

ASYMPTOTIC BEHAVIOR OF THE EMPIRICAL PROCESS FOR GAUSSIAN DATA PRESENTING SEASONAL LONG-MEMORY

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Abstract. We study the asymptotic behavior of the empirical process when the underlying data are Gaussian and exhibit *seasonal* long-memory. We prove that the limiting process can be quite different from the limit obtained in the case of *regular* long-memory. However, in both cases, the limiting process is degenerated. We apply our results to von-Mises functionals and U -Statistics.

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1. INTRODUCTION

Let $(Y_n)_{n \geq 1}$ be an ergodic stationary process and let

$$F_N(x) = \frac{1}{N} \sum_{j=1}^N \mathbb{I}_{\{Y_j \leq x\}}, \quad x \in \mathbb{R}$$

be the associated empirical process.

The importance of this process for statistical applications is well known. For every x , $F_N(x)$ converges almost surely to $F(x)$, the distribution function of the variable Y_1 .

When $(Y_n)_{n \geq 1}$ is i.i.d., $N^{1/2}(F_N - F)$ converges in the space $D([-\infty, +\infty])$, endowed with the uniform topology, towards the generalized Brownian bridge, a zero-mean Gaussian process with covariance function $F(x \wedge y) - F(x)F(y)$ (see [26]).

This result still holds, with a different limiting covariance, under weak dependence conditions. Billingsley [3] proves the convergence in $D([-\infty, +\infty])$ under ϕ -mixing. Newman [18] obtains the convergence of the finite-dimensional distributions of $N^{1/2}(F_N - F)$ for associated sequences satisfying the condition

$$\sum_{n=1}^{\infty} \text{Cov}(Y_1, Y_n) < \infty.$$

Shao and Yu [25], Doukhan and Louhichi [8] prove the convergence of $N^{1/2}(F_N - F)$, in the above mentioned space, for second order strong-mixing or for associated processes having a conveniently decreasing covariance

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sequence. For Gaussian subordinated sequences, Czörge and Mielniczuk [4] prove the convergence under the rather weak condition

$$\sum_{n=1}^{\infty} |\text{Cov}(Y_1, Y_n)| < \infty.$$

Doukhan and Surgailis [9] consider weakly dependent linear sequences. In all the above cases, the limit is a Gaussian process with covariance function

$$F(x \wedge y) - F(x)F(y) + \sum_{k \geq 2} \{ \text{Cov}(\mathbb{I}_{Y_1 \leq x}, \mathbb{I}_{Y_k \leq y}) + \text{Cov}(\mathbb{I}_{Y_k \leq x}, \mathbb{I}_{Y_1 \leq y}) \}.$$

The first results under strong dependence, obtained by Dehling and Taqqu [5], concern subordinated Gaussian sequences, that is $Y_n = G(X_n)$, where (X_n) is a zero mean stationary Gaussian process with $\mathbb{E}X_1^2 = 1$. The results are completely different from the previous ones. The random variable

$$\Delta_x(X_j) = \mathbb{I}_{\{G(X_j) \leq x\}} - F(x) \tag{1}$$

admits the expansion

$$\Delta_x(X_j) = \sum_{k=1}^{\infty} \frac{J_k(x)}{k!} H_k(X_j) \tag{2}$$

with H_k the Hermite polynomial of degree k and

$$J_k(x) = \int H_k(u) \Delta_x(u) \phi(u) du,$$

where ϕ is the standard Gaussian density. Denote by τ the Hermite rank of the family $(\Delta_x(\cdot), x \in \mathbb{R})$, *i.e.*

$$\tau = \inf\{k \geq 1, \exists x \text{ such that } J_k(x) \neq 0\}.$$

Assume that the covariance function $r(n)$ of $(X_n)_{n \geq 1}$ is regularly varying at infinity. More precisely the covariance has the form

$$r(n) = n^{-\alpha} L(n), \quad 0 < \alpha\tau < 1, \tag{3}$$

where L is slowly varying at infinity.

This condition implies that $\sum |r(n)|^\tau = \infty$, which means that, for at least one x , the sequence $(\Delta_x(X_j))_j$ has long-memory. Throughout this paper, condition (3) shall be referred to as *regular* long-memory. Dehling and Taqqu [5] consider, under the above conditions, the doubly indexed empirical process

$$F_{[Nt]}(x) = \frac{1}{[Nt]} \sum_{j=1}^{[Nt]} \mathbb{I}_{\{G(X_j) \leq x\}} \quad x \in \mathbb{R}, t \in [0, 1], \tag{4}$$

and prove that, with

$$v_N^2 = \frac{2\tau!}{(1 - \tau\alpha)(2 - \tau\alpha)} N^{2-\tau\alpha} L^\tau(N),$$

$$v_N^{-1} [Nt] (F_{[Nt]}(x) - F(x)) \implies \frac{J_\tau(x)}{\tau!} Z_{\tau, 1-\tau\alpha/2}(t). \tag{5}$$

The convergence takes place in $D([-\infty, +\infty] \times [0, 1])$ endowed with the sup norm.

The process $Z_{\tau,1-\tau\alpha/2}(t)$, is the so called Hermite process of order τ and self-similarity parameter $1 - \tau\alpha/2$, a zero mean process with covariance

$$\text{Cov}(Z_{\tau,1-\tau\alpha/2}(s), Z_{\tau,1-\tau\alpha/2}(t)) = \frac{1}{2}(s^{2-\tau\alpha} + t^{2-\tau\alpha} - |t - s|^{2-\tau\alpha}). \tag{6}$$

This process admits the random harmonic representation

$$(D(\tau, \alpha))^{-1} \int_{\mathbb{R}^\tau} \frac{e^{it(x_1+\dots+x_\tau)} - 1}{i(x_1 + \dots + x_\tau)} |x_1|^{\frac{\alpha-1}{2}} \dots |x_\tau|^{\frac{\alpha-1}{2}} W(dx_1) \dots W(dx_\tau), \tag{7}$$

where $D(\tau, \alpha)$ is a normalizing constant, actually

$$D(\tau, \alpha) = \sqrt{\frac{2\tau!(2\Gamma(\alpha) \cos(\alpha\pi/2))^\tau}{(1 - \tau\alpha)(2 - \tau\alpha)}}, \tag{8}$$

and W is the spectral random measure of the standard Gaussian white noise (see [7]).

In particular $Z_{1,1-\alpha/2}(t)$ is the fractional Brownian motion (fBm) with parameter $1 - \alpha/2$. For $\tau \geq 2$, $Z_{\tau,1-\tau\alpha/2}(t)$ is a non Gaussian process. Recall that $Z_{2,1-\alpha}(t)$ is the celebrated Rosenblatt process with parameter $1 - \alpha$.

Ho and Hsing [15] and Giraitis and Surgailis [13] consider linear processes $(Y_n)_{n \geq 1}$ under the condition of *regular* long memory (3) with $\tau = 1$. They obtain the convergence of $F_N(x) - F(x)$, suitably normalized, towards a degenerated process $\psi(x)Z$, where ψ is the distribution density of Y_1 , and Z is a standard Gaussian random variable.

