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MINIMAX NONPARAMETRIC HYPOTHESIS TESTING FOR ELLIPSOIDS AND BESOV BODIES *, **

Yuri I. Ingster[1] and Irina A. Suslina [2]

Abstract. We observe an infinitely dimensional Gaussian random vector $x = \xi + v$ where ξ is a sequence of standard Gaussian variables and $v \in l_2$ is an unknown mean. We consider the hypothesis testing problem $H_0 : v = 0$ *versus* alternatives $H_{\varepsilon,\tau} : v \in V_\varepsilon$ for the sets $V_\varepsilon = V_\varepsilon(\tau, \rho_\varepsilon) \subset l_2$. The sets V_ε are l_q-ellipsoids of semi-axes $a_i = i^{-s}R/\varepsilon$ with l_p-ellipsoid of semi-axes $b_i = i^{-r}\rho_\varepsilon/\varepsilon$ removed or similar Besov bodies $B_{q,t;s}(R/\varepsilon)$ with Besov bodies $B_{p,h;r}(\rho_\varepsilon/\varepsilon)$ removed. Here $\tau = (\kappa, R)$ or $\tau = (\kappa, h, t, R)$; $\kappa = (p, q, r, s)$ are the parameters which define the sets V_ε for given radii $\rho_\varepsilon \to 0$, $0 < p, q, h, t \le \infty$, $-\infty < r, s < \infty$, $R > 0$; $\varepsilon \to 0$ is the asymptotical parameter. We study the asymptotics of minimax second kind errors $\beta_\varepsilon(\alpha) = \beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon))$ and construct asymptotically minimax or minimax consistent families of tests $\psi_{\alpha;\varepsilon,\tau,\rho_\varepsilon}$, if it is possible. We describe the partition of the set of parameters κ into regions with different types of asymptotics: classical, trivial, degenerate and Gaussian (of various types). Analogous rates have been obtained in a signal detection problem for continuous variant of white noise model: alternatives correspond to Besov or Sobolev balls with Besov or Sobolev balls removed. The study is based on an extension of methods of constructions of asymptotically least favorable priors. These methods are applicable to wide class of "convex separable symmetrical" infinite-dimensional hypothesis testing problems in white Gaussian noise model. Under some assumptions these methods are based on the reduction of hypothesis testing problem to convex extreme problem: to minimize specially defined Hilbert norm over convex sets of sequences $\bar{\pi}$ of measures π_i on the real line. The study of this extreme problem allows to obtain different types of Gaussian asymptotics. If necessary assumptions do not hold, then we obtain other types of asymptotics.

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[1] St. Petersburg Transport University, Department of Applied Mathematics, Moskowskii Av. 9, 190031 St. Petersburg, Russia; e-mail: ingster@pdmi.ras.ru

[2] St. Petersburg Institute of Exact Mechanics and Optics, Technical University, Sablinskaya Str. 14, 197101 St. Petersburg, Russia; e-mail: SuslinaI@mkk.ifmo.ru

1. Introduction

1.1. **Setting**

Let an infinitely-dimensional Gaussian random vector $x = \xi + v$ be observed where ξ is a sequence of standard independent Gaussian random variables with zero mean and unit variance, $v \in l_2$ is an unknown mean sequence.

We consider the problem of testing null hypothesis $H_0 : v = 0$ on a sequence v and consider families of alternatives $H_\varepsilon : v \in V_\varepsilon$ for a given families of the sets V_ε of unknown v in the sequence space l_2, $\varepsilon \to 0$ is an asymptotical parameter. Certainly this problem is equivalent to the well known problem of the testing $H_0 : s = 0$ *versus* the family of alternatives $H_\varepsilon : s \in S_\varepsilon \subset L_2(0,1)$ in Gaussian white noise model:

$$dX_\varepsilon(t) = s(t)dt + \varepsilon dW(t),\ t \in [0,1],\ s \in L_2(0,1),\ \varepsilon > 0.$$

In fact, for a fixed orthonormal basis $\{\zeta_n\}$ we consider the sequences of normalized empirical Fourier coefficients x_i and the sets $V_\varepsilon = \{v_\varepsilon(s),\ s \in S_\varepsilon\}$ of the normalized Fourier coefficients:

$$x_i = \varepsilon^{-1} \int_0^1 \zeta_i(t) dX_\varepsilon(t), \quad v_{i,\varepsilon}(s) = \varepsilon^{-1} \int_0^1 \zeta_i(t) s(t) dt.$$

