^\gamma(t)}\big\|_{L^p(\Omega;H_T^{\otimes k})}\right), \\
&\sup_{\|h\|_{H_T} \leq c} \sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \big\|D^k\big(X_n(t,x) + X_n^-(t,x)\big)\; 1_{A_n^\gamma(t)}\big\|_{L^p(\Omega;H_T^{\otimes k})} < \infty. \qquad (3.11)
\end{aligned}$$

*Consequently,*

$$\sup_{\|h\|_{H_T} \leq c} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \big\|D^k\big(X_n(t,x) - X_n^-(t,x)\big)\; 1_{A_n^\gamma(t)}\big\|_{L^p(\Omega;H_T^{\otimes k})} \leq C\, n\, 2^{-n\frac{1+\beta}{2}}.$$

**Lemma 3.** *We assume the hypothesis of Theorem 2. Then, for any $c \geq 0$, $p \in [1,\infty)$, $0 \leq t \leq t' \leq T$, $k \in \mathbb{Z}_+$, $\alpha \in (0, \frac{1}{2}(\beta_0 \wedge 1))$, we have*

$$\sup_{\|h\|_{H_T} \leq c} \big\|D^k(X(t,x) - X(t',x))\big\|_{L^p(\Omega;H_T^{\otimes k})} \leq C\; |t-t'|^\alpha. \qquad (3.12)$$

*Proof.* Fix $0 \leq t \leq t' \leq T,\ x \in \mathbb{R}^2$; set $\gamma(t,t',x;s,y) = S(t-s,x-y) - S(t'-s,x-y), (s,y) \in [0,T]\times\mathbb{R}^2$. Notice that $(s,y,z) \longmapsto \gamma(t,t',x;s,y) f(|y-z|) \gamma(t,t',x;s,z)$ defines a density on $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$.

We first prove (3.12) for $k = 0$. For any fixed $p \in [1,\infty)$ we set

$$E\big(|X(t,x) - X(t',x)|^p\big) \leq C \sum_{i=1}^4 R_i(t,t';x),$$

with

$$R_1(t,t';x) = \left|X^0(t,x) - X^0(t',x)\right|^p,$$

$$R_2(t,t';x) = E\left(\left|\sum_{j\geq 0}\int_0^T \left\langle \gamma(t,t',x;s,*)(A+B)(X(s,*)),\, e_j(*)\right\rangle_{\mathcal{H}} W_j(\mathrm{d}s)\right|^p\right),$$

$$R_3(t,t';x) = E\left(\left|\sum_{j\geq 0}\int_0^T \left\langle \gamma(t,t',x;s,*)\, G(X(s,*)),\, e_j(*)\right\rangle_{\mathcal{H}} h_j(s)\,\mathrm{d}s\right|^p\right),$$

$$R_4(t,t';x) = E\left(\left|\int_0^T \gamma(t,t',x;s,y)\, b(X(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|^p\right).$$

In the proof of Proposition 1.4 in [10] we have established that

$$R_1(t,t';x) \leq C\left\{\|\, v_0\,\|_{q_0}^p\; |\, t-t'\,|^{p(\frac{1}{q}-\frac{1}{2})} + |\, t-t'\,|^{\frac{p}{2}(\beta_0\wedge 1)}\right\},$$

where $\frac{1}{q}+\frac{1}{q_0}=1$ and we have used (C2). The restriction on $q_0$ yields, for $\alpha\in(0,\frac{1}{2}(\beta_0\wedge 1))$,

$$R_1(t,t';x) \leq C\;|\, t-t'\,|^{\alpha p}. \tag{3.13}$$

Burkholder's inequality, then Fubini's theorem and finally Hölder's inequality with respect to the measure whose density has been described at the begining of the proof, yield

$$R_2(t,t';x) \leq C\left\|\gamma(t-t',x;\cdot,*)\right\|^p_{L^2([0,T];\mathcal{H})} \sup_{(s,x)\in[0,T]\times\mathbb{R}^2} E\big(|(A+B)(X(s,x))|^p\big).$$

In [11], Lemma 2.5 (see the proof of (2.42) with $x=\bar{x}$), it is proved that

$$\left\|\gamma(t,t',x,\cdot,*)\right\|^2_{L^2([0,T];\mathcal{H})} \leq \mu_{t,t'-t} + \tilde{\mu}_{t,t'-t} + 2(\mu_{t,t'-t}\,\tilde{\mu}_{t,t'-t})^{1/2},$$

with $\mu_{t,h},\tilde{\mu}_{t,h}$ are defined in (A.2, A.3), respectively. The bound (A.5) yields

$$R_2(t,t';x) \leq C\big(|\, t-t'\,|^{\alpha p}\big), \tag{3.14}$$

with $\alpha\in(0,\frac{1}{2}(\beta_0\wedge 1))$.

The same bound is obtained for the remaining terms $R_i(t,t';x),\ i=3,4$, following similar ideas.

The proof for any integer $k\geq 1$ is completed using induction on $k$. □

We continue by setting the additional ingredients needed in the application of Proposition 1 in our example.