The result (5) displays three main features. Firstly, as it is the case for the Donsker line (see [7] and [27]), the limiting process is not necessarily Gaussian. More precisely, it is Gaussian only if $\tau = 1$. Secondly, the rate convergence of $F_N(x)$ to $F(x)$ is always slower than the classical rate \sqrt{N} . Finally, this process is degenerated, being the product of a deterministic function of the variable x and a random function of the variable t .

In this paper we chose the framework of *seasonal* long-memory. There are two basic papers concerning the effects of *seasonal* long memory on the limit theorems. Both concern the convergence of the (suitably normalized) partial sums $\sum_{j=1}^{[nt]} Y_j$ for a zero-mean Gaussian subordinated sequence $(Y_n = G(X_n))_{n \geq 1}$. Rosenblatt [24] assumes that the underlying Gaussian process X_n has a covariance of the form

$$r(n) = n^{-\alpha}(a_0 \cos n\lambda_0 + \dots + a_m \cos n\lambda_m)L(n), \quad 0 < \alpha < 1, \tag{9}$$

where L is slowly varying at infinity. Giraitis [11] extends this result to Gaussian processes having a spectral density of type

$$f(\lambda) = \sum_{j=-m}^m s_j L\left(\frac{1}{|\lambda - \lambda_j|}\right) |\lambda - \lambda_j|^{\alpha_j-1}, \quad 0 < \alpha_j < 1, \tag{10}$$

where $s_j = s_{-j} > 0$, $\lambda_j = -\lambda_{-j}$ and $0 = \lambda_0 < \lambda_1 < \dots < \lambda_m < \pi$.

Condition (10) allows the frequencies λ_j to have different contributions to the asymptotic behavior of the covariance sequence. More precisely, when L is slowly varying at infinity in the sense of Zygmund [28], equation (10) implies that

$$r(n) = L(n) \sum_{j=1}^m a_j n^{-\alpha_j} \cos n\lambda_j.$$

In a recent work [1], Arcones treats the case of functions G having Hermite rank $\tau = 1$. His hypotheses concern the moments of partial sums built on the two first polynomials of the underling Gaussian process and his

method for proving the results is based on an expansion of the characteristic function, as in Taqqu paper [27]. The conditions of Arcones are rather fitted to situations where an exact form of the covariance is available. For instance in [20], we used the same type of methods for the same Hermite rank when the covariance of the underlying Gaussian sequence has the form (9).

Hereafter we consider again a Gaussian subordinated sequence $Y_n = G(X_n)$ and we study the doubly indexed empirical process (4) when the spectral density of $(X_n)_{n \geq 1}$ has the form (11) below. This form, slightly different from (10), is more adapted to the context of the two main parametric families of long memory: the generalized fractional ARIMA, and the aggregated processes (see [19] and [22] for a review on seasonal long-memory models). We prove that the three main features appearing in the *regular* long memory case are preserved in the *seasonal* situation. In particular, the limiting process of $F_{[Nt]}(x) - F(x)$ suitably normalized is still degenerated and always has the typical form $J(x)Z(t)$ obtained in (5). However, the first Hermite polynomial H_τ in the expansion (2) is not necessarily dominant, and the random factor $Z(t)$ can be quite different from $Z_{\tau, 1-\tau\alpha/2}(t)$ in (5). In particular, when $\tau = 1$, as it is the case for instance when $G(x) \equiv x$, $Z(t)$ is not necessarily Gaussian.

The paper is organized as follows. In Section 2, we give the limit of the (normalized) doubly indexed empirical process and an outline of the proof of the theorem. In Section 3, we propose statistical applications. The appendix contains a basic result concerning the limiting law of partial sums under seasonal long-memory and some proofs. Through this result we see that no more 2 chaos can contribute to the limit of the partial sums $\sum_{j=1}^{[Nt]} G(X_j)$, independently of the Hermit rank of G (see Rem. 2 after the end of Sect. 4.2).

2. CONVERGENCE OF THE EMPIRICAL PROCESS UNDER SEASONAL LONG-MEMORY

Let $(X_n)_{n \geq 1}$ be a zero-mean stationary Gaussian process such that $EX_1^2 = 1$ and admitting a spectral density of the form

$$g(\lambda) = h(\lambda) \prod_{j=-m}^m |\lambda - \lambda_j|^{\alpha_j - 1}, \tag{11}$$

with $0 = \lambda_0 < \lambda_1 < \dots < \lambda_m < \pi$, and

$$0 < \alpha_j < 1, \quad \alpha_j = \alpha_{-j}, \quad \lambda_{-j} = -\lambda_j, \quad j \in \{-m, \dots, m\},$$

and where h is an even function, continuous on $[0, \pi]$, such that $h(\lambda_j) \neq 0$ for every j .

Let us denote

$$\alpha = \min\{\alpha_j : j \in \{0, \dots, m\}\}, \quad \text{and} \quad J = \{j \in \{-m, \dots, m\} : \alpha_j = \alpha\}. \tag{12}$$

We know from [12] that the covariance function of (X_n) has asymptotically the form

$$r(n) = n^{-\alpha} \left(\sum_{j \in J} a_j \cos n\lambda_j + o(1) \right). \tag{13}$$

This covariance is not *regularly* varying as n tends to infinity as soon as there exists $j \neq 0$ such that $\alpha_0 \geq \alpha_j$.

For the sake of comparison with the *regular* long-memory case, we suppose here that, in (2), the functions $J_1(x)$ and $J_2(x)$ do not identically vanish. Consequently, the results below are to be compared to (5) with $\tau = 1$.

Let us denote

$$c_j = (1 + \mathbb{1}_{\{j=0\}})^{-1} h(\lambda_j) \prod_{i \neq j} |\lambda_i - \lambda_j|^{\alpha_i - 1}, \quad \forall j = 0, \dots, m, \tag{14}$$

and

$$C = 8c_0 \int_{\mathbb{R}} |\lambda|^{2\alpha_0 - 2} \sin^2(\lambda/2) \tau d\lambda. \tag{15}$$

Theorem 2.1. *Assume that $2\alpha < 1$ and let*

$$d_N^2 = N^{2 - (\alpha_0 \wedge 2\alpha)}.$$

Then, with $F_{[Nt]}$ defined in (4), we have

- the convergence takes place in the space $D([-\infty, +\infty] \times [0, 1])$ endowed with the sup norm and the sigma field generated by the open balls;
- the process $Z(x, t)$ is defined by

$$\begin{aligned} \text{i) } Z(x, t) &= J_1(x)B(t) && \text{if } \alpha_0 < 2\alpha, \\ \text{ii) } Z(x, t) &= J_1(x)B(t) + \frac{J_2(x)}{2}R(t) && \text{if } \alpha_0 = 2\alpha, \\ \text{iii) } Z(x, t) &= \frac{J_2(x)}{2}R(t) && \text{if } \alpha_0 > 2\alpha, \end{aligned}$$

where

- * $B(t)$ and $R(t)$ are independent;
- * $C^{-1/2}B(t)$ is the fBm with parameter $1 - \alpha_0/2$. The constant C is given in (15);
- * the process $R(t)$ is defined by

$$R(t) = D(2, \alpha) \sum_{j \in J} c_j \left(R_j^{(1)}(t) + R_j^{(2)}(t) \right). \tag{16}$$

In (16), $D(2, \alpha)$ is defined in (8), and for fixed j , the processes $R_j^{(1)}(t)$ and $R_j^{(2)}(t)$ are independent, except if $j = 0$ in which case $R_0^{(1)}(t) = R_0^{(2)}(t)$, and the $(R_j^{(1)}(t), R_j^{(2)}(t))_{j \in J}$ are independent. All the $R_j^{(i)}(t)$, $i = 1, 2$ and $j \in J$, are Rosenblatt processes with parameter $1 - \alpha$, having the representation (7) with $\tau = 2$.

