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PENULTIMATE APPROXIMATION FOR THE DISTRIBUTION OF THE EXCESSES

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Abstract. Let F be a distribution function (d.f) in the domain of attraction of an extreme value distribution H_γ; it is well-known that $F_u(x)$, where F_u is the d.f of the excesses over u, converges, when u tends to $s_+(F)$, the end-point of F, to $G_\gamma(\frac{x}{\sigma(u)})$, where G_γ is the d.f. of the Generalized Pareto Distribution. We provide conditions that ensure that there exists, for $\gamma > -1$, a function Λ which verifies $\lim_{u\to s_+(F)} \Lambda(u) = \gamma$ and is such that $\Delta(u) = \sup_{x\in[0,s_+(F)-u[} |\bar{F}_u(x) - \bar{G}_{\Lambda(u)}(x/\sigma(u))|$ converges to 0 faster than $d(u) = \sup_{x\in[0,s_+(F)-u[} |\bar{F}_u(x) - \bar{G}_\gamma(x/\sigma(u))|$.

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Introduction

Let F be a distribution function in the domain of attraction of an extreme value distribution H_γ, where

$$\begin{aligned} H_\gamma(x) &= \exp(-(1+\gamma x)^{-1/\gamma}) && \text{for } x \text{ such that } 1+\gamma x > 0, && \text{if } \gamma \neq 0,\\ &= \exp(-\mathrm{e}^{-x}) && \text{for } x \in \mathbb{R} && \text{if } \gamma = 0, \end{aligned}$$

which means there exist sequences (α_n) and (σ_n) such that

$$\hat{d}_n = \sup_{x\in\mathbb{R}} \left| F^n(x) - H_\gamma\left(\frac{x-\alpha_n}{\sigma_n}\right)\right| \longrightarrow 0, \text{ as } n \to +\infty. \tag{1}$$

It is well-known (see [8] and [1]) that this is equivalent to the existence of $\sigma(u) > 0$ such that

$$d(u) = \sup_{x\in[0,s_+(F)-u[} \left| \bar{F}_u(x) - \bar{G}_\gamma\left(\frac{x}{\sigma(u)}\right)\right| \longrightarrow 0, \text{ as } u \to s_+(F), \tag{2}$$

where $s_+(F) = \sup\{x, F(x) < 1\}$ is the upper endpoint of F (and $s_-(F) = \inf\{x, F(x) > 0\}$ its lower endpoint), $\bar{F}_u(x)$ $(= \bar{F}(x+u)/\bar{F}(u) = (1-F(x+u))/(1-F(u)))$ is the survival function of the excess over u and $\bar{G}_\gamma$ the

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survival function of the Generalized Pareto Distribution, with

$$\begin{aligned} \bar{G}_\gamma(x) &= (1+\gamma x)^{-1/\gamma} && \text{for } x \geq 0 \text{ such that } 1+\gamma x > 0, && \text{if } \gamma \neq 0, \\ &= \mathrm{e}^{-x} && \text{for } x \in \mathbb{R}_+ && \text{if } \gamma = 0. \end{aligned}$$

H_γ will be called in this paper the "ultimate" approximation of F^n and, similarly, $\bar{G}_\gamma$ the "ultimate" approximation of $\bar{F}_u$. In 1928, Fisher and Tippet [4] showed empirically, for the normal distribution (for which $\gamma = 0$), that there exists a sequence of extreme value distributions H_{γ_n} of Weibull type ($\gamma_n < 0$ and $\gamma_n \to 0$), which is a better approximation of F^n than its limiting distribution H_γ. They called H_{γ_n} a "penultimate" approximation of F^n.

Cohen, in [3], studied the case of the Gumbel domain of attraction ($\gamma = 0$). He exhibited a penultimate approximation of F^n, for distributions in a sub-class of this domain and compared the rates of convergence to 0 of

$$\hat{d}_n = \sup_{x \in \mathbb{R}} |F^n(\sigma_n x + \alpha_n) - H_\gamma(x)|$$

and

$$\hat{\Delta}_n = \sup_{x \in \mathbb{R}} |F^n(\sigma_n x + \alpha_n) - H_{\gamma_n}(x)|,$$

for appropriate normalizing sequences α_n and σ_n.

In [5] and [7], Gomes studied the rates of convergence of $\hat{d}_n$ and $\hat{\Delta}_n$, in the case $\gamma = 0$ and then in the other cases ($\gamma > 0$ and $\gamma < 0$).

More recently, Gomes and de Haan, in [6], gave a necessary condition for the existence of a penultimate approximation for F^n. In other words, they provided a condition for the existence of a sequence γ_n tending to γ such that the rate of convergence to 0 of $\hat{\Delta}_n$ is better than the rate of $\hat{d}_n$.