**Lemma 4.** *Let* $\underline{z}=(z_1,\dots,z_d)$. *For any integer* $k\geq 0$ *and any* $h\in H_T$ *set*

$$\rho_n^{k,h,\underline{z}}(t,\underline{x})(\omega) = D^k u(t,\underline{x})\left(W - W^n + \int_0^{\cdot}\left(h + \sum_{i=1}^d z_i\;\bar{D}\;\Phi^h(t,x_i)\right)(s)\mathrm{d}s\right),$$

*where* $\bar{D}$ *denotes the Fréchet derivative operator. Then, for any* $p\geq 1$,

$$\lim_{M\to\infty}\ \sup_n\ P\left\{\int_{|\underline{z}|\leq 1}\|\rho_n^{k,h,\underline{z}}(t,\underline{x})\|^p_{H_T^{\otimes k}}\,\mathrm{d}z > M\right\} = 0.$$

*Proof.* Fix $\varepsilon > 0$. By property (2.17) there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $\gamma > 2T\,\ell n\, 2$, $P\big(A_n^\gamma(t)^c\big) < \varepsilon$. Then, Chebychev's inequality yields, for any $n \geq n_0$,

$$P\left\{\int_{|\underline{z}|\leq 1} \|\rho_n^{k,h,\underline{z}}(t,\underline{x})\|^p_{H^{\otimes k}} \mathrm{d}z > M\right\} \leq \varepsilon + \frac{1}{M} \sup_{|\underline{z}|\leq 1} E\Big( \|\rho_n^{k,h,\underline{z}}(t,\underline{x})\|^p_{H_T^{\otimes k}} 1_{A_n^\gamma(t)}\Big).$$

Thus, it suffices to check that

$$\sup_n \sup_{|\underline{z}|\leq 1} E\Big( \|\rho_n^{k,h,\underline{z}}(t,\underline{x})\|^p_{H_T^{\otimes k}} 1_{A_n^\gamma(t)}\Big) \leq C < \infty. \tag{3.15}$$

The equation satisfied by $D^k u(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}^2$, is obtained from equation (3.8) with $A = G = 0$, $B = \sigma$. Set $\tilde{W}^{n,h,\underline{z}} = W - W^n + \dot{\int_0} \big(h + \sum_{i=1}^d\ z_i\ \bar{D}\ \Phi^h(t,x_i)\big)(s)\mathrm{d}s$; then, each component of the $d$-dimensional random vector $\rho_n^{k,h,\underline{z}}(t,\underline{x})$ satisfies the following equation, where for simplicity we have omitted the index of the variable $x$:

$$\begin{aligned}
\rho_{n,\alpha}^{k,h,\underline{z}}(t,x) \ =\ & \sum_{i=1}^k \langle D_{\hat{\alpha}_i}^{k-1}\big(\sigma(u(r_i,*))\big)\ (\tilde{W}^{n,h,\underline{z}})\ S(t-r_i,x-*),\ e_{j_i}(*)\rangle_{\mathcal{H}} \\
&+\sum_{j\geq 0}\int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\,\Delta_\alpha(\sigma,u(s,*))\,(\tilde{W}^{n,h,\underline{z}}),\ e_j(*)\rangle_{\mathcal{H}}\, W_j(\mathrm{d}s) \\
&+\sum_{j\geq 0}\int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\,\Delta_\alpha(\sigma,u(s,*))\,(\tilde{W}^{n,h,\underline{z}}),\ e_j(*)\rangle_{\mathcal{H}}\,\big(h_j(s) \\
&-\dot{W}_j^n(s) + \sum_{i=1}^d\ z_i\ \bar{D}_{s,j}\ \Phi^h(t,x_i)\big)\,\mathrm{d}s \\
&+\int_{\bigvee_i r_i}^t\int_{\mathbb{R}^2} S(t-s,x-y)\ \Delta_\alpha(b,\ u(s,y))\,(\tilde{W}^{n,h,\underline{z}})\,\mathrm{d}s\,\mathrm{d}y \\
&+\sum_{j\geq 0}\int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\,\sigma'(\rho_n^{0,h,\underline{z}}(s,*))\ \rho_{n,\alpha}^{k,h,\underline{z}}(s,*),\ e_j(*)\rangle_{\mathcal{H}}\, W_j(\mathrm{d}s) \\
&+\sum_{j\geq 0}\int_{\bigvee_i r_i}^t \langle S(t-s,x-*)\,\sigma'(\rho_n^{0,h,\underline{z}}(s,*))\ \rho_{n,\alpha}^{k,h,\underline{z}}(s,*),\ e_j(*)\rangle_{\mathcal{H}}\,\big(h_j(s) \\
&-\dot{W}_j^n(s) + \sum_{i=1}^d\ z_i\ \bar{D}_{s,j}\ \Phi^h(t,x_i)\big)\,\mathrm{d}s \\
&+\int_{\bigvee_i r_i}^t\int_{\mathbb{R}^2} S(t-s,x-y)\ b'(\rho_n^{0,h,z}(s,y))\,\rho_{n,\alpha}^{k,h,\underline{z}}(s,y)\,\mathrm{d}s\,\mathrm{d}y.
\end{aligned} \tag{3.16}$$