Firstly, we see that, in all situations, the limiting process is degenerated. Secondly, from (13), we have $r(n) = O(n^{-\alpha})$. Comparing with (5), we remark that, when $\alpha_0 < 2\alpha$, the limit $J_1(x)B(t)$ is exactly the limit obtained in the case of *regular* long-memory (3) with $\tau = 1$, while the normalizing coefficients which are respectively $N^{1-\alpha_0/2}$ and $N^{1-\alpha/2}$ are different. In fact, in this case, only the singularity $\lambda_0 = 0$ plays a role in the asymptotic behavior of the empirical process. The situation changes when $\alpha_0 \geq 2\alpha$. In this case, the singularity $\lambda_0 = 0$ is not enough marked so that, due to the presence of the oscillating terms in the covariance, the dominant term in the Hermite expansion (2) is H_2 . Then, the limiting process is the sum of independent Rosenblatt processes, each one being produced by one of the pairs $(-\lambda_j, \lambda_j)$ corresponding to the minimal exponent α . The normalizing coefficient becomes $N^{1-\alpha}$. In this case, despite the fact that $\tau = 1$, the asymptotic behavior of the empirical process is similar to its behavior under *regular* long-memory when $\tau = 2$. In the general case, for arbitrary τ , only one or two chaos can contribute to the limit of the empirical process (see Appendix, Rem. 2 after Th. 4.1).

Sketch of the proof:

First step: Reduction of the problem

Theorem 4.1 in the Appendix implies that, for fixed x , the finite dimensional distributions of the process $d_N^{-1}[Nt](F_{[Nt]}(x) - F(x))$ and those of $J_1(x)X_{N,1}(t) + (J_2(x)/2)X_{N,2}(t)$ have the same limits (see the remark

at the end of Sect. 4.1). Hence, in the Hermite expansion (2), only the two first terms play a role. In order to take account of the parameter x and to reduce tightness of the first process to that of the second one we need, as in [5], an uniform weak reduction principle. More precisely, with Δ_x given in (1), there exist $C > 0$, $\delta > 0$ such that for every $N \geq 1$ and $\epsilon \in (0, 1]$

$$P \left\{ \max_{n \leq > N} \sup_{\infty \leq x \leq +\infty} d_N^{-1} \left| \sum_{j=1}^n \left(\Delta_x(X_j) - J_1(x)H_1(X_j) - \frac{J_2(x)}{2}H_2(X_j) \right) \right| > \epsilon \right\} \leq CN^{-\delta}(1 + \epsilon^{-3}). \tag{17}$$

This inequality relies on the fact that $r(n) = O(n^{-\alpha})$ which is a consequence of (13). Its proof follows the same lines as in [5] and is omitted.

Consider the sequences $(X_{N,1}(t))$ and $(X_{N,2}(t))$ defined by

$$X_{N,1}(t) = d_N^{-1} \sum_{j=1}^{[Nt]} H_1(X_j) \quad \text{and} \quad X_{N,2}(t) = d_N^{-1} \sum_{j=1}^{[Nt]} H_2(X_j). \tag{18}$$

From (17), the proof of the theorem reduces to the proof of the convergence of $J_1(x)X_{N,1}(t) + (J_2(x)/2)X_{N,2}(t)$ to the process $Z(x, t)$ in the announced space.

Second step: Variances

As N tends to infinity,

$$\text{Var} \left(\sum_{j=1}^N H_1(X_j) \right) \sim C_1 N^{2-\alpha_0} \quad \text{and} \quad \text{Var} \left(\sum_{j=1}^N H_2(X_j) \right) \sim C_2 N^{2-2\alpha}. \tag{19}$$

The first equivalence is proved in [17]. For the second one, we have from (13), as N tends to infinity,

$$\begin{aligned} \text{Var} \left(\sum_{j=1}^N H_2(X_j) \right) &= 2 \sum_{i,j=1}^N r^2(i-j) = 2N \\ &\quad + 2 \sum_{i \neq j} \sum_{h,h' \in J} a_h a_{h'} \frac{(\cos((i-j)\lambda_h) + o(1))(\cos((i-j)\lambda_{h'}) + o(1))}{|i-j|^{2\alpha}} \\ &\sim 2N + \sum_{i \neq j} \sum_{h \in J} a_h^2 |i-j|^{-2\alpha} \sim C_2 N^{2-2\alpha}. \end{aligned}$$

From (19), we get, if $\alpha_0 < 2\alpha$ (resp. if $\alpha_0 > 2\alpha$)

$$d_N^{-1} \sum_{j=1}^N H_2(X_j) \xrightarrow{L^2} 0, \quad \left(\text{resp. } d_N^{-1} \sum_{j=1}^N H_1(X_j) \xrightarrow{L^2} 0 \right), \tag{20}$$

and in all cases,

$$\text{Var}(X_{N,1}(t)) + \text{Var}(X_{N,2}(t)) = O(d_N^2). \tag{21}$$

The convergences (20) provide a simple explanation to the presence of the zero components in the limit (22) below, and to the contrast between the two situations $\alpha_0 < 2\alpha$ and $\alpha_0 > 2\alpha$.

Third step: Convergence of the finite-dimensional distributions

As it is proved in the Appendix, the next proposition is a corollary of a general result on the partial sums. This result is stated and commented in the Appendix (Th. 4.1). Denote by $\xrightarrow{\mathcal{D}}$ the convergence of the finite-dimensional distributions.

Proposition 2.2. *Assume the spectral density has the form (11). Then, we have*

$$(X_{N,1}(t), X_{N,2}(t)) \xrightarrow{\mathcal{D}} \begin{cases} (B(t), 0) & \text{if } \alpha_0 < 2\alpha, \\ (B(t), R(t)), & \text{if } \alpha_0 = 2\alpha, \\ (0, R(t)) & \text{if } \alpha_0 > 2\alpha, \end{cases} \tag{22}$$

where $B(t)$ and $R(t)$ are as in Theorem 2.1.