The problems are studied in asymptotical minimax setting (as $\varepsilon \to 0$). For a family of alternatives $H_\varepsilon : v \in V_\varepsilon$ a family of (randomized) tests $\psi_\varepsilon = \psi_\varepsilon(x),\ \psi_\varepsilon(x) \in [0,1]$ is characterized by the families of the first kind errors $\alpha(\psi_\varepsilon) = E_0(\psi_\varepsilon)$ and by the supremum of the second kind errors

$$\beta(\psi_\varepsilon, V_\varepsilon) = \sup_{v \in V_\varepsilon} \beta(\psi_\varepsilon, v),\ \beta(\psi_\varepsilon, v) = E_v(1 - \psi_\varepsilon),$$

where E_v stands for the mean value with respect to the measure P_v which corresponds to the observation $x = \xi + v,\ v \in l_2$. For fixed $\alpha \in (0,1)$ the minimax distinguishability is characterized by the asymptotics of the values

$$\beta(\alpha, V_\varepsilon) = \inf_{\psi \in \Psi_\alpha} \beta(\psi, V_\varepsilon), \quad \Psi_\alpha = \{\psi :\ \alpha(\psi) \le \alpha\}\cdot$$

It is clear that

$$0 \le \beta(\alpha, V_\varepsilon) \le 1 - \alpha.$$

The problem is called *trivial*, if $\beta(\alpha, V_\varepsilon) = 1 - \alpha$ for any $\alpha \in (0,1)$.

The problem of sharp asymptotics is to investigate asymptotics of the values $\beta(\alpha, V_\varepsilon)$ (up to vanishing term, as $\varepsilon \to 0$) and to construct *asymptotically minimax families of tests* $\psi_{\varepsilon,\alpha}$ such that, as $\varepsilon \to 0$,

$$\alpha(\psi_\varepsilon, \alpha) = \alpha + o(1), \quad \beta(\psi_{\varepsilon,\alpha}, V_\varepsilon) = \beta(\alpha, V_\varepsilon) + o(1).$$

The problem of rate asymptotics is to obtain conditions of *distinguishability*:

$$\beta(\alpha, V_\varepsilon) \to 0$$

and to construct *minimax consistent families of tests* $\psi_{\varepsilon,\alpha}$:

$$\alpha(\psi_\varepsilon, \alpha) = \alpha + o(1), \quad \beta(\psi_{\varepsilon,\alpha}, V_\varepsilon) = o(1),$$

or to obtain conditions of *indistinguishability (asymptotical triviality)*:

$$\beta(\alpha, V_\varepsilon) \to 1 - \alpha.$$

1.2. **Alternatives**

It is clear that it is not possible to distinguish null-hypothesis and alternatives which are too close to hypothesis. Thus it is necessary to remove some small neighborhoods of null hypothesis. Typically these neighborhoods can be defined in the form

$$f_1(v) < H_{\varepsilon,1},$$

where f_1 is some norm or sub-norm on the sequence space.

Also often (with exception of "classical" case, see Ingster [12] and Sect. 2.1 later) it is necessary to restrict nonparametrical alternatives to obtain nontrivial problem. Restrictions of such type also may be given by some other norms or sub-norms f_2 in sequence space:

$$f_2(v) \le H_{\varepsilon,2}.$$

Thus alternatives may be defined by constraints

$$V_\varepsilon = \{v \in l_2 : \ f_1(v) \ge H_{\varepsilon,1}, \ f_2(v) \le H_{\varepsilon,2}\}\cdot$$

The objects of our interest are two cases: ellipsoids and Besov bodies.

1.2.1. *Ellipsoidal case*

In this case we consider simplest (as it seems) variant of norms: $f_1 = f_{r,p}, \ f_2 = f_{s,q}$, where

$$f_{r,p}(v) = \left(\sum_{i=1}^{\infty} |v_i|^p i^{rp}\right)^{1/p} \ ; \ f_{r,\infty}(v) = \sup_{1\le i<\infty} |v_i| i^r$$

and $-\infty < r, s < \infty, \ 0 < p, q \le \infty$ (if $0 < p, q < 1$, then this relation define quasi-norm). Also we consider thresholds H_k of the form $H_1 = \rho_\varepsilon/\varepsilon, \ H_2 = R/\varepsilon, \ \rho_\varepsilon \to 0$.