Now, regarding the Generalized Pareto approximation for the distribution of the excesses, we have studied, in [9], the rate of convergence to 0 of

$$d(u) = \sup_{x \in [0, s_+(F)-u[} \left| \bar{F}_u(x) - \bar{G}_\gamma\left(\frac{x+u-\alpha(u)}{\sigma(u)}\right) \right|,$$

for appropriate normalizing functions α and σ.

The aim of this paper is to study the existence of a penultimate approximation for this distribution of the excesses. In other words, we look for conditions under which there exists a function Λ such that $\Lambda(u) \longrightarrow \gamma$, as $u \to s_+(F)$, and the rate of convergence to 0 of

$$\Delta(u) = \sup_{x \in [0, s_+(F)-u[} \left| \bar{F}_u(x) - \bar{G}_{\Lambda(u)}\left(\frac{x+u-\alpha(u)}{\sigma(u)}\right) \right|$$

is better than the rate of $d(u)$, for appropriate normalizing functions α and σ.

In Section 1, we present the framework. In Section 2, we provide a necessary condition for the existence of a penultimate approximation, with the appropriate normalizing functions. In Section 3, we state our results. In Section 4, we prove the main result.

1. Assumptions and preliminary properties

In the sequel, we suppose that F is four times differentiable and that its inverse F^{-1} exists. We define the mapping V from $\mathbb{R}_+^*$ onto $]s_-(F), s_+(F)[$, by $V(t) = \bar{F}^{-1}(\mathrm{e}^{-t}) = \bar{F}^{-1}(\bar{G}_0(t))$ and we note $A(t) = \frac{V''(\ln t)}{V'(\ln t)} - \gamma$.

We have established, in [9] (and [10]), the rate of convergence to 0 of $d(u)$, under the following first and second order conditions:

$$\lim_{t\to+\infty} A(t) = 0, \tag{3}$$

and

$$A \text{ is of constant sign at } +\infty \text{ and there exists } \rho \le 0 \text{ such that } |A| \in RV_\rho.^2 \tag{4}$$

Here, we use a second order condition which (see [2]) is stronger than (4):

$$A \text{ is of constant sign at } +\infty \text{ and there exists } \rho \le 0 \text{ such that } \lim_{t\to+\infty} \frac{tA'(t)}{A(t)} = \rho. \tag{5}$$

We shall see in the following section that, under assumptions (3) and (4), a necessary condition for the existence of a penultimate approximation for the distribution of the excesses is $\rho = 0$. In order to compute the rate of convergence of $\Delta(u)$, we introduce the following condition:

$$A' \text{ is of constant sign at } +\infty \text{ and } \lim_{t\to+\infty} \frac{tA''(t)}{A'(t)} = -1, \tag{6}$$

which, by an immediate application of the l'Hospital rule, implies $\rho = 0$.

Remark 1. Gomes and de Haan, in [6], make use of conditions (3), (5) and (6), where A is replaced by $\hat{A}(t) = \frac{\hat{V}''(\ln t)}{\hat{V}'(\ln t)} - \gamma$, with $\hat{V}(t) = F^{-1}(\exp(-\mathrm{e}^{-t}))$.

The results stated in the following proposition using functions V and A are stated in [6] (see 1.14 and Lem. 2.1) using functions $\hat{V}$ and $\hat{A}$. This proposition has to be compared with Proposition 1 in [9] (or Prop. 1 in [10]).

Proposition 1. *Under conditions* (3), (5) *and* (6)*, for* $\gamma \in \mathbb{R}$*,*
(i)

$$\lim_{t\to+\infty} \left[\frac{V(t+x)-V(t)}{V'(t)} - \frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)}\right] \Big/ \lambda'(t) = M_\gamma(x),$$

where

$$M_\gamma(x) = \frac{1}{2}\int_0^x u^2 \mathrm{e}^{\gamma u} \mathrm{d}u.$$

(ii) $\forall \epsilon > 0,\ \exists t_0 \ge 0,\ \forall t \ge t_0, \forall x \ge t_0 - t$,

- *if* $x \ge 0$,

$$(1-\epsilon)\mathrm{e}^{-\epsilon x} M_\gamma(x) < \frac{\frac{V(t+x)-V(t)}{V'(t)} - \frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)}}{\lambda'(t)} < (1+\epsilon)\mathrm{e}^{\epsilon x} M_\gamma(x),$$

- *if* $x \le 0$,

$$(1+\epsilon)\mathrm{e}^{-\epsilon x} M_\gamma(x) < \frac{\frac{V(t+x)-V(t)}{V'(t)} - \frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)}}{\lambda'(t)} < (1-\epsilon)\mathrm{e}^{\epsilon x} M_\gamma(x).$$

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[2] $f \in RV_\alpha$ (*i.e.* f is regularly varying with index α) means that for all $x > 0$, $\lim_{t\to+\infty} \frac{f(tx)}{f(t)} = x^\alpha$ (see [2]).