Notice that, by the chain rule (2.20)

$$D_{\hat{\alpha}_i}^{k-1}\left(\sigma\left(u(s,x)\right)\right)(\tilde{W}^{n,h,\underline{z}}) = \sum_{m=1}^{k-1}\sum_{P_m} c_m\ \sigma^{(m)}\left(\rho_n^{0,h,\underline{z}}(s,x)\right)\ \prod_{j=1}^m\ \rho_{n,p_i}^{|p_i|,h,\underline{z}}(s,x),$$

where if $\alpha = (\alpha_1,\ldots,\alpha_k)$, $P_m$ is the set of partitions of the set $\{(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1}\ldots\alpha_k) \in (\mathbb{R}_+\times\mathbb{N})^{k-1}\}$ consisting of $m$ disjoint subsets $p_1,\ldots,p_m$, $m = 1,\ldots,k-1$. Moreover, for $g = \sigma, b$,

$$\Delta_\alpha\big(g,\ u(s,x)\big)\,(\tilde{W}^{n,h,\underline{z}}) = D_\alpha^k\big(g\left(u(s,x)\right)\big)\,(\tilde{W}^{n,h,\underline{z}}) - g'\big(\rho_n^{0,h,\underline{z}}(s,x)\big)\ \rho_{n,\alpha}^{k,h,\underline{z}}(s,x).$$

Therefore equation (3.16) has the same structure than equation (3.7) with $h_j$ replaced by $h_j + \sum_{i=1}^d z_i \bar{D}_{\cdot,j} \Phi^h(t, x_i)$. We point out in the Appendix that, for any $a \geq 0$, $(t,x) \in [0,T] \times \mathbb{R}^2$, $\sup_{\|h\|_{H_T} \leq a} \|\bar{D}\, \Phi^h(t,x)\|_{H_T} \leq C < \infty$. Thus, Lemma 2 (see (3.11)) shows (3.15) and completes this proof. □

**Remark 1.** *Consider the transformation defined in (2.15),* $T_n^h(\omega) = (W_j - W_j^n + \int_0^{\cdot} h_j(s)\,\mathrm{d}s)_{j\geq 0}$, $h \in H_T$. *Using the notation introduced in Lemma 4 we have that* $u(t,x)\,(T_n^h) = \rho_n^{0,h,\underline{0}}(t,x)$ *and* $D\,u(t,x)\ (T_n^h) = \rho_n^{1,h,\underline{0}}(t,x)$. *Hence (3.16) yields*

$$\begin{aligned}
\rho_{n,\alpha}^{1,h,\underline{0}}(t,x) \quad &= \quad \langle S(t-r, x-*)\,\sigma\big(\rho_n^{0,h,\underline{0}}(r,*)\big),\, e_j(*)\rangle_{\mathcal{H}} \\
&\quad + \sum_{k\geq 0} \int_r^t \langle S(t-s,x-*)\,\sigma'\big(\rho_n^{0,h,\underline{0}}(r,*)\big),\, \rho_{n,\alpha}^{1,h,0}(s,*)\, e_k(*)\rangle_{\mathcal{H}} \\
&\quad \times \{W_k(\mathrm{d}s) - \dot{W}_k^n(s)\,\mathrm{d}s + h_k(s)\,\mathrm{d}s\} \\
&\quad + \int_r^t \int_{\mathbb{R}^2} S(t-s,x-y)\, b'\big(\rho_n^{0,h,0}(s,y)\big)\ \rho_{n,\alpha}^{1,h,0}(s,y)\,\mathrm{d}s\,\mathrm{d}y,
\end{aligned}$$

*with* $\alpha = (r,j) \in [0,T] \times \mathbb{N}$.

Then, Theorem 2 with $k = 1$ and the coefficients $A = -B = G = \sigma$ yields

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \sup_{\underline{x}\in K^d} E\Big(\|(D\,u(t,\underline{x}))\circ\ T_n^h - \bar{D}\,\Phi^h(t,\underline{x})\|_{H_T(\mathbb{R}^d)}^p\ 1_{A_n^\gamma(t)}\Big) = 0. \tag{3.17}$$

We conclude this section by completing Step 1 of our programme, as has been described in Section 1. Recall that $p_{t,\underline{x}}(y)$ denotes the density of the random vector $(u(t,x_1),\ldots,u(t,x_d))$ at $y \in \mathbb{R}^d$.

**Theorem 3.** *We assume*

- (i') *there exist* $0 < a_2 \leq a_1 < 2$ *such that* $2(a_1 - a_2) < a_2 \wedge 1$, *and a positive constant* $C_1$ *such that for* $t \in [0,T]$, $C_1\, t^{a_1} \leq \int_0^t y\, f(y)\, \ell n\, \left(1 + \frac{t}{y}\right)\mathrm{d}y$ *and* $\int_{0+} y^{1-a_2}\ f(y)\,\mathrm{d}y < \infty$;
- (ii') $u_0 : \mathbb{R}^2 \to \mathbb{R}$ *is of class* $\mathcal{C}^1$, *bounded, with* $\frac{1}{2}\,(a_2 \wedge 1)$*-Hölder continuous partial derivatives,* $\nabla u_0 \in L^{q_1}(\mathbb{R}^2)$ *for some* $q_1 > 2$; $v_0 : \mathbb{R}^2 \to \mathbb{R}$ *belongs to* $L^{q_0}(\mathbb{R}^2)$ *for some* $q_0 \in \left]4 \vee \frac{2}{1-(a_2\wedge 1)}, \infty\right]$;
- (iii) *the coefficients* $\sigma$ *and* $b$ *are* $\mathcal{C}^\infty$ *functions with bounded derivatives of any order* $i \geq 1$;
- (iv) *there exists* $a > 0$ *such that* $|\sigma\big(u(t,x_j)\big)| \geq a$, *for any* $j = 1,\ldots,d$, *a.s.*

*Then the next two statements on* $y \in \mathbb{R}^d$ *are equivalent: (a)* $p_{t,\underline{x}}\,(y) > 0$ *and (b) there exists* $h \in H_T$ *such that* $\Phi^h(t,\underline{x}) = y$ *and* $\det\, \gamma_{\Phi^h(t,\underline{x})} > 0$.

