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LONG MEMORY PROPERTIES AND COVARIANCE STRUCTURE OF THE EGARCH MODEL

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Abstract. The EGARCH model of Nelson [29] is one of the most successful ARCH models which may exhibit characteristic asymmetries of financial time series, as well as long memory. The paper studies the covariance structure and dependence properties of the EGARCH and some related stochastic volatility models. We show that the large time behavior of the covariance of powers of the (observed) ARCH process is determined by the behavior of the covariance of the (linear) log-volatility process; in particular, a hyperbolic decay of the later covariance implies a similar hyperbolic decay of the former covariances. We show, in this case, that normalized partial sums of powers of the observed process tend to fractional Brownian motion. The paper also obtains a (functional) CLT for the corresponding partial sums' processes of the EGARCH model with short and moderate memory. These results are applied to study asymptotic behavior of tests for long memory using the R/S statistic.

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1. Introduction

Autoregressive Conditionally Heteroskedastic (ARCH) models of Engle [10] are widely recognized as being instrumental for modeling temporal variation in financial market volatility. Generally, by ARCH model one means a strictly stationary time series $X_t, t \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ of the form

$$X_t = \zeta_t V_t, \tag{1}$$

where $\zeta_t, t \in \mathbb{Z}$ is an i.i.d. sequence with zero mean and unit variance, and V_t ("volatility") is a general function of the "past information" up to time $t-1$. Among various forms and parametrizations of volatility, one of the most successful has been the *Exponential Generalized ARCH (EGARCH)* model proposed by Nelson [29]. In the EGARCH model, the volatility is given by

$$V_t = \exp\left\{a + \sum_{j=1}^{\infty} b_j g\left(\zeta_{t-j}\right)\right\}, \tag{2}$$

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where $a \in \mathbb{R}$ is a constant, $b_j, j = 1, 2, \ldots$ are deterministic weights satisfying $\sum_{j=1}^{\infty} b_j^2 < \infty$, and

$$g(z) = \theta z + \gamma \left[|z| - E|\zeta| \right], \tag{3}$$

where $\theta, \gamma \in \mathbb{R}$ are parameters which account for certain asymmetries observed in financial data (see Nelson [29]). In the particular case of the EGARCH(p, q) model, $\log V_t$ satisfies the ARMA equation

$$\phi(L) \log V_t = \psi(L) g(\zeta_t), \tag{4}$$

where L is the lag operator and $\phi(z), \psi(z)$ are autoregressive polynomials of order p and q, respectively, satisfying the usual root requirements for the existence of a stationary solution of (4).

An important stylized fact of asset returns and some other financial data is the presence of long memory, or long-range persistence (Bollerslev and Mikkelsen [4], Baillie [2], Ding and Granger [9], Lobato and Savin [23]). To model this phenomenon, Bollerslev and Mikkelsen [4] introduced the *Fractionally Integrated Exponential GARCH (FIEGARCH)* model. The FIEGARCH(p, d, q) model is defined by (1, 2), where $b_j, j \geq 1$ are ARFIMA(p, d, q) weights, p, q are nonnegative integers, and $-1/2 < d < 1/2$. In the FIEGARCH model, $\log V_t$ satisfies the equation

$$\phi(L)(1 - L)^d \left(\log V_t - a \right) = \psi(L) g\left(\zeta_t \right),$$

where $(1 - L)^d$ is the fractional differencing operator, see *e.g.* Hosking [19]. In particular, for $0 < d < 1/2$ one has $\sum_{j=1}^{\infty} |b_j| = \infty, \sum_{j=1}^{\infty} b_j^2 < \infty$, and

$$b_j \sim c_0 j^{d-1}, \qquad j \to \infty, \tag{5}$$

where $c_0 = |\psi(1)|/(|\phi(1)|\Gamma(d))$ and $\sim$ denotes the fact that the ratio of both sides tends to 1 as $j \to \infty$.

The aim of the present paper is to study the covariance structure and dependence properties of a general stochastic volatility model which includes the EGARCH and the FIEGARCH models. Let $(\zeta_s, \xi_s), s \in \mathbb{Z}$ be an i.i.d. sequence of random vectors with values in $\mathbb{R}^2$, with zero means $E\zeta = E\xi = 0$ and finite variances (here and below (ξ, ζ) stands for a generic vector). We do not assume any particular form of dependence between ζ and ξ. Let

$$X_t = \zeta_t V_t, \qquad V_t = \exp\left\{ a + \sum_{j=1}^{\infty} b_j \xi_{t-j} \right\}, \tag{6}$$

where $a \in \mathbb{R}$ and $\sum_{j=1}^{\infty} b_j^2 < \infty$. In the special case $\xi = g(\zeta)$, equation (6) becomes the EGARCH model (1–2), while in the case when ζ, ξ are independent and b_j are ARFIMA weights, equation (6) is known as the *long memory stochastic volatility (LMSV)* model, introduced in Breidt *et al.* [5] and Harvey [16]. Related stochastic volatility models were studied by Robinson and Zaffaroni [32], Robinson [31], Ghysels *et al.* [11].

Let us briefly describe the main results of the paper. Theorem 1 obtains long memory asymptotic of covariances $\mathrm{cov}(|X_0|^u, |X_t|^u)$, $\mathrm{cov}(V_0^u, V_t^u)$, for arbitrary $u > 0$, under the regular decay condition (5) with $0 < d < 1/2$, and assuming finiteness of all moments of $|\zeta|$ and $\mathrm{e}^{|\xi|}$. Namely, we show that the above covariances decay as t^{2d-1} with an asymptotic proportionality constant depending on u. Theorem 2 states that, under the same conditions, suitably normalized partial sums' processes $\sum_{s=1}^{[Nt]} |X_s|^u$ and $\sum_{s=1}^{[Nt]} V_s^u$ tend as $N \to \infty$, in the Skorohod space $D[0, 1]$, to a $(d + 1/2)$-fractional Brownian motion. These results are contrasted in Theorem 3 which refers to the short memory case $\sum_{j=1}^{\infty} |b_j| < \infty$: in the latter case, the above covariances are summable, for any $u > 0$, and the corresponding partial sums' processes converge in $D[0, 1]$ to a standard Brownian motion. The proof of Theorem 2.6 is based on Lemma 2.8 which says that, in the short memory case, cumulants $\mathrm{cum}(|X_0|^u, |X_{t_2}|^u, \ldots, |X_{t_n}|^u)$ and $\mathrm{cum}(V_0^u, V_{t_2}^u, \ldots, V_{t_n}^u)$ are absolutely summable in $t_2, \ldots, t_n \in \mathbb{Z}$, for any $n = 2, 3, \ldots$ Corollary 3.1 obtains the asymptotic distribution of the modified R/S statistic of Lo [22] for the stochastic volatility model (6) under short and long memory alternatives.

The paper is organized as follows. In Section 2 we formulate the main results, see Theorems 2.1, 2.2 and 2.6. Section 3 discusses the R/S statistic. Sections 4 and 5 are devoted to proofs; in particular, Section 5 contains the proof of Lemma 2.8 and the cumulant analysis of the model (6).

2. Main results

Let X_t, V_t be given by (6). We shall assume in the rest of the paper that $\sum_{j=1}^{\infty} b_j^2 < \infty$ and, for each $u > 0$,

$$E|\zeta|^u < \infty, \qquad E\mathrm{e}^{u|\xi|} < \infty. \tag{7}$$

Moreover, we assume that $E\xi = E\zeta = 0$ and $E\zeta^2 = E\xi^2 = 1$. Put $Y_t = \log V_t$, then

$$Y_t = a + \sum_{j=1}^{\infty} b_j \xi_{t-j} \tag{8}$$

is a strictly stationary linear process with mean a and covariance

$$r_t = \operatorname{cov}(Y_0, Y_t) = \sum_{j=1}^{\infty} b_j b_{t+j}.$$

Theorem 2.1. *Let b_j satisfy condition (5), where $c_0 \in \mathbb{R}, c_0 \neq 0$ and $0 < d < 1/2$. Then for any $u_1, u_2 > 0$, as $t \to \infty$,*

$$\operatorname{cov}(|X_0|^{u_1}, |X_t|^{u_2}) = u_1 u_2 |\mu|_{u_1} |\mu|_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{9}$$

$$\operatorname{cov}(V_0^{u_1}, V_t^{u_2}) = u_1 u_2 \nu_{u_1} \nu_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{10}$$

where $|\mu|_u = E|X_0|^u, \nu_u = EV_0^u \ (u > 0)$ and $\lambda = \min(d, 1-2d) > 0$.

It is well-known that (5) implies the hyperbolic decay

$$r_t \sim c_1^2 t^{2d-1}, \quad t \to \infty, \tag{11}$$

where $c_1 = c_0 B^{1/2}(d, 1-2d)$ and $B(\cdot,\cdot)$ is the beta-function. Theorem 2.1 together with (11) implies that for each $u > 0$

$$\operatorname{cov}(|X_0|^u, |X_t|^u) \sim (u|\mu|_u c_1)^2 t^{2d-1}, \qquad \operatorname{cov}(V_0^u, V_t^u) \sim (u\nu_u c_1)^2 t^{2d-1}.$$

In other words, the autocovariances of $|X_t|^u$ and V_t^u exhibit the characteristic long memory decay which is asymptotically proportional to the decay of the autocovariance r_t of the linear process Y_t (8). In the case when ζ and ξ are independent, and ξ is normally distributed, one can find the covariances in (9), (10) explicitly, *e.g.*

$$\operatorname{cov}(|X_0|^u, |X_t|^u) = |\mu|_u^2 \left(\mathrm{e}^{u^2 r_t} - 1\right) \sim (u|\mu|_u)^2 r_t, \quad t \to \infty,$$

which agrees with (9). The autocorrelation function $\operatorname{corr}(|X_0|^u, |X_t|^u)$ for the above case was found by Harvey [16], who also discusses its shape and other features.