Proof. We prove only (i) (for (ii), see Lem. 2.1 in [6]).

$$\begin{aligned}\frac{V(t+x)-V(t)}{V'(t)}-\frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)} &= \int_0^x \frac{V'(t+s)}{V'(t)}\mathrm{d}s - \int_0^x \exp(s\frac{V''(t)}{V'(t)})\mathrm{d}s,\\ &= \int_0^x \exp(s\frac{V''(t)}{V'(t)})\left[\exp\left(\ln V'(t+s)-\ln V'(t)-s\frac{V''(t)}{V'(t)}\right)-1\right]\mathrm{d}s,\\ &= \int_0^x \exp(s\frac{V''(t)}{V'(t)})\left[\exp\left(\int_0^s\left(\frac{V''(t+z)}{V'(t+z)}-\frac{V''(t)}{V'(t)}\right)\mathrm{d}z\right)-1\right]\mathrm{d}s.\end{aligned}$$

Now $\frac{V''(t)}{V'(t)}=\lambda(t)$ and $\frac{V''(t+z)}{V'(t+z)}-\frac{V''(t)}{V'(t)}=\int_0^z \lambda'(t+q)\mathrm{d}q$, hence

$$\left[\frac{V(t+x)-V(t)}{V'(t)}-\frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)}\right]/\lambda'(t)=\int_0^x \exp(s\lambda(t))\left[\frac{\exp\left(\lambda'(t)\int_0^s\int_0^z \frac{\lambda'(t+q)}{\lambda'(t)}\mathrm{d}q\mathrm{d}z\right)-1}{\lambda'(t)}\right]\mathrm{d}s.$$

When $t\to+\infty$, $\lambda(t)\longrightarrow\gamma$ (from (3)), $\lambda'(t)\longrightarrow 0$ (from (5)), and $\frac{\lambda'(t+q)}{\lambda'(t)}\longrightarrow 1$ uniformly on every interval $[0,X]$ (from Lem. 1). Hence, for all $x\in\mathbb{R}$,

$$\left[\frac{V(t+x)-V(t)}{V'(t)}-\frac{\mathrm{e}^{\lambda(t)x}-1}{\lambda(t)}\right]/\lambda'(t)\longrightarrow\int_0^x \mathrm{e}^{\gamma s}\frac{s^2}{2}\mathrm{d}s=M_\gamma(x).$$

□

Remark 2. Note that $M_\gamma(0)=0$, M_γ is positive on $\mathbb{R}_+^*$ and negative on $\mathbb{R}_-^*$.

2. Necessary condition for penultimate approximation

Our aim is to find conditions under which there exists a function Λ such that $\Lambda(u)\longrightarrow\gamma$, as $u\to s_+(F)$, and the rate of convergence to 0 of

$$\Delta(u)=\sup\left(\left|\bar F_u(x)-\bar G_{\Lambda(u)}\left(\frac{x+u-\alpha(u)}{\sigma(u)}\right)\right|\ ; x\in[0,s_+(F)-u[\right) \tag{7}$$

$$=\sup\left(\left|\bar F_u(\sigma(u)x-u+\alpha(u))-\bar G_{\Lambda(u)}(x)\right|\ ; x\in\left[\frac{u-\alpha(u)}{\sigma(u)},\frac{s_+(F)-\alpha(u)}{\sigma(u)}\right[\right) \tag{8}$$

is faster than $A(\mathrm{e}^{V^{-1}(u)})$, which was the rate in the ultimate case (see [9] or [10]). A method for this is to express[3] $\bar F_u(x)-\bar G_{\Lambda(u)}(\frac{x+u-\alpha(u)}{\sigma(u)})$ using $\bar G_0$ (*i.e.* $x\mapsto\mathbb{I}_{]-\infty,0[}(x)+\mathbb{I}_{[0,+\infty[}(x)\mathrm{e}^{-x}$).

For this purpose, we introduce the function ϕ_u, defined from $[0,s_+(F)-u[$ onto $\mathbb{R}_+$, by

$$\phi_u(x)=V^{-1}(x+u)-V^{-1}(u).$$

We obtain

$$\bar F_u(x)=\bar G_0(\phi_u(x)).$$

[3]As we did in [9] or [10] in order to study $\mathrm{d}(u)$.