*Proof.* First we establish $(b) \Rightarrow (a)$. With this purpose, we apply Proposition 1 to $F = \big(u(t,x_1),\ldots,u(t,x_d)\big)$. Let $v_n = u \circ T_n^h$ with $T_n^h$ defined by (2.15). The process $\{v_n(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^2\}$ satisfies the equation

$$\begin{aligned}
v_n(t,x) = X^0(t,x) + \sum_{j\geq 0} \int_0^t \langle S(t-s,x-*)\,\sigma\big(v_n(s,*)\big)\,, e_j(*)\rangle_{\mathcal{H}}\ \{W_j\,(\mathrm{d}s) \\
- \dot{W}_j^n(s)\,ds + h_j(s)\,ds\} + \int_0^t \int_{\mathbb{R}^2} S(t-s,x-y)\, b\big(v_n(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y,
\end{aligned}$$

which is a particular case of equation (3.1) with $A = -B = G = \sigma$. For this choice of coefficients, equation (3.2) coincides with that satisfied by the skeleton $\Phi^h(t,x)$ (see (1.9)). Then, Theorem 2 yields

$$\lim_{n\to\infty} \sup_{0\le t\le T} \sup_{\underline{x}\in K^d} E\Big(\|(u\circ T_n^h)(t,\underline{x}) - \Phi^h(t,x)\|^p \; 1_{A_n^\gamma(t)}\Big) = 0,$$

for any $p\in(1,\infty)$, $\gamma > 2T\,\ell n\,2$ and every compact set $K\subset\mathbb{R}^2$. By (2.17) this ensures condition (2.2).

The validity of (2.3) follows from (3.17) and (2.17). Finally (2.4) has been proved in Lemma 4.

Let us now check $(a)\Rightarrow(b)$. Consider the process defined by the evolution equation

$$\begin{aligned} u_n(t,x) &= X^0(t,x) + \sum_{j\ge 0}\int_0^t \langle S(t-s,x-*)\;\sigma\big(u_n(s,*)\big),\, e_j(*)\rangle_{\mathcal{H}} \dot{W}_j^n(s)\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)\; b\big(u_n(s,y)\big)\,\mathrm{d}s\,\mathrm{d}y. \end{aligned}$$

This is a particular case of equation (3.1) with $A = G = 0$, $B=\sigma$. Moreover, the process $\{u_n(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^2\}$ coincides with the skeleton $\{\Phi^{\dot{W}^n}(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^2\}$; with this choice of coefficients, equation (3.2) coincides with (1.3). Theorem 2 shows that the assumptions of Proposition 2 are satisfied with $\Phi(h) = \Phi^h(t,\underline{x})$, $F = u(t,\underline{x})$ and the localizing sequence $(A_n^\gamma(t))_{n\in\mathbb{N}}$ defined in (2.16). This completes de proof of the theorem. □

## 4. Positivity of the density

The purpose of this section is to analize under which conditions property (b) of Theorem 3 is satisfied for any $y\in\mathbb{R}^d$, that is, the density $p_{t,\underline{x}}(y)$ is strictly positive everywhere.

**Theorem 4.** *Suppose that the assumptions (i', ii', iii) of Theorem 3 are satisfied. Moreover, we assume*

(iv') $\{\inf\ |\sigma(z)|, z\in\mathbb{R}\} > C$, *for some constant* $C>0$*;*

(v) *the covariance functional* $J$ *defined in (1.8) is positive.*

*Then, for any* $y\in\mathbb{R}^d$ *there exists* $h\in H_T$ *such that* $\Phi^h(t,\underline{x}) = y$ *and* $\det\gamma_{\Phi^h(t,\underline{x})} > 0$.

*Proof.* The nondegeneracy of the matrix $\gamma_{\Phi^h(t,\underline{x})}$ has been established in Proposition 3 of the Appendix. Hence it only remains to prove that each $y\in\mathbb{R}^d$ can be reached, with an appropriate choice of $h\in H_T$, through the skeleton.

Set $y = (y_1,\dots,y_d)$ and let $\lambda_1,\dots,\lambda_d\in\mathbb{R}$ satisfying the linear system

$$y_\ell - X^0(t,x_\ell) = \sum_{i=1}^d \lambda_i\;\gamma_S^{i,\ell},\quad \ell = 1,\dots,d,$$

where $\gamma_S^{i,\ell} = \langle S(t-\cdot,x_i-*),\ S(t-\cdot,x_\ell-*)\rangle_{L^2([0,T];\mathcal{H})}$.

Let $k(s,z) = \sum_{i=1}^d \lambda_i\; S(t-s,x_i-z)$, $K(s,y) = \int_{\mathbb{R}^2} f(|y-z|)\; k(s,z)\,\mathrm{d}z$, $s\in[0,T]$, $z,y\in\mathbb{R}^2$. Notice that $k\in L^2([0,T];\mathcal{H})$ and by Schwarz's inequality

$$\int_0^T \mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}y\;|K(s,y)|\;S(t-s,x-y) < \infty. \tag{4.1}$$

For any $\tau\in[0,T]$, $x\in\mathbb{R}^2$, set

$$\varphi(\tau,x) = X^0(\tau,x) + \int_0^\tau \mathrm{d}s\int_{\mathbb{R}^2}\mathrm{d}y\;K(s,y)\;S(\tau-s,x-y). \tag{4.2}$$

By construction, $\varphi(t, x_\ell) = y_\ell,\ \ell = 1, \dots, d$.