The relation between long memory properties of (6) and (8) extends to the limit distributions of the corresponding partial sums' processes. Let $B_H(t), t \in [0,1]$ be a *fractional Brownian motion*, *i.e.* a Gaussian process with zero mean and the covariance

$$\operatorname{cov}(B_H(s), B_H(t)) = (1/2)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,1].$$

Write $\Longrightarrow$ for weak convergence of stochastic processes in the Skorohod space $D[0,1]$ (see *e.g.* Billingsley [3] for the definition).

Theorem 2.2. *Let conditions of Theorem 2.1 be fulfilled. Then for any $u > 0$, as $N \to \infty$,*

$$N^{-d-1/2} \sum_{s=1}^{[Nt]} (|X_s|^u - |\mu|_u) \Longrightarrow c_2 u |\mu|_u B_{d+1/2}(t), \tag{12}$$

$$N^{-d-1/2} \sum_{s=1}^{[Nt]} (V_s^u - \nu_u) \Longrightarrow c_2 u \nu_u B_{d+1/2}(t), \tag{13}$$

where $c_2 = c_1/d(2d+1)$.

Remark 2.3. The rather restrictive moment conditions (7) can be weakened; in fact, the statements of Theorems 2.1 and 2.2 remain valid for $u, u_i < \bar{u}$, if assumption (7) is replaced by

$$E|\zeta|^{2\bar{u}} < \infty, \quad E\mathrm{e}^{2\bar{u}\|b\|_\infty |\xi|} < \infty, \tag{14}$$

where $\|b\|_\infty = \sup_{j\ge 1} |b_j|$. Note (14) is close to $E|X_0|^{2\bar{u}} < \infty$. The proofs of the corresponding statements under condition (14) use the same ideas but are technically more involved.

Remark 2.4. A more general class of log volatility processes Y_t (9) corresponds to weights of the form

$$b_j = L(j) j^{d-1}, \qquad j \ge 1, \tag{15}$$

where $L(s), s \in [1, \infty)$ is a function slowly varying at infinity. We expect that for b_j of (15), Theorems 2.1 and 2.2 continue to hold, with $N^{-d-1/2}$ in (12, 13) replaced by $(L(N))^{-1} N^{-d-1/2}$, and with $c_2 = B^{1/2}(d, 1-2d)/(d(2d+1))$. The question of what happens when $L(j)$ in (15) is not slowly varying (*e.g.*, is an oscillating function as in seasonal ARIMA) is open.

Remark 2.5. A notable aspect of Theorems 2.1 and 2.2, as well as of Theorem 2.6 below, is to avoid distributional assumptions on the innovations (ξ_t, ζ_t), which can be largely explained by the exponential nonlinearity of the volatility. In contrast, Robinson [31] established the long memory, in the sense of the asymptotic behavior of the autocovariance function, for a class of stochastic volatility models of an arbitrary form of the model nonlinearity, but imposing Gaussianity of the innovations. Under certain distributional assumptions, exact expressions of the autocovariance function of $|X_t|^u$ were obtained by Demos [8] and He *et al.* [19].

Let us also note that results similar to Theorems 2.1 and 2.2 were recently obtained by Giraitis *et al.* [15] for a different ARCH (called *Linear ARCH (LARCH)* model, first introduced by Robinson [30]). However, the last results refer to integer powers $u = 2, 3, \ldots$ only.

Theorems 2.1 and 2.2 refer to the long memory case $\sum_{j=1}^\infty |b_j| = \infty$. In the case $\sum_{j=1}^\infty |b_j| < \infty$, the linear process Y_t (8) has short memory in the sense that $\sum_{t=-\infty}^\infty |r_t| < \infty$. If, in addition, $\sigma^2 := \sum_{t=-\infty}^\infty r_t \ne 0$, it is well-known that $N^{-1/2} \sum_{s=1}^{[Nt]} Y_s$ converge to $\sigma B(t)$, where $B(t)$ is a standard Brownian motion with $\operatorname{cov}(B(s), B(t)) = \min(t, s)$. (In the case when $\sigma = 0$ as in ARFIMA(p, d, q) with $d \in (-1/2, 0)$, the partial sums converge to a fractional Brownian motion under a normalization which grows slower than $N^{1/2}$; see Davydov [7].) A similar result holds also for the stochastic volatility model (6).

Theorem 2.6. *Let*

$$\sum_{j=1}^{\infty} |b_j| < \infty. \tag{16}$$

Then for any $u_1, u_2 > 0$

$$\sum_{t=-\infty}^{\infty} |\mathrm{cov}\,(|X_0|^{u_1}, |X_t|^{u_2})| \;<\; \infty, \tag{17}$$

$$\sum_{t=-\infty}^{\infty} |\mathrm{cov}\,(V_0^{u_1}, V_t^{u_2})| \;<\; \infty. \tag{18}$$

Moreover, for any $u > 0$,

$$N^{-1/2} \sum_{s=1}^{[Nt]} (|X_s|^u - |\mu|_u) \Longrightarrow \sigma_{u,X} B(t), \tag{19}$$

$$N^{-1/2} \sum_{s=1}^{[Nt]} (V_s^u - \nu_u) \Longrightarrow \sigma_{u,V} B(t), \tag{20}$$

where $\sigma^2_{u,X} = \sum_{t=-\infty}^{\infty} \mathrm{cov}(|X_0|^u, |X_t|^u)$ *and* $\sigma^2_{u,V} = \sum_{t=-\infty}^{\infty} \mathrm{cov}(V_0^u, V_t^u)$.

Remark 2.7. Some of the results of Theorems 2.2 and 2.6 referring to the volatility process V_t follow from Ho and Hsing [18], who studied limit theorems for sums of general instantenous functionals of moving averages. However, their paper does not discuss functional convergence nor the asymptotics of covariance functions as in Theorem 2.1. The proofs of the present paper (with the exception of Lem. 2.8) are also much simpler than those of Ho and Hsing [18].

Note Theorem 2.6 does not rely on any mixing conditions which are usually required to prove functional central limit theorems for dependent sequences. The proof of Theorem 2.6, including the verification of the tightness condition, is based on the following lemma:

Lemma 2.8. *Let condition (16) hold. Then for any* $u > 0$ *and any* $n = 2, 3, \ldots$

$$\sum_{t_2,\ldots,t_n=-\infty}^{\infty} |\mathrm{cum}\,(|X_0|^u, |X_{t_2}|^u, \ldots, |X_{t_n}|^u)| \;<\; \infty, \tag{21}$$

$$\sum_{t_2,\ldots,t_n=-\infty}^{\infty} |\mathrm{cum}\left(V_0^u, V_{t_2}^u, \ldots, V_{t_n}^u\right)| \;<\; \infty. \tag{22}$$

Conditions (21, 22) play important role in time series analysis; see Anderson [1] and Brillinger [6]. In Section 3 we use them to obtain the limit distribution of the modified R/S statistic of Lo [22].

As mentioned above, the specific form $\xi_s = g(\zeta_s)$ with $g(z)$ of (3) allows for certain asymmetries observed in financial data. One of such asymmetries known as *leverage effect* is the observation that X_t and $V_s, s > t$ ("present returns and future volatilities") are negatively correlated. Although the leverage effect has been discussed for the EGARCH model (Nelson [29], Bollerslev and Mikkelsen [4]), we have not found a mathematical proof. The leverage effect in the LARCH model is studied in Giraitis *et al.* [14].

Proposition 2.9. *Consider the EGARCH model given by (1–3). Assume that the distribution of* ζ *is symmetric:*

$$P(\zeta \in \mathrm{d}x) = P(\zeta \in -\mathrm{d}x), \quad x > 0. \tag{23}$$

Then for any $t = 1, 2, \ldots,$ $u > 0,$ $\operatorname{cov}(X_0, V_t^u)$ *has the sign of the product* θb_t *and* $\operatorname{cov}(X_0, V_t^u) = 0$ *if* $\theta b_t = 0$.

3. Asymptotic behavior of R/S statistic

Let $Z_t, t \in \mathbb{Z}$ be a strictly stationary time series. The classical R/S statistic of Hurst [20] is defined as

$$Q_N = R_N / S_N, \tag{24}$$

where

$$R_N = \max_{1\le t\le N} \sum_{s=1}^{t} \left(Z_s - \bar{Z}_N\right) - \min_{1\le t\le N} \sum_{s=1}^{t} \left(Z_s - \bar{Z}_N\right)$$

is the "adjusted range" of the observations $Z_1, \ldots, Z_N$, $\bar{Z}_N = N^{-1} \sum_{t=1}^{N} Z_t$, and

$$S_N^2 = N^{-1} \sum_{t=1}^{N} \left(Z_t - \bar{Z}\right)^2$$

is the sample variance. The R/S statistic provides one of the oldest techniques for detecting long memory and measuring its intensity. The R/S analysis was developpped by Mandelbrot and his collaborators, see Mandelbrot and Wallis [28], Mandelbrot [25, 26] and Mandelbrot and Taqqu [27].