We note that, for $\gamma \neq 0$ and $x \in \mathbb{R}_+$,

$$\bar{G}_\gamma(x) = \bar{G}_0\left(\frac{1}{\gamma}\ln(1+\gamma x)\right). \tag{9}$$

If $\Lambda(u) \neq 0$, for u sufficiently large[4], it follows that, for all $x \in [0, s_+(F) - u[$,

$$\bar{F}_u(x) - \bar{G}_{\Lambda(u)}\left(\frac{x+u-\alpha(u)}{\sigma(u)}\right) = \bar{G}_0(\phi_u(x)) - \bar{G}_0\left(\frac{1}{\Lambda(u)}\ln\left(1+\Lambda(u)\frac{V(\phi_u(x)+V^{-1}(u))-\alpha(u)}{\sigma(u)}\right)\right).$$

Then we can write

$$\Delta(u) = \sup_{s\in\mathbb{R}_+}|B_u(s)|, \tag{10}$$

where

$$B_u(s) = \bar{G}_0(s) - \bar{G}_0(s+q_u(s)),$$

and

$$q_u(s) = \frac{1}{\Lambda(u)}\ln\left(1+\Lambda(u)\frac{V(s+V^{-1}(u))-\alpha(u)}{\sigma(u)}\right) - s. \tag{11}$$

Now, let us choose

$$\sigma(u) = V'(V^{-1}(u)) \text{ and } \alpha(u) = V(V^{-1}(u)) = u. \tag{12}$$

These are the normalizing functions used in the ultimate case for $\gamma > -1$ (see [9] or [10]).

Theorem 1. *Under (3) and (4), if $\alpha(u)$ and $\sigma(u)$ are as in (12), a necessary condition to have, for all $\gamma \in \mathbb{R}$ and all $s \in \mathbb{R}_+$,*

$$B_u(s) = \bar{G}_0(s) - \bar{G}_0\left(\frac{1}{\Lambda(u)}\ln\left\{1+\Lambda(u)\frac{V(s+V^{-1}(u))-\alpha(u)}{\sigma(u)}\right\}\right) = o(A(\mathrm{e}^{V^{-1}(u)}))$$

is: $\rho = 0$ and $\Lambda(u) = \gamma + A(\mathrm{e}^{V^{-1}(u)})$.

Proof. The idea of the proof is due to Gomes and de Haan (see [6]).

Let $\bar{G}_0'(0)$ be the right derivative of $\bar{G}_0$ at 0. As $\frac{1}{\Lambda(u)}\ln\left\{1+\Lambda(u)\frac{V(s+V^{-1}(u))-u}{V'(V^{-1}(u))}\right\} \geq 0$, for all $s \in \mathbb{R}_+$,

$$\bar{G}_0(s) - \bar{G}_0\left(\frac{1}{\Lambda(u)}\ln\left\{1+\Lambda(u)\frac{V(s+V^{-1}(u))-\alpha(u)}{\sigma(u)}\right\}\right) = \bar{G}_0'(s)\,(s - J_u(s)) + O(s-J_u(s))^2,$$

[4]If $\gamma \neq 0$, this is obvious because $\Lambda(u)$ tends to γ, but if $\gamma = 0$, we have to suppose that $\Lambda(u)$ is of constant sign for u sufficiently large.

where

$$J_u(s)=\frac{1}{\gamma+z_u}\ln(1+K_u(z_u)),$$

with $z_u=\Lambda(u)-\gamma$ and $K_u(z_u)=(\gamma+z_u)\frac{V(s+V^{-1}(u))-u}{V'(V^{-1}(u))}$.

We know from [9] (or [10]) that conditions (3) and (4) imply that for all $\gamma\in\mathbb{R}$ and all $x\in\mathbb{R}$,

$$\left[\frac{V(t+s)-V(t)}{V'(t)}-\int_0^s \mathrm{e}^{\gamma x}\mathrm{d}x\right]\Big/A(\mathrm{e}^t)\overset{t\to+\infty}{\longrightarrow}\int_0^s \mathrm{e}^{\gamma x}\int_0^x \mathrm{e}^{\rho z}\mathrm{d}z\ \mathrm{d}x=I_{\gamma,\rho}(s). \tag{13}$$

From now on, we treat only the case $\gamma\neq 0$ (the case $\gamma=0$ is similar).

$$\begin{aligned}1+K_u(z_u)&=1+(\gamma+z_u)\left(\frac{\mathrm{e}^{\gamma s}-1}{\gamma}+A(\mathrm{e}^{V^{-1}(u)})I_{\gamma,\rho}(s)+o(A(\mathrm{e}^{V^{-1}(u)}))\right)\\&=\mathrm{e}^{\gamma s}+\frac{\mathrm{e}^{\gamma s}-1}{\gamma}z_u+\gamma A(\mathrm{e}^{V^{-1}(u)})I_{\gamma,\rho}(s)+o(A(\mathrm{e}^{V^{-1}(u)})).\end{aligned}$$

