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GOODNESS OF FIT TEST FOR ISOTONIC REGRESSION

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Abstract. We consider the problem of hypothesis testing within a monotone regression model. We propose a new test of the hypothesis H_0: "$f = f_0$" against the composite alternative H_a: "$f \neq f_0$" under the assumption that the true regression function f is decreasing. The test statistic is based on the $\mathbb{L}_1$-distance between the isotonic estimator of f and the function f_0, since it is known that a properly centered and normalized version of this distance is asymptotically standard normally distributed under H_0. We study the asymptotic power of the test under alternatives that converge to the null hypothesis.

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1. Introduction

We consider the following regression model

$$Y_i = f(x_i) + \varepsilon_i, \qquad 1 \leqslant i \leqslant n,$$

where $Y_1, \ldots, Y_n$ are the observations, f is an unknown regression function with support $[0,1]$, $\varepsilon_1, \ldots, \varepsilon_n$ are independent, identically distributed random variables with zero mean and for every i, $x_i = i/n$. The regression function f is assumed to be monotone, say decreasing. We wish to test the hypothesis H_0: "$f = f_0$" where f_0 is a given decreasing function. For this purpose, we use the $\mathbb{L}_1$-distance between the function f_0 and the isotonic estimator of f defined in Section 2. The test procedure is based on a central limit theorem, proved by Durot [4], for the $\mathbb{L}_1$-distance between the isotonic estimator and the true regression function f: a centered version of this $\mathbb{L}_1$-distance converges at the $n^{-1/2}$ rate to a Gaussian variable with variance independent of f. This result provides a test statistic which is asymptotically standard normally distributed under the hypothesis H_0.

The nonparametric theory of hypothesis testing in regression models is now well developed. Many of the test procedures proposed in that context are based on either a kernel estimator or an estimator obtained by projection (on a polynomial or spline basis for example), see Eubank and Spiegelman [6], Härdle and Mammen [10] and Staniswalis and Severini [17]. The main drawback of these methods is that they require the choice of a smoothing parameter. Several authors have proposed approaches that avoid this arbitrary choice, see Barry and Hartigan [2], Eubank and Hart [5], Hart and Wehrly [11], Stute [18]. Most of them consider a

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test statistic which is itself a smoothing parameter selected from the data by minimizing some empirical risk criterion.

The test procedures mentioned above apply in particular for a regression model where the regression function f is known to be decreasing. However, it is more relevant in that case to use a procedure that takes into account the monotonicity assumption. That is the reason why we propose a new test that involves the isotonic estimator, the construction of which is based on the monotonicity assumption. The isotonic estimator is entirely data driven and does not require any arbitrary choice of parameter. Our test procedure can thus be easily implemented. Moreover the isotonic estimator is known to be locally adaptive, in the sense that it works as well as the best regressogram estimator (with arbitrary partition). (A precise meaning of this property in terms of $\mathbb{L}_1$-risk is to be found in Reboul [12], Prop. 2.2.) One can thus expect that our test has a good ability for detecting local perturbations of the null hypothesis. We thus focus here on the study of the asymptotic power of the test under alternatives H_n: "$f = f_0 + c_n\Lambda_n$" where $\{c_n\}$ is a sequence of numbers that tends to zero as n goes to infinity and where the function Λ_n may depend on n and if so is defined as a local bump. More precisely, we are interested in the minimal rate of convergence for c_n so that the test has a prescribed asymptotic power. We prove that the minimal rate is $n^{-5/12}$ if Λ_n does not depend on n and $n^{-3/8}$ if Λ_n is a local bump.

This article is organized as follows. We describe the test procedure in Section 2 and state the result concerning the asymptotic power of the test in Section 3. Section 4 is devoted to a simulation study comparing the power of the test developed in Section 2 with that of the likelihood ratio test. Proofs of theoretical results are given in Section 5.

2. Model and test procedure

We consider the following regression model

$$Y_i = f(x_i) + \varepsilon_i, \qquad 1 \leqslant i \leqslant n. \tag{2.1}$$

The regression function f is decreasing over $[0,1]$ and for every $i \in \{1,...,n\}$, $x_i = i/n$. The errors $\varepsilon_1, \ldots, \varepsilon_n$ are independent, identically distributed random variables with zero mean and $\mathbb{E}|\varepsilon_i|^p < \infty$ for some $p > 12$.

Our objective is to test the hypothesis H_0: "$f = f_0$" where f_0 is a given decreasing function defined over $[0,1]$. The test procedure is based on the isotonic estimator $\hat{f}_n$ of f, defined as the left-continuous slope of the least concave majorant of $\widehat{F}_n$, where

$$\forall t \in [0,1], \qquad \widehat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} Y_i \mathbb{1}_{x_i \leqslant t}. \tag{2.2}$$

More specifically, our test statistic is a properly centered and normalized version of the $\mathbb{L}_1$-distance between $\hat{f}_n$ and f_0. The asymptotic distribution of the $\mathbb{L}_1$-distance between $\hat{f}_n$ and the true regression function depends on a process defined as the location of the maximum of a drifted Brownian motion. This location process is called *Groeneboom's process* and is defined as follows:

Definition 2.1. *Let W be a standard two-sided Brownian motion originating from zero. Then, the Groeneboom process V associated with W is defined by:*

$$\forall a \in \mathbb{R}, V(a) = \sup\left\{u \in \mathbb{R}, W(u) - (u-a)^2 \text{ is maximal}\right\} \cdot$$

If the errors' variance σ^2 is known, we propose to use the test statistic

$$S_n = \frac{n^{1/6}}{\sigma\sqrt{8k}}\left\{n^{1/3}\int_0^1 |\hat{f}_n(t) - f_0(t)|\mathrm{d}t - \mathcal{C}_{f_0,\sigma}\right\} \tag{2.3}$$

where k and $\mathcal{C}_{f_0,\sigma}$ are defined from Groeneboom's process V as follows:

$$k = \int_0^\infty \operatorname{cov}(|V(0)|, |V(b) - b|)\mathrm{d}b \quad \text{and} \quad \mathcal{C}_{f_0,\sigma} = 2\mathbb{E}|V(0)| \int_0^1 \left|\sigma^2 f_0'(t)/2\right|^{1/3} \mathrm{d}t.$$

From Theorem 2 of Durot [4], S_n is asymptotically standard normally distributed under the null hypothesis H_0, provided f_0 is decreasing over $[0,1]$ and twice differentiable with non vanishing first derivative and bounded second derivative (that is f_0 satisfies regularity condition $\mathcal{R}_0$ defined below):

$$\text{Under } H_0, \quad S_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n \to \infty.$$

This theorem suggests the following testing procedure:

Definition 2.2. *Assume we are given the regression model (2.1) where $x_i = i/n$ and the ε_i's are i.i.d. with mean zero and $\mathbb{E}|\varepsilon_1|^p < \infty$ for some $p > 12$. Suppose $\hat{f}_n$ is the isotonic estimator of f, f_0 satisfies $\mathcal{R}_0$ and S_n is defined by (2.3). The isotonic test for goodness of fit with asymptotic level α rejects the null hypothesis H_0: "$f = f_0$" if $|S_n| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.*

We present the test with σ^2 known for the sake of simplicity. If σ^2 is unknown and bounded away from 0, then one can use our results for testing $f = f_0$ with a plug-in estimator $\hat{\sigma}_n^2$ for σ^2, provided $n^{1/6}(\hat{\sigma}_n^2 - \sigma^2) = o_{\mathbb{P}}(1)$. One can find in the literature many estimators of σ^2 that satisfy this last property. For example, one can consider the following estimator defined by Rice [14]:

$$\hat{\sigma}_n^2 = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (Y_{i+1} - Y_i)^2. \tag{2.4}$$

Its bias and its variance are in our setting respectively of order $O(n^{-2})$ and $O(n^{-1})$. One can also use generalizations of this estimator as defined by Hall *et al.* [9].

The isotonic test for goodness of fit can easily be implemented since the constants $2\mathbb{E}|V(0)|$ and $8k$ are known to be approximately equal to 0.82 and 0.17 respectively (see Groeneboom [7]). Moreover, the isotonic estimator is entirely data driven and easily computable *via* the "Pool-Adjacent-Violators" algorithm or a similar device (see Barlow *et al.* [1]).

Note that the isotonic estimator $\hat{f}_n$ is comparable to the estimator defined by Brunk [3], which is the nonparametric least-squares estimator obtained under the constraint $f(x_1) \geqslant ... \geqslant f(x_n)$. Brunk's estimator is indeed the left-continuous slope of the least concave majorant of the so-called cumulative sum diagram. This diagram is composed of the points of the Cartesian plane $P_0 = (0,0)$ and for $i = 1, \dots, n$, $P_i = (x_i, \widehat{F}_n(x_i))$ where $\widehat{F}_n$ is given by (2.2). If $\widehat{F}_n$ is non decreasing, then the isotonic estimator is exactly equal to Brunk's estimator. In Durot's paper [4], the distinction between Brunk's estimator and the isotonic estimator, and thus which of these two estimators is used, does not appear clearly. Considering the proof, one can however easily check that Durot's Theorem 2 holds for both estimators.

3. Asymptotic power

The aim of this section is to study the asymptotic power of the isotonic test for goodness of fit, under the alternative H_n:"$f = f_n$", where

$$f_n = f_0 + c_n \Lambda_n \tag{3.1}$$

$\{c_n\}$ is a sequence of positive numbers that converges to zero as n goes to infinity and $\{\Lambda_n\}$ is a sequence of functions with $||\Lambda_n||_2 = 1$. We consider functions Λ_n of the form $\Lambda_n(.) = \delta_n^{-1/2} \phi\left(\frac{. - x_0}{\delta_n}\right)$ where ϕ is defined from $\mathbb{R}$ to $\mathbb{R}$ with support $[0,1]$, $\|\phi\|_2 = 1$, x_0 lies in $[0,1[$ and δ_n is positive. The support of Λ_n is then $[x_0, x_0+\delta_n]$. The sequence $\{\delta_n\}$ is taken such that either Λ_n does not depend on n (we simply take $\delta_n = 1$ and $x_0 = 0$) or Λ_n is a local bump which simply means that δ_n tends to zero as n goes to infinity.