Lo [22] introduced the modified R/S statistic

$$Q_N(q) = R_N / S_N(q),$$

where

$$S_N^2(q) = N^{-1} \sum_{t=1}^{N} \left(Z_t - \bar{Z}_N\right)^2 + 2 \sum_{j=1}^{q} \omega_j(q) \hat{\gamma}_j$$

is an estimator of $\sum_{t=-\infty}^{\infty} \operatorname{cov}(Z_0, Z_t)$. Here, $\omega_j(q) = 1 - \frac{j}{q+1}$ are Bartlett's weights, and

$$\hat{\gamma}_j = N^{-1} \sum_{t=1}^{N-j} \left(Z_t - \bar{Z}_N\right)\left(Z_{t+j} - \bar{Z}_N\right), \quad 0 \le j < N$$

are the sample covariances. The classical R/S statistic corresponds to $q = 0$. Contrary to the classical R/S statistic, the modified R/S statistic is asymptotically distribution free provided q increases slowly with the sample size, and can be used to test statistical hypotheses about the presence of long memory (Lo [22]). Further modifications of the classical R/S statistics were proposed in Kwiatkowski *et al.* [21] (the KPSS statistic) and Giraitis *et al.* [12] (the V/S statistic).

In the remaining part of this section, $R_N, S_N(q), Q_N(q)$ will denote the corresponding statistics based on observations $Z_t = |X_t|^u, t = 1, \ldots, N$, where X_t is given by our stochastic volatility model (6), with ζ, ξ satisfying (7), and $u > 0$ is a fixed number. Put

$$\gamma_j = \operatorname{cov}\left(|X_0|^u, |X_j|^u\right), \quad \sigma^2 = \sum_{j=-\infty}^{\infty} \gamma_j.$$

Corollary 3.1. (i) ("short memory") *Assume* $\sum_{j=1}^{\infty} |b_j| < \infty$ *and* $\sigma \neq 0$. *Then, as* $q \to \infty$ *and* $q/N \to 0$,

$$N^{-1/2} Q_N(q) \Rightarrow \max_{0 \le t \le 1} B^0(t) - \min_{0 \le t \le 1} B^0(t), \tag{25}$$

where $B^0(t) = B(t) - tB(1), t \in [0,1]$ *is a Brownian bridge, and* $\Rightarrow$ *denotes the convergence in distribution.*

(ii) ("long memory") *Assume condition (5), with* $0 < d < 1/2$, *and let* $q \to \infty, q/N \to 0$. *Then*

$$N^{-1/2} |Q_N(q)| \to \infty \tag{26}$$

in probability.

Proof. (i) According to Anderson [1] (Th. 9.3.4), relations (21) for $n = 2, 4$ imply

$$S_N^2(q) \to \sigma^2,$$

in probability. This, together with Theorem 2.6 implies (25), see Lo [22], also Giraitis *et al.* [12].

(ii) Rewrite $N^{-1/2} Q_N(q) = N^{-d-1/2} R_N / (N^{-d} S_N(q))$. By Theorem 2.2, $N^{-d-1/2} R_N$ tends in distribution to a non-degenerate limit, while $N^{-d} S_N(q) \to 0$ in probability; see the references right above for details. Hence (26) holds. □

In a similar way, one can obtain from Theorems 2.1, 2.2 and 2.6 the limit distribution of the original R/S statistic Q_N (24), as well as of the KPSS statistic and the V/S statistic mentioned above.

4. Proofs of Theorems 2.1, 2.2, 2.6 and of Proposition 2.9

Without loss of generality, we put below $a = 0$. Let $L^2(\mathbb{Z})$ be the space of all real sequences $f = f(s), s \in \mathbb{Z}$ with finite norm $\|f\|_2 = \left(\sum_s f^2(s)\right)^{1/2}$. Let

$$h_i = \sum_{s \in \mathbb{Z}} f_i(s) \xi_s, \tag{27}$$

where $f_i \in L^2(\mathbb{Z}), i = 1, 2$. Note that $\operatorname{cov}(h_1, h_2) = \sum_s f_1(s) f_2(s)$. Given h_i of (27) and a set $A \subset \mathbb{Z}$, define $h_{i,A}$ as $h_{i,A} = \sum_{s \in A} f_i(s) \xi_s$, $i = 1, 2$, and let $\mathcal{F}_A$ be the σ-field generated by $(\zeta_s, \xi_s) : s \in A$.

Lemma 4.1. *(i) For any* $C_1 < \infty$ *there exists a constant* $C < \infty$, *independent of* $f_i, i = 1, 2$, *such that for all* $f_i \in L^2(\mathbb{Z}), \|f_i\|_2 \le C_1, i = 1, 2$

$$\left|\operatorname{cov}\left(\mathrm{e}^{h_1}, \mathrm{e}^{h_2}\right) - \operatorname{cov}(h_1, h_2)\, E\mathrm{e}^{h_1} E\mathrm{e}^{h_2}\right| \le C \left(\operatorname{cov}^2(h_1, h_2) + \sum_s \left(f_1^2(s) |f_2(s)| + |f_1(s)| f_2^2(s) \right) \right). \tag{28}$$

(ii) Let $A_i \subset \mathbb{Z}, i = 1, 2, A_1 \cap A_2 = \emptyset$ *be arbitrary disjoint subsets, and* F_i *be* $\mathcal{F}_{A_i}$*-measurable random variables,* $i = 1, 2$, *with finite variance. Let* $f_i(s) = 0, s \in A_i$ $(i = 1, 2)$. *Then*

$$\left|\operatorname{cov}\left(F_1 \mathrm{e}^{h_1}, F_2 \mathrm{e}^{h_2}\right)\right| \le C \left(\sum_s |f_1(s) f_2(s)| + \sum_{s \in A_2} |f_1(s)| + \sum_{s \in A_1} |f_2(s)| \right). \tag{29}$$

The constant $C < \infty$ *in (29) depends on* C_1 *and does not depend on* $f_i, A_i, F_i, i = 1, 2$ *provided* $\|f_i\|_2 \le C_1, EF_i^2 \le C_1, i = 1, 2$, *for any fixed* $C_1 < \infty$. *In particular,*

$$\left|\operatorname{cov}\left(\mathrm{e}^{h_1}, \mathrm{e}^{h_2}\right)\right| \le C \sum_s |f_1(s) f_2(s)|. \tag{30}$$

Proof. (i) By independence of $\xi_s, s \in \mathbb{Z}$,

$$\operatorname{cov}\left(\mathrm{e}^{h_1}, \mathrm{e}^{h_2}\right) = \prod_s \phi\left(f_1(s) + f_2(s)\right) - \prod_s \phi\left(f_1(s)\right) \phi\left(f_2(s)\right),$$

where the function

$$\phi(u) = E\mathrm{e}^{u\xi}, \quad u \in \mathbb{R}$$

is well-defined and infinitely differentiable on $\mathbb{R}$, according to assumption (7). Then

$$\operatorname{cov}\left(\mathrm{e}^{h_1}, \mathrm{e}^{h_2}\right) - \operatorname{cov}\left(h_1, h_2\right) E\mathrm{e}^{h_1} E\mathrm{e}^{h_2} = E\mathrm{e}^{h_1} E\mathrm{e}^{h_2} \left(\mathrm{e}^{\sum_s \psi(f_1(s), f_2(s))} - 1 - \operatorname{cov}\left(h_1, h_2\right)\right), \tag{31}$$

where

$$\psi\left(u_1, u_2\right) = \log \phi\left(u_1 + u_2\right) - \log \phi\left(u_1\right) - \log \phi(u_2).$$

The function $\psi(u_1, u_2)$ is infinitely differentiable on $\mathbb{R}^2$, which follows from (7) and the fact that ϕ is bounded from below by a strictly positive constant on each bounded subset of $\mathbb{R}$. Furthermore, note $\psi(u_1, 0) = \psi(0, u_2) = 0, \partial^2\psi(u_1, u_2)/\partial u_1 \partial u_2 = (\log \phi)''(u_1 + u_2)$ so that

$$\psi\left(u_1, u_2\right) = \int_0^{u_1} \int_0^{u_2} \left(\log \phi\right)'' \left(v_1 + v_2\right) \mathrm{d}v_1 \mathrm{d}v_2.$$

By expanding the last integral in a neighborhood of $u_1 = u_2 = 0$, similarly as in Giraitis *et al.* [13] we obtain

$$\psi\left(u_1, u_2\right) = \int_0^{u_1} \int_0^{u_2} \left[\left(\log \phi\right)''(0) + \left(v_1 + v_2\right) \left(\log \phi\right)^{(3)}(z)\right] \mathrm{d}v_1 \mathrm{d}v_2 = u_1 u_2 + \epsilon\left(u_1, u_2\right),$$

where $|\epsilon(u_1, u_2)| \le C(u_1^2|u_2| + |u_1|u_2^2)$ for all $|u_i| \le C_1, i = 1, 2$. Moreover, $E\mathrm{e}^{h_i} \le C, i = 1, 2$, where C depends on $\|f_i\|$ only. Therefore from (31) we obtain

$$\begin{aligned}
&\left|\operatorname{cov}\left(\mathrm{e}^{h_1}, \mathrm{e}^{h_2}\right) - \operatorname{cov}\left(h_1, h_2\right) E\mathrm{e}^{h_1} E\mathrm{e}^{h_2}\right| \\
&= E\mathrm{e}^{h_1} E\mathrm{e}^{h_2} \left|\exp\left\{\operatorname{cov}\left(h_1, h_2\right) + \sum_s \epsilon\left(f_1(s), f_2(s)\right)\right\} - 1 - \operatorname{cov}\left(h_1, h_2\right)\right| \\
&\le C \left(\operatorname{cov}^2\left(h_1, h_2\right) + \sum_s \left|\epsilon\left(f_1(s), f_2(s)\right)\right|\right),
\end{aligned}$$

thereby proving part (i) by the above bound on $|\epsilon(u_1, u_2)|$.