Thus in this case we consider the sets $V_\varepsilon = V_\varepsilon(\tau, \rho_\varepsilon)$ which are ellipsoids with "small" ellipsoids removed:

$$V_\varepsilon(\tau, \rho_\varepsilon) = E_{q,s}(R_{\varepsilon,2}) \setminus E_{p,r}(R_{\varepsilon,1}); \quad R_{\varepsilon,2} = R/\varepsilon, \ R_{\varepsilon,1} = \rho_\varepsilon/\varepsilon, \tag{1.1}$$

where $E_{p,r}(R)$ is l_p-ellipsoid of semi-axes $a_i = i^{-r}R$:

$$E_{p,r}(R) = \left\{ v \in l_2 : \sum_{i=1}^{\infty} i^{rp} |v_i|^p < R^p \right\}$$

with evident modification for $p = \infty$. Here $\tau = (\kappa, R), \ \kappa \in \Xi$ where we determine the set $\Xi \subset R^4$ as

$$\Xi = \{(p, q, r, s) : \ 0 < p, q \le \infty, \ -\infty < r, s < \infty\};$$

$R > 0$, the values $\rho_\varepsilon > 0, \ \rho_\varepsilon \to 0$ are given.

The thresholds H_k of such form correspond to the normalization in a signal detection problem (for fixed orthonormal basis in $L_2(0,1)$ these sets correspond to ellipsoids of radii R with small ellipsoids of radii ρ_ε removed).

Observe evident inequality

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \le \beta(\alpha, V_\varepsilon(\tau', \rho'_\varepsilon)),$$

which follows from the natural inclusions

$$V_\varepsilon(\tau, \rho_\varepsilon) \subset V_\varepsilon(\tau', \rho'_\varepsilon),$$

when $p \ge p', r \le r', R \le R' \ q \le q', \ s \ge s', \ \rho_\varepsilon \ge \rho'_\varepsilon$.

1.2.2. *Besov bodies case*

In this case we consider the norms (or quasi-norms) $f_1 = f_{r,p,h},\ f_2 = f_{s,q,t}$ of Besov type, where if $p, h < \infty$, then

$$f_{r,p,h}(v) = \left(\sum_{j=1}^{\infty} \left(2^{jr} \left(\sum_{l=1}^{2^j} |v_{lj}|^p \right)^{1/p} \right)^h \right)^{1/h},$$

if $p < h = \infty$, then

$$f_{r,p,h}(v) = \left(\sup_{1 \le j < \infty} \left(2^{jr} \left(\sum_{l=1}^{2^j} |v_{lj}|^p \right)^{1/p} \right) \right),$$

if $h \le p = \infty$, then we have the analogous modifications. Here we consider $x = \{x_i\},\ v = \{v_i\} \in l_2$ as a pyramidal sequences: $x_i = x_{l,j},\ v_i = v_{l,j},\ j = 1, \dots,\ l = 1, \dots, 2^j,\ i = 2^j + l$. Note that there are some different definitions of Besov norm in sequence space (up to some finite-dimensional subspace); this difference is not essential to our study.

The sets $V_\varepsilon = V_\varepsilon(\tau, \rho_\varepsilon)$ are Besov bodies with "small" Besov bodies removed:

$$V_\varepsilon = B_{q,t;s}(R_{\varepsilon,2}) \setminus B_{p,h;r}(R_{\varepsilon,1});\quad R_{\varepsilon,2} = R/\varepsilon,\ R_{\varepsilon,1} = \rho_\varepsilon/\varepsilon, \tag{1.2}$$

where

$$B_{p,h;r}(R) = \{v \in l_2 : f_{r,p,h}(v) \le R\},\ \tau = (\kappa, R, t, h),\ 0 < t, h \le \infty,\ \kappa \in \Xi,$$

the values $\rho_\varepsilon > 0,\ \rho_\varepsilon \to 0$ are given.