Then

$$\ln\big(1+K_u(z_u)\big)=\gamma s+\frac{1-\mathrm{e}^{-\gamma s}}{\gamma}z_u+\gamma A(\mathrm{e}^{V^{-1}(u)})\mathrm{e}^{-\gamma s}I_{\gamma,\rho}(s)+o(z_u)+o(A(\mathrm{e}^{V^{-1}(u)}))$$

and

$$J_u(s)=\frac{1}{\gamma}\left(1-z_u/\gamma+o(z_u)\right)\left(\gamma s+\frac{1-\mathrm{e}^{-\gamma s}}{\gamma}z_u+\gamma A(\mathrm{e}^{V^{-1}(u)})\mathrm{e}^{-\gamma s}I_{\gamma,\rho}(s)+o(z_u)+o(A(\mathrm{e}^{V^{-1}(u)}))\right),$$

whence

$$s-J_u(s)=\frac{1}{\gamma^2}(\gamma s-1+\mathrm{e}^{-\gamma s})z_u-A(\mathrm{e}^{V^{-1}(u)})\mathrm{e}^{-\gamma s}I_{\gamma,\rho}(s)+o(z_u)+o(A(\mathrm{e}^{V^{-1}(u)})).$$

The rate of convergence of $B_u(s)$ will be better than $O(A(\mathrm{e}^{V^{-1}(u)}))$ (the rate in the ultimate case), for all $s\in\mathbb{R}_+$, if we can choose z_u tending to 0 as $u\to s_+(F)$, such that, for all $s\in\mathbb{R}_+$,

$$\frac{1}{\gamma^2}(-\gamma s+1-\mathrm{e}^{-\gamma s})z_u+A(\mathrm{e}^{V^{-1}(u)}))\mathrm{e}^{-\gamma s}I_{\gamma,\rho}(s)=0$$

and $z_u=O(A(\mathrm{e}^{V^{-1}(u)}))$.

This is possible only if $\rho=0$. Indeed, $I_{\gamma,0}(s)=\int_0^s y\mathrm{e}^{\gamma y}\mathrm{d}y=-\frac{\mathrm{e}^{\gamma s}}{\gamma^2}(-\gamma s+1+\mathrm{e}^{-\gamma s})$ and then it suffices to choose $z_u=A(\mathrm{e}^{V^{-1}(u)})$. □

Remark 3. We know from [9] (or [10]) that another choice of normalizing functions $(\alpha^*(u),\sigma^*(u))$ is possible for $\gamma<0$. However, one can check that the same kind of computations as above shows that this choice does not lead to a penultimate approximation.

3. Rates of convergence

In order to determine the rate of convergence to 0 of

$$\begin{aligned}\Delta(u)&=\sup\left(\left|\bar{F}_u(x)-\bar{G}_{\Lambda(u)}(\frac{x+u-\alpha(u)}{\sigma(u)})\right|\ ;x\in[0,s_+(F)-u[\right)\\&=\sup\nolimits_{s\in\mathbb{R}_+}|B_u(s)|,\end{aligned}$$

where $B_u(s)$ is defined in (10), we begin (Th. 2, whose proof is given in Sect. 4) by giving the rate of uniform convergence to 0 of $B_u(s)$, as u tends to $s_+(F)$, where the normalizing functions are chosen as in (12) and Λ is defined by $\Lambda(u) = \gamma + A(\mathrm{e}^{V^{-1}(u)})$, as in Theorem 1.

Theorem 2. *Let F be a distribution function satisfying (3), (5) and (6). Define C_γ by*

$$C_\gamma(s) = \mathrm{e}^{-\gamma s}|\bar{G}_0'(s)|M_\gamma(s) = \exp[-(1+\gamma)s]M_\gamma(s).$$

Then, for $\gamma > -1$, $\frac{B_u}{\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})}$ converges to C_γ, uniformly on $\mathbb{R}_+$, as u tends to $s_+(F)$.

The rate of convergence of $\Delta(u)$ to 0 follows straightforwardly:

Corollary 1. *Under the same hypothesis and notations as Theorem 2, if $\gamma > -1$, as u tends to $s_+(F)$,*

$$\Delta(u) = \sup_{x\in[0,s_+(F)-u[} \left| \bar{F}_u(x) - \bar{G}_{\Lambda(u)}\left(\frac{x+u-\alpha(u)}{\sigma(u)}\right)\right| = O\left(\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})\right).$$

Remark 4. Note that $\mathrm{e}^{V^{-1}(u)} = 1/\bar{F}(u)$.

Remark 5. The condition $\lim_{t\to+\infty}\frac{tA'(t)}{A(t)} = 0$ (see (5)) ensures that the rate of convergence to 0 of $\Delta(u)$ is better than the rate of $d(u)$, which is of order $A(\mathrm{e}^{V^{-1}(u)})$ (see Cor. 1 in [9]). However, it follows from Lemma 1 below that the rate of convergence in the penultimate case remains a slowly varying function ($\in RV_0$).

Lemma 1. *If we note*

$$\lambda(t) = \frac{V''(t)}{V'(t)} = A(\mathrm{e}^t) + \gamma,$$

then, under (6), *for all q in $\mathbb{R}$,*

$$\lim_{t\to+\infty}\frac{\lambda'(t+q)}{\lambda'(t)} = 1.$$

In other words, $t \mapsto |\lambda'(\ln t)| = |tA'(t)|$ is RV_0. This convergence is uniform on every compact set of the form $[0,T]$.