(ii) Note (30) follows from (28), as the right hand side of (28) is bounded by $C \sum_s |f_1(s) f_2(s)|$. Let us prove (29). Using the independence of $\mathcal{F}_{A_1}$ and $\mathcal{F}_{A_2}$, one has

$$\begin{aligned}
\operatorname{cov}\left(F_1\mathrm{e}^{h_1}, F_2\mathrm{e}^{h_2}\right) &= E\left[F_1\mathrm{e}^{h_{2,A_1}}\right] E\left[F_2\mathrm{e}^{h_{1,A_2}}\right] E\left[\mathrm{e}^{h_1' + h_2'}\right] \\
&\quad - E\left[F_1\right] E\left[F_2\right] E\left[\mathrm{e}^{h_{1,A_2}}\right] E\left[\mathrm{e}^{h_{2,A_1}}\right] E\left[\mathrm{e}^{h_1'}\right] E\left[\mathrm{e}^{h_2'}\right],
\end{aligned}$$

where $h'_1 = h_{1,A_2^c}, h'_2 = h_{2,A_1^c}$, $A_i^c = \mathbb{Z}\backslash A_i, i = 1,2$. Therefore,

$$\left|\operatorname{cov}\left(F_1 \mathrm{e}^{h_1}, F_2 \mathrm{e}^{h_2}\right)\right| \leq |\alpha_1| \left|E\mathrm{e}^{h'_1+h'_2} - E\mathrm{e}^{h'_1} E\mathrm{e}^{h'_2}\right| + \alpha_2|\beta|,$$

where

$$\alpha_1 = E\left[F_1 \mathrm{e}^{h_{2,A_1}}\right] E\left[F_2 \mathrm{e}^{h_{1,A_2}}\right], \qquad \alpha_2 = E\left[\mathrm{e}^{h'_1}\right] E\left[\mathrm{e}^{h'_2}\right],$$

and

$$\beta = E\left[F_1 \mathrm{e}^{h_{2,A_1}}\right] E\left[F_2 \mathrm{e}^{h_{1,A_2}}\right] - E\left[F_1\right] E\left[F_2\right] E\left[\mathrm{e}^{h_{1,A_2}}\right] E\left[\mathrm{e}^{h_{2,A_1}}\right].$$

By (30), $|E\mathrm{e}^{h'_1+h'_2} - E\mathrm{e}^{h'_1} E\mathrm{e}^{h'_2}| \leq C\sum_s |f_1(s) f_2(s)|$, and, by Cauchy–Schwarz, the constants α_1, α_2 depend only on $C_1 < \infty$ provided $\|f_i\|_2 \leq C_1, EF_i^2 \leq C_1, i = 1,2$. To bound β, write $\beta = \beta_1\beta_2 + \alpha_3\beta_2 + \alpha_4\beta_1$, where $\beta_1 = E[F_1(\mathrm{e}^{h_{2,A_1}} - E\mathrm{e}^{h_{2,A_1}})]$, $\beta_2 = E[F_2(\mathrm{e}^{h_{1,A_2}} - E\mathrm{e}^{h_{1,A_2}})]$, $\alpha_3 = EF_1 E\mathrm{e}^{h_{2,A_1}}$, $\alpha_4 = EF_2 E\mathrm{e}^{h_{1,A_2}}$. Then

$$|\beta_1| \leq \left(EF_1^2\right)^{1/2} \left(\operatorname{var}\left(\mathrm{e}^{h_{2,A_1}}\right)\right)^{1/2} \leq C\left(\sum_{s\in A_2} f_1^2(s)\right)^{1/2} \leq C\sum_{s\in A_2} |f_1(s)|,$$

according to (30). By estimating β_2 in a similar way, we obtain

$$|\beta| \leq C\left(\sum_{s\in A_1} |f_2(s)| + \sum_{s\in A_2} |f_1(s)|\right).$$

Together with the argument above, this proves (*ii*) and the lemma, too. □

Lemma 4.2. *Under conditions and notation of Theorem 2.1, for any $u_1, u_2 > 0$, as $t \to \infty$,*

$$\operatorname{cov}\left(V_0^{u_1} - u_1\nu_{u_1} Y_0, V_t^{u_2} - u_2\nu_{u_2} Y_t\right) = O\left(|r_t|\, t^{-\lambda}\right), \tag{32}$$

$$\operatorname{cov}\left(|X_0|^{u_1} - u_1 |\mu|_{u_1} Y_0, |X_t|^{u_2} - u_2 |\mu|_{u_2} Y_t\right) = O\left(|r_t|\, t^{-\lambda}\right). \tag{33}$$

Proof. To verify (32), it suffices to show

$$\operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) = u_1 u_2 \nu_{u_1}\nu_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{34}$$

$$\operatorname{cov}\left(V_0^{u_1}, Y_t\right) = u_1\nu_{u_1} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{35}$$

$$\operatorname{cov}\left(Y_0, V_t^{u_2}\right) = u_2\nu_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{36}$$

and use the fact that $r_t = \operatorname{cov}(Y_0, Y_t)$.

Let us first show the bound

$$\left|\operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) - u_1 u_2 \nu_{u_1}\nu_{u_2} r_t\right| \leq C u_1 u_2 |r_t|\, t^{-\lambda}, \tag{37}$$

where the constant C does not depend on t, u_1, u_2 for $0 \leq u_1, u_2 \leq C_1$ and $C_1 < \infty$ fixed.

To show (37), let us apply Lemma 4.1(i), with $f_1(s) = u_1 b_{-s}, f_2(s) = u_2 b_{t-s}$. According to (28), with $h_1 = u_1 Y_0, h_2 = u_2 Y_t$, $\operatorname{cov}(h_1, h_2) = u_1 u_2 r_t$, $E\mathrm{e}^{h_i} = \nu_{u_i}, i = 1,2$, one obtains

$$\left|\operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) - u_1 u_2 \nu_{u_1}\nu_{u_2} r_t\right| \leq C\left(u_1^2 u_2^2 r_t^2 + u_1 u_2 \sum_s \left(b_s^2\, |b_{t+s}| + |b_s|\, b_{t+s}^2\right)\right).$$

From (11) we have $r_t^2 = O(|r_t|t^{2d-1}) = O(|r_t|t^{-\lambda})$. Relation (5) also implies

$$\sum_s b_s^2 \left|b_{t+s}\right| = O\left(t^{d-1}\right) = O\left(\left|r_t\right| t^{-d}\right) = O\left(\left|r_t\right| t^{-\lambda}\right).$$

This proves (37) and (34), too. Let us prove (35). We have

$$\operatorname{cov}\left(V_0^{u_1}, Y_t\right) = \lim_{u_2 \to 0} u_2^{-1} \operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right),$$

so that

$$\begin{aligned}
&\operatorname{cov}\left(V^{u_1}, Y_t\right) - u_1 \nu_{u_1} r_t \\
&= \lim_{u_2 \to 0} u_2^{-1} \left[\operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) - u_1 u_2 \nu_{u_1} \nu_{u_2} r_t\right] + r_t u_1 \nu_{u_1} \lim_{u_2 \to 0} \left[\nu_{u_2} - 1\right] \\
&= \lim_{u_2 \to 0} u_2^{-1} \left[\operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) - u_1 u_2 \nu_{u_1} \nu_{u_2} r_t\right].
\end{aligned}$$

Hence (35) follows in view of the bound (37). The proof of (36) is completely analogous. This proves (32).