By natural inclusions: if $p \ge p',\ r \le r',\ R \le R',\ q \le q',\ s \ge s',\ h' \le h,\ t' \ge t,\ \rho_\varepsilon \ge \rho'_\varepsilon$,, then

$$V_\varepsilon(\tau, \rho_\varepsilon) \subset V_\varepsilon(\tau', \rho'_\varepsilon),$$

one has evident inequality

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \le \beta(\alpha, V_\varepsilon(\tau', \rho'_\varepsilon)).$$

1.2.3. *Discussion*

It is well known that ellipsoids for $p = 2$ and for standard Fourier basis correspond to Sobolev balls of periodical r-smooth functions in L_2-norm.

There are no simple relations between Besov bodies $B_{p,h;r}(R)$ and ellipsoids $E_{p,r}(R)$. However note that if $p = h$, then the Besov body $B_{p,p;r}(R)$ is an ellipsoid of semi-axes $a_i = a_{lj} = R2^{-jr},\ l = 1, \dots, 2^j,\ i = 2^j + l$. This implies the inclusions $E_{p,r}(C_1 R) \subset B_{p,p;r}(R) \subset E_{p,r}(C_2 R)$ for positive constants $C_{1,2} = C_{1,2}(p, r)$.

Note also that Besov bodies $B_{p,h;r}$ for specific regular "wavelet"-basis correspond to Besov balls $B^\sigma_{p,h}$ of σ-smooth functions in the functional space $L_2(0,1)$ with $r = \sigma + 1/2 - 1/p$ (up to factors in radii and up to finite-dimensional balls), at least for $\sigma \ge 0; p, h \ge 1$; see Meyer [21], Cohen *et al.* [2]. These relations provide translations of the rate results from the case of the alternatives defined by Besov bodies in sequence space l_2 to the case of the alternatives defined by Besov balls in functional space $L_2(0,1)$ (see Donoho *et al.* [4,5]; Spokoiny [24, 25]).

The main subject of our interest is the sharp asymptotics for ellipsoidal case. Also we show that (excepted some "boundary" cases) the same (as for ellipsoids) rates hold for the case of Besov bodies with the same κ and do not depend on the R (it is assumed fixed or $R \asymp 1$) and on the parameters h and t which define "thin" structure of Besov norms.

Well known inclusions

$$B^\sigma_{p,\min\{p,2\}}(C_1 R) \subset S^\sigma_p(R) \subset B^\sigma_{p,\max\{p,2\}}(C_2 R)$$

(where $C_{1,2} = C_{1,2}(p, \sigma)$ are positive constants) provide the translation of rate results to Sobolev balls $S^\sigma_p(R)$ in functional space in these cases. The case of Sobolev ball with $r = 1/2 - 1/p$ corresponds to L_p-balls removed.

These facts provide the translation of results bellow from ellipsoidal and Besov bodies cases to the cases of alternatives defined by Besov or Sobolev balls in functional space $L_2(0,1)$.

There are some reasons to consider cases when we remove ellipsoids or Besov bodies and Besov or Sobolev balls with $r \neq 0$ and $\sigma \neq 0$. First, if $p \neq 2$, then L_p-ball in the functional space ($\sigma = 0$) roughly corresponds not to l_p-ball in sequence space but to ellipsoid or Besov body with $r = 1/2 - 1/p$. Next, the cases $\sigma \neq 0$ correspond to hypothesis testing on derivatives or on integrals of a signal of interest in many problems. Particularly, for the model of the sample from the interval $[0,1]$ with unknown probability density the case $\sigma = -1$ corresponds to hypothesis testing problem on uniformity of a density and alternative corresponds to the set of distribution functions on $[0,1]$ bounded away in L_p-norm from linear distribution function $F_0(t) = t$. It is well known, that in estimation and in hypothesis testing the we have classical rates in this case: the accuracy of estimation and the rate of testing is $n^{-1/2}$ where n is the sample size. If $\sigma = 0$, then it does not hold. It is of interest to describe the "boundary" between classical and nonclassical asymptotics (see Ingster [12]).

The problem of sharp asymptotics for ellipsoids were studied by Ermakov [6], Ingster [11–13] and by Suslina [26,27] for different values of $\tau, s > r$. In Ermakov [6] the case $p = q = 2$ had been investigated. In Ingster [11,12] the results for the cases $0 < p = q < \infty$ and $q \leq p = \infty$ had been obtained. In Suslina [26, 27] the cases $p \neq q,\ r = 0,\ s > 0$ had been studied.