Proof. For q in $\mathbb{R}$ and $t \geq \max(0,-q)$,

$$\ln\frac{\lambda'(t+q)}{\lambda'(t)} = \ln|\lambda'(t+q)| - \ln|\lambda'(t)| = \int_t^{t+q}\frac{\lambda''(r)}{\lambda'(r)}\mathrm{d}r.$$

Condition (6) implies that $\frac{\lambda''(t)}{\lambda'(t)} \longrightarrow 0$, as $u \to s_+(F)$. It follows that $\ln\frac{\lambda'(t+q)}{\lambda'(t)}$ tends to 0 uniformly on every compact set $[0,T]$. Hence $\frac{\lambda'(t+q)}{\lambda'(t)}$ tends to 1 uniformly on $[0,T]$. □

As a consequence of Theorem 2, we can also give the uniform rate of convergence to 0 of $\bar{F}_u(\sigma(u)y) - \bar{G}_{\Lambda(u)}$:

Theorem 3. *Under the same hypothesis and notations as in Theorem 2, if $\gamma > -1$,*

$$\frac{\bar{F}_u(V'(V^{-1}(u))y) - \bar{G}_{\Lambda(u)}(y)}{\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})},$$

converges, when $u \to s_+(F)$, *to*

$$\begin{aligned} D_\gamma(y) &= C_\gamma\left(\frac{1}{\gamma}\ln(1+\gamma y)\right) && \text{if } \gamma \neq 0,\\ D_0(y) &= C_0(y) && \text{if } \gamma = 0. \end{aligned}$$

This convergence is uniform:

- *on* $\mathbb{R}_+$ *if* $s_+(F) = +\infty$ *(particularly if* $\gamma > 0$*);*
- $[0, X]$ *(where* $X > 0$*) if* $\gamma = 0$ *and* $s_+(F) < +\infty$*;*
- $[0, X]$ *(where* $0 < X < -\frac{1}{\gamma}$*), if* $\gamma < 0$.

The proof of this theorem is similar to the proof of theorem 3 in [9] (or Th. 4 in [10]). The only difference is that, if we define ψ_u by $s = \psi_u(y) = V^{-1}(\sigma(u)y + \alpha(u)) - V^{-1}(u)$, an important step is to prove the uniform convergence to 0 of $\bar{G}_0(\psi_u(y)) - \bar{G}_\gamma(y)$, on $[0, s_{+,\gamma}[$ (where $s_{+,\gamma}$ is the upper end point of $\bar{G}_\gamma$); here, Theorem 2 ensures the uniform convergence to 0 of $\bar{G}_0(\psi_u(y)) - \bar{G}_{\Lambda(u)}(y)$. Besides, using the fact that $\bar{G}_\gamma$ is decreasing and that $\lim_{y\to s_+(F)} \bar{G}_\gamma(y) = 0$, we prove that $\bar{G}_{\Lambda(u)}(y) - \bar{G}_\gamma(y)$ converges to 0, uniformly on $[0, s_{+,\gamma}[$.

Example 1. *Distribution functions defined by*

$$1 - F(x) = \exp(-x^\beta \tilde{l}(x)),$$

where $\beta > 0$ *and* $\tilde{l}$ *is a smooth slowly varying function*[5], *are called of Weibull type. They are in the Gumbel domain of attraction. For these distributions, if* $\beta \notin \{\frac{1}{2}, 1\}$,

$$A(t) = \frac{V''(\ln t)}{V'(\ln t)} \sim \left(\frac{1}{\beta} - 1\right)\frac{1}{\ln t}, \text{ and } tA'(t) \sim \left(1 - \frac{1}{\beta}\right)\frac{1}{(\ln t)^2}, \text{ as } t \to +\infty. \tag{14}$$

Assumptions of Theorem 2 are satisfied. It follows that we can get a penultimate approximation for this type of distributions, as the Normal distribution ($\beta = 2$*) and the Weibull distribution (*$\beta > 0$ *and* $\tilde{l} = 1$*).*

We deduce from (14) that the rate of convergence is of order $\frac{1}{-\ln(1-F(u))}$ *in the ultimate case and* $\frac{1}{(-\ln(1-F(u)))^2}$ *in the penultimate one.*

4. Proof of Theorem 2

The proof of Theorem 2 follows the same lines as the proof of Theorem 1 in [9] (the ultimate case). Here are the two main steps of the proof:

(i) We find a positive function S satisfying $S(u) \to +\infty$ and such that, when $u \to s_+(F)$,

$$\frac{B_u}{e^{V^{-1}(u)}A'(e^{V^{-1}(u)})} \text{ converges to } C_\gamma$$

uniformly on $[0, S(u)]$, and

$$\frac{1}{e^{V^{-1}(u)}A'(e^{V^{-1}(u)})}\bar{G}_0(S(u)) \text{ converges to } 0.$$

(ii) We extend the convergence established in (i) to $\mathbb{R}_+$.