Let us turn to the proof of (33), which follows from the relations

$$\operatorname{cov}\left(\left|X_0\right|^{u_1}, \left|X_t\right|^{u_2}\right) = u_1 u_2 |\mu|_{u_1} |\mu|_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{38}$$

$$\operatorname{cov}\left(\left|X_0\right|^{u_1}, Y_t\right) = u_1 |\mu|_{u_1} r_t \left(1 + O\left(t^{-\lambda}\right)\right), \tag{39}$$

$$\operatorname{cov}\left(Y_0, \left|X_t\right|^{u_2}\right) = u_2 |\mu|_{u_2} r_t \left(1 + O\left(t^{-\lambda}\right)\right). \tag{40}$$

Consider (38). Write

$$\left|X_t\right|^u = \left|\zeta_t\right|^u V_t^u = |z|_u V_t^u + Q_{t,u},$$

where $|z|_u = E|\zeta|^u$ and $Q_{t,u} = (|\zeta_t|^u - |z|_u)V_t^u$. Note $Q_{t,u}, t = 0, 1, \ldots$ is a martingale difference sequence: $E[Q_{t,u}|\mathcal{F}_{t-1}] = 0$, where $\mathcal{F}_t = \sigma\{(\zeta_s, \xi_s) : s \le t\}$, hence also uncorrelated. Therefore

$$\begin{aligned}
\operatorname{cov}\left(\left|X_0\right|^{u_1}, \left|X_t\right|^{u_2}\right) = {} & |z|_{u_1} |z|_{u_2} \operatorname{cov}\left(V_0^{u_1}, V_t^{u_2}\right) \\
& + |z|_{u_2} \operatorname{cov}\left(Q_{0,u_1}, V_t^{u_2}\right) + |z|_{u_1} \operatorname{cov}\left(Q_{0,u_2}, V_t^{u_1}\right).
\end{aligned}$$

Hence (38) follows from (37), $|\mu|_u = |z|_u \nu_u$, and

$$\left|\operatorname{cov}\left(Q_{0,u_1}, V_t^{u_2}\right)\right| \le C u_2 \left|r_t\right| t^{-\lambda}, \tag{41}$$

which we show right below. Similarly as in the proof of Lemma 4.1, one obtains

$$\begin{aligned}
\left|\operatorname{cov}\left(Q_{0,u_1}, V_t^{u_2}\right)\right| &= \prod_{j=1}^{\infty} \phi\left(u_1 b_j + u_2 b_{t+j}\right) \prod_{j=1}^{t-1} \phi\left(u_2 b_j\right) \left|E\left[\left|\zeta_0\right|^{u_1} \left(\mathrm{e}^{u_2 b_t \xi_0} - E \mathrm{e}^{u_2 b_t \xi_0}\right)\right]\right| \\
&\le C \left(E\left[\left|\zeta\right|^{u_1} \left|\mathrm{e}^{u_2 b_t \xi} - E \mathrm{e}^{u_2 b_t \xi}\right|\right]\right. \\
&\le C \left(E|\zeta|^{2u_1}\right)^{1/2} \left(\phi\left(2u_2 b_t\right) - \phi^2\left(u_2 b_t\right)\right)^{1/2} \le C u_2 \left|b_t\right|,
\end{aligned}$$

where the last inequality follows from $|\phi(u) - 1| = O(u^2)$. As $b_t = O(|r_t|t^{-\lambda})$, this proves (41), hence also (38).

To show (39), write $\mathrm{cov}(|X_0|^{u_1}, Y_t) = \mathrm{cov}(Q_{0,u_1}, Y_t) + |z|_{u_1}\mathrm{cov}(V_0^{u_1}, Y_t)$. Then (40) follows from (35) and (41), as

$$|\mathrm{cov}\,(Q_{0,u_1}, Y_t)| = \left|\lim_{u_2\to 0} u_2^{-1}\mathrm{cov}\,(Q_{0,u_1}, V_t^{u_2})\right| \le \limsup_{u_2\to 0} u_2^{-1}\,|\mathrm{cov}\,(Q_{0,u_1}, V_t^{u_2})| \le C\,|r_t|\,t^{-\lambda}.$$

Relation (40) is immediate from (36), as $\mathrm{cov}(Q_{u_1,t}, Y_t) = 0$. □

Proof of Theorem 2.1. Relations (9, 10) were proved in (38, 34), respectively. □

Proof of Theorem 2.2. We shall prove (12) only, as the proof of (13) is completely analogous. Write $H_t = |X_t|^u - |\mu|_u - u|\mu|_u Y_t$. By (9), as $N\to\infty$,

$$\mathrm{var}\left(\sum_{t=1}^{N}(|X_t|^u - |\mu|_u)\right) \sim (u|\mu|_u)^2 \sum_{t,s=1}^{N} r_{t-s}\left(1 + O\left(|t-s|^{-\lambda}\right)\right) \sim (c_2 u|\mu|_u)^2\, N^{1+2d}, \tag{42}$$

and, by (33), $\mathrm{var}\left(\sum_{t=1}^{N} H_t\right) = O(N^{1+2d-\lambda})$. Therefore

$$N^{-d-1/2}\sum_{s=1}^{[Nt]}(|X_s|^u - |\mu|_u) = u|\mu|_u N^{-d-1/2}\sum_{s=1}^{[Nt]} Y_s + o_P(1). \tag{43}$$

It is well-known (Davydov [7]) that

$$N^{-d-1/2}\sum_{s=1}^{[Nt]} Y_s \Longrightarrow c_2 B_{d+1/2}(t). \tag{44}$$

From (42) it also follows that the sequence of processes $N^{-d-1/2}\sum_{s=1}^{[Nt]}(|X_s|^u - |\mu|_u),\ t\in[0,1],\ N = 1,2,\dots$ is tight in $D[0,1]$. Together with (43, 44), this proves (12). □

Proof of Theorem 2.6. Relations (17, 18) (for $u_1 = u_2$) and (19, 20) follow from Lemma 2.8, see *e.g.* Brillinger [6] (Th. 4.4.1). In particular, the tightness of the corresponding sequences of random elements with values in $D[0,1]$ follows from the bounds $E(\sum_{t=1}^{N}(|X_t|^u - |\mu|_u))^4 \le CN^2$, $E(\sum_{t=1}^{N}(V_t - \nu_u))^4 \le CN^2$, both of which are easy consequences of Lemma 2.8. Lemma 2.8 can be generalized so as to include the case different powers $u_i, i = 1,\dots,n$; however, its proof is rather involved. The convergences (17, 18) easily follow from Lemma 2 whose proof is more simpler. Let us prove (17) (the proof of (18) is analogous). To that end, use Lemma 4.1(ii) with $f_1(s) = u_1 b_{-s},\ f_2(s) = u_2 b_{t-s},\ A_1 = \{0\},\ A_2 = \{t\},\ F_1 = |\zeta_0|^{u_1},\ F_2 = |\zeta_t|^{u_2}$. Then $h_1 = u_1 Y_0,\ h_2 = u_2 Y_t$ and, from (28) we obtain

$$|\mathrm{cov}\,(|X_0|^{u_1}, |X_t|^{u_2})| = \left|\mathrm{cov}\left(F_1 \mathrm{e}^{h_1}, F_2 \mathrm{e}^{h_2}\right)\right| \le C\left(\sum_s |b_s b_{t+s}| + |b_t|\right), \tag{45}$$

where the constant C does not depend on t, due to $E(F_2\mathrm{e}^{h_2})^2 = E|\zeta_0|^{2u_2} E\mathrm{e}^{2u_2 Y_0}$, which follows from stationarity and the independence of ζ_t and Y_t. Clearly, equations (45) and (16) imply (17). □

Proof of Proposition 2.9. We have

$$\mathrm{cov}\,(X_0, V_t^u) = \prod_{i=1}^{t-1}\phi\,(ub_i)\prod_{j=1}^{\infty}\phi\,(b_j + ub_{t+j}) \times E\left[\zeta_0 \mathrm{e}^{ub_t g(\zeta_0)}\right],$$

so that the proposition follows from

$$\operatorname{sgn}(D) = \operatorname{sgn}(\theta b_t), \tag{46}$$

where $D = E[\zeta \exp\{ub_t g(\zeta)\}]$. By the symmetry (23),

$$\begin{aligned} D &= \int_{\{x>0\}} \left(x \exp\left\{ub_t\left(\theta x + \gamma\left(x - |z|_1\right)\right\} - x \exp\left\{ub_t\left(-\theta x + \gamma\left(x - |z|_1\right)\right\}\right) P\left(\zeta \in \mathrm{d}x\right) \\ &= \int_{\{x>0\}} x \exp\left\{ub_t \gamma\left(x - |z|_1\right)\right\} \left(\exp\left\{u\theta b_t x\right\} - \exp\left\{-u\theta b_t x\right\}\right) P\left(\zeta \in \mathrm{d}x\right). \end{aligned}$$

As $\operatorname{sgn}(\exp\{u\theta b_t x\} - \exp\{-u\theta b_t x\}) = \operatorname{sgn}(\theta b_t)$ for any $x, u > 0$, this proves (46). □

5. Cumulants (proof of Lem. 2.8)

Let us introduce some definitions. Let I be a finite set, $|I| < \infty$ the number of points in I, and let $\{\eta_i, i \in I\}$ be a system of random variables indexed by elements of I, such that $E|\eta_i|^{|I|} < \infty$, $\forall i \in I$. The joint cumulant $\operatorname{cum}(\eta_i : i \in I)$ is the partial derivative

$$\operatorname{cum}(\eta_i : i \in I) = \left(-\sqrt{-1}\right)^{|I|} \partial^{|I|} \log E \exp\left\{\sqrt{-1} \sum_{i \in I} z_i \eta_i\right\} / \prod_{i \in I} \partial z_i \Big|_{z_i = 0, i \in I}.$$

We shall use the notation $\langle \eta_i : i \in I\rangle = \operatorname{cum}(\eta_i : i \in I)$, $\eta^I = \prod_{i \in I} \eta_i$. Let us mention some basic properties of cumulants.