We are interested in evaluating the minimal rate of convergence for c_n (that is the smallest c_n up to some constant) for which the test has a prescribed asymptotic power. The choice of the $\mathbb{L}_2$-distance c_n between f_n and f_0 to measure the gap between H_n and H_0 is motivated by the relationship between this distance and the Hellinger distance in the case of Gaussian errors and by the role of the Hellinger distance for hypothesis testing *via* the testing affinity.

We need some regularity assumptions, namely

- $\mathcal{R}_0$: f_0 is decreasing on $[0,1]$ and twice differentiable with non vanishing first derivative and bounded second derivative;
- $\mathcal{R}_n$: f_n is decreasing on $[0,1]$ and twice differentiable with bounded second derivative. Moreover, there exists some positive k_1 that does not depend on n such that $\inf_{t\in[0,1]} |f_n'(t)| > k_1$.

It is worth noticing that the monotonicity assumption on f_n requires $c_n\delta_n^{-3/2}$ to be bounded whenever $x_0 \neq 0$, and that the assumption $\mathcal{R}_n$ is fulfilled whenever for example ϕ is twice differentiable with a bounded second derivative and $c_n\delta_n^{-3/2}$ is small enough. We state the following result concerning the asymptotic power of the isotonic test for goodness of fit defined in Definition 2.2.

Theorem 3.1. *Assume we are given the regression model (2.1). Let f_0 be some function that satisfies $\mathcal{R}_0$ and let f_n be defined by (3.1). Let H_0 and H_n be the hypothesis defined respectively by H_0: "$f = f_0$" and H_n: "$f = f_n$". Assume f_n to satisfy $\mathcal{R}_n$ and $c_n\delta_n^{-3/2}$ to be bounded. Then for every $\alpha \in (0,1)$ and $\beta \in (\alpha, 1)$, there exists some positive γ such that the isotonic test for goodness of fit with asymptotic level α has an asymptotic power greater than β (that is $\liminf_{n\to\infty} \mathbb{P}_{f_n}\left(|S_n| > z_{\alpha/2}\right) \geqslant \beta$) whenever*

$$c_n \geqslant \gamma\left(n^{-5/12} \vee n^{-1/2}\delta_n^{-1/2}\right)$$

for all large enough n.

Assume $c_n\delta_n^{-3/2}$ to be bounded. Assume moreover $c_n \geqslant \gamma' n^{-3/8}$ for some positive γ' and all large enough n. Then, $c_n \geqslant \gamma'^{4/3} n^{-1/2} c_n^{-1/3}$. For all $\gamma > 0$, there thus exists some $\gamma' > 0$ such that $c_n \geqslant \gamma' n^{-3/8}$ implies $c_n \geqslant \gamma n^{-1/2}\delta_n^{-1/2}$. The following corollary is thus a straightforward consequence of Theorem 3.1:

Corollary 3.1. *Assume we are given the regression model (2.1). Let f_0 be some function that satisfies $\mathcal{R}_0$ and let f_n be defined by (3.1). Let H_0 and H_n be the hypothesis defined respectively by H_0: "$f = f_0$" and H_n: "$f = f_n$". Assume f_n to satisfy $\mathcal{R}_n$ and $c_n\delta_n^{-3/2}$ to be bounded. Then for every $\alpha \in (0,1)$ and $\beta \in (\alpha, 1)$, there exists some positive γ such that*

$$\liminf_{n\to\infty} \mathbb{P}_{f_n}\left(|S_n| > z_{\alpha/2}\right) \geqslant \beta$$

whenever $c_n \geqslant \gamma n^{-3/8}$ for all large enough n.

The meaning of Theorem 3.1 and Corollary 3.1 is that the minimal rate of convergence for the distance c_n so that the test has a prescribed asymptotic power depends on δ_n but is anyway smaller than or equal to $n^{-3/8}$.

For the sake of simplicity, we assume $c_n\delta_n^{-3/2}$ to be bounded. If the perturbation is added to f_0 at the point zero (that is if $x_0 = 0$), then the assumption $\mathcal{R}_n$ is fulfilled whenever for example ϕ is decreasing, non negative and twice differentiable with a bounded second derivative. In this case, $c_n\delta_n^{-3/2}$ does not have to be bounded. If $x_0 = 0$ and if $\mathcal{R}_n$ is fulfilled with $c_n\delta_n^{-3/2}$ not necessarily bounded, one can calculate the minimal rate of convergence c_n so that the test has a prescribed asymptotic power. This minimal rate is given in Tocquet [19] for Gaussian errors and bounded $||f_n - f_0||_\infty$. It still depends on δ_n, is smaller than or equal to $n^{-5/12} \vee n^{-1/2}\delta_n^{-1/2}$,

TABLE 1. Empirical levels (in %) of the tests using the 5% bilateral Gaussian critical value 1.96.

		σ			
n	Test	0.5	0.7	1	1.5
200	$\hat{S}_n$	9.73	9.43	9.40	9.77
	$\hat{S}_n^B$	10.73	10.57	10.33	10.73
	$\hat{T}_n$	7.20	6.70	6.13	5.33
	$\hat{T}_n^B$	7.33	6.83	5.67	5.23
1000	$\hat{S}_n$	7.57	7.67	8.23	8.60
	$\hat{S}_n^B$	8.00	7.90	8.50	8.80
	$\hat{T}_n$	5.70	5.40	4.90	4.80
	$\hat{T}_n^B$	5.97	5.67	5.10	4.70

and equals $n^{-1/2}$ whenever $n^{1/2}\delta_n$ and $(n\delta_n)^{-1}$ are bounded (the width of the perturbation has to be larger than n^{-1} in order that the perturbation can be detected). From the relationship between the $\mathbb{L}_2$-distance, the Hellinger distance and the testing affinity within the Gaussian regression experiment, the rate $n^{-1/2}$ is known to be the optimal rate. The arguments of the proof are closely related to those developed in the proof of Theorem 3.1, but additional technical difficulties, because $c_n\delta_n^{-3/2}$ can go to infinity with n, make the proof cumbersome.

4. Simulation study

In this section, we study the behavior of the isotonic test for goodness of fit in a simulation experiment, in the case where the errors are normally distributed. We first study the level of the test, comparing the asymptotic level (which is fixed *a priori*) to the level obtained for finite n. We then study the power of our test, comparing it with the likelihood ratio test's. Part of the results, which reflects the observed behavior over the entire experiment, are presented in Tables 1 and 2. The sample size n is set at 200 and 1000 and the asymptotic level α is set at 0.05.

Let us study first the level. For completeness, we study the test procedure described in Section 2 and also the test procedure that involves Brunk's estimator instead of our isotonic estimator (see the end of Sect. 2). Moreover, it emerges from simulation studies that the actual levels of these two test procedures significantly differ from the asymptotic level. We thus propose two other test procedures, the levels of which are closer to the fixed asymptotic level.

We fix $n \in \{200, 1000\}$ and draw a sample $(\varepsilon_1, \ldots, \varepsilon_n)$ from the Gaussian distribution $\mathcal{N}(0,1)$. We fix $\sigma \in \{0.5, 0.7, 1, 1.5\}$ and for every i, we generate

$$Y_i = f_0(x_i) + \sigma\varepsilon_i,$$

where $x_i = i/n$ and $f_0(x) = 5 - 10x$. We then build from $Y_1, \ldots, Y_n$ the test statistics $\hat{S}_n$, $\hat{T}_n$, $\hat{S}_n^B$ and $\hat{T}_n^B$ as follows. $\hat{S}_n$ is the test statistic studied in Section 2:

$$\hat{S}_n = \frac{n^{1/6}}{\hat{\sigma}_n\sqrt{0.17}}\left(n^{1/3}\int_0^1 |\hat{f}_n(t) - f_0(t)|\mathrm{d}t - 0.82\int_0^1 |\hat{\sigma}_n^2 f_0'(t)/2|^{1/3}\mathrm{d}t\right),$$

TABLE 2. Percentage of rejections of H_0 in 3000 samples using simulated 5% critical values, when the regression function f is given by (4.1) and the test is based on $\hat{T}_n$ (straight) or LR (italic).

	$\sigma = 0.5,\ n = 200$			$\sigma = 1,\ n = 200$		
(δ, x_0)	$c' = 10$	$c' = 20$	$c' = 30$	$c' = 10$	$c' = 20$	$c' = 30$
$(1, 0)$	23.90	87.90	99.97	8.97	29.57	67.57
	18.40	*75.23*	*99.43*	*8.50*	*23.47*	*54.33*
$(0.5, 0.25)$	22.50	83.90	99.93	8.80	26.20	61.50
	18.43	*74.23*	*99.50*	*8.50*	*22.97*	*52.90*
$(0.3, 0.35)$	21.03	77.17	99.40	8.23	22.20	50.77
	18.07	*73.53*	*99.30*	*8.27*	*21.87*	*50.07*
	$\sigma = 0.5,\ n = 1000$			$\sigma = 1,\ n = 1000$		
(δ, x_0)	$c' = 5$	$c' = 10$	$c' = 20$	$c' = 5$	$c' = 10$	$c' = 20$
$(1, 0)$	22.30	90.40	100	9.00	30.47	95.53
	19.93	*81.50*	*100*	*8.80*	*26.47*	*90.70*
$(0.5, 0.25)$	22.03	88.10	100	8.37	29.30	95.57
	19.80	*81.47*	*100*	*8.77*	*25.93*	*91.13*
$(0.3, 0.35)$	21.33	83.63	100	8.23	26.33	90.07
	19.30	*81.53*	*100*	*8.77*	*26.13*	*91.37*
$(0.1, 0.45)$	15.57			7.07		
	18.47			*8.00*		

where $\hat{\sigma}_n^2$ is Rice's estimator of σ^2 given by (2.4); $\hat{T}_n$ is given by

$$\hat{T}_n = \frac{n^{1/6}}{\hat{\sigma}_n \sqrt{0.17}} \left(n^{1/3} \int_{f_0(1)}^{f_0(0)} |V_n(a) - g_0(a)| \mathrm{d}a - 0.82 \int_0^1 |\hat{\sigma}_n^2 f_0'(t)/2|^{1/3} \mathrm{d}t \right),$$

where V_n is the generalized inverse function of $\hat{f}_n$ and g_0 is the inverse function of f_0; $\hat{S}_n^B$ and $\hat{T}_n^B$ are defined in the same way as $\hat{S}_n$ and $\hat{T}_n$ respectively, but where the isotonic estimator $\hat{f}_n$ is replaced by Brunk's estimator $\hat{f}_n^B$. One can prove by using the same arguments as Durot that under the null hypothesis, all of these test statistics converge to a Gaussian distribution $\mathcal{N}(0,1)$ as n goes to infinity, provided f_0 satisfies the regularity condition $\mathcal{R}_0$ (see Sect. 2). Therefore, for every $\mathcal{S}_n \in \{\hat{S}_n, \hat{T}_n, \hat{S}_n^B, \hat{T}_n^B\}$, the test procedure that rejects H_0: "$f = f_0$" if $|\mathcal{S}_n| > z_{\alpha/2}$ (where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution) is of asymptotic level α. The empirical levels (percentage of rejection of H_0: "$f = f_0$" using the 5% bilateral Gaussian critical value 1.96) computed from 3000 samples are reported in Table 1 for the four tests.