For similar problems in functional space the rates were studied by Ingster [9,10] for Sobolev balls $S_2^\eta(R), p = 2$ and for Sobolev or Nikol'ski balls S_q^η with L_p-balls removed; $p \leq 2,\ q \geq p$ or $2 \leq p = q \leq \infty$; by Lepski and Spokoiny [19] for Sobolev balls $S_2^\eta(R)$ with L_2-balls removed, $p < 2,\ q\eta > 1$; by Spokoiny [25] for Besov balls $B_{q,t}^\eta(R)$ with L_p-balls removed for all $p, q \geq 1, \eta > 0, q\eta > 1$. Sharp asymptotics for Besov bodies $B_{q,t}^\eta(R)$ with L_2-balls removed were studied in Ingster and Suslina [16]. The results of these papers show that different asymptotics arise in these problems.

1.3. **Structure of the paper**

The main result of the paper is the classification of the types of asymptotics. We call these types *classical, trivial, degenerate and Gaussian* (of two main and some "boundary" types). In Sections 2 and 3 we describe sharp asymptotics for these types (except for "classical" type) for ellipsoidal case and the rates for Besov bodies (with the exception of "boundary" types). Also we describe the partitions of the set $\Xi = \{\kappa\} \subset R^4$ onto regions of different types of asymptotics. This partition is drowning on the plane $\{s, r\}$ for different values p, q (see Figs. 1–8 in Sect. 3.3).

In Section 4 we describe the asymptotical minimax or consistent test procedures for the cases of degenerate and Gaussian asymptotics.

In Sections 5–9 we give the proofs.

The main part of this paper deals with Gaussian asymptotics.

The study is based on reduction of the problem of finding asymptotically least favorable priors to specific convex extreme problem: to minimize Hilbert norm $\|\bar{\pi}\|$ of sequences $\bar{\pi}$ of measures π_i on the real line over specific convex sets (Sect. 5). We study these extreme problems for ellipsoids (Sect. 6) and for Besov bodies (Sect. 7). These studies are difficult enough and for Besov bodies cases we obtain only the rates. It seems very probable that if $p \geq h, q \leq t$, then for Besov bodies case analogous sharp asymptotics hold also (which depend on h, t). However the proof seems to include hard enough calculations and we do not consider this problem here. The same is for "boundary" problems in Besov bodies case.

Note that these methods seem to be close enough to the methods by Donoho and Johnstone [4] and Donoho *et al.* [5].

The proofs for degenerate and trivial types of the asymptotics are more simple. They are given in Sections 8 and 9.

Remark 1.1. One can regard as inconvenient the removing of alternatives too close to null-hypothesis and the restrictions on alternatives. Other variant of minimax setting is possible where these constraints are replaced by the introduction of some loss functions $r_\varepsilon(v)$ which characterize losses of an statistician to accept the null-hypothesis whenever the alternative v holds. The traditional setting corresponds to $r(v) = \mathbf{1}_{V_\varepsilon}(v)$. This setting is considered in Ingster [13] for the losses type of $r_\varepsilon(v) = g(f_1^p(v)/H_{\varepsilon,1}, f_2^q(v)/H_{\varepsilon,2})$ where f_k correspond to the ellipsoidal case (for Besov bodies case the analogous consideration is possible). Under some assumptions (the main is that the function $\log g(x,y)$ is concave) one can translate the results of these paper onto this setting.

Remark 1.2. It follows from results later that there is essential dependence of test procedures on the parameters τ for the case of Gaussian asymptotics. In the paper Ingster [13] we consider different (adaptive) variant of the problem which corresponds to the case of unknown parameters τ. In this case we assume that $\tau \in K$ for a given compact K and we consider the alternatives of the type

$$H_{\varepsilon,K} : v \in V_\varepsilon(K)$$

corresponding to all $\tau \in K$:

$$V_\varepsilon(K) = \bigcup_{\tau\in K} V_\varepsilon(\tau, \rho_\varepsilon(\tau))$$

with $\rho_\varepsilon = \rho_\varepsilon(\tau)$ be a given functions on K.