(c.1) $$\langle \eta_i : i \in I\rangle = \sum_{\{W_1, \ldots, W_r\}} (-1)^r (r-1)!\, E\eta^{W_1} \ldots E\eta^{W_r},$$

(c.2) $$E\eta^I = \sum_{\{W_1, \ldots, W_r\}} \langle \eta_i : i \in W_1\rangle \ldots \langle \eta_i : i \in W_r\rangle\cdot$$

In (c.1, c.2), the sums are taken over all partitions $\{W_1, \ldots, W_r\}, r = 1, 2, \ldots$ of I by nonempty subsets $W_i \subset I$.

(c.3) Let $I = I' \cup I''$ be a partition of I, and let $\{\eta_i, i \in I'\}$, $\{\eta_i'', i \in I''\}$ be independent. Then $\langle \eta_i : i \in I\rangle = 0$.
Let now $\{U_1, \ldots, U_p\}$, $p = 1, 2, \ldots$ be a system of subsets of I such that $\cup_{i=1}^p U_i = I$ (the U_i's need not be disjoint). Any such system can be identified with a graph G whose vertices are $U_1, \ldots, U_p$. A pair (U_i, U_j) $(i \neq j)$ forms an edge of G if and only if $U_i \cap U_j \neq \emptyset$. Call the system $\{U_1, \ldots, U_p\}$ *connected* if the graph G is connected; in other words, $\{U_1, \ldots, U_p\}$ is connected if for any partition $I = I' \cup I'', I' \cap I'' = \emptyset$ there exists $U_\ell, 1 \leq \ell \leq p$ such that $U_\ell \cap I' \neq \emptyset, U_\ell \cap I'' \neq \emptyset$.

(c.4) Let $\eta_i = \eta_i' \cdot \eta_i'', i \in I$, where the systems $\{\eta_i', i \in I\}$, $\{\eta_i'', i \in I\}$ of random variables are mutually independent. Then

$$\langle \eta_i : i \in I\rangle = \sum_{\{W_1', \ldots, W_{r'}'\}} \sum_{\{W_1'', \ldots, W_{r''}''\}} \prod_{j=1}^{p} \langle \eta_i' : i \in W_j'\rangle \prod_{\ell=1}^{r''} \langle \eta_i'' : i \in W_\ell''\rangle,$$

where the double sum is taken over all partitions $\{W_1', \ldots, W_{r'}'\}, \{W_1'' \ldots, W_{r''}''\}$, $r', r'' = 1, 2, \ldots$ of I such that the system $\{W_1', \ldots, W_{r'}', W_1'', \ldots, W_{r''}''\}$ is connected.

Properties (c.1–c.4) are well-known and can be found *e.g.* in Brillinger [6] (Chap. 2.3) or Malyshev and Minlos [24]. We shall also need

Lemma 5.1. *Let* $0 \le \chi_W \equiv \chi_W(t_i, i \in W)$, $W \subset I$ *be a system of functions,* $t_i \in \mathbb{Z}$, *such that for any* $W \subset I, |W| \ge 2$, *and any* $j \in W$,

$$\sup_{t_j \in \mathbb{Z}} \sum_{t_i \in \mathbb{Z}: i \neq j} \chi_W \left(t_i : i \in W\right) < \infty. \tag{47}$$

Let $\{U_1, \ldots, U_p\}$, $p \ge 1$ *be a connected system of subsets* $U_q \subset I, \cup_{q=1}^p U_q = I$, *and let* $j \in I$. *Then*

$$\sum_{t_i \in \mathbb{Z}: i \in I, i \neq j} \chi_{U_1}\left(t_i : i \in U_1\right) \ldots \chi_{U_p}\left(t_i : i \in U_p\right) < \infty. \tag{48}$$

Proof. For $p = 1$ (48) follows from (47). Let $p > 1$. Without loss of generality, we may assume $j \in U_1$, and $U_q \cap U_{q+1} \neq \emptyset, q = 1, \ldots, p-1$. Put $\tilde{U}_p = U_p \backslash \cup_{q=1}^{p-1} U_q$, $\tilde{I} = I \backslash \tilde{U}_p = \cup_{q=1}^{p-1} U_q$. Then $\{U_1, \ldots, U_{p-1}\}$ is a connected system of subsets of $\tilde{I}$. According to (47),

$$\sum_{t_i \in \mathbb{Z}: i \in \tilde{U}_p} \chi_{U_p}\left(t_i : i \in U_p\right) \le C,$$

where $C < \infty$ does not depend on $t_i, i \in U_p \backslash \tilde{U}_p$. Hence

$$\sum_{t_i \in \mathbb{Z}: i \in I, i \neq j} \prod_{q=1}^{p} \chi_{U_q} \le C \sum_{t_i \in \mathbf{Z}: i \in \tilde{I}, i \neq j} \prod_{q=1}^{p-1} \chi_{U_q},$$

and (48) follows by induction on p. $\square$

Next we derive a combinatorial formula for joint cumulants of exponents of linear sequences, which might also present an independent interest. Let

$$h_i = \sum_s f_i(s) \xi_s, \quad i \in I \tag{49}$$

be a finite system of linear random variables, $I = \{1, 2, \ldots, n\}$; *i.e.*, each h_i is a linear combination of i.i.d. random variables $\xi_s, s \in \mathbb{Z}$ having zero mean and finite exponential moments, as in Section 2. We shall assume moreover that all sums (49) are finite, *i.e.*, that all $f_i(s)$ vanish for all sufficiently large $|s|$. As in Section 4, put

$$\phi(u) = E\mathrm{e}^{u\xi}, \quad u \in \mathbb{R}.$$

For any $W \subset I$, $|W| \ge 2$ put

$$\psi_W\left(u_i : i \in W\right) = \sum_{U \subset W} (-1)^{|W \backslash U|} \log \phi\left(\sum_{i \in U} u_i\right), \tag{50}$$

where $u_i \in \mathbb{R}$, $\forall i$. In particular,

$$\begin{aligned}
\psi_{12}\left(u_1, u_2\right) &= \log \phi\left(u_1 + u_2\right) - \log \phi\left(u_1\right) - \log \phi\left(u_2\right), \\
\psi_{123}\left(u_1, u_2, u_3\right) &= \log \phi\left(u_1 + u_2 + u_3\right) - \log \phi\left(u_1 + u_2\right) - \log \phi\left(u_1 + u_3\right) \\
&\quad - \log \phi\left(u_2 + u_3\right) + \log \phi\left(u_1\right) + \log \phi\left(u_2\right) + \log \phi\left(u_3\right).
\end{aligned}$$

Also put, for $U \subset I$,

$$\Delta_U = \exp\left\{\sum_s \psi_U\left(f_i(s) : i \in U\right)\right\} - 1,$$

$$c_U = \prod_{i\in U} E\mathrm{e}^{Y_i} = \exp\left\{\sum_s \sum_{i\in U} \log\phi\left(f_i(s)\right)\right\}.$$

Lemma 5.2. *For any finite set I, $|I| \geq 2$,*

$$\langle \mathrm{e}^{h_i} : i \in I\rangle = c_I \sum_{\{U_1,\ldots,U_p\}} \Delta_{U_1}\ldots\Delta_{U_p}, \tag{51}$$

where the sum is taken over all connected systems $\{U_1,\ldots,U_p\}, p = 1,2,\ldots$ of subsets of I. Moreover $\langle \mathrm{e}^{h_i} : i \in I\rangle = c_I$ if $|I| = 1$.

In the Gaussian case $\xi \sim N(0,1)$, we have $\phi(u) = e^{u^2/2}$, and the function

$$\psi_W\left(u_i : i \in W\right) = \frac{1}{2}\sum_{U\subset W} (-1)^{|W\setminus U|}\left(\sum_{i\in U} u_i\right)^2$$

vanishes unless $|W| = 2$. In this case,

$$\Delta_W = \begin{cases} e^{r_{ij}} - 1, \; W = \{i,j\}, \; |W| = 2, \\ 0, \qquad\qquad \text{otherwise,}\end{cases}$$

where $r_{ij} = \sum_s f_i(s)f_j(s) = \mathrm{cov}(Y_i, Y_j)$. This leads to the following

Corollary 5.3. *Let $(h_i, i \in I) \sim N(0, (r_{ij}))$ be a Gaussian vector. Then*

$$\langle \mathrm{e}^{h_i} : i \in I\rangle = c_I \sum_G \prod_{(i,j)\in\mathcal{E}(G)} \left(\mathrm{e}^{r_{ij}} - 1\right),$$

where $c_I = \exp\{(1/2)\sum_{i\in I} r_{ii}\}$, the sum $\sum_G$ is taken over all connected graphs G whose set of vertices is the set I, and the product is taken over the set $\mathcal{E}(G)$ of edges of G.