It is seen in Table 1 that the level of the test procedure is only slightly affected by the choice of the estimator of f (isotonic or Brunk's estimator). The normal approximation for the distributions of $\hat{S}_n$ and $\hat{S}_n^B$ is misleading: the actual levels are significantly greater than 5% even for $n = 1000$. That is the reason why we considered the tests based on $\hat{T}_n$ and $\hat{T}_n^B$. The replacement of

$$\int_0^1 |\hat{f}_n(t) - f_0(t)|\mathrm{d}t = \int_{\mathbb{R}} |V_n(a) - g_0(a)|\mathrm{d}a$$

by $\int_{f_0(1)}^{f_0(0)} |V_n(a) - g_0(a)|\mathrm{d}a$ (and the analogue with Brunk's estimator) indeed allows to correct this failing; the empirical levels for $\hat{T}_n$ and $\hat{T}_n^B$ are rather close to 5%.

The most natural test to compare our test's power with in the case of Gaussian errors is the likelihood ratio test. Since Brunk's estimator $\hat{f}_n^B$ is the maximum likelihood estimator under order restriction $f(x_1) \geqslant \ldots \geqslant f(x_n)$, the likelihood ratio test rejects the null hypothesis H_0: "$f = f_0$" if

$$LR = \frac{\sum_{i=1}^{n} (Y_i - f_0(x_i))^2}{\sum_{i=1}^{n} (Y_i - \hat{f}_n^B(x_i))^2}$$

exceeds a critical value. The aim of the simulation study reported here is thus to compare with each others the powers of the test procedures based on LR, $\hat{S}_n$, $\hat{T}_n$, $\hat{S}_n^B$ and $\hat{T}_n^B$. One can prove (by using the same arguments as in the proof of Th. 3.1) that the result of Theorem 3.1 concerning the asymptotic power of the test procedure based on $\hat{S}_n$ still holds for the test procedures based on $\hat{T}_n$, $\hat{S}_n^B$ or $\hat{T}_n^B$. But no distribution theory is available for LR, so we use 5% critical values obtained by simulations for the five test procedures considered here. These critical values have been obtained from those 3000 samples used for the computation of the empirical levels of Table 1.

The regression function considered under the alternative is

$$f(x) = f_0(x) + c'\delta^{-1/2} \left(0.5^2 - \left(\frac{x - x_0}{\delta} - 0.5\right)^2\right)^3 \mathbb{1}_{x_0 \leqslant x \leqslant x_0 + \delta}. \tag{4.1}$$

The monotonicity constraint here is $c' < \delta^{3/2} 250\sqrt{5}/3$. In the results we report here, four values for (δ, x_0): (1, 0), (0.5, 0.25), (0.3, 0.35), (0.1, 0.45) and two values for σ: 0.5, 1 have been selected. The choice of c' depends on n: c' is taken to be 10, 20, 30 for $n = 200$ and 5, 10, 20 for $n = 1000$.

We fix n, c' and (δ, x_0) and draw a sample $(\varepsilon_1, \ldots, \varepsilon_n)$ from the Gaussian distribution $\mathcal{N}(0, 1)$. We fix σ and for every $i \in \{1, \ldots, n\}$ we generate

$$Y_i = f(x_i) + \sigma\varepsilon_i.$$

We compute then the five test statistics and the percentages of rejection of H_0 (using simulated critical values) over 3000 samples. Considering the previous results about the behavior of $\hat{S}_n$ and $\hat{S}_n^B$ under H_0, it seemed not relevant to present results on the power of the tests based on these statistics. Moreover, the test based on $\hat{T}_n^B$ is in most cases less powerful than the test based on $\hat{T}_n$. For the sake of simplicity, we thus only present in Table 2 the results concerning the power of both tests based on $\hat{T}_n$ and LR.

The test based on $\hat{T}_n$ is better than the likelihood ratio test in general. The likelihood ratio test is marginally better in the case where $\delta = 0.1$. The parameter x_0 has been chosen here in order that the bump is centered. The results obtained over the entire experiment show that when $x_0 = 0$, the test based on $\hat{T}_n$ is slightly less powerful comparing to the case $x_0 \neq 0$. However the general behavior observed here carries on: test based on $\hat{T}_n$ is better than the likelihood ratio test except in the case of very thin bumps (δ and c' small). These results show also that the test based on $\hat{S}_n$ is in most cases the most powerful among the five tests.

5. Derivation of results

If h_n denotes the isotonic estimator of some regression function h and C is a positive number, then the isotonic estimator of Ch is Ch_n. It thus suffices to prove Theorem 3.1 in the case where $\sigma = 1$, so we assume in the sequel $\sigma = 1$.

To study the asymptotic behavior of S_n under H_n, we consider both inverse functions of $\hat{f}_n$ and f_0. This key idea comes from Groeneboom [7]. We recall that the (generalized) inverse h^{-1} of a non increasing, left-continuous function h defined on $[0,1]$ is given by

$$\forall a \in \mathbb{R} \quad h^{-1}(a) = \inf\{t \in [0,1], h(t) < a\}$$

with the convention that the infimum of an empty set is 1. Let $\widehat{F}_n$ be the empirical process defined by (2.2) and let define the "argmax" of a process $\{X(u),\ u \in I\}$, $I \subset \mathbb{R}$ by

$$\underset{u \in I}{\operatorname{argmax}}\{X(u)\} = \sup\{u \in I, X(u) \text{ is maximal}\}$$

with the convention that the supremum of an empty set is the infimum of I. The inverse function V_n of $\hat{f}_n$ is given by: $\forall a \in \mathbb{R}$, $V_n(a) = \operatorname{argmax}_{u\in[0,1]}\{\widehat{F}_n(u) - au\}$. Therefore,

$$\forall a \in \mathbb{R}, \quad V_n(a) = \underset{u\in[0,1]}{\operatorname{argmax}}\left\{\frac{1}{n}\sum_{i=1}^n \varepsilon_i \mathbb{1}_{x_i \leqslant u} + \frac{1}{n}\sum_{i=1}^n f_n(x_i)\mathbb{1}_{x_i\leqslant u} - au\right\} \tag{5.1}$$

under the hypothesis H_n. Because V_n is more tractable than the isotonic estimator $\hat{f}_n$, we use the following identity, where g_0 stands for the inverse function of f_0:

$$\int_0^1 |\hat{f}_n(t) - f_0(t)|\mathrm{d}t = \int_{\mathbb{R}} |V_n(a) - g_0(a)|\mathrm{d}a. \tag{5.2}$$

Moreover, it is proved in Lemma 5.2 that V_n can be approached by a process U_n defined by

$$\forall a \in \mathbb{R}, \quad U_n(a) = \underset{u\in[0,1]}{\operatorname{argmax}}\left\{W(u) + \sqrt{n}\left(\int_0^u f_n(s)\ \mathrm{d}s - au\right)\right\}, \tag{5.3}$$

where W is a standard two-sided Brownian motion. So we deal with U_n instead of $\hat{f}_n$ in the proof of Theorem 3.1. The proof is organized as follows. Probabilistic tools are given in Section 5.1. The proofs of these probabilistic tools are postponed to Section 5.4. In Section 5.2, Theorem 3.1 is proved in the case where $n^{1/3}c_n\delta_n^{1/2} > \eta$ for all n and some large enough $\eta > 0$. Theorem 3.1 is finally proved in the case where $n^{1/3}c_n\delta_n^{1/2}$ is bounded in Section 5.3. Throughout the proof, we will use the following notation.

Notation 1.

- $[m, M]$ (resp. $[m_n, M_n]$) stands for the range of f_0 (resp. f_n);
- g_0 (resp. g_n, V_n) stands for the inverse function of f_0 (resp. f_n, $\hat{f}_n$);
- $I = n^{-1/3}\log n$;

– the functions d_n and L_n are defined by: for all real number a,

$$d_n(a) = \left|\frac{f_n'(g_n(a))}{2}\right|^{2/3} \quad \text{and} \quad L_n(a) = \sup_{t\in[g_n(a)-I,g_n(a)+I]\cap[0,1]} |f_n''(t)|\,; \tag{5.4}$$

– $\mathbb{P}$ (resp. $\mathbb{E}$, resp. var) stands for the probability (resp. the expectation, resp. the variance) under the hypothesis H_n.