First adaptive setting had been considered by Spokoiny [24, 25] for Besov or Sobolev balls with L_p-balls removed. It was shown that it is not possible to distinguish hypothesis and alternative without losses in efficiency (type of $\log\log$-factor). The lower bounds for $p = 2$ and upper bounds for fixed $p \geq 1$ had been obtained in these papers.

We would like to obtains sharp adaptive asymptotics for ellipsoidal case and exact adaptive rates for the case of Besov bodies with Besov bodies removed, which imply rate adaptive asymptotics for Besov or Sobolev balls.

These studies will be based on the results of this paper and we give here the results in more general form to use ones later for investigation of adaptive setting.

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2. Non-Gaussian asymptotics

2.1. Classical type (C)

Denote $\Xi_C = \{\kappa \in \Xi : r < r_p\}$ where

$$r_p = \begin{cases} 1/4 - 1/p, & \text{if } p \leq 2, \\ -1/2p, & \text{if } 2 < p < \infty, \\ 0, & \text{if } p = \infty, \end{cases}$$

Theorem 1. *Let $\kappa \in \Xi_C$. Then*

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \to 0 \quad \textit{iff} \quad \rho_\varepsilon/\varepsilon \to \infty$$

and

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \to 1 - \alpha \quad \textit{iff} \quad \rho_\varepsilon/\varepsilon \to 0.$$

If $\rho_\varepsilon/\varepsilon \to \infty$, then minimax consistent families of tests $\psi_\varepsilon = \mathbf{1}_{\{L_{\varepsilon,p,r} > T_\varepsilon\}}$, $T_\varepsilon \to \infty$ are based on statistics

$$L_{\varepsilon,p,r} = \begin{cases} \sum_i i^{2rp/(4-p)}(x_i^2 - 1), & \textit{if } p \leq 2, \\ \sum_i i^{rp}|x_i|^p, & \textit{if } 2 < p < \infty, \\ \sup_i(|x_i|\varepsilon)/(\rho_\varepsilon(i^r - 1/2)), & \textit{if } p = \infty \end{cases}$$

for ellipsoids with evident modification of sums and changing r to any $r' \in (r, r_p)$ for Besov bodies.

The proof of Theorem 1 follows directly from Ingster [12], Theorem 2.5 and from the proofs in this paper for ellipsoids (one can make simple modifications for Besov bodies) and we omit it.

Thus the classical type (C) of the rates is defined by "minimum signal-noise ratio $\rho_\varepsilon/\varepsilon$" only. It is the same as for the case of the simple or finite dimensional alternatives. We do not consider this type later on and assume bellow that $\kappa \in \Xi^C = \{\kappa : r \geq r_p\}$.

2.2. **Trivial type (T)**

It was shown by Ibragimov and Khasminkii [7] that the problem is trivial for $S_\varepsilon = L_2(0,1) \setminus D_p(\rho)$ with $p = 2$ and any $\rho > 0,\ \varepsilon > 0$; for $p \neq 2$ this result follows from Burnashev [1]. The same holds for $V_\varepsilon = l_2 \setminus E_{p,r}(\rho)$ for any $\rho > 0$ and $r \geq r_p$ (see Ingster [12], Th. 2.5; for $p = 2$ it follows from Ermakov [6]).

It means the necessity of restrictions on alternatives for $r \geq r_p$ and for "ball-shaped" neighborhoods removed. It is easy to see that the problem is trivial for ellipsoidal case with $s \leq r,\ r \geq 0$, however it was shown by Suslina [26, 27] that the problem is also trivial for ellipsoidal case $V_\varepsilon(\tau, \rho_\varepsilon))$ if $p < q,\ r = 0$ and $s \leq s_{p,q}$ with

$$s_{p,q} = \begin{cases} (q-p)/pq, & \text{if } p \leq 4, \\ (q-p)/2q(p-2), & \text{if } p > 4. \end{cases}$$

This means that the restrictions are not enough to obtains nontrivial problem. Now we describe the regions $\Xi_T \subset \Xi^C$ of the trivial type.