With the reduction principle, Proposition 2.2 implies the convergence of the finite dimensional distributions of $d_N^{-1}[Nt](F_{[Nt]}(x) - F(x))$ to those of $Z(x, t)$.

Last step: Convergence in the space $D([-\infty, +\infty] \times [0, 1])$ endowed with the sup-norm.

As this space is not separable, it shall be equipped with the sigma field generated by the open balls instead of the Borel sigma field which is too large for the empirical process to be measurable [23]. From Lemma 2.1. of [27] the convergences (22) and the equivalence (21) imply that the sequences $X_{N,1}(t)$ and $X_{N,2}(t)$ converge in the space $D[0, 1]$ endowed with the Skorohod metric and the induced Borel sigma field. Now, the form of the covariances of $B(t)$ and $R(t)$ given in (6), imply, using Kolmogorov–Čentsov theorem [16], that the sample paths of the limiting processes are continuous. It follows (see [3]), that the convergence takes place in $D[0, 1]$ endowed with the uniform metric and the induced Borel sigma field. This of course yields the convergence in this space equipped with the sigma field \mathcal{E} generated by the open balls. In the sequel, all the spaces are equipped with their sup-norm, which shall always be denoted by $\|\cdot\|$. We need the following lemma where $\xrightarrow{D[0,1],\mathcal{E}}$ denotes the weak convergence in the space $(D[0, 1], \mathcal{E})$.

Lemma 2.3. *Suppose that*

$$X_n \xrightarrow{D[0,1],\mathcal{E}} X, \quad Y_n \xrightarrow{D[0,1],\mathcal{E}} Y, \quad \text{and} \quad (X_n, Y_n) \xrightarrow{\mathcal{D}} (X, Y),$$

and that

$$P\{(X, Y) \in C[0, 1] \times C[0, 1]\} = 1.$$

Then (X_n, Y_n) converges to (X, Y) in the space $(D[0, 1] \times D[0, 1], \mathcal{E} \otimes \mathcal{E})$.

The proof is in the Appendix.

It remains to use this lemma to prove the convergence of $J_1(x)X_{N,1}(t) + \frac{J_2(x)}{2}X_{N,2}(t)$ in the space $D([-\infty, +\infty] \times [0, 1])$ endowed with the sigma field generated by the open balls. For instance suppose that $\alpha_0 = 2\alpha$ (the proof is even simpler in the other cases). The two sequences $X_{N,1}(t)$ and $X_{N,2}(t)$ respectively converge to $B(t)$ and $R(t)$, and hence, using (22), Lemma 2.3 implies that $(X_{N,1}(t), X_{N,2}(t))$ converges to $(B(t), R(t))$ in the space $(D[0, 1] \times D[0, 1], \mathcal{E} \otimes \mathcal{E})$. Now, from the almost sure representation theorem of Skorohod and Dudley ([23], p. 71), there exist a sequence of vectors $(\tilde{X}_{N,1}(t), \tilde{X}_{N,2}(t))$ having the same distribution as $(X_{N,1}(t), X_{N,2}(t))$ and a vector $(\tilde{B}(t), \tilde{R}(t))$ having the same distribution as $(B(t), R(t))$ such that

$$\|(\tilde{X}_{N,1}(\cdot), \tilde{X}_{N,2}(\cdot)) - (\tilde{B}(\cdot), \tilde{R}(\cdot))\| \xrightarrow{\text{a.s.}} 0.$$

As $J_1(x)$ and $J_2(x)$ are bounded, the sequence $J_1(x)\tilde{X}_{N,1}(t) + J_2(x)/2\tilde{X}_{N,2}(t)$ almost surely converges to $J_1(x)\tilde{B}(t) + (J_2(x)/2)\tilde{R}(t)$ in the space $D([-\infty, +\infty] \times [0, 1])$ endowed with the sigma field generated by the open balls. The convergence of $J_1(x)X_{N,1}(t) + (J_2(x)/2)X_{N,2}(t)$ in this space follows.

3. APPLICATIONS TO VON-MISES FUNCTIONALS AND U -STATISTICS

3.1. General case

The context is the same as in Section 2. For $m \geq 1$ we consider $U_N(h)$, the (non-normalized) U -statistic defined by

$$U_N(h) = \sum_{\substack{1 \leq j_1, \dots, j_m \leq N \\ j_\mu \neq j_\nu, \mu \neq \nu}} h(Y_{j_1}, \dots, Y_{j_m}),$$

where $h : \mathbb{R}^m \rightarrow \mathbb{R}$ is integrable with respect to $\prod_{j=1}^m F(dx_j)$ and invariant with respect to any permutation of the variables. If h satisfies

$$\int_{\mathbb{R}} h(x_1, \dots, x_m) F(dx_1) = 0, \quad \forall x_2, \dots, x_m \tag{23}$$

the U -statistic is called degenerated, and the associated von-Mises functionals defined by

$$V_N(h) = \sum_{1 \leq j_1, \dots, j_m \leq N} h(Y_{j_1}, \dots, Y_{j_m})$$

writes

$$V_N(h) = N^m \int_{\mathbb{R}^m} h(x_1, \dots, x_m) \prod_{j=1}^m (F_N(dx_j) - F(dx_j)).$$

Moreover, if the total variation of h is bounded, and if h has no common discontinuities with $\prod_{i=1}^m F(x_i)$, then an integration by parts leads to

$$\frac{V_{[Nt]}(h)}{d_N^m} = \int_{\mathbb{R}^m} \prod_{i=1}^m \frac{[Nt]}{d_N} (F_{[Nt]}(x_i) - F(x_i)) h(dx_1, \dots, dx_m). \tag{24}$$

The application of $D([-\infty, +\infty] \times [0, 1])$ into $D[0, 1]$ defined by

$$Q \longrightarrow \int_{\mathbb{R}^m} Q(x_1, t) \cdots Q(x_m, t) h(dx_1, \dots, dx_m),$$

is continuous with respect to the sup-norm. Using (24) and the convergence of $d_N^{-1}[Nt](F_{[Nt]}(x) - F(x))$, we obtain the convergence of $d_N^{-m}V_{[Nt]}(h)$. The convergence of $d_N^{-m}U_{[Nt]}(h)$ to the same limiting process is immediate since $V_N(h) - U_N(h) = o(d_N^m)$ (see [5]).

These results are collected in the corollary below, where $J_1(x)$ and $J_2(x)$ are defined in (2).