5.1. Some probabilistic tools

Probability inequalities are provided in the following lemma:

Lemma 5.1. *Let g_n be the inverse function of f_n, where f_n satisfies the regularity condition $\mathcal{R}_n$. Let V_n be the process defined by (5.1), where $x_i = i/n$, the ε_i's are independent and identically distributed random variables such that $\mathbb{E}|\varepsilon_i|^p$ is finite for some $p \geqslant 2$. If $\sup_{t,n}|f_n(t)|$ is finite then there exists some $c_p > 0$ such that for all $t > 0$, $a \in \mathbb{R}$, $n \in \mathbb{N}^*$*

$$\mathbb{P}\left(|V_n(a) - g_n(a)| > t\right) \leqslant c_p t^{-3p/2} n^{-p/2}.$$

Let U_n be the process defined by (5.3) where W is a standard two-sided Brownian motion. There exists some $C > 0$ that only depends on k_1 such that for all $t > 0$, $a \in \mathbb{R}$, $n \in \mathbb{N}^*$

$$\mathbb{P}\left(|U_n(a) - g_n(a)| > t\right) \leqslant 2\exp\left(-nCt^3\right).$$

Since for all positive random variable X, $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$, this lemma ensures on the one hand that for all $q \in (0, 3p/2)$ there exists some positive constant c_q such that

$$\sup_{n\in\mathbb{N}}\sup_{a\in\mathbb{R}}\mathbb{E}\left(n^{1/3}|V_n(a) - g_n(a)|\right)^q \leqslant c_q \tag{5.5}$$

and on the other hand that for all $q \geqslant 0$, there exists some positive constant c_q' such that

$$\sup_{n\in\mathbb{N}}\sup_{a\in\mathbb{R}}\mathbb{E}\left(n^{1/3}|U_n(a) - g_n(a)|\right)^q \leqslant c_q'. \tag{5.6}$$

It is stated in Lemma 5.2 below that the process V_n can be approached by the process U_n. Moreover, it is stated there that the integration range of $\int_{\mathbb{R}}|V_n(a) - g_0(a)|\mathrm{d}a$ can be restricted to a well chosen bounded interval providing an error of order $O_{\mathbb{P}}(n^{-1/2})$. This lemma is the starting point of the proof of Theorem 3.1.

Lemma 5.2. *Let g_0 denote the inverse function of f_0, where f_0 is decreasing on $[0,1]$ and let f_n be defined by (3.1). Suppose that f_n satisfies $\mathcal{R}_n$ and that $\sup_{n,t}|f_n(t)|$ is finite. Let $[m, M]$, $[m_n, M_n]$ denote the range of f_0 and f_n respectively.*

Let V_n be the process defined by (5.1), where the ε_i's are independent and identically distributed random variables with mean zero and variance one, $x_i = i/n$. Then

$$\mathbb{E}\int_{\mathbb{R}} |V_n(a) - g_0(a)|\ \mathrm{d}a = \mathbb{E}\int_{m_n\wedge m}^{M_n\vee M} |V_n(a) - g_0(a)|\ \mathrm{d}a + O(n^{-2/3}). \tag{5.7}$$

Assume that $\mathbb{E}|\varepsilon_i|^p$ is finite for some $p > 12$ and the ε_i's are defined on some rich probability space. Assume *moreover $c_n\delta_n^{-3/2}$ and $n^{1/6}c_n\delta_n^{-1/2}$ to be bounded. Then there exists some standard Brownian motion W such that*

$$\mathbb{E}\int_{\mathbb{R}} |V_n(a) - g_0(a)|\, \mathrm{d}a = \mathbb{E}\int_{m_n\wedge m}^{M_n\vee M} |U_n(a) - g_0(a)|\, \mathrm{d}a + O(n^{-1/2}),$$

where $U_n(a)$ is given by (5.3).

The following lemma is a technical lemma that will be useful to study the asymptotic expectation of the test statistic S_n. It allows to approach the random variable $n^{1/3}(U_n(a) - g_n(a))$ by a normalized Groeneboom process at time zero.

Lemma 5.3. *Let f_0 be a decreasing function and let f_n be defined by (3.1). Let U_n be the process defined by (5.3) where W is a standard two-sided Brownian motion. Suppose $c_n\delta_n^{-3/2}$ to be bounded and f_n to satisfy condition $\mathcal{R}_n$. Let g_n be the inverse function of f_n, L_n and d_n be defined by (5.4) and let a be a real number such that for all n, $[-\log n, \log n] \subset [-n^{1/3}g_n(a), n^{1/3}(1 - g_n(a))]$ and $4n^{-1/4}(\log n)^{7/2}L_n(a) \leqslant 1$. Then, there exist some positive constants D_1 and D_2, some integer n_0 that all do not depend on a and some Groeneboom process $V_{a,n}$ such that*

$$n^{1/6}\mathbb{E}\left|n^{1/3}(U_n(a) - g_n(a)) - d_n(a)^{-1}V_{a,n}(0)\right| \leq D_1 n^{-1/12}(\log n)^{9/2}L_n(a) + \frac{D_2}{\log n}$$

whenever $n \geq n_0$.

5.2. Proof of Theorem 3.1 in the case where $n^{1/3}c_n\delta_n^{1/2} > \eta$

We use Notation 1.

Let S_n be the random variable defined by (2.3) where $\sigma = 1$ and let η be some large enough positive number. Fix $\alpha \in (0, 1)$ and $\beta \in (\alpha, 1)$. Suppose that $n^{1/3}c_n\delta_n^{1/2} > \eta$ for all large enough n. Since $\|f_n - f_0\|_1 = c_n\delta_n^{1/2}\|\phi\|_1$, we get for large enough η and n,

$$\begin{aligned}\mathbb{P}(|S_n| \leqslant z_{\alpha/2}) &\leqslant \mathbb{P}\left(n^{1/6}\left\{n^{1/3}c_n\delta_n^{1/2}\|\phi\|_1 - n^{1/3}\|\hat{f}_n - f_n\|_1 - \mathcal{C}_{f_0,1}\right\} \leqslant z_{\alpha/2}\sqrt{8k}\right)\\ &\leqslant \mathbb{P}\left(n^{1/3}\|\hat{f}_n - f_n\|_1 \geqslant \frac{\eta}{2}\|\phi\|_1\right).\end{aligned}$$

One can easily check that $\|\hat{f}_n - f_n\|_1 = \|V_n - g_n\|_1$. Moreover $g_n(a) = g_0(a)$ for all $a \notin [m \wedge m_n, M \vee M_n]$. Therefore (5.5) and (5.7) prove that $n^{1/3}\mathbb{E}\|\hat{f}_n - f_n\|_1$ is bounded. Markov's inequality then yields

$$\limsup_{n\to\infty} \mathbb{P}(|S_n| \leqslant z_{\alpha/2}) \leqslant 1 - \beta,$$

whenever η is large enough.

5.3. Proof of Theorem 3.1 in the case where $n^{1/3}c_n\delta_n^{1/2}$ is bounded

We use Notation 1. We assume without loss of generality that the ε_i's are defined on some rich enough probability space so that Hungarian constructions hold (see Lem. 5.2). We assume moreover $c_n\delta_n^{-3/2}$ and $n^{1/3}c_n\delta_n^{1/2}$ to be bounded.

Let S_n be the random variable defined by (2.3) where $\sigma = 1$. Let decompose S_n into the sum of a bias term B_n and a centered random term S'_n. More precisely, $n^{1/6}c_n\delta_n^{-1/2}$ is bounded so we define B_n and S'_n by

$$B_n = n^{1/6}\left\{\mathbb{E}\left(n^{1/3}\int_{m\wedge m_n}^{M\vee M_n}|U_n(a)-g_0(a)|\mathrm{d}a\right)-\mathcal{C}_{f_0,1}\right\}, \tag{5.8}$$

$$S'_n = n^{1/6}\left\{n^{1/3}\int_{m\wedge m_n}^{M\vee M_n}|U_n(a)-g_0(a)|\mathrm{d}a-\mathbb{E}\left(n^{1/3}\int_{m\wedge m_n}^{M\vee M_n}|U_n(a)-g_0(a)|\mathrm{d}a\right)\right\} \tag{5.9}$$

where U_n satisfies the second assertion in Lemma 5.2. By identity (5.2) and the second assertion in Lemma 5.2 we then have $\sqrt{8k}S_n = B_n + S'_n + O_{\mathbb{P}}(1)$. The main issue to prove Theorem 3.1 is thus to prove the following proposition:

Proposition 5.1. *Let f_0 be some function satisfying $\mathcal{R}_0$ and let f_n be defined by (3.1). Assume we are given the regression model (2.1) under the hypothesis H_n: "$f = f_n$". Let B_n be defined by (5.8) and S'_n be defined by (5.9). Assume f_n to satisfy $\mathcal{R}_n$ and suppose $n^{1/3}c_n\delta_n^{1/2}$ and $c_n\delta_n^{-3/2}$ to be bounded. We have the following results:*

1. *suppose $n^{1/3}c_n\delta_n^{-1/2}$ to be bounded. Then there exists some positive B such that for all n, $B_n \geqslant O(1) + Bn^{5/6}c_n^2$. Moreover,* $\mathrm{var}(S'_n) = O(1)$;
2. *suppose δ_n to converge to zero as n goes to infinity and suppose $n^{1/3}c_n\delta_n^{-1/2} > 1$ for all n. Then there exists some positive B such that for all n, $B_n \geqslant O(1)+Bn^{1/2}c_n\delta_n^{1/2}$. Moreover,* $\mathrm{var}(S'_n) = O(1+n^{2/3}c_n^2)$.

Theorem 3.1 follows from Proposition 5.1 in the case where $n^{1/3}c_n\delta_n^{1/2}$ is bounded. Assume indeed $c_n\delta_n^{-3/2}$ and $n^{1/3}c_n\delta_n^{1/2}$ to be bounded. Assume moreover $c_n \geqslant \gamma(n^{-5/12}\vee n^{-1/2}\delta_n^{-1/2})$ for some large enough γ. Fix $\alpha \in (0,1)$ and $\beta \in (\alpha, 1)$. We have $\sqrt{8k}S_n = B_n + S'_n + O_{\mathbb{P}}(1)$, so

$$\mathbb{P}(|S_n| \leqslant z_{\alpha/2}) \leqslant \mathbb{P}\left(|S'_n| \geqslant B_n + O_{\mathbb{P}}(1) - z_{\alpha/2}\sqrt{8k}\right).$$

By Proposition 5.1, for all positive C we have $\liminf_{n\to\infty} B_n > C$ provided γ is large enough. Fix $\varepsilon > 0$. It thus follows from Markov's inequality that there exists some γ_0 such that

$$\limsup_{n\to\infty}\mathbb{P}(|S_n| \leqslant z_{\alpha/2}) \leqslant \limsup_{n\to\infty}\frac{4\,\mathrm{var}(S'_n)}{B_n^2}+\varepsilon$$

whenever $\gamma \geqslant \gamma_0$. There exists some $A > 0$ such that $\delta_n \geqslant An^{-1/4}\sqrt{\gamma}$ since $c_n \geqslant \gamma n^{-1/2}\delta_n^{-1/2}$ and $c_n\delta_n^{-3/2}$ is bounded. Using again Proposition 5.1 we obtain $\limsup_{n\to\infty}\mathbb{P}(|S_n| \leqslant z_{\alpha/2}) \leqslant 1-\beta$ whenever γ is large enough.