Put for $p, q < \infty$:

$$\lambda = \lambda(\kappa) = qs - pr,\ \mu = \mu(\kappa) = pq(s-r),\ I = I(\kappa) = 2q(p-2)s - 2p(q-2)r + p - q$$

and if $q = \infty$, then $I = I(\kappa) = 2s(p-2) - 2rp - 1$. Define the set Ξ_T by the inequality $r \geq r_p$ as well as by the following inequalities. If $p, q < \infty$, then

$$\begin{cases} \mu \leq 0 \ \&\ \lambda \leq 0 \ \&\ I \geq 0, & \text{if } 2 > p > q, \\ \mu \leq q - p \ \&\ I \leq 0, & \text{if } 2 < p < q, \\ \mu \leq 0 \ \&\ \lambda \leq 0, & \text{if } p \geq 2,\ p > q, \\ \mu \leq q - p, & \text{if } p \leq 2,\ p \leq q \text{ or } p = q > 2. \end{cases}$$

If $q = \infty,\ p < \infty$, then

$$\begin{cases} s - r \leq 1/p, & \text{if } p < 2, \\ s - r \leq 1/p \ \&\ I \leq 0, & \text{if } p \geq 2, \end{cases}$$

and if $p = \infty,\ q \leq \infty$, then $s \leq r$ and $r \geq 0$.

For boundary case $r = r_p$ these inequalities are equivalent to the following:

$$s \leq s^*_{pq} = \begin{cases} 1/4 - 1/q, & \text{if } p < 2 \text{ or } p = 2,\ q \geq 2, \\ -1/2q, & \text{if } p > 2 \text{ or } p = 2,\ q < 2\ . \end{cases}$$

Theorem 2. *Let $\kappa \in \Xi_T$ and if $\mu = 0$, then $R > \rho_\varepsilon$. Then the problem is trivial for ellipsoidal and Besov bodies case:*

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) = (1 - \alpha). \tag{2.1}$$

The proof of Theorem 2 is given in Section 9.

2.3. **Degenerate type**

This type is characterized by the asymptotics

$$\beta(\alpha, V_\varepsilon(\tau,\rho_\varepsilon)) = (1-\alpha)\Phi(R_\varepsilon(\tau,\rho_\varepsilon)) + o(1)$$

where

$$R_\varepsilon(\tau,\rho_\varepsilon) = \sqrt{2\log n_\varepsilon} - n_\varepsilon^{-r}\rho_\varepsilon/\varepsilon, \quad n_\varepsilon = n_\varepsilon(\tau,\rho_\varepsilon) = (R/\rho_\varepsilon)^{1/(s-r)} \tag{2.2}$$

(note that $s > r \geq 0,\ p > q$ for this type) which implies

$$\beta(\alpha, V_\varepsilon(\tau,\rho_\varepsilon)) = (1-\alpha)\Phi\left(\sqrt{\frac{2\log(R/\rho_\varepsilon)}{s-r}} - \rho_\varepsilon^{s/(s-r)}R^{-r/(s-r)}\varepsilon^{-1}\right) + o(1). \tag{2.3}$$

Here and later Φ stands for standard Gaussian distribution function.

This type had been described by Ingster [12], Theorem 3.3 for $p=\infty,\ q \leq p;\ r=0$ and follows from Ingster [12], Theorem 3.4 for $s \geq p \geq 0$. The asymptotically minimax tests are based on simple thresholding in this case.

We use the term "degenerate" in this case by likelihood ratio $L_\varepsilon(\tau,\rho_\varepsilon) = dP_{\pi^\varepsilon}/dP_0$ for asymptotically least favorable prior π^ε has asymptotically degenerate distribution for null-hypothesis: $L_\varepsilon(\tau,\rho_\varepsilon) - \Phi(R_\varepsilon(\tau,\rho_\varepsilon)) \to 0$ under P_0-probability.

This type allows boundary between distinguishability and indistinguishability. Put the critical radii of removing sets and constants:

$$\rho_\varepsilon^*(\tau) = R\left((\varepsilon/R)^2\log\varepsilon^{-1}\right)^{(s-r)/2s},\ \Lambda(\tau) = \Lambda_1(\tau) = \Lambda_2(\tau) = (2/s)^{(s-r)/2s}.$$

This corresponds to the relation

$$\Lambda^{r/(s-r)}(\tau)\rho_\varepsilon^*(\tau) \sim \varepsilon(n_\varepsilon(\tau,\rho_\varepsilon^*(\tau)))^r\sqrt{2\log n_\varepsilon(\tau,\rho_\varepsilon^*(\tau))}.$$