Corollary 3.1. *Let h have bounded total variation and satisfy condition (23). In addition, suppose that $h(x_1, \dots, x_m)$ has no common discontinuities with $\prod_{i=1}^m F(x_i)$. Then $d_N^{-m}V_{[Nt]}(h)$ and $d_N^{-m}U_{[Nt]}(h)$ converge weakly in the space $D[0, 1]$ to*

- i) $C(1)(B(t))^m$ if $\alpha_0 < 2\alpha$,
- ii) $\int_{\mathbb{R}^m} h(x_1, \dots, x_m) \prod_{i=1}^m \left(dJ_1(x_i)B(t) + \frac{dJ_2(x_i)}{2}R(t) \right)$ if $\alpha_0 = 2\alpha$,
- iii) $C(2)(R(t))^m$, if $\alpha_0 > 2\alpha$,

where, for $k = 1, 2$,

$$C(k) = (k!)^{-m} \int_{\mathbb{R}^m} h(x_1, \dots, x_m) dJ_k(x_1) \cdots dJ_k(x_m).$$

Note that in [5], Dehling and Taqqu proved that

$$C(k) = (k!)^{-m} \int_{\mathbb{R}^m} h(G(x_1), \dots, G(x_m)) \prod_{i=1}^m H_k(x_i) \phi(x_i) dx_1 \cdots dx_m. \tag{25}$$

This relation shall be used later.

3.2. Particular case $m = 2$

For applications, the condition of finite total variation is too restrictive. For example, it rules out the polynomial functions. In the particular case $m = 2$, it turns out that the results of Corollary 1 remain valid for a larger class of locally bounded functions, which allows to obtain the convergence of some standard statistics. For a locally bounded function h , let μ_h be the measure generated by the increments of h (for more details, see [6]). This measure admits a Hahn–Jordan decomposition $\mu_h = \mu_h^+ - \mu_h^-$. Let h^+ and h^- be the functions such that $h^+(c) = h^-(c) = 0$ where c is a median of F and whose increments define respectively the measures μ_h^+ and μ_h^- . Define on $D_L(\mathbb{R}^2)$ the semi-norm $\|\cdot\|_F$ by

$$\|h\|_F = \int_{\mathbb{R}^2} (|h^+(x, y)| + |h^-(x, y)|) |d\rho(x)| |d\rho(y)|,$$

where

$$\rho(x) = \sqrt{F(x)(1 - F(x))} \quad \text{and} \quad |d\rho(x)| = \left| \frac{d\rho(x)}{dF(x)} \right| dF(x). \tag{26}$$

Corollary 3.2. *Let h be a locally bounded function such that $\|h\|_F < \infty$ and having no common discontinuities with $F(x)F(y)$. Then the conclusions of Corollary 1 hold with $m = 2$.*

The proof is similar to that given by Dehling and Taqqu ([6], proof of the theorem). It consists in approximating h by compactly supported functions for which Corollary 1 applies. We omit the details.

3.3. Example of the empirical variance

Let us compare our results to those obtained by Dehling and Taqqu [6], in the *regular* long-memory setting.

We shall focus on the interesting situation of a weakly marked singularity at 0, that is $\alpha_0 > 2\alpha$ (with the notations of Th. 2.1).

Let $G(x) = \sigma x + \mu$, and $Y_j = G(X_j)$, where, as in Section 2, X_j is a standard Gaussian variable. Put

$$S_N^2 = \frac{1}{N-1} \sum_{j=1}^N (Y_j - \bar{Y})^2.$$

In the case of i.i.d. variables it is well known that

$$\sqrt{N} \left(\frac{S_N^2 - \sigma^2}{\sqrt{2}\sigma^2} \right) \implies \mathcal{N}(0, 1), \quad \text{as } N \rightarrow \infty, \tag{27}$$

where $\mathcal{N}(0, 1)$ is the standard Gaussian law.

In the case of *regular* long memory, *i.e.* when the covariance of X_n has the form $r(n) = n^{-\alpha}L(n)$, $0 < \alpha < 1$, Dehling and Taqqu [6], proved that as $N \rightarrow \infty$

$$\frac{N^\alpha}{\sqrt{L(N)}} (S_N^2 - \sigma^2) \implies \sigma^2 \left(\frac{\sqrt{(1-2\alpha)(2-2\alpha)}}{2} Z_2 - \frac{(1-\alpha)(2-\alpha)}{2} Z_1^2 \right), \tag{28}$$

where Z_1 is a standard Gaussian variable and Z_2 is a Rosenblatt variable of parameter $1 - \alpha$.

In order to study the asymptotic behavior of $S_N^2 - \sigma^2$ under long memory with *seasonal* effects, define the U -statistic

$$U_N = N(N - 1)(S_N^2 - \sigma^2),$$

whose Hoeffding-decomposition is

$$U_N = (N - 1) \sum_{j=1}^N (Y_j - \mu)^2 - \sum_{i \neq j} (Y_i - \mu)(Y_j - \mu) := U_N^{(1)} + U_N^{(2)}.$$

The first term is

$$U_N^{(1)} = \sigma^2(N - 1) \sum_{j=1}^N H_2(X_j),$$

and hence,

$$\frac{U_N^{(1)}}{(N - 1)d_N} = \sigma^2 X_{N,2}(1),$$

where $X_{N,2}(1)$ is defined in (18). According to (22), as N tends to infinity,

$$\frac{U_N^{(1)}}{Nd_N} \implies \sigma^2 R(1),$$

where $R(1)$ is the combination of independent Rosenblatt variables defined in (16).

The second term $U_N^{(2)}$ is a degenerated U -statistic with kernel

$$h(x, y) = (x - \mu)(y - \mu).$$

Since h is differentiable, and $F = \Phi$, the standard Gaussian distribution function, we have, with ρ defined in (26),

$$\|h\|_F = \int_{\mathbb{R}^2} \rho(x)\rho(y)dx dy = \left(\int_{\mathbb{R}} \left(\Phi \left(\frac{x - \mu}{\sigma} \right) \left(1 - \Phi \left(\frac{x - \mu}{\sigma} \right) \right) \right)^{1/2} dx \right)^2 < \infty.$$

Hence the conditions of Corollary 2 are satisfied and, as N goes to infinity,

$$d_N^{-2} U_N^{(2)} \implies C(2)R(1)^2.$$

Thus, as N goes to infinity,

$$N^{\alpha-2} U_N^{(2)} \longrightarrow 0$$

in probability.

From this it follows that, as it is the case in the i.i.d. situation, and contrarily to what happens in the *regular* long-memory case, the second component in the Hoeffding-decomposition is negligible with respect to the first one. We obtain

Proposition 3.3. *Let X_n be a zero mean stationary Gaussian process with variance σ^2 , admitting a spectral density of the form (11), with $\alpha_0 > 2\alpha$. The empirical variance*

$$S_N^2 = \frac{1}{N-1} \sum_{j=1}^N (X_j - \bar{X})^2$$

satisfies

$$N^\alpha (S_N^2 - \sigma^2) \implies \sigma^2 R(1), \quad (29)$$

where, $R(1)$ is the combination of independent Rosenblatt variables defined in (16).

Remark. As the Rosenblatt process is centered and non-Gaussian, we see from (27, 28) and (29) that the limit of the normalized empirical variance is no more Gaussian for strongly-dependent data, and that in this case, the non Gaussian limit has a zero mean only in the *seasonal* situation. We illustrate these two remarks by some simulations.