The end of this section is devoted to the proof of Proposition 5.1. The proof is organized as follows. We first build a sequence Z_n in such a way that

$$B_n = n^{1/6}\left(Z_n - \mathcal{C}_{f_0,1}\right) + O(1). \tag{5.10}$$

The results stated in Proposition 5.1 concerning the asymptotic expectation B_n of the test statistic are derived from the asymptotic behavior of $n^{1/6}(Z_n - \mathcal{C}_{f_0,1})$. Results concerning the asymptotic variance of the test statistic are finally proved. For the sake of simplicity, we assume throughout the proof x_0 to be zero. Recall we assume also $\sigma = 1$.

• Construction of Z_n

Suppose first that δ_n converges to zero as n goes to infinity. We assume without loss of generality that $m_n = m$ and $m + n^{-1/6} < f_0(\delta_n + I)$. Then $[m \wedge m_n, M \vee M_n] = \mathcal{I}_n^{(1)} \cup \mathcal{I}_n^{(2)} \cup \mathcal{I}_n^{(3)}$ where

$$\begin{aligned}
\mathcal{I}_n^{(1)} &= [m, m + n^{-1/6}] \cup [f_n(n^{-1/3}\log^2 n), M \vee M_n],\\
\mathcal{I}_n^{(2)} &= [m + n^{-1/6}, f_0(\delta_n + I)],\\
\mathcal{I}_n^{(3)} &= [f_0(\delta_n + I), f_n(n^{-1/3}\log^2 n)],
\end{aligned}$$

$\mathcal{I}_n^{(3)}$ being an empty set whenever $f_0(\delta_n + I) > f_n(n^{-1/3}\log^2 n)$. Since $c_n\delta_n^{-3/2}$ and $n^{1/3}c_n\delta_n^{1/2}$ are bounded, $n^{1/6}c_n\delta_n^{-1/2}$ is also bounded and the length of both intervals defining $\mathcal{I}_n^{(1)}$ is of order of magnitude $O(n^{-1/6})$. If $V(0)$ is the location of the maximum of $\{W(u) - u^2,\ u \in \mathbb{R}\}$ then for every $x > 0$, $|V(0)|$ can be larger than x only if there exists some u with $|u| \geqslant x$ for which $W(u) - u^2 \geqslant W(0)$. Time-inversion property of Brownian motion thus implies

$$\mathbb{P}\left(|V(0)| \geqslant x\right) \leqslant 2\exp(-x^3/2) \tag{5.11}$$

for all $x > 0$. Therefore, $\mathbb{E}|V(0)|$ is finite. By (5.6) we thus get for all Groeneboom process V:

$$n^{1/6}\int_{\mathcal{I}_n^{(1)}} \mathbb{E}\left|n^{1/3}\left(U_n(a) - g_n(a)\right) - V(0)d_n(a)^{-1}\right| \mathrm{d}a = O(1).$$

Fix now $a \in \mathcal{I}_n^{(2)}$. Since $a \leqslant f_0(\delta_n + I)$, we have $g_n(a) = g_0(a)$ and $L_n(a) \leqslant \|f_0''\|_\infty$. Assumptions of Lemma 5.3 are thus fulfilled for large enough n. Therefore, for all $a \in \mathcal{I}_n^{(2)}$, there exists some Groeneboom process $V_{a,n}$ such that

$$n^{1/6}\int_{\mathcal{I}_n^{(2)}} \mathbb{E}\left|n^{1/3}\left(U_n(a) - g_n(a)\right) - V_{a,n}(0)d_n(a)^{-1}\right| \mathrm{d}a = O(1).$$

Suppose $n^{1/6}\delta_n$ to be bounded. Then, the length of the interval $\mathcal{I}_n^{(3)}$ is of order of magnitude $O(n^{-1/6})$ and for all $a \in [m \wedge m_n, M \vee M_n]$, there exists some Groeneboom process $V_{a,n}$ such that

$$n^{1/6}\int_{m\wedge m_n}^{M\vee M_n} \mathbb{E}\left|n^{1/3}\left(U_n(a) - g_n(a)\right) - V_{a,n}(0)d_n(a)^{-1}\right| \mathrm{d}a = O(1). \tag{5.12}$$

Suppose now that there exists some positive constant C such that $n^{1/6}\delta_n > C$ for all large enough n. Then, $\sup_{a\in\mathcal{I}_n^{(3)}} L_n(a) = O(1 + n^{1/2}c_n\delta_n^{1/2})$ and the assumptions of Lemma 5.3 are fulfilled for all $a \in \mathcal{I}_n^{(3)}$ whenever n is large enough (recall that $n^{1/3}c_n\delta_n^{1/2}$ is bounded). We also have $\sup_{a\in\mathcal{I}_n^{(3)}} L_n(a) = O(1 + c_n\delta_n^{-5/2})$ and the length of $\mathcal{I}_n^{(3)}$ is of order of magnitude $O(\delta_n + n^{-1/3}(\log n)^2)$. So (5.12) still holds for some $V_{a,n}$ in the case where $n^{1/6}\delta_n > C$. If δ_n converges to zero as n goes to infinity, there thus exists for all a some Groeneboom process $V_{a,n}$ such that (5.12) holds. One can easily prove that (5.12) still holds for some $V_{a,n}$ whenever $\delta_n = 1$

for all n since in that case $L_n(a)$ is bounded uniformly in a and n. Let thus define

$$Z_n = \mathbb{E}\int_{m\wedge m_n}^{M\vee M_n} |V_{a,n}(0)+\psi_n(a)|\, d_n(a)^{-1}\mathrm{d}a, \tag{5.13}$$

where $\psi_n(a) = d_n(a)n^{1/3}(g_n(a)-g_0(a))$. Then (5.10) holds.

- **Asymptotic expectation of the test statistic**

Let Z_n be defined by (5.13). We get

$$Z_n \geqslant 2\mathbb{E}\int_0^1 |V_{t,n}(0)+\psi_n\circ f_n(t)| \left|\frac{f_n'(t)}{2}\right|^{1/3} \mathrm{d}t$$

where for every real number t, $V_{t,n}$ is a Groeneboom process. Since f_0 and f_n satisfy the regularity conditions $\mathcal{R}_0$ and $\mathcal{R}_n$ respectively, there exists some positive constant C such that

$$\left| |f_n'(t)|^{1/3} - |f_0'(t)|^{1/3} \right| \leqslant Cc_n\delta_n^{-3/2}\left|\phi'\left(\frac{t}{\delta_n}\right)\right|.$$

Let V be a Groeneboom process. Inequality (5.11) holds for all $x>0$ so $\mathbb{E}|V(0)|$ is finite and

$$n^{1/6}\mathbb{E}|V(0)|\int_0^1 \left(|f_n'(t)|^{1/3} - |f_0'(t)|^{1/3}\right)\mathrm{d}t = O(1).$$

Therefore (recall $\sigma=1$ and $x_0=0$) $n^{1/6}(Z_n - \mathcal{C}_{f_0,1}) \geqslant O(1) + B_n'$ where

$$B_n' = 2n^{1/6}\int_0^1 \left|\frac{f_n'(t)}{2}\right|^{1/3} \left(\mathbb{E}|V(0)+\psi_n\circ f_n(t)| - \mathbb{E}|V(0)|\right)\mathrm{d}t.$$

The distribution of $V(0)$ is symmetric about zero, so

$$\mathbb{E}|V(0)+\psi_n\circ f_n(t)| - \mathbb{E}|V(0)| = \int_0^{|\psi_n\circ f_n(t)|}\mathbb{P}(|V(0)|<u)\mathrm{d}u.$$

But $\psi_n\circ f_n$ has support included in the support $[0,\delta_n]$ of $f_n - f_0$ and therefore

$$B_n' = 2n^{1/6}\int_0^{\delta_n}\left|\frac{f_n'(t)}{2}\right|^{1/3}\int_0^{|\psi_n\circ f_n(t)|}\mathbb{P}\left(|V(0)|<u\right)\mathrm{d}u\mathrm{d}t.$$

Suppose first $n^{1/3}c_n\delta_n^{-1/2}$ to be bounded. Since the density h of $V(0)$ has a bell shape curve (see Groeneboom [8]), we have:

$$\int_0^{|\psi_n\circ f_n(t)|}\mathbb{P}\left(|V(0)|<u\right)\mathrm{d}u \geqslant \left(\psi_n\circ f_n(t)\right)^2 h\left(|\psi_n\circ f_n(t)|\right).$$

The function h is continuous and $||\psi_n\circ f_n||_\infty$ is bounded. Moreover, there exist some subinterval I_n of $[0,\delta_n]$ with length of order δ_n and some positive constant C such that $\inf_{t\in I_n}|\psi_n\circ f_n(t)| > Cn^{1/3}c_n\delta_n^{-1/2}$. By (5.10) there thus exists some $B>0$ such that $B_n > O(1) + Bn^{5/6}c_n^2$.

Suppose now that there exists some positive constant D such that $Cn^{1/3}c_n\delta_n^{-1/2} > 2D$ for all large enough n. Then

$$B_n' \geqslant 2n^{1/6} \int_{I_n} \left|\frac{f_n'(t)}{2}\right|^{1/3} \int_D^{|\psi_n \circ f_n(t)|} \mathbb{P}\left(|V(0)| < D\right) \mathrm{d}u\mathrm{d}t.$$

By (5.10) there thus exists some $B > 0$ such that $B_n > O(1) + Bn^{1/2}c_n\delta_n^{1/2}$.