Let $\mathcal{Y}$ be the linear space consisting of functions $H : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\int_0^T \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y\ |H(s,y)|\ S(t-s, x-y) < \infty$. Consider the linear operator

$$\mathcal{T} = L^2([0,T];\mathcal{H}) \longrightarrow \mathcal{Y}$$
$$h \longmapsto \int_{\mathbb{R}^2} f(|y-z|)\ h(s,z)\,\mathrm{d}z.$$

Hypothesis (v) ensures that $\mathcal{T}$ admits an inverse operator $\mathcal{T}^{-1}$. Indeed, 0 is not an eigenvalue of $\mathcal{T}$.

Set

$$H_0(s,y) = -\Big[b\Big(\varphi(s,y)\Big) - K(s,y)\Big]\,\sigma\Big(\varphi(s,y)\Big)^{-1}, (s,y) \in [0,T] \times \mathbb{R}^2. \tag{4.3}$$

It is not difficult to check that $H_0 \in \mathcal{Y}$. Therefore

$$H_0(s,y) = \int_{\mathbb{R}^2} f(|y-z|)\ h_0(s,z)\,\mathrm{d}z,$$

for some $h_0 \in L^2([0,T];\mathcal{H})$. Let $h_0^j(s) = \langle h_0(s,*), e_j(*)\rangle_{\mathcal{H}},\ j \geq 0$. Then $(h_0^j,\ j \geq 0) \in H_T$.

Substituting the function $K = H_0$ given by (4.3) in (4.2) we conclude that $\Phi^{h_0} \equiv \varphi$, by uniqueness of solution, and consequently $\Phi^{h_0}(t, x_\ell) = y_\ell,\ \ell = 1, \dots, d$, as we wanted to prove. □

## 5. Appendix

Let $S(t,x), (t,x) \in [0,\infty[\times R^2$ be the fundamental solution of the stochastic wave equation. We start this section by quoting some notations and results concerning $S$ that have been proved in [10] and [8] and used along the proofs.

For any $t \in [0,T]$, $h \geq 0$, set

$$J(t) = \int_{|y|<|x|<t} \frac{1}{\sqrt{t^2 - |x|^2}}\, f(|x-y|) \frac{1}{\sqrt{t^2-|y|^2}}\,\mathrm{d}x\,\mathrm{d}y, \tag{A.1}$$

$$\mu(t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y\, S(s,x) f(|x-y|)\, S(s,y) = \frac{1}{2\pi^2} \int_0^t J(s)\,\mathrm{d}s,$$

$$\nu(t) = \frac{1}{2\pi} \int_0^t \mathrm{d}s \int_{|x|<s} \frac{\mathrm{d}x}{\sqrt{s^2-|x|^2}} = \frac{t^2}{2},$$

$$\begin{aligned} \mu_{t,h} = &\int_0^t \mathrm{d}s \int_{|y|<s} \mathrm{d}y \int_{|z|<s} \mathrm{d}z \big[S(s,y) - S(s+h,y)\big]\ f(|y-z|) \\ &\times \big[S(s,z) - S(s+h,z)\big]\,, \end{aligned} \tag{A.2}$$

$$\tilde{\mu}_{t,h} = \int_0^t \mathrm{d}s \int_{s\leq|y|<s+h} \mathrm{d}y \int_{s\leq|z|<s+h} \mathrm{d}z\ S(s+h,y) f(|y-z|) S(s+h,z). \tag{A.3}$$

Assume that $f$ satisfies the assumption (C1) of Section 3; then Lemma A.1 in [10] implies

$$J(t) \le C\, t^{\beta},\ \mu(t) \le C\, t^{\beta+1},\ t \in [0,T],\quad \beta < \beta_0. \tag{A.4}$$

Moreover, for $h$ small enough and $0 < \delta < \beta_0 \wedge 1$,

$$\mu_{t,h} + \tilde{\mu}_{t,h} \le C\, h^{\delta} \tag{A.5}$$

(see Lem. A.5 in [10] and [8], Rem. A.6).

For fixed distinct points $x_1, \dots, x_d \in \mathbb{R}^2$, set

$$\gamma_S = \Big( \langle S(t-\cdot, x_i - *),\ S(t-\cdot, x_j - *)\rangle_{L^2([0,T];\mathcal{H})} \Big)_{1\le i,\, j\le d}.$$

The next lemma provides one of the ingredients in the proofs of Section 4.

**Lemma 5.** *Assume that there exist constants $C_1 > 0$ and $a_1 \in (0,2)$ such that for $t \in [0,T]$,*

$$C_1\, t^{a_1} \le \int_0^t y\ f(y)\ \ell n\Big(1 + \frac{t}{y}\Big) \mathrm{d}y.$$

*Then,* $\det \gamma_S > 0$.