3.4. Simulations

We consider three different situations:

1. $X_n^{(1)}$ is an i.i.d. standard Gaussian random sequence;
2. $X_n^{(2)}$ is a zero-mean long-memory Gaussian sequence with *regular* long memory, having spectral density

$$f_2(\lambda) = \frac{1}{2\pi} |1 - e^{i\lambda}|^{-0.6};$$

3. $X_n^{(3)}$ is a zero mean Gaussian process exhibiting *seasonal* long-memory, with spectral density

$$f_3(\lambda) = \frac{1}{2\pi} |1 - e^{i(\lambda+1.4)}|^{-0.6} |1 - e^{i(\lambda-1.4)}|^{-0.6}.$$

For each of the three models above, we have simulated 500 independent sample paths of length $N = 10\,000$, $(X_{1,k}^{(j)}, \dots, X_{N,k}^{(j)})_{k \in \{1, \dots, 500\}}$. The algorithms for simulating such processes are detailed in Bardet *et al.* [2].

From the corresponding sample paths, we have computed the values of the statistics

$$\delta_{N,k}^{(j)} = N^{\alpha(j)} (S_{N,k}^2 - \sigma_j^2)$$

where, according to (27, 28) and (29), $\alpha_{(1)} = \frac{1}{2}$, $\alpha_{(2)} = \alpha_{(3)} = 1 - 0.6 = 0.4$. The variance $\sigma_j^2 = \text{Var}(X_n^{(j)})$ is the integral of the spectral density. We know from Gradshcheyn and Ryzhik ([14], p. 511) that $\sigma_2^2 = \Gamma(0.4)/(\Gamma(0.7))^2$. A numerical method is used to approximate σ_3^2 .

The empirical distributions of the statistics $\delta_N^{(j)}$ ($j = 1, 2, 3$) are depicted in Figure 1(top). We clearly see that the distribution has mean zero only in the *seasonal* and independent cases.

Moreover the lack of symmetry indicates that the limiting distributions of $\delta_N^{(2)}$ and $\delta_N^{(3)}$ are not Gaussian. To display non-Gaussianity, we use the graphical method introduced by Ghosh [10]. This method is based on the properties of the third derivative of the logarithm of the empirical moment generating function (called

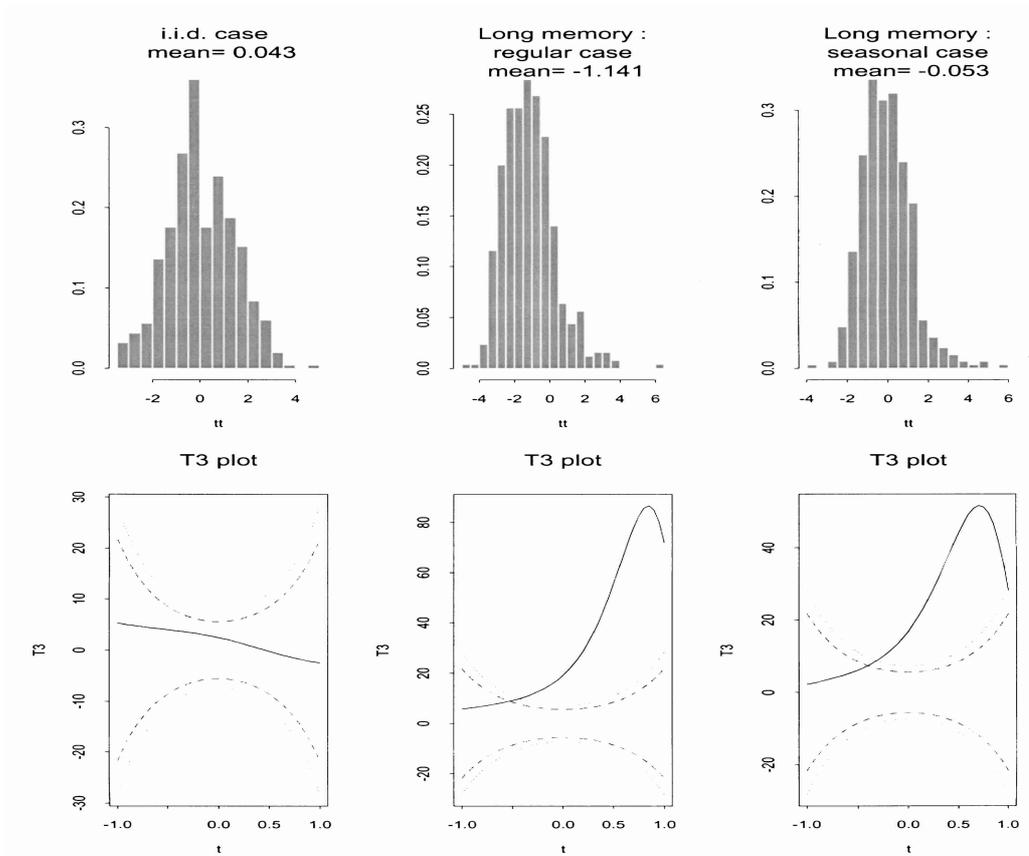


FIGURE 1. Estimated distribution of $\delta_N^{(j)}$ (top). Curve of the T_3 -function (plain) with the confidence band at significant level 1% (dots) 5% (dashes) (bottom). The simulations are based on 500 replications of sample path of length 10 000 in the following set-up: i.i.d. (left), *regular* long memory (middle) and *seasonal* long memory (right).

T_3 -function in [10]).

$$T_3^{(j)}(t) = \frac{d^3}{dt^3} \ln \left(\frac{1}{500} \sum_{k=1}^{500} \exp(t\delta_{N,k}^{(j)}) \right) \quad t \in [-1, 1].$$

Deviation of the curve of the T_3 -function from the horizontal zero line indicates a lack of normality. A Central Limit Theorem for the T_3 -function provides approximated confidence bands for $t \in [-1, 1]$. We thus reject the normality when the curve of the T_3 -function crosses the upper or lower bounds anywhere in the interval $[-1, 1]$. According to this procedure, we reject the normality at significant level 1% and 5% in both cases of long memory processes (see Fig. 1(bottom)). On the opposite, in the i.i.d. case, the curve of the T_3 -function is inside the confidence bands in the interval $[-1, 1]$ and thus we accept normality.