• **Asymptotic variance of the test statistic**

In the sequel we use the following convention: for all real valued function h, all $x > y$, $\int_x^y h(t)\mathrm{d}t = 0$. Let S_n' be the random variable defined by (5.9). We write $\operatorname{var}(S_n') = I_n^{(1)} + I_n^{(2)}$, where

$$I_n^{(1)} = 2n \int_{m\wedge m_n}^{M\vee M_n} \int_{a+n^{-1/3}}^{M\vee M_n} \operatorname{cov}\left(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|\right) \mathrm{d}b\mathrm{d}a,$$
$$I_n^{(2)} = 2n \int_{m\wedge m_n}^{M\vee M_n} \int_{a}^{a+n^{-1/3}} \operatorname{cov}\left(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|\right) \mathrm{d}b\mathrm{d}a.$$

We first state several covariance inequalities that will be used to provide upper bounds for both $I_n^{(1)}$ and $I_n^{(2)}$. It is assumed that both f_n' and f_0' are bounded away from zero so

$$\sup_{a\in[m\wedge m_n, M\vee M_n]} |g_n(a) - g_0(a)| = O(c_n\delta_n^{-1/2}). \tag{5.14}$$

Cauchy–Schwarz inequality and (5.6) then yield

$$\sup_{a,b\in[m\wedge m_n, M\vee M_n]} \left|\operatorname{cov}\left(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|\right)\right| = O(n^{-2/3} + c_n^2\delta_n^{-1}). \tag{5.15}$$

The second covariance inequality we use in order to estimate $I_n^{(1)}$ and $I_n^{(2)}$ brings independent variables in. The idea is to use the following consequence of Cauchy–Schwarz inequality. Let D, D', E and E' be random variables. If E and D are independent, then

$$|\operatorname{cov}(D', E')| \leqslant \mathbb{E}^{1/2}({D'}^2)\mathbb{E}^{1/2}(E - E')^2 + \mathbb{E}^{1/2}(E^2)\mathbb{E}^{1/2}(D - D')^2. \tag{5.16}$$

For every real numbers a and b such that $a < b$ let $D_n(a, b)$ and $E_n(a, b)$ be defined by

$$D_n(a, b) = \operatorname*{argmax}_{u\in\left[\frac{3g_n(b)-g_n(a)}{2}, \frac{g_n(b)+g_n(a)}{2}\right]\cap[0,1]} \left\{W(u) + \sqrt{n}(F_n(u) - bu)\right\},$$
$$E_n(a, b) = \operatorname*{argmax}_{u\in\left[\frac{g_n(b)+g_n(a)}{2}, \frac{3g_n(a)-g_n(b)}{2}\right]\cap[0,1]} \left\{W(u) + \sqrt{n}(F_n(u) - au)\right\}$$

where $F_n(u) = \int_0^u f_n(s)\ \mathrm{d}s$. Since Brownian motion's increments are independent, $D_n(a, b)$ and $E_n(a, b)$ are independent for all $a < b$. Moreover, we will prove that the $\mathbb{L}_2$-distance between $D_n(a, b)$ and $U_n(b)$ (resp. between $E_n(a, b)$ and $U_n(a)$) is small. From the definition of $D_n(a, b)$ and $U_n(b)$ we have $|D_n(a, b) - g_n(b)| \leqslant |U_n(b) - g_n(b)|$. Moreover, either $D_n(a, b) = U_n(b)$ or $|U_n(b) - g_n(b)| > (g_n(a) - g_n(b))/2$. It thus follows from triangular inequality and the Cauchy–Schwarz inequality that

$$\begin{aligned}\mathbb{E}^{1/2} \left| |D_n(a,b) - g_0(b)| - |U_n(b) - g_0(b)| \right|^2 \\ \leqslant 2\mathbb{E}^{1/4} |U_n(b) - g_n(b)|^4 \, \mathbb{P}^{1/4} \left(|U_n(b) - g_n(b)| > \frac{g_n(a) - g_n(b)}{2} \right). \end{aligned} \tag{5.17}$$

One can obtain the same kind of upper bound for $\mathbb{E}^{1/2} \left| |E_n(a,b) - g_0(a)| - |U_n(a) - g_0(a)| \right|^2$. This implies by (5.16, 5.14) and Lemma 5.1 that there exist some positive A and C such that for all $a < b$,

$$|\operatorname{cov}(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|)| \leqslant An^{-1/3}(n^{-1/3} + c_n\delta_n^{-1/2}) \exp\left(-nC(g_n(a) - g_n(b))^3\right). \tag{5.18}$$

We propose a last covariance inequality that applies in the case where both a and b lie in $[m \vee m_n, f_0(\delta_n)]$ and $a < b$. In that case, $g_n(a) = g_0(a)$ and $g_n(b) = g_0(b)$. Moreover, there exists some positive K such that $g_n(a) - g_n(b) \geqslant K(b-a)$ since $c_n\delta_n^{-3/2}$ is bounded. By using the same arguments as above combined with Lemma 5.1, one can easily check that there exist some positive A and C such that

$$|\operatorname{cov}(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|)| \leqslant An^{-2/3} \exp(-Cn(b-a)^3). \tag{5.19}$$

We now provide upper bounds for $I_n^{(1)}$ and $I_n^{(2)}$. Suppose first $n^{1/3}c_n\delta_n^{-1/2}$ to be bounded. There exists some positive constant K such that $g_n(a) - g_n(b) \geqslant K(b-a)$ whenever a and b lie in $[m_n, M_n]$. By (5.15) and (5.18), there thus exist some positive constants A and C such that

$$\begin{aligned}\left| I_n^{(1)} \right| &\leqslant An^{1/3} \int_{m \vee m_n}^{M \wedge M_n} \int_{a + n^{-1/3}}^{M \wedge M_n} \exp\left(-Cn(b-a)^3\right) \mathrm{d}b\mathrm{d}a + O(1) \\ &\leqslant A \int_m^M \int_1^\infty \exp\left(-Cx^3\right) \mathrm{d}x\mathrm{d}a + O(1).\end{aligned}$$

Hence $I_n^{(1)}$ is bounded. It follows from (5.15) that $I_n^{(2)}$ is also bounded, so $\operatorname{var}(S_n')$ is bounded whenever $n^{1/3}c_n\delta_n^{-1/2}$ is bounded.

Suppose now that δ_n converges to zero as n goes to infinity and that $n^{1/3}c_n\delta_n^{-1/2} > 1$ for all large enough n. Then we can assume $m_n = m$. On the one hand, inequality (5.19) holds whenever both a and b lie in $[m, f_0(\delta_n)]$ and $a < b$. Therefore,

$$n \int_m^{f_0(\delta_n)} \int_{a+n^{-1/3}}^{f_0(\delta_n)} \operatorname{cov}(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|) \, \mathrm{d}b\mathrm{d}a = O(1).$$

We have $M \vee M_n - f_0(\delta_n) = O(\delta_n)$. Moreover, g_n is strictly decreasing from $[m, M_n]$ onto $[0,1]$. Therefore, $b \mapsto n^{1/3}(g_n(a) - g_n(b))$ is one-to-one over $[m, M_n]$ for all fixed $a \in \mathbb{R}$. By (5.18) and change of variables, we thus obtain (recall that $n^{1/3}c_n\delta_n^{1/2}$ is bounded)

$$n \int_{f_0(\delta_n)}^{M \vee M_n} \int_{a+n^{-1/3}}^{M \wedge M_n} \operatorname{cov}(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|) \, \mathrm{d}b\mathrm{d}a = O(1).$$

Likewise, $a \mapsto n^{1/3}(g_n(a) - g_n(b))$ is one-to-one over $[m, M_n]$ for all fixed $b \in \mathbb{R}$. It thus follows from (5.18) and change of variables that

$$n \int_m^{f_0(\delta_n)} \int_{f_0(\delta_n)}^{M \vee M_n} \operatorname{cov}(|U_n(a) - g_0(a)|, |U_n(b) - g_0(b)|) \, \mathrm{d}b\mathrm{d}a = O(1).$$

Finally, the exponential factor in (5.18) is no more than one so

$$n\int_{f_0(\delta_n)}^{M\vee M_n}\int_{M\wedge M_n}^{M\vee M_n}\operatorname{cov}\left(|U_n(a)-g_0(a)|,|U_n(b)-g_0(b)|\right)1\!\!1_{b>a}\mathrm{d}b\mathrm{d}a=O(n^{2/3}c_n^2).$$

Therefore, $I_n^{(1)}=O(1+n^{2/3}c_n^2)$. On the other hand, $g_n(a)=g_0(a)$ and $g_n(b)=g_0(b)$ for all $a\in[m,f_0(\delta_n)-n^{-1/3}]$ and $b\in[a,a+n^{-1/3}]$. By the Cauchy–Schwarz inequality and (5.6) we thus have

$$|I_n^{(2)}|\leqslant O(1)+\,2n\int_{f_0(\delta_n)-n^{-1/3}}^{M\vee M_n}\int_a^{a+n^{-1/3}}\left|\operatorname{cov}\left(|U_n(a)-g_0(a)|,|U_n(b)-g_0(b)|\right)\right|\mathrm{d}b\mathrm{d}a. \tag{5.20}$$

Conditions $n^{1/3}c_n\delta_n^{-1/2}>1$ and $c_n\delta_n^{-3/2}=O(1)$ imply $\delta_n^{-1}=O(n^{1/3})$. So by (5.15) we have $I_n^{(2)}=O(1+n^{2/3}c_n^2)$. Therefore $\operatorname{var}(S_n')=O(1+n^{2/3}c_n^2)$ whenever $n^{1/3}c_n\delta_n^{-1/2}>1$ for all large enough n and $\delta_n=o(1)$, which completes the proof of Proposition 5.1.

5.4. **Proof of the probabilistic tools**

Notation 1 is used throughout this section.