*Proof.* It suffices to prove that for any $\underline{v} = (v_1, \dots, v_d) \in \mathbb{R}^d,\ |v| = 1$,

$$\Gamma(s,v) := v^t\, \gamma_S\, v = \int_0^t \mathrm{d}r \Big\| \sum_{i=1}^d v_i\ S(t-r, x_i - *) \Big\|_{\mathcal{H}}^2 > 0.$$

Let $\varepsilon \in (0,t)$ to be determined later. Then $\Gamma(s,v) \ge I_1 - |I_2|$ with

$$I_1 = \sum_{i=1}^d \int_{t-\varepsilon}^t v_i^2\ \|S(t-r, x_i - *)\|_{\mathcal{H}}^2\ \mathrm{d}r = \mu(\varepsilon),$$

$$I_2 = \sum_{\substack{i,j=1 \\ i\neq j}}^d \int_{t-\varepsilon}^t v_i\, v_j\ \langle S(t-r, x_i - *),\ S(t-r, x_j - *)\rangle_{\mathcal{H}}\ \mathrm{d}r.$$

Lemma A.1 [10] together with the assumption of the lemma yield $I_1 \ge C\, \varepsilon^{a_1+1}$.

Let $m = \inf\{|x_i - x_j|,\ i \ne j\}$, $M = \sup\{|x_i - x_j|,\ i \ne j\}$. Then, if $4\varepsilon < m$, for any $y \in \mathbb{R}^2,\ |y - x_i| \le \varepsilon,\ z \in \mathbb{R}^2,\ |z - x_j| \le \varepsilon$, we have $\frac{m}{2} \le |y - z| \le 2\varepsilon + M$. Therefore $\sup\{f(|y-z|),\ |y - x_i| \le \varepsilon,\ |z - x_j| \le \varepsilon\} \le C$. Then

$$\begin{aligned} &\int_{t-\varepsilon}^t \langle S(t-r, x_i - *),\ S(t-r, x_j - *)\rangle_{\mathcal{H}}\ \mathrm{d}r \\ &= \int_{t-\varepsilon}^t \mathrm{d}r \int_{|x_i - y|\le t-r} \mathrm{d}y \int_{|x_j - z| \le t-r} \mathrm{d}t\ S(t-r, x_i - y)\ f(|y-z|)\ S(t-r, x_j - z) \\ &\le C \int_{t-\varepsilon}^t \mathrm{d}r \Big( \int_{|x-y|\le t-r} S(t-r, x-y) \mathrm{d}y \Big)^2 \le C\ \varepsilon^3. \end{aligned}$$

Since $a_1 + 1 < 3$, taking $\varepsilon$ small enough we obtain $I_1 - |I_2| > 0$. □

Fix $(t,x) \in [0,T] \times \mathbb{R}^2$ and assume (1.5) and the hypotheses (ii', iii) of Theorem 3. Then the map $h \in H_T \to \Phi^h(t,x)$ is infinitely Fréchet differentiable. For any integer $k \geq 1$, the $k$-th order Fréchet derivative $\bar{D}^k \Phi^h(t,x)$ satifies a differentiable equation like (3.8) with $A = -B = G = \sigma$. In particular, for $k = 1$.

$$\begin{aligned}\bar{D}_{r,j}\Phi^h(t,x) &= \left\langle S(t-r, x-*)\, \sigma(\Phi^h(r,*)),\ e_j(*)\right\rangle_{\mathcal{H}} \\ &\quad + \sum_{k\geq 0} \int_0^t \mathrm{d}s \left\langle S(t-s, x-*)\, \sigma'(\Phi^h(s,*))\, D_{r,j}\Phi^h(s,*),\ e_k(*)\right\rangle_{\mathcal{H}} h_k(s) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s,\ x-y)\ b'(\Phi^h(s,y))\, D_{r,j}\ \Phi^h(s,y)\ \mathrm{d}s\, \mathrm{d}y,\end{aligned}$$

if $r \leq t$ and $\bar{D}_{r,j}\Phi^h(t,x) = 0$, if $t < r$.

It is easy to check that for any $(t,x) \in [0,T] \times \mathbb{R}^2$,

$$\sup_{\|h\|_{H_T} \leq a} \left\|\bar{D}\, \Phi^h(t,x)\right\|_{H_T} \leq C < \infty.$$

We now prove the nondegeneracy of the deterministic Malliavin matrix

$$\gamma_{\Phi^h(t,\underline{x})} = \left(\langle \bar{D}\, \Phi^h(t,x_i),\ \bar{D}\, \Phi^h(t,x_j)\rangle_{H_T}\right)_{1\leq i,j\leq d}.$$

**Proposition 3.** *Assume (i', ii', iii) and (iv'). Then* $\det \gamma_{\Phi^h(t,\underline{x})}$ *is strictly positive.*

*Proof.* Set $\bar{D}_{r,j}\, \Phi^h(t,x) = \varphi_{r,j}(t,x) + \psi_{r,j}(t,x)$, with $\varphi_{r,j}(t,x) = \langle S(t-r, x-*)\ \sigma\Big(\Phi^h(r,*)\Big),\ e_j(*)\rangle_{\mathcal{H}}$. Let $v \in \mathbb{R}^d,\ |v| = 1$.