4. APPENDIX

4.1. Convergence of the Donsker line

In this section we examine the asymptotic behavior of the partial sums in presence of *seasonal* long-memory with effects of type (11). For a function H satisfying the conditions

$$\int_{\mathbb{R}} H(x)e^{-\frac{x^2}{2}} dx = 0, \quad \int_{\mathbb{R}} H^2(x)e^{-\frac{x^2}{2}} dx < \infty,$$

we consider the convergence of the processes $Y_N(t)$, $0 \leq t \leq 1$ defined by

$$Y_N(t) = d_N^{-1} \sum_{j=1}^{[Nt]} H(X_j),$$

where the normalizing coefficient is defined below. Let

$$H(x) = \sum_{k=1}^{\infty} \frac{J_k}{k!} H_k(x),$$

be the Hermite expansion of H . The following quantities shall be of central interest:

$$\begin{aligned} \gamma_k &= \min\{\alpha_{j_1} + \dots + \alpha_{j_k} \mid \lambda_{j_1} + \dots + \lambda_{j_k} = 0 \pmod{2\pi}\} \quad k \geq 1, \\ \gamma &= \min\{\gamma_k, J_k \neq 0\}. \end{aligned} \tag{30}$$

Denote

$$d_N = N^{1-\gamma/2},$$

and for $j = 0, \dots, m$,

$$s_j = s_{-j} = h(\lambda_j) \prod_{i \neq j} |\lambda_i - \lambda_j|^{\alpha_i - 1}.$$

Theorem 4.1. *Let (X_n) be a zero mean Gaussian process with $\mathbb{E}X_1^2 = 1$, having a spectral density of the form (11). If $\gamma < 1$, we have*

$$Y_N(t) \xrightarrow{\mathcal{D}} \sum_{k|\gamma_k=\gamma} \frac{J_k}{k!} Y^{t,k} \quad \text{as } N \rightarrow \infty, \tag{31}$$

where

$$Y^{t,k} = \sum_k (s_{j_1} \dots s_{j_k})^{1/2} \int_{\mathbb{R}^k} \frac{e^{it(x_1+\dots+x_k)} - 1}{i(x_1 + \dots + x_k)} \prod_{i=1}^k |x_i|^{(\alpha_{j_i} - 1)/2} Z_{j_i}(dx_i),$$

formula in which \sum_k is over all $j_1, \dots, j_k \in \{-m, \dots, m\}$ such that $\alpha_{j_1} + \dots + \alpha_{j_k} = \gamma_k$ and $\lambda_{j_1} + \dots + \lambda_{j_k} = 0 \pmod{2\pi}$, and where $Z_{-m}, \dots, Z_0, \dots, Z_m$ are complex random measures with the following properties: Z_0 is a spectral measure of a standard Gaussian white noise i.e. Z_0 is a Gaussian random measure such that

- for every interval Δ ,

$$\begin{aligned} Z_0(\Delta) &= \overline{Z_0(-\Delta)} \\ \mathbb{E}|Z_0(\Delta)|^2 &= \frac{1}{2\pi} |\Delta|; \end{aligned} \tag{32}$$

- for every positive interval Δ , $\text{Re} Z_0(\Delta)$ and $\text{Im} Z_0(\Delta)$ are zero-mean i.i.d. variables;
 - for every disjoint intervals $\Delta_1, \dots, \Delta_n$, $Z_0(\Delta_1), \dots, Z_0(\Delta_n)$ are independent.
- The others measures $Z_{\pm 1}, \dots, Z_{\pm m}$ have the same properties that Z_0 except (32), which is replaced by

$$Z_{-j}(\Delta) = \overline{Z_j(-\Delta)}, \quad \forall j \in \{1, \dots, m\}.$$

Finally the random measures Z_0, Z_1, \dots, Z_m are independent.

Complex random measures such that The proof is omitted, because it is very close to that of Theorem 1 in [11]. Indeed, our setting is only slightly different from the one treated by Giraitis: spectral densities of the form (11) can also be written as

$$g(\lambda) = \sum_{j=-m}^m s_j L_j \left(\frac{1}{\lambda - \lambda_j} \right) |\lambda - \lambda_j|^{\alpha_j - 1},$$

where $L_j(x) \rightarrow 1$ as x goes to infinity. This form differs from (10) only by the fact that the slowly varying functions L_j are not necessarily the same.

For the details of the proof, the reader is referred to [21].

Let us notice that condition $\gamma < 1$ extends to the seasonal situation the condition of long-memory (3). In order to make this clear, let us suppose that the covariance (13) is regularly varying at infinity, that is

$$\alpha_0 < \alpha_j \quad \forall j \in \{1, \dots, m\}.$$

Then, $\alpha_{j_1} + \dots + \alpha_{j_k} > k\alpha_0$ for all k , so that $\gamma = \gamma_\tau = \tau\alpha_0$, where τ is the Hermite rank of H . Hence, as $r(n) = a_0 n^{-\alpha_0} (1 + o(1))$, condition $\gamma < 1$ is exactly (3).

Remark 1. When $J_1 J_2 \neq 0$, there are only two possible values for γ , according to the position of α_0 . Let α be the parameter defined in (12). It is easy to check that

$$\begin{aligned} \gamma &= \gamma_1 = \alpha_0 && \text{if } \alpha_0 < 2\alpha \\ \gamma &= \gamma_2 = 2\alpha && \text{if } \alpha_0 > 2\alpha \\ \gamma &= \gamma_1 = \gamma_2 = 2\alpha && \text{if } \alpha_0 = 2\alpha. \end{aligned}$$

As for the set of integers $E = \{k | \gamma_k = \gamma\}$, it is reduced to one single element, respectively $E = \{1\}$ and $E = \{2\}$, in the two first cases. In the situation where $\alpha_0 = 2\alpha$, $E = \{1, 2\}$. This explains the three forms taken by the limiting process in Theorem 2.1. Of course, it also proves that the finite dimensional distributions of $d_N^{-1} [Nt] (F_{[Nt]}(x) - F(x))$ and those of $J_1(x) X_{N,1}(t) + (J_2(x)/2) X_{N,2}(t)$ are the same.

Remark 2. More generally, the number of chaos appearing in the limit process (31) is exactly that of indices k such that $\gamma_k = \gamma$, and it happens that this number is never greater than 2. The basic facts to explain this are firstly that $(\gamma_{2k})_k$ and $(\gamma_{2k+1})_k$ are increasing sequences, secondly that $\gamma_{2k} = \min\{\alpha_{j_1} + \dots + \alpha_{j_{2k}}\} = 2k\alpha$ in (30), because it is always possible to satisfy the condition $\lambda_{j_1} + \dots + \lambda_{j_{2k}} = 0$ by taking pairwise opposite frequencies and thirdly that $\gamma_{2k} < \gamma_{2k+1}$ while it is not always true that $\gamma_{2k+1} < \gamma_{2k+2}$.

Consequently, when the Hermite rank τ is even, the asymptotic behavior of $Y_N(t)$ only rely on the Hermite polynomial H_τ . In other words, the limit process in (31) is simply $(J_\tau/\tau!) Y^{t,\tau}$.

Suppose now that $\tau = 2p - 1$. Denote by H_{2k} the first even polynomial if any, in the Hermite expansion of H (by convention, we take $2k = \infty$ when H is odd). Then, the limit process in (31) is given by

$$\begin{cases} \frac{J_{2p-1}}{(2p-1)!} Y^{t,2p-1} & \text{if } \gamma_{2p-1} < 2k\alpha \\ \frac{J_{2k}}{(2k)!} Y^{t,2k} & \text{if } \gamma_{2p-1} > 2k\alpha \\ \frac{J_{2p-1}}{(2p-1)!} Y^{t,2p-1} + \frac{J_{2k}}{(2k)!} Y^{t,2k} & \text{if } \gamma_{2p-1} = 2k\alpha. \end{cases}$$

4.2. Proof of Proposition 2.2

From Theorem 4.1, we have (22) with $B(t)$ and $R(t)$ replaced by $Y^{t,1}$ and $Y^{t,2}$ given by

$$Y^{t,1} = \sqrt{s_0} \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} |x|^{(\alpha_0-1)/2} Z_0(dx),$$

$$Y^{t,2} = \sum_{j \in J} s_j \int_{\mathbb{R}^2} \frac{e^{it(x+y)} - 1}{i(x+y)} |x|^{(\alpha-1)/2} |y|^{(\alpha-1)/2} Z_{-j}(dx) Z_j(dy),$$

where J is defined in (12).