[5]This means that $\tilde{l}$ is C^∞ and is slowly varying.

• We begin by showing how (ii) is derived from (i). As $\lim_{x\to+\infty} C_\gamma(x) = 0$, for $\gamma > -1$, it is sufficient to show that $\sup_{s\geq S(u)} \frac{B_u(s)}{\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})}$ tends to 0 when u tends to $s_+(F)$. As $\bar{G}_0$ is a decreasing function and $s \mapsto s + q_u(s)$ is an increasing one, we get

$$\sup_{s\geq S(u)} |B_u(s)| \leq \bar{G}_0(S(u)) + \bar{G}_0(S(u) + q_u(S(u))) \leq 2\bar{G}_0(S(u)) + |B_u(S(u))|.$$

It follows from (i) that $\frac{\bar{G}_0(S(u))}{\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})} \to 0$ and $\frac{B_u(S(u))}{\mathrm{e}^{V^{-1}(u)}A'(\mathrm{e}^{V^{-1}(u)})} \to 0$.

• In order to prove (i), we give the following lemmas:

Lemma 2. *Let a be equal to 2 or 3.*
For all $\epsilon > 0$, we define on $\mathbb{R}_+$ $\tilde{f}_{\gamma,\epsilon}$ and $\tilde{g}_{\gamma,\epsilon}$ by

$$\tilde{f}_{\gamma,\epsilon}(x) = ((1+\epsilon)^a \mathrm{e}^{2\epsilon x} - 1) M_\gamma(x) \mathrm{e}^{-(\gamma+1)x},$$
$$\tilde{g}_{\gamma,\epsilon}(x) = ((1-\epsilon)^a \mathrm{e}^{-2\epsilon x} - 1) M_\gamma(x) \mathrm{e}^{-(\gamma+1)x}.$$

If $\gamma > -1$,

$$\lim_{\epsilon\to 0} \sup_{x\geq 0} |\tilde{f}_{\gamma,\epsilon}(x)| = 0 \quad \textit{and} \quad \lim_{\epsilon\to 0} \sup_{x\geq 0} |\tilde{g}_{\gamma,\epsilon}(x)| = 0.$$

The proof of this lemma uses Lemma 3 and follows the same lines as the proof of Lemma 4 in [9] (or Lem. 4 in [10]).

Lemma 3. *For any $\epsilon > 0$,*
if s tends to $+\infty$, then, if $\gamma \geq 0$, $M_\gamma(s) = o(\mathrm{e}^{(\gamma+\epsilon)s})$ and, if $\gamma < 0$, $M_\gamma(s) \to -\frac{1}{\gamma^3}$;
if s tends to $-\infty$, then, if $\gamma \geq 0$, $M_\gamma(s) \to -\frac{1}{\gamma^3}$ and, if $\gamma < 0$, $M_\gamma(s) = o(\mathrm{e}^{(\gamma-\epsilon)s})$.

If we let

$$p_u(s) = \frac{V(s + V^{-1}(u)) - \alpha(u)}{\sigma(u)} - \frac{\mathrm{e}^{\lambda(V^{-1}(u))s} - 1}{\lambda(V^{-1}(u))}, \tag{15}$$

we derive the following "Potter-type" bounds, consequences of Proposition 1.

Lemma 4. *For all $\gamma \in \mathbb{R}$,*
(i) $\forall\epsilon > 0, \exists u_0, \forall u \geq u_0, \forall s \in \mathbb{R}_+$,

$$(1-\epsilon)\mathrm{e}^{-\epsilon s} M_\gamma(s) \leq \frac{p_u(s)}{\lambda'(V^{-1}(u))} \leq (1+\epsilon)\mathrm{e}^{\epsilon s} M_\gamma(s).$$

(ii) $\lim_{u\to s_+(F)} p_u(s) = 0$.

Remark 6. This lemma establishes that, for u sufficiently large, $p_u(s)$ has the same sign as $\lambda'(V^{-1}(u))$ for all $s > 0$.

q_u (see (11)) and p_u (see (15)) are linked in the following way:

- if $\gamma = 0$, $q_u(s) = p_u(s)$,
- if $\gamma \neq 0$ and $1 + \Lambda(u)\frac{V(s+V^{-1}(u))-\alpha(u)}{\sigma(u)} > 0$ (which is true for u sufficiently large) there exists $\theta = \theta(u, s) \in [0, 1]$ such that

$$q_u(s) = \mathrm{e}^{-\Lambda(u)s} p_u(s)(1 + \theta\Lambda(u)\mathrm{e}^{-\Lambda(u)s} p_u(s))^{-1}. \tag{16}$$

The proof of the following lemma is similar to the proof of Lemma 6 in [9] (or Lem. 8 in [10]), the only difference being that γ is replaced by $\Lambda(u)$, which tends to γ as u tends to $s_+(F)$.