The triangle inequality yields

$$v^* \gamma_{\Phi^h(t,\underline{x})} v = \sum_{k=0}^{\infty} \int_0^t \mathrm{d}r \left|\sum_{i=1}^d v_i\ \bar{D}_{r,k}\ \Phi^h(t,x_i)\right|^2 \geq \frac{1}{2}\ J_1 - J_2,$$

with

$$\begin{aligned}J_1 &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t \mathrm{d}r \left|\sum_{i=1}^d v_i\ \varphi_{r,k}(t,x_i)\right|^2, \\ J_2 &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t \mathrm{d}r \left|\sum_{i=1}^d v_i\ \psi_{r,k}(t,x_i)\right|^2,\end{aligned}$$

$\gamma \in (0,t)$ to be determined later.

**Lower bound for** $J_1$: We write $J_1 \geq \frac{1}{2}\ J_{11} - J_{12}$, where

$$\begin{aligned}J_{11} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t \mathrm{d}r \left|\sum_{i=1}^d v_i\ \sigma\Big(\Phi^h(r,x_i)\Big) \langle S(t-r, x_i-*), e_k(*)\rangle_{\mathcal{H}}\right|^2, \\ J_{12} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t \mathrm{d}r \left|\sum_{i=1}^d v_i\ \langle S(t-r, x_i-*)\Big[\sigma\Big(\Phi^h(r,*)\Big) - \sigma\Big(\Phi^h(r,x_i)\Big)\Big], e_k(*)\rangle_{\mathcal{H}}\right|^2.\end{aligned}$$

Furthermore, $J_{11} \geq J_{111} - |J_{112}|$, with

$$J_{111} = \sum_{k=0}^{\infty} \int_{t-\gamma}^{t} \mathrm{d}r \sum_{i=1}^{d} v_i^2 \ \sigma^2\Big(\Phi^h(r,x_i)\Big) \langle S(t-r, x_i - *), e_k(*)\rangle_{\mathcal{H}}^2,$$

$$J_{112} = \sum_{k=0}^{\infty} \int_{t-\gamma}^{t} \mathrm{d}r \sum_{\substack{i,j=1 \\ i \neq j}}^{d} v_i \ v_j \ \sigma\Big(\Phi^h(r,x_i)\Big) \ \sigma\Big(\Phi^h(r,x_j)\Big)$$
$$\times \langle S(t-r, x_i - *), e_k(*)\rangle_{\mathcal{H}} \ \langle S(t-r, x_j - *), e_k(*)\rangle_{\mathcal{H}}.$$

The lower bound assumption in (i') and (iv') yield

$$J_{111} \geq C \ \gamma^{1+a_1}. \tag{A.6}$$

Parseval's identity and the growth condition on $\sigma$ yield

$$|J_{112}| \leq C \sup_{(s,x)\in[0,T]\times\mathbb{R}^2} \Big(1 + |\Phi^h(s,x)|\Big)^2 \int_{t-\gamma}^{t} \langle S(t-r, x_i - *), S(t-r, x_j - *)\rangle_{\mathcal{H}} \ \mathrm{d}r,$$

with $i \neq j$. Notice that $\sup_{(s,x)\in[0,T]\times\mathbb{R}^2} |\Phi^h(t,x)| \leq C$,

Therefore, as has been checked in the proof of Lemma 5,

$$|J_{112}| \leq C \ \gamma^3. \tag{A.7}$$

Then (A.6) and (A.7) yield

$$|J_{11}| \geq C \ (\gamma^{1+a_1} - \gamma^3). \tag{A.8}$$

Following the same ideas as in the proof of Theorem 2.2 in [8] one can prove that $\sup_{t\in[0,T]}$ $\sup_{|y-z|\leq\xi} \Big(|\Phi^h(t,y) - \Phi^h(t,z)|\Big) \leq C \ \xi^\alpha$, with $\alpha \in \Big(0, \ \frac{1}{2} \ (a_2 \wedge 1)\Big)$. Hence

$$\begin{aligned} J_{12} &\leq C\gamma^{2\alpha} \ \sup\left\{\int_{t-\gamma}^{t} \langle S(t-r, x_i - *), \ S(t-r, x_j - *)\rangle_{\mathcal{H}} \ \mathrm{d}r; \ i,j = 1,\ldots,d\right\} \\ &\leq C \ \gamma^{2\alpha+1+a_2}. \end{aligned} \tag{A.9}$$

Consequently (A.8) and (A.9) imply

$$J_1 \geq C(\gamma^{1+a_1} - \gamma^3 - \gamma^{2\alpha+1+a_2}). \tag{A.10}$$

**Upper bound for** $J_2$: By the definition of $\psi_{r,k}(t,x)$ and $J_2$, we clearly have $J_2 \le C(J_{21}+J_{22})$ with

$$J_{21} = \sum_{k=0}^{\infty} \int_{t-\gamma}^{t} \mathrm{d}r \sum_{i=1}^{d} v_i^2 \left( \sum_{j=0}^{\infty} \int_r^t \Big\langle S(t-s, x_i - *)\ \sigma'\Big(\Phi^h(s,*)\Big)\, D_{r,k}\ \Phi^h(s,*),\right.$$
$$\left. e_j(*) \Big\rangle_{\mathcal{H}}\ h_j(s)\ \mathrm{d}s \right)^2,$$
$$J_{22} = \sum_{k=0}^{\infty} \int_{t-\gamma}^{t} \mathrm{d}r \sum_{i=1}^{d} v_i^2 \left( \int_r^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y\ S(t-s, x_i - y)\, b'\Big(\Phi^h(s,y)\Big)\, D_{r,k}\ \Phi^h(s,y) \right)^2.$$