5.4.1. *Proof of Lemma 5.1*

For all $a\in\mathbb{R}$, $V_n(a)-g_n(a)$ is the location of the maximum of the process $Z_{a,n}-D_{a,n}$ over $I_n(a)=[-g_n(a),1-g_n(a)]$, where

$$Z_{a,n}(v)=\frac{1}{n}\sum_{i=1}^n\varepsilon_i(1\!\!1_{x_i\leqslant v+g_n(a)}-1\!\!1_{x_i\leqslant g_n(a)}), \tag{5.21}$$

and

$$D_{a,n}(v)=-\frac{1}{n}\sum_{i=1}^n f_n(x_i)(1\!\!1_{x_i\leqslant v+g_n(a)}-1\!\!1_{x_i\leqslant g_n(a)})+av. \tag{5.22}$$

Fix $v\in[0,1-g_n(a)]$. By definition of k_1 we have $f_n(x_i)\leqslant f_n(g_n(a))-k_1(x_i-g_n(a))$ for all $x_i\in]g_n(a),v+g_n(a)]$. Moreover $\operatorname{card}\{i,x_i\in]g_n(a),v+g_n(a)]\}\in[nv-1,nv+1]$ since $x_i=i/n$. Therefore,

$$\begin{aligned}D_{a,n}(v)\geqslant(a-f_n(g_n(a)))v-\frac{|f_n(g_n(a))|}{n}+k_1\Bigg[\frac{1}{n}\sum_{i=1}^n(x_i-g_n(a))1\!\!1_{x_i\in]g_n(a),v+g_n(a)]}\\-\int_{g_n(a)}^{v+g_n(a)}(x-g_n(a))\mathrm{d}x\Bigg]+k_1\int_{g_n(a)}^{v+g_n(a)}(x-g_n(a))\mathrm{d}x\end{aligned}$$

which implies

$$D_{a,n}(v)\geqslant(a-f_n(g_n(a)))\,v-\frac{|f_n(g_n(a))|}{n}-\frac{k_1}{n}+\frac{k_1v^2}{2}\,\cdot$$

By assumption f_n is continuous so $(a-f_n(g_n(a)))\,v\geqslant0$ for all $v\in[-g_n(a),1-g_n(a)]$. But $\sup_{t,n}|f_n(t)|$ is finite so there exists some positive c_0 that does not depend on a or n such that

$$D_{a,n}(v)\geqslant\frac{k_1}{4}v^2 \tag{5.23}$$

whenever $v^2 \geqslant c_0/n$ and $v \in [0, 1 - g_n(a)]$. One can prove using the same arguments that (5.23) still holds whenever $v^2 \geqslant c_0/n$ and $v \in [-g_n(a), 0]$. Fix $t > 0$. The first inequality in Lemma 5.1 is trivial whenever $t^2 < c_0/n$ so we assume $t^2 \geqslant c_0/n$ and we get

$$\begin{aligned}\mathbb{P}\left(|V_n(a) - g_n(a)| > t\right) &\leqslant \mathbb{P}\left(\sup_{|v|>t}\{Z_{a,n}(v) - k_1 v^2/4\} \geqslant 0\right)\\ &\leqslant \sum_{k\geqslant 0} \mathbb{P}\left(\sup_{t2^k<|v|\leqslant t2^{k+1}}\{Z_{a,n}(v)\} \geqslant k_1 t^2 2^{2(k-1)}\right).\end{aligned}$$

By Doob's and Rosenthal's inequalities (see Revuz and Yor [13] and Rosenthal [15]) there thus exists some positive K_p that only depends on $\mathbb{E}|\varepsilon_1|^p$ and k_1 such that

$$\mathbb{P}\left(|V_n(a) - g_n(a)| > t\right) \leqslant K_p \sum_{k\geqslant 0} \frac{(nt2^{k+1}+1)^{p/2}}{(nt^2 2^{2k})^p},$$

which proves the first assertion of the lemma.

Expanding $u \mapsto \int_0^u f_n(s)\,\mathrm{d}s$ in the neighborhood of $g_n(a)$, one can prove in the same way that for all $t > 0$,

$$\mathbb{P}\left(|U_n(a) - g_n(a)| > t\right) \leqslant \mathbb{P}\left(\sup_{1\geqslant|v|>t}\left\{\frac{1}{\sqrt{n}}\left(W(v+g_n(a)) - W(g_n(a))\right) - \frac{k_1 v^2}{2}\right\} \geqslant 0\right).$$

By time-homogeneity property of Brownian motion, the latter probability does not depend on a. Moreover, the process $\left\{ut^{-1/2}W(t/u),\ u \in \mathbb{R}\right\}$ has the same distribution as $\{W(u),\ u \in \mathbb{R}\}$ so change of variables $u = t/v$ yields

$$\mathbb{P}\left(|U_n(a) - g_n(a)| > t\right) \leqslant 2\mathbb{P}\left(\sup_{0\leqslant v\leqslant 1}\{W(v)\} \geqslant \frac{\sqrt{n}t^{3/2}k_1}{2}\right).$$

The second assertion of the lemma now follows from exponential inequality.

5.4.2. *Proof of Lemma 5.2*

Fix $a > M_n$. Let g_n denote the inverse function of f_n. We have $g_n(a) = 0$ and $V_n(a) - g_n(a)$ lies in $[0, 1]$ so for all $t > 0$,

$$\mathbb{P}(|V_n(a) - g_n(a)| > t) \leqslant \mathbb{P}\left(\sup_{t<v\leqslant 1}\{Z_{a,n}(v) - D_{a,n}(v)\} \geqslant 0\right),$$

where $Z_{a,n}$ and $D_{a,n}$ are given by (5.21) and (5.22) respectively. For all $v \in [0, 1]$, $\mathrm{Card}\{i, x_i \in]0, v]\} \in [nv - 1, nv]$. Thus

$$D_{a,n}(v) \geqslant v(a - M_n) - \sup_{n,t}|f_n(t)|/n.$$

Assume $2\sup_{n,t}|f_n(t)| \leqslant nt(a - M_n)$ and $t \geqslant 1/n$. Then, $D_{a,n}(v) \geqslant v(a - M_n)/2$ for all $v > t$. Moreover, $\mathbb{E}|\varepsilon_1|^2$ is finite so one can obtain by using the same arguments as in the proof of Lemma 5.1 that there exists some positive constant c such that

$$\mathbb{P}\left(|V_n(a) - g_n(a)| > t\right) \leqslant \frac{c}{nt(a - M_n)^2} \tag{5.24}$$

for all $t \geqslant 1/n$, $a > M_n$ and $n \geqslant 1$ with $2\sup_{n,t}|f_n(t)| \leqslant nt(a-M_n)$. The latter inequality is trivial whenever $2\sup_{n,t}|f_n(t)| > nt(a-M_n)$ and $t \geqslant 1/n$ (provided c is large enough) so it holds for all $t \geqslant 1/n$ and $a > M_n$. By definition, $\hat{f}_n$ can jump only at times i/n, $i \in \{1,\dots,n\}$ so for all $t > 0$, we can have $V_n(a) > t$ only if $V_n(a) \geqslant 1/n$. By (5.24), there thus exists some positive c such that

$$\mathbb{P}\left(|V_n(a) - g_n(a)| > t\right) \leqslant \frac{c}{(a-M_n)^2}$$

for all $t \in (0, 1/n)$, $a > M_n$. There thus exists some positive c such that (5.24) holds for all positive t, n and all $a > M_n$. Combined with Lemma 5.1 with $p = 2$, it proves that there exists some c' such that for all $a > M_n$,

$$\mathbb{E}|V_n(a) - g_n(a)| \leqslant c'\left(\frac{1}{n(a-M_n)^2} + \int_{n^{-1}(a-M_n)^{-2}}^{a-M_n} \frac{1}{nt(a-M_n)^2}\,\mathrm{d}t + \int_{a-M_n}^{\infty} \frac{1}{nt^3}\,\mathrm{d}t\right).$$

So by (5.5),

$$\mathbb{E}\int_{M_n}^{\infty} |V_n(a) - g_n(a)|\,\mathrm{d}a = O(n^{-2/3}).$$

One can obtain in the same way that $\mathbb{E}\int_{-\infty}^{m_n} |V_n(a) - g_n(a)|\,da = O(n^{-2/3})$. Equality (5.7) then follows since $g_n(a) = g_0(a)$ for all $a \notin [m_n \wedge m, M_n \vee M]$.

By Sakhanenko's theorem, see Theorem 5 of Sakhanenko [16], we may assume (provided the ε_i's are defined on some rich enough probability space) that there exists some Brownian motion W_0 such that

$$\mathbb{E}\left(\sup_{k\leqslant n}\left|\sum_{i=1}^{k}\varepsilon_i - W_0(k)\right|^p\right) \leqslant n\mathbb{E}|\varepsilon_1|^p.$$

Let W be the Brownian motion defined by $W(u) = n^{-1/2}W_0(nu)$, $u \in \mathbb{R}$. By exponential inequality and Markov's inequality there exists some c_p such that for all $t > 0$,

$$\mathbb{P}\left(\sup_{u\in[0,1]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i \mathbb{1}_{x_i\leqslant u} - W(u)\right| > t\right) \leqslant c_p n^{1-p/2} t^{-p}. \tag{5.25}$$

Let U_n be defined by (5.3) where W is some Brownian motion that satisfies (5.25). Let $I_{a,n} = [-n^{1/3}g_n(a), n^{1/3}(1-g_n(a))]$. By definition, $n^{1/3}(U_n(a) - g_n(a))$ is the location of the maximum of the process $\{Z(u),\ u \in I_{a,n}\}$ where

$$Z(u) = n^{1/6}W(n^{-1/3}u + g_n(a)) + n^{2/3}\int_0^{n^{-1/3}u+g_n(a)} f_n(s)\,\mathrm{d}s - an^{1/3}u.$$