These two processes are independent when $\alpha_0 = 2\alpha$, since in this case Z_0 does not appear in the construction of $Y^{t,2}$. It appears only when $\alpha_0 = \alpha$.

It remains to prove that $Y^{t,1}$ and $Y^{t,2}$ have respectively the same distribution as $B(t)$ and $R(t)$ of Theorem 2.1. It is clear that, with C defined in (15), $\sqrt{C}Y^{t,1}$ is a fractional Brownian motion, with parameter $1 - \alpha_0/2$. Let us see why $Y^{t,2}$ has the same distribution as $R(t)$.

For $j = 1, \dots, m$ and for any interval Δ , put

$$\text{for } j = 1, \dots, m, \text{ let } W_j^{(1)}(\Delta) = \frac{Z_j(\Delta) + Z_{-j}(\Delta)}{\sqrt{2}}, \quad W_j^{(2)}(\Delta) = i \frac{Z_j(\Delta) - Z_{-j}(\Delta)}{\sqrt{2}}.$$

It is easy to check that $W_j^{(1)}$ and $W_j^{(2)}$ are random spectral measures of independent standard Gaussian white noises and that

$$Z_j(\Delta)Z_{-j}(\Delta') + Z_j(\Delta')Z_{-j}(\Delta) = W_j^{(1)}(\Delta)W_j^{(1)}(\Delta') + W_j^{(2)}(\Delta)W_j^{(2)}(\Delta'). \tag{33}$$

Define then for $h = 1, 2, j = 1, \dots, m$,

$$R_j^{(h)}(t) = (D(2, \alpha))^{-1} \int_{\mathbb{R}^2} \frac{e^{it(x+y)} - 1}{i(x+y)} |x|^{(\alpha-1)/2} |y|^{(\alpha-1)/2} W_j^{(h)}(dx) W_j^{(h)}(dy),$$

and when $\alpha_0 = \alpha$,

$$R_0^{(1)}(t) = R_0^{(2)}(t) = (D(2, \alpha))^{-1} \int_{\mathbb{R}^2} \frac{e^{it(x+y)} - 1}{i(x+y)} |x|^{(\alpha-1)/2} |y|^{(\alpha-1)/2} Z_0(dx) Z_0(dy),$$

where $D(2, \alpha)$ is given by (8).

It is clear, from representation (16), that $R_j^{(h)}(t)$, $h = 1, 2, j \in J$ are Rosenblatt processes with the same parameter $1 - \alpha$. Finally, from (14) and (33),

$$Y^{t,2} = D(2, \alpha) \sum_{j \in J} c_j \left(R_j^{(1)}(t) + R_j^{(2)}(t) \right) = R(t).$$

4.3. Proof of Lemma 2.3

Let $T = \{0 = t_0 < t_1 < \dots < t_m = 1\}$ be a subdivision of $[0, 1]$. For x in $D[0, 1]$, let $A_T x$ denote its piecewise linear approximation built from T . According to Pollard ([23], Th. 3, p. 92), relative compactity of X_n and of Y_n in the space $(D[0, 1], \mathcal{E})$ is equivalent to the fact that, for every $\epsilon > 0$ and $\delta > 0$ there exist subdivisions T and S such that

$$\limsup_{n \rightarrow \infty} P\left\{\|A_T X_n - X_n\| > \delta\right\} \leq \epsilon, \quad \limsup_{n \rightarrow \infty} P\left\{\|A_S Y_n - Y_n\| > \delta\right\} \leq \epsilon. \quad (34)$$

Since the sample paths of X and Y are continuous, there exist subdivisions T' and S' such that

$$P\left\{\|A_{T'} X - X\| > \delta\right\} \leq \epsilon, \quad P\left\{\|A_{S'} Y - Y\| > \delta\right\} \leq \epsilon. \quad (35)$$

Without loss of generality, we can suppose that all these subdivisions are the same.

We have to prove that, for any bounded uniformly continuous measurable function f on $D[0, 1] \times D[0, 1]$,

$$\mathbb{E}(f(X_n, Y_n)) \longrightarrow \mathbb{E}(f(X, Y)), \quad \text{as } n \rightarrow \infty.$$

Since $A_T x$ depends on x continuously through $x(t_0), \dots, x(t_m)$, we can write

$$f \circ (A_T, A_T) = g \circ (\pi_T, \pi_T)$$

where g is a bounded continuous function on \mathbb{R}^{2m} and π_T is the projection induced by T from $D[0, 1]$ on \mathbb{R}^m . As f is uniformly continuous, for $\epsilon > 0$, there exists $\delta > 0$ such that for every x, y, x', y' in $D[0, 1]$,

$$\|(x, y) - (x', y')\| \leq \delta \implies |f(x, y) - f(x', y')| < \epsilon. \quad (36)$$

Using (34–36) and the fact that for any Z, Z' in $D[0, 1]$

$$\left\{\|(Z, Z')\| > \delta\right\} = \left\{\max\{\|Z\|, \|Z'\|\} > \delta\right\} = \left\{\|Z\| > \delta\right\} \cup \left\{\|Z'\| > \delta\right\},$$

we obtain that

$$\begin{aligned} |\mathbb{E}(f(X_n, Y_n)) - \mathbb{E}(f(X, Y))| &\leq \mathbb{E}|f(X_n, Y_n) - f(A_T X_n, A_T Y_n)| + |\mathbb{E}(f(A_T X_n, A_T Y_n)) - \mathbb{E}(f(A_T X, A_T Y))| \\ &\quad + \mathbb{E}|f(A_T X, A_T Y) - f(X, Y)| \\ &\leq \epsilon + 2\|f\|P\left\{\|(X_n, Y_n) - (A_T X_n, A_T Y_n)\| > \delta\right\} \\ &\quad + |\mathbb{E}(g(\pi_T X_n, \pi_T Y_n)) - \mathbb{E}(g(\pi_T X, \pi_T Y))| \\ &\quad + \epsilon + 2\|f\|P\left\{\|(X, Y) - (A_T X, A_T Y)\| > \delta\right\} \\ &\leq 2\epsilon(1 + 4\|f\|) + |\mathbb{E}(g(\pi_T X_n, \pi_T Y_n)) - \mathbb{E}(g(\pi_T X, \pi_T Y))|. \end{aligned}$$

This last term converges to 0 as $n \rightarrow \infty$ because of the finite-dimensional distribution convergence of (X_n, Y_n) to (X, Y) .

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