Proof of Lemma 5.2. Observe that

$$\log\phi\left(\sum_{i\in I} u_i\right) = \sum_{i\in I}\log\phi(u_i) + \sum_{U\subset I, |U|\geq 2} \psi_U\left(u_i : i \in U\right).$$

Therefore

$$\begin{aligned} E\prod_{i\in I} e^{h_i} &= \exp\left\{\sum_s \log\phi\left(\sum_{i\in I} f_i(s)\right)\right\} \\ &= c_I \prod_{U\subset I} \left(1 + \Delta_U\right) \\ &= c_I \sum_{\{U_1,\ldots,U_p\}} \Delta_{U_1}\ldots\Delta_{U_p}, \end{aligned} \tag{52}$$

where the last sum is taken over all systems $\{U_1, \ldots, U_p\}, p = 1, 2, \ldots$ of subsets $U_i \subset I, |U_i| \geq 2$, including the empty system (in the latter case, we put $\Delta_{U_1} \ldots \Delta_{U_p} = 1$ by definition). As such systems $\{U_1, \ldots, U_p\}$ are not necessarily connected, the last sum in (52) can be rewritten by considering, first, all possible partitions $\{W_1, \ldots, W_r\}$ of I, and then, summing up over all connected subsystems $\{U'_1, \ldots, U'_{p'}\}$ within each W_j ($j = 1, \ldots, r$). This yields

$$E \prod_{i \in I} \mathrm{e}^{h_i} = \sum_{\{W_1, \ldots, W_r\}} \Gamma_{W_1} \ldots \Gamma_{W_r}, \tag{53}$$

where the sum is taken over all partitions $\{W_1, \ldots, W_r\}, r = 1, 2, \ldots$ of I, and where

$$\Gamma_W = c_W \sum_{\{U_1, \ldots, U_p\}} \Delta_{U_1} \ldots \Delta_{U_p}, \quad |W| \geq 2, \tag{54}$$

the sum in (54) being taken over all connected systems $\{U_1, \ldots, U_p\}, p = 1, 2, \ldots$ of subsets of W, and $\Gamma_W = c_W$ when $|W| = 1$. A similar representation (with Γ_W's replaced by cumulants $\langle \mathrm{e}^{h_i} : i \in W \rangle$) holds by virtue of property (c.2) for the expectation $E \prod_{i \in I} \mathrm{e}^{h_i}$. It is well-known (see *e.g.* Malyshev and Minlos [24]) that (53) determines Γ_W's uniquely. Therefore, $\Gamma_W = \langle \mathrm{e}^{h_i} : i \in W \rangle$ for each $W \subset I$, thereby proving (51). □

Proof of Lemma 2.8. Let us prove (19). To simplify the notation, we put $u = 1$. Let $I = \{1, 2, \ldots, n\}$. According to Lemma 5.2,

$$\langle \mathrm{e}^{Y_{t_i}} : i = 1, \ldots, n \rangle = c_I \sum_{\{U_1, \ldots, U_p\}} \Delta_{U_1} \ldots \Delta_{U_p},$$

where

$$\Delta_W \equiv \Delta_W (t_i : i \in W) = \exp \left\{ \sum_s \psi_W (b_{t_i - s} : i \in W) \right\} - 1,$$

$W \subset I, |W| \geq 2$. The functions ψ_W (50) satisfy the following inequality. For any $C_1 < \infty$ there exists a constant $C < \infty$ such that for all $|u_i| \leq C_1, i \in W$

$$|\psi_W (u_i : i \in W)| \leq C \prod_{i \in W} |u_i|. \tag{55}$$

The bound (55) follows from

$$\psi_W (u_1, \ldots, u_q) = \int_0^{u_1} \cdots \int_0^{u_q} \frac{\partial^q \psi_W (x_1, \ldots, x_q)}{\partial x_1 \ldots \partial x_q} \mathrm{d}x_1 \ldots \mathrm{d}x_q, \tag{56}$$

where $W = \{1, \ldots, q\}$ and

$$\frac{\partial^q \psi_W (u_1, \ldots, u_q)}{\partial u_1 \ldots \partial u_q} = (\log \phi)^{(q)} (u_1 + \cdots + u_q), \tag{57}$$

and the fact that the derivative $(\log \phi)^{(q)}(u) = d^q \log \phi(u) / \mathrm{d}u^q$ is bounded on each compact set $\{u \in \mathbf{R} : |u| \leq C_1\}$. Both (56) and (57) easily follow from the definition of ψ_W (50), similarly to the case $|W| = 2$ discussed in Section 4. In particular, (56) is a consequence of the fact that ψ_W vanishes on each hyperplane $\{u_i = 0\}, i \in W$.

Inequality (55) implies the bound

$$|\Delta_W\,(t_i : i \in W)| \,\le\, C \sum_s \prod_{i \in W} |b_{t_i - s}|\,, \tag{58}$$

where the constant $C < \infty$ does not depend on $t_i \in \mathbf{Z}, i \in W$. But (58) and (14) imply that $\Delta_W(t_i : i \in W)$ is summable with respect to $t_i \in \mathbf{Z}, i \in W \backslash \{i^*\}$, for arbitrary $i^* \in W$. Indeed, let $W = \{1, 2, \ldots, q\}$, $i^* = 1$, then

$$\sum_{t_2, \ldots, t_q} \sum_s \left| b_{t_1 - s} b_{t_2 - s} \ldots b_{t_q - s} \right| \,\le\, \left(\sum_s |b_s| \right)^q < \infty,$$

where the right hand side does not depend on t_1. Therefore the functions $|\Delta_W(t_i : i \in W)| = \chi_W(t_i : i \in W)$ satisfy condition (47) of Lemma 5.1. As the cumulant $\langle \mathrm{e}^{Y_{t_i}} : i \in I \rangle$ admits the representation (51), from Lemma 5.1 we obtain that for any $i^* \in I$

$$\sum_{t_i \in \mathbf{Z} : i \in I \backslash \{i^*\}} \left| \langle \mathrm{e}^{Y_{t_i}} : i \in I \rangle \right| \,<\, \infty,$$

thereby proving the statement (22) of Lemma 2.8.

Next we turn to the proof of (21). To simplify the notation, we shall again consider the case $u = 1$ only, and, moreover, we shall restrict ourselves to the proof of the convergence of the "off-diagonal" part; more precisely, to the proof of

$$\sum_{t_2, \ldots, t_n}{}' |\langle |X_{t_i}| : i = 1, 2, \ldots, n \rangle| \,<\, \infty, \tag{59}$$

where the sum is taken over all $t_i \in \mathbb{Z}, 2 \le i \le n$ such that $t_i \ne t_j\,(i \ne j), i, j = 1, \ldots, n$. Put $I = \{1, 2, \ldots, n\}$, $T = \{t_1, \ldots, t_n\} = \{t_i : i \in I\}$, and write

$$|X_{t_i}| = |\zeta_{t_i}| \mathrm{e}^{Y_{t_i}} = F_i \mathrm{e}^{h_i},$$

where

$$h_i = \sum_{s \notin T} b_{t_i - s} \xi_s, \quad F_i = |\zeta_{t_i}| \prod_{s \in T} \mathrm{e}^{b_{t_i - s} \xi_s}.$$

As $\{F_i, i \in I\}$ and $\{h_i, i \in I\}$ are independent collections of random variables, by property (c.4) we obtain

$$\langle |X_{t_i}| : i \in I \rangle| \,=\, \sum_{\{W'_1, \ldots, W'_{r'}\}} \sum_{\{W''_1, \ldots, W''_{r''}\}} \prod_{j=1}^{r'} \langle F_i : i \in W'_j \rangle \prod_{\ell=1}^{r''} \langle \mathrm{e}^{h_i} : i \in W''_\ell \rangle, \tag{60}$$

where the double sum is taken over all partitions $\{W'_1, \ldots, W'_{r'}\}, \{W''_1, \ldots, W''_{r''}\}, r', r'' = 1, 2, \ldots$ of I, such that the system $\{W'_1, \ldots, W'_{r'}, W''_1, \ldots, W''_{r''}\}$ is connected. We claim that for each $W \subset I$ there exists a constant $C < \infty$ such that, for any $i^* \in W$,

$$\sum_{t_i \in \mathbf{Z} : i \in W \backslash \{i^*\}} |\langle F_i : i \in W \rangle| \le C, \tag{61}$$

$$\sum_{t_i \in \mathbf{Z} : i \in W \backslash \{i^*\}} |\langle \mathrm{e}^{h_i} : i \in W \rangle| \le C. \tag{62}$$

From (60–62) the desired convergence (59) follows by Lemma 5.1.