Lemma 5. *Let S be a positive function defined on $\mathbb{R}_+$ and satisfying $\lim_{u\to s_+(F)} S(u) = +\infty$.*

(i) *If $\gamma \geq 0$ and there exists $\epsilon > 0$ such that*

$$\lim_{u\to s_+(F)} \ln|\lambda'(V^{-1}(u))| + 3\epsilon\, S(u) = -\infty, \tag{17}$$

for u sufficiently large, $q_u(s)$ and $p_u(s)$ have the same sign and, as u tends to $s_+(F)$,

$$\sup_{s\in[0,S(u)]} \mathrm{e}^{-\Lambda(u)s}|p_u(s)| \longrightarrow 0 \text{ and } \sup_{s\in[0,S(u)]} |q_u(s)| \longrightarrow 0.$$

(ii) *If $\gamma < 0$ and there exists $\epsilon > 0$ such that*

$$\lim_{u\to s_+(F)} \ln|\lambda'(V^{-1}(u))| + (2\epsilon - \gamma)S(u) = -\infty, \tag{18}$$

for u sufficiently large, $q_u(s)$ and $p_u(s)$ have the same sign and, as u tends to $s_+(F)$,

$$\sup_{s\in[0,S(u)]} \mathrm{e}^{-\Lambda(u)s}|p_u(s)| \longrightarrow 0 \text{ and } \sup_{s\in[0,S(u)]} |q_u(s)| \longrightarrow 0.$$

(iii) *Condition (17) is fulfilled with $S(u) = -\alpha \ln|\lambda'(V^{-1}(u))|$, for $\alpha > 0$ and condition (18) is fulfilled with $S(u) = -\alpha \ln|\lambda'(V^{-1}(u))|$, for $0 < \alpha < -\frac{1}{\gamma}$.*

• Here are the main steps of the proof of (i):
It is easy to check that if we choose (see Lem. 5 (iii))

$$S(u) = -\alpha \ln|\mathrm{e}^{V^{-1}(u)} A'(\mathrm{e}^{V^{-1}(u)})|,$$

with $\alpha > 1$, then

$$\frac{1}{\mathrm{e}^{V^{-1}(u)} A'(\mathrm{e}^{V^{-1}(u)})} \bar{G}_0(S(u)) \longrightarrow 0, \text{ as } u \to s_+(F).$$

Now, relation (16) and Lemma 5 yield bounds for $q_u(s)$ and then we deduce that there exists $u_0 \in [0, s_+(F)[$ such that, for $u \in [u_0, s_+(F)[$ and $s \in R_+$,

$$(1-\epsilon)^2 \frac{\mathrm{e}^{-\gamma s} p_u(s)|\bar{G}_0'(s+\epsilon)|}{\mathrm{e}^{V^{-1}(u)} A'(\mathrm{e}^{V^{-1}(u)})} \leq \frac{B_u(s)}{\mathrm{e}^{V^{-1}(u)} A'(\mathrm{e}^{V^{-1}(u)})} \leq (1+\epsilon) \frac{\mathrm{e}^{-\gamma s} p_u(s)|\bar{G}_0'(s)|}{\mathrm{e}^{V^{-1}(u)} A'(\mathrm{e}^{V^{-1}(u)})} \cdot \tag{19}$$

According to Lemma 4, there exists $u_1 \geq u_0$ such that, for $u \in [u_1, s_+(F)[$ and $s \in \mathbb{R}_+$,

$$(1-\epsilon)\mathrm{e}^{-\epsilon s} M_\gamma(s) \leq \frac{p_u(s)}{\exp(V^{-1}(u)) A'(\exp(V^{-1}(u)))} \leq (1+\epsilon)\mathrm{e}^{\epsilon s} M_\gamma(s). \tag{20}$$

It follows from (19) and (20) that, for $u \in [u_1, s_+(F)[$ and $s \in [0, S(u)]$,

$$[(1-\epsilon)^3 \mathrm{e}^{-\epsilon s} - 1]\mathrm{e}^{-\gamma s} M_\gamma(s)|\bar{G}_0'(s)| \leq \frac{B_u(s)}{\mathrm{e}^{V^{-1}(u)} A(\mathrm{e}^{V^{-1}(u)})} - C_\gamma(s) \leq [(1+\epsilon)^2 \mathrm{e}^{\epsilon s} - 1]\mathrm{e}^{-\gamma s} M_\gamma(s)|\bar{G}_0'(s)|. \tag{21}$$

The bounds in (21) being continuous functions tending to 0 when ϵ tends to 0, the convergence of $\frac{B_u(s)}{e^{V^{-1}(u)}A'(e^{V^{-1}(u)})}$ towards $C_\gamma(s)$ is established on every compact set $[0, T]$ $(T > 0)$.

We extend the convergence to $[0, S(u)]$ using Lemma 2.

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