Schwarz's inequality and Parseval's identity ensure

$$J_{21} \le C \sum_{k=0}^{\infty} \int_{t-\gamma}^{t} \mathrm{d}r\ \sum_{i=1}^{d}\ v_i^2 \int_r^t \Big\| S(t-s, x_i - *)\ \sigma'\Big(\Phi^h(s,*)\Big)\, D_{r,k}\ \Phi^h(s,*) \Big\|_{\mathcal{H}}^2\ \mathrm{d}s.$$

Then, applying Fubini's theorem and Schwarz's inequality this term is bounded by

$$C \left( \sup_{(s,y)\in[t-\gamma,t]\times\mathbb{R}^2}\ \sum_{k=1}^{\infty} \int_{t-\gamma}^{t} \|D_{r,k}\ \Phi^h(s,y)\|_{\mathcal{H}}^2 \mathrm{d}r \right)\ \mu(\gamma).$$

Following the proof of Theorem 2.2 in [10] we obtain $J_{21} \le C\ \mu(\gamma)^2$. Then, by (A.4),

$$J_{2,1} \le C\ \gamma^{2(a_2+1)} \tag{A.11}$$

Jensen's inequality, Fubini's theorem and similar arguments as those used to obtain (A.11) yield

$$J_{22} \le C\ \mu(\gamma)\ \nu(\gamma) \le C\ \gamma^{3+a_2}. \tag{A.12}$$

Therefore, by (A.11, A.12)

$$J_2 \le C\Big(\gamma^{2(a_2+1)} + \gamma^{3+a_2}\Big). \tag{A.13}$$

Finally (A.10) and (A.13) yield

$$v^* \gamma_{\Phi^h(t,\underline{x})} v \ge C\Big(\gamma^{1+a_1} - \gamma^3 - \gamma^{2\alpha+1+a_2} - \gamma^{2(a_2+1)} - \gamma^{3+a_2}\Big). \tag{A.14}$$

Set $\gamma = \varepsilon^\delta$, $\delta > 0$, such that

$$\delta(1+a_1) < 1,\ 3\delta > 1,\ \delta(2\alpha+1+a_2) > 1,\ 2\delta(a_2+1) > 1,\ \delta(3+a_2) > 1,$$

where $\alpha \in (0, \frac{1}{2}\,(a_2 \wedge 1))$ and $a_1, a_2$ satisfy the restrictions stated in (i'). It is easy to check that such a choice is possible. Then, the right hand-side of (A.14) is strictly positive. This finishes the proof of the proposition.□

## References


[1] S. Aida, S. Kusuoka and D. Stroock, *On the support of Wiener functionals*, edited by K.D. Elworthy and N. Ikeda, Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics. Longman Scient. and Tech., New York, *Pitman Res. Notes in Math. Ser.* **284** (1993) 3-34.

[2] V. Bally and E. Pardoux, Malliavin Calculus for white-noise driven parabolic spde's. *Potential Anal.* **9** (1998) 27-64.
[3] G. Ben Arous and R. Léandre, Décroissance exponentielle du noyau de la chaleur sur la diagonale (II). *Probab. Theory Related Fields* **90** (1991) 377-402.
[4] R. Dalang and N. Frangos, The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26** (1998) 187-212.
[5] O. Lévêque, *Hyperbolic stochastic partial differential equations driven by boundary noises.* Thèse EPFL, Lausanne, 2452 (2001).
[6] D. Mrquez–Carreras, M. Mellouk and M. Sarrà, On stochastic partial differential equations with spatially correlated noise: Smoothness of the law. *Stochastic Proc. Appl.* **93** (2001) 269-284.
[7] M. Métivier, *Semimartingales.* De Gruyter, Berlin (1982).
[8] A. Millet and P.-L. Morien, On a stochastic wave equation in two dimensions: Regularity of the solution and its density. *Stochastic Proc. Appl.* **86** (2000) 141-162.
[9] A. Millet and M. Sanz–Solé, Points of positive density for the solution to a hyperbolic spde. *Potential Anal.* **7** (1997) 623-659.
[10] A. Millet and M. Sanz–Solé, A stochastic wave equations in two space dimension: Smoothness of the law. *Ann. Probab.* **27** (1999) 803-844.
[11] A. Millet and M. Sanz–Solé, Approximation and support theorem for a two space-dimensional wave equation. *Bernoulli* **6** (2000) 887-915.
[12] P.-L. Morien, Hölder and Besov regularity of the density for the solution of a white-noise driven parabolic spde. *Bernoulli* **5** (1999) 275-298.
[13] D. Nualart, *Malliavin Calculus and Related Fields.* Springer-Verlag (1995).
[14] D. Nualart, Analysis on the Wiener space and anticipating calculus, in *École d'été de Probabilités de Saint-Flour.* Springer-Verlag, *Lecture Notes in Math.* **1690** (1998) 863-901.
[15] M. Sanz–Solé and M. Sarrà, Path properties of a class of Gaussian processes with applications to spde's, in *Stochastic Processes, Physics and Geometry: New interplays*, edited by F. Gesztesy *et al.* American Mathematical Society, *CMS Conf. Proc.* **28** (2000) 303-316.
[16] J.B. Walsh, An introduction to stochastic partial differential equations, in *École d'été de Probabilités de Saint-Flour*, edited by P.L. Hennequin. Springer-Verlag, *Lecture Notes in Math.* **1180** (1986) 266-437.