As h_i's are linear variables, relation (62) follows similarly to the proof of (22) given above. It remains to show (61). We shall consider the case $W = I$ only. Put $Y_{ij} = b_{t_i - t_j}\xi_{t_j}$, $i, j \in I$, then $Y_{ii} = 0$ by $b_0 = 0$, and $\sum_{j\in I} Y_{ij} = Y_{t_i} - h_i = \sum_{s\in T} b_{t_i - s}\xi_s$. Now write

$$F_i = |\zeta_{t_i}| \prod_{j\in I} \left(1 + \left(\mathrm{e}^{Y_{ij}} - 1\right)\right) = \sum_{U\subset I} F_{i,U}, \tag{63}$$

where

$$F_{i,U} = |\zeta_{t_i}| \prod_{j\in U\setminus\{i\}} \left(\mathrm{e}^{Y_{ij}} - 1\right),$$

and the sum in (63) is taken over all subsets $U \subset I, i \in U$ (for $U = \{i\}$, we put $F_{i,U} = |\zeta_{t_i}|$ by definition). Thus, by multilinearity of joint cumulant, we obtain

$$\langle F_i : i \in I\rangle = \sum_{\{U_i : i\in I\}} \langle F_{i,U_i} : i \in I\rangle, \tag{64}$$

where the sum is taken over all collections $\{U_i : i \in I\} = \{U_1, \ldots, U_n\}$ of subsets $U_i \in I, i \in U_i$. Moreover, as for any partition $I' \cup I'' = I, I' \cap I'' = \emptyset$ the families $\{\zeta_{t_i}, Y_{ij}, i, j \in I'\}$ and $\{\zeta_{t_i}, Y_{ij}, i, j \in I''\}$ are independent, by property (c.3) we see that the cumulant $\langle F_{i,U_i} : i \in I\rangle = 0$ unless $\{U_i : i \in I\}$ is connected. Thus, equation (61) will follow from (64) by Lemma 5.1, provided we can show that

$$|\langle F_{i,U_i} : i \in I\rangle| \le \prod_{i\in I} \chi_{U_i}, \tag{65}$$

where functions $\chi_U = \chi_U(t_i : i \in U), U \subset I$ satisfy condition (47).

This last step of the proof of Lemma 2.8 can be obtained as follows. Observe, for any subset $W \subset I, |W| = q$

$$\begin{aligned} E \prod_{i\in W} |F_{i,U_i}| &\le E \prod_{i\in W} |\zeta_{t_i}| \prod_{j\in U_i\setminus\{i\}} \left|\mathrm{e}^{Y_{ij}} - 1\right| \le \prod_{i\in W} E^{1/q} \left[|\zeta_{t_i}|^q \prod_{j\in U_i\setminus\{i\}} \left|\mathrm{e}^{Y_{ij}} - 1\right|^q\right] \\ &= \prod_{i\in W} \|\zeta_{t_i}\|_q \prod_{j\in U_i\setminus\{i\}} \left\|\mathrm{e}^{Y_{ij}} - 1\right\|_q, \end{aligned}$$

where $\|\cdot\|_q$ stands for the L^q-norm, and where we used the independence of random variables $\zeta_{t_i}, Y_{ij}, j \in I\setminus\{i\}$, for each $i \in I$. Next, observe the bound

$$\left\|\mathrm{e}^{Y_{ij}} - 1\right\|_q = E^{1/q} \left|\mathrm{e}^{b_{t_i - t_j}\xi_{t_j}} - 1\right|^q \le C \left|b_{t_i - t_j}\right|,$$

with $C < \infty$ independent of t_i, t_j. Hence we obtain

$$E\left[\prod_{i\in W} |F_{i,U_i}|\right] \le C \prod_{i\in W} \prod_{j\in U_i\setminus\{i\}} \left|b_{t_i - t_j}\right|.$$

Let now $\{W_1, \ldots, W_r\}$ be any partition of I, then from the above inequality we obtain

$$\prod_{q=1}^{r} E\left[\prod_{i\in W_q} |F_{i,U_i}|\right] \le C \prod_{q=1}^{r} \prod_{i\in W_q} \prod_{j\in U_i\setminus\{i\}} \left|b_{t_i - t_j}\right| = C \prod_{i\in I} \prod_{j\in U_i\setminus\{i\}} \left|b_{t_i - t_j}\right|.$$

According to property (c.1), a similar bound (with a possibly different constant C) follows for the cumulant $\langle F_{i,U_i} : i \in I \rangle$:

$$|\langle F_{i,U_i} : i \in I \rangle| \le C \prod_{i \in I} \prod_{j \in U_i \setminus \{i\}} \left| b_{t_i - t_j} \right|.$$

Hence we obtain (65) with

$$\chi_{U_i} \equiv \chi_{U_i}(t_j : j \in U_i) = C \prod_{j \in U_i \setminus \{i\}} \left| b_{t_i - t_j} \right|, \quad i \in I.$$

It is easy to check that the above functions χ_{U_i} satisfy condition (47). Indeed, let *e.g.* $U_1 = \{1, 2, \ldots, q\}, i^* = 1$, then, uniformly in t_1,

$$\sum_{t_2, \ldots, t_q} \chi_{U_1}(t_1, \ldots, t_q) \le C \sum_{t_2, \ldots, t_q} \left| b_{t_1 - t_2} \ldots b_{t_1 - t_q} \right| \le C \left(\sum_s |b_s| \right)^q < \infty.$$

This proves (65) and Lemma 2.8, too. □

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References

[1] T.W. Anderson, *The Statistical Analysis of Time Series.* Wiley, New York (1971).
[2] R.T. Baillie, Long memory processes and fractional integration in econometrics. *J. Econometrics* **73** (1996) 5-59.
[3] P. Billingsley, *Convergence of Probability Measures.* Wiley, New York (1968).
[4] T. Bollerslev and H.O. Mikkelsen, Modeling and pricing long memory in stock market volatility. *J. Econometrics* **73** (1996) 151-184.
[5] F.J. Breidt, N. Crato and P. de Lima, On the detection and estimation of long memory in stochastic volatility. *J. Econometrics* **83** (1998) 325-348.
[6] D.R. Brillinger, *Time Series. Data Analysis and Theory.* Holt, Rinehart and Winston, New York (1975).
[7] Yu. Davydov, The invariance principle for stationary processes. *Theory Probab. Appl.* **15** (1970) 487-489.
[8] A. Demos, *Moments and dynamic structure of a time-varying-parameter stochastic volatility in mean model.* Preprint (2001).
[9] Z. Ding and C.W.J. Granger, Modeling volatility persistence of speculative returns: A new approach. *J. Econometrics* **73** (1996) 185-215.
[10] R.F. Engle, Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica* **50** (1982) 987-1008.
[11] E. Ghysels, A.C. Harvey and E. Renault, *Stochastic volatility*, edited by G.S. Maddala and C.R. Rao. North Holland, Amsterdam, *Handb. Statist.* **14** (1993) 119-191.
[12] L. Giraitis, P. Kokoszka and R. Leipus, *Rescaled variance and related tests for long memory in volatility and levels.* Preprint (1999).
[13] L. Giraitis, H.L. Koul and D. Surgailis, Asymptotic normality of regression estimators with long memory errors. *Statist. Probab. Lett.* **29** (1996) 317-335.
[14] L. Giraitis, R. Leipus, P.M. Robinson and D. Surgailis, *LARCH, leverage and long memory.* Preprint (2000).
[15] L. Giraitis, P.M. Robinson and D. Surgailis, A model for long memory conditional heteroskedasticity. *Ann. Appl. Probab.* (2000) (forthcoming).
[16] A. Harvey, *Long memory in stochastic volatility*, edited by J. Knight and S. Satchell, *Forecasting Volatility in the Financial Markets.* Butterworth & Heineman, Oxford (1998).
[17] C. He, T. Teräsvirta and H. Malmsten, Fourth moment structure of a family of first order exponential GARCH models, Preprint. *Econometric Theory* (to appear).
[18] H.-C. Ho and T. Hsing, Limit theorems for functionals of moving averages. *Ann. Probab.* **25** (1997) 1636-1669.
[19] J.R.M. Hosking, Fractional differencing. *Biometrika* **68** (1981) 165-176.
[20] H. Hurst, Long term storage capacity of reservoirs. *Trans. Amer. Soc. Civil Engrg.* **116** (1951) 770-799.
[21] D. Kwiatkowski, P.C. Phillips, P. Schmidt and Y. Shin, Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root? *J. Econometrics* **54** (1992) 159-178.

[22] A.C. Lo, Long memory in stock market prices. *Econometrica* **59** (1991) 1279-1313.
[23] I.N. Lobato and N.E. Savin, Real and spurious long-memory properties of stock market data (with comments). *J. Business Econom. Statist.* **16** (1998) 261-283.
[24] V.A. Malyshev and R.A. Minlos, *Gibbs Random Fields*. Kluwer, Dordrecht (1991).
[25] B.B. Mandelbrot, Statistical methodology for non-periodic cycles: From the covariance to R/S analysis. *Ann. Econom. Social Measurement* **1** (1972) 259-290.
[26] B.B. Mandelbrot, Limit theorems of the self-normalized range for weakly and strongly dependent processes. *Z. Wahrsch. Verw. Geb.* **31** (1975) 271-285.
[27] B.B. Mandelbrot and M.S. Taqqu, Robust R/S analysis of long run serial correlation. *Bull. Int. Statist. Inst.* **48** (1979) 59-104.
[28] B.B. Mandelbrot and J.M. Wallis, Robustness of the rescaled range R/S in the measurement of noncyclic long run statistical dependence. *Water Resources Research* **5** (1969) 967-988.
[29] D.B. Nelson, Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* **59** (1991) 347-370.
[30] P.M. Robinson, Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. *J. Econometrics* **47** (1991) 67-84.
[31] P.M. Robinson, The memory of stochastic volatility models. *J. Econometrics* **101** (2001) 195-218.
[32] P.M. Robinson and P. Zaffaroni, Nonlinear time series with long memory: A model for stochastic volatility. *J. Statist. Plan. Inf.* **68** (1998) 359-371.
[33] S. Taylor, *Modelling Financial Time Series*. Wiley, Chichester (1986).