Moreover $n^{1/3}(V_n(a) - g_n(a))$ is the location of the maximum of $\{Z(u) + R(u),\ u \in I_{a,n}\}$ where

$$\sup_{u\in I_{a,n}}|R(u)| \leqslant cn^{-1/3} + n^{1/6}\sup_{t\in[0,1]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\mathbb{1}_{x_i\leqslant t} - W(t)\right|$$

for some $c > 0$, since $\sup_{n,t}|f_n'(t)|$ is finite. By (5.25)

$$\mathbb{P}\left(2\sup_{u\in I_{a,n}}|R(u)| > x\eta^{3/2}\right) \leqslant 4^p c_p n^{1-p/3} x^{-p}\eta^{-3p/2} \tag{5.26}$$

whenever $4cn^{-1/3} \leqslant x\eta^{3/2}$. Let $z = n^{1/3}(U_n(a) - g_n(a))$ denote the location of the maximum of Z and for every $\eta > 0$ let

$$\mathbb{P}_a(\eta) = \mathbb{P}\left(n^{1/3}|V_n(a) - U_n(a)| > \eta\right).$$

For all positive x, η

$$\mathbb{P}_a(\eta) \leqslant \mathbb{P}\left(Z(z) - Z(n^{1/3}(V_n(a) - g_n(a))) > x\eta^{3/2}\right) + \mathbb{P}\left(Z(z) - \sup_{|t-z|>\eta} Z(t) \leqslant x\eta^{3/2}\right),$$

where

$$Z(z) - Z(n^{1/3}(V_n(a) - g_n(a))) \leqslant 2 \sup_{u \in I_{a,n}} |R(u)|.$$

Upper bounds for the last two probabilities are obtained using respectively (5.26) and Proposition 1 of Durot [4]. From now on, T denotes $\log(n)$. By Lemma 5.1 there exist some positive A, C such that

$$\mathbb{P}\left(n^{1/3}|V_n(a) - U_n(a)| > \eta\right) \leqslant ATx + An^{1-p/3}x^{-p}\eta^{-3p/2} + 2\exp(-CT^3)$$

whenever $a \in [m_n, M_n]$, $4cn^{-1/3} \leqslant x\eta^{3/2}$, $\eta \in (0, 1)$ and $c'T^2 \leqslant -1/\eta \log(2x\eta)$ for some large enough, positive c'. Fix $a \in [m_n, M_n]$ and for all $\eta > 0$ let define

$$x_\eta = n^{(3-p)/3(p+1)}\eta^{-3p/2(p+1)}T^{-1/(p+1)}.$$

The latter inequality holds whenever $x = x_\eta$, $\eta \in [n^{2(3-p)/9p}, n^{-\alpha}]$ for some $\alpha > 0$ and $n \geqslant n_0$ for some large enough n_0 that does not depend on a. So for large enough n

$$n^{1/6}\int_0^{n^{-\alpha}} \mathbb{P}_a(\eta)\ \mathrm{d}\eta \leqslant n^{(12-p)/18p}T^{2/3} + 2ATn^{1/6}\int_{n^{2(3-p)/9p}T^{2/3}}^{n^{-\alpha}} x_\eta\ \mathrm{d}\eta + 2n^{1/6-\alpha}\mathrm{e}^{-CT^3}.$$

This upper bound does not depend on a and converges to zero as n goes to infinity since $p > 12$. Fix $\beta \in (1/3(3p-2); (p-7)/6(p+1))$. We have $\mathbb{P}_a(\eta) \leqslant \mathbb{P}_a(n^{-\alpha})$ for all $\eta \geqslant n^{-\alpha}$. The integral $n^{1/6}\int_{n^{-\alpha}}^{n^\beta} \mathbb{P}_a(\eta)\ \mathrm{d}\eta$ thus also converges to zero as n goes to infinity uniformly in $a \in [m_n, M_n]$ whenever α is small enough. Finally it follows from Lemma 5.1 that $n^{1/6}\int_{n^\beta}^\infty \mathbb{P}_a(\eta)\ \mathrm{d}\eta$ uniformly converges to zero as n goes to infinity. Since we have $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x)\ \mathrm{d}x$ for all positive random variable X it follows that

$$\lim_{n\to\infty} n^{1/2} \sup_{a \in [m_n, M_n]} \mathbb{E}|U_n(a) - V_n(a)| = 0.$$

By (5.5, 5.6) this implies

$$n^{1/2}\int_{m_n \wedge m}^{M_n \vee M} \mathbb{E}|U_n(a) - V_n(a)|\ \mathrm{d}a = O(1)$$

whenever $M_n - M = O(n^{-1/6})$ and $m_n - m = O(n^{-1/6})$. Lemma 5.2 then follows from (5.7).

5.4.3. *Proof of Lemma 5.3*

Let first define a process $\widetilde{U}_n$ that approaches the process U_n: for all $a \in \mathbb{R}$ let

$$\widetilde{U}_n(a) = \underset{u \in [g_n(a)-I, g_n(a)+I] \cap [0,1]}{\operatorname{argmax}} \left\{ W(u) + \sqrt{n}(F_n(u) - au) \right\}$$

where $F_n(u) = \int_0^u f_n(s)\,\mathrm{d}s$ and $I = n^{-1/3}\log n$. For every $a \in \mathbb{R}$, either $\widetilde{U}_n(a) = U_n(a)$ or $|U_n(a) - g_n(a)| > I$. By Lemma 5.1 there thus exists some positive C such that

$$\mathbb{E}|\widetilde{U}_n(a) - U_n(a)| \leqslant 2\exp(-C(\log n)^3)$$

for all $a \in \mathbb{R}$. Therefore, it suffices to prove that there exist some positive constants D_1 and D_2 such that for large enough n,

$$n^{1/6}\mathbb{E}\left| n^{1/3}(\widetilde{U}_n(a) - g_n(a)) - d_n(a)^{-1}V_{a,n}(0) \right| \leq D_1 n^{-1/12}(\log n)^{9/2} L_n(a) + \frac{D_2}{\log n} \cdot \tag{5.27}$$

Fix a with $[-\log n, \log n] \subset [-n^{1/3}g_n(a), n^{1/3}(1-g_n(a))]$. Let W_1 be the Brownian motion defined for all $v \in \mathbb{R}$ by $W_1(v) = n^{1/6}(W(n^{-1/3}v + g_n(a)) - W(g_n(a)))$. Let Z be the process defined by

$$\forall v \in \mathbb{R}, \qquad Z(v) = W_1(v) - v^2 d_n(a)^{3/2},$$

and let z be the location of the maximum of Z. It is worth noticing that there exists some Groeneboom process $V_{a,n}$ such that $z = d_n(a)^{-1}V_{a,n}(0)$. Moreover, $n^{1/3}(\widetilde{U}_n(a) - g_n(a))$ is the location of the maximum of $\{Z(v) + R(v),\ |v| \leqslant \log n\}$ where

$$\sup_{|v| \leqslant \log n} |R(v)| \leqslant L_n(a) n^{-1/3}(\log n)^3/6. \tag{5.28}$$

For every positive η, let $\mathbb{P}_a(\eta)$ denote the following probability:

$$\mathbb{P}_a(\eta) = \mathbb{P}\left(\left| n^{1/3}(\widetilde{U}_n(a) - g_n(a)) - d_n(a)^{-1}V_{a,n}(0) \right| > \eta \right)$$

where by definition, $d_n(a)^{-1}V_{a,n}(0) = z$. For every positive x and η,

$$\mathbb{P}_a(\eta) \leqslant \mathbb{P}\left(Z(z) - Z(n^{1/3}(\widetilde{U}_n(a) - g_n(a))) > x\eta^{3/2} \right) + \mathbb{P}\left(Z(z) - \sup_{|t-z|>\eta} Z(t) \leqslant x\eta^{3/2} \right).$$

Moreover, if $|z| \leqslant \log n$ then

$$Z(z) - Z(n^{1/3}(\widetilde{U}_n(a) - g_n(a))) \leqslant 2 \sup_{|v| \leqslant \log n} |R(v)|.$$

Fix $\eta \in (0,1]$, $x > 0$ and suppose

$$(\log n)^3 \leqslant \left(\eta \log\left(\frac{1}{2x\eta} \right) \right)^{-1}.$$

By (5.11) and Proposition 1 of Durot [4] there exists some positive A that does not depend on a or n such that for large enough n,

$$\mathbb{P}_a(\eta) \leqslant \mathbb{P}\left(2 \sup_{|v|\leqslant \log n} |R(v)| > x\eta^{3/2}\right) + Ax\log n + 4\mathrm{e}^{-k_1^2(\log n)^3/8}.$$

Let ε be a positive real number such that $\varepsilon < 1/18$ and for all $\eta \in [(\log n)^{-1}n^{-1/6}, n^{-\varepsilon}]$ let define $x_\eta = n^{-1/3}\eta^{-3/2}(\log n)^3(L_n(a)+1)$. If $4n^{-1/4}(\log n)^{7/2}L_n(a) \leqslant 1$ then for every $\eta \in [(\log n)^{-1}n^{-1/6}, n^{-\varepsilon}]$, we have $\log(2x_\eta\eta) < 0$ whenever n is large enough. If n is large enough, every pair (η, x_η) thus satisfy the above conditions. By (5.28) we thus have

$$\mathbb{P}_a(\eta) \leqslant An^{-1/3}(\log n)^4\eta^{-3/2}(L_n(a)+1) + 4\exp(-k_1^2(\log n)^3/8) \tag{5.29}$$

for all $\eta \in [(\log n)^{-1}n^{-1/6}, n^{-\varepsilon}]$. We thus get for large enough n:

$$n^{1/6}\int_0^{n^{-\varepsilon}} \mathbb{P}_a(\eta)\mathrm{d}\eta \leqslant \frac{2}{\log n} + 2An^{-1/12}(\log n)^{9/2}(L_n(a)+1).$$

Since $\mathbb{P}_a$ is a decreasing function, and since for all real number a, $n^{1/3}|\widetilde{U}_n(a) - g_n(a)|$ is less than or equal to $n^{1/3}|U_n(a) - g_n(a)|$, equation (5.11) and Lemma 5.1 yield

$$n^{1/6}\int_{n^{-\varepsilon}}^{\infty} \mathbb{P}_a(\eta)\mathrm{d}\eta \leqslant n^{1/6}(\log n)\mathbb{P}_a(n^{-\varepsilon}) + n^{1/6}\int_{\log n}^{\infty} 4\exp(-C'\eta^3)\mathrm{d}\eta$$

for some positive constant C'. By assumption, $\varepsilon < 1/18$. Moreover (5.29) is available for $\eta = n^{-\varepsilon}$ and $EX = \int_0^\infty \mathbb{P}(X > x)\ \mathrm{d}x$ for all positive random variable X. The last two inequalities thus imply (5.27), which completes the proof of the lemma.

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