Then for any $\alpha \in (0,1)$ one has

$$\beta(\alpha, V_\varepsilon) \to 0 \quad \text{if} \quad \liminf \rho_\varepsilon/\rho_\varepsilon^*(\tau) > \Lambda_1(\tau) \tag{2.4}$$

and

$$\beta(\alpha, V_\varepsilon) \to 1-\alpha \quad \text{if} \quad \limsup \rho_\varepsilon/\rho_\varepsilon^*(\tau) < \Lambda_2(\tau). \tag{2.5}$$

Using the translation

$$r = \sigma + 1/2 - 1/p,\ \ s = \eta + 1/2 - 1/q \tag{2.6}$$

we can rewrite the rates in terms of smoothness parameters σ, η for white Gaussian noise model:

$$\rho_\varepsilon^* = (\varepsilon^2\log\varepsilon^{-1})^{(\eta-\sigma-1/q+1/p)/(2\eta-2/q+1)}. \tag{2.7}$$

The conditions close to (2.4) and (2.5) arise in functional space for the balls of Holder η- smooth function with L_∞-balls removed, see Ingster [10, 12], where the relations (2.4) and (2.5) with different values $\Lambda_1(\tau) > \Lambda_2(\tau)$ had been obtained (note that this case corresponds to $s = \eta + 1/2,\ r = 1/2$). Lepski [17] had shown that there is the equality: $\Lambda_1(\tau) = \Lambda_2(\tau)$ for $\eta \le 1$; Lepski and Tsybakov [20]) had shown that this equality holds for $\eta > 1$ also. For finite p this type of rate asymptotics (with different $\Lambda(\tau) = \Lambda_{1,2}(\tau)$) arises in Spokoiny [25].

Note that the rates (2.7) in the region Ξ_D (possibly, excepted the boundary) are the same that in minimax signal estimation problem in white Gaussian noise model (assuming the losses are defined by Sobolev or Besov norm with parameters (p,σ) and signal set is the ball in Sobolev or Besov norm with parameters (q,η)); see Donoho *et al.* [4, 5] and Lepski *et al.* [18].

In our problem for ellipsoidal case we get the asymptotics of degenerate type in the region

$$\Xi_D = \{\kappa \in \Xi^T : \ s > r > 0, \ \lambda \le 0\}, \quad \Xi^T = \{\kappa \in \Xi^C : \ \kappa \notin \Xi_T\}.$$

For Besov bodies with $q < t$ we consider the "interior" of Ξ_D only: we assume $\kappa \in \Xi_D, \ \lambda < 0$.

Note that there exists common family of tests $\psi_{\varepsilon,\alpha}$ which asymptotically minimax for any $\kappa \in \Xi_D, \ R > 0$. These test procedures are described in Section 4.1.

Theorem 3. *Let $\kappa \in \Xi_D$. Then*

1. *For ellipsoidal case the sharp asymptotics (2.3) hold.*
2. *For Besov bodies case let $\lambda < 0$, if $q < t$. Then there exist such constants $c_1 = c_1(\tau) > 0, \ c_2 = c_2(\tau) > 0$ which are bounded away from 0 and ∞ on any compact in Ξ_D that*

$$(1-\alpha)\Phi(R_\varepsilon(\tau,\rho_\varepsilon,c_1)) + o(1) \le \beta(\alpha, V_\varepsilon(\tau,\rho_\varepsilon)) \le (1-\alpha)\Phi(R_\varepsilon(\tau,\rho_\varepsilon,c_2)) + o(1)$$

where

$$R_\varepsilon(\tau,\rho_\varepsilon,c) = \sqrt{2\log n_\varepsilon(\tau)} - (cn_\varepsilon(\tau))^{-r}\rho_\varepsilon/\varepsilon, \quad n_\varepsilon(\tau) = (R/\rho_\varepsilon)^{1/(s-r)}.$$

These relations imply the rates (2.4), (2.5) with some (different) constants $\Lambda_1(\tau) > \Lambda_2(\tau)$ which are bounded away from 0 and ∞ on any compact in Ξ_D.

Proof of Theorem 3 is given in Section 8.

3. Gaussian asymptotics

3.1. Types G_1 and G_2

These types of asymptotics seem to be the most important and interesting. For ellipsoidal case these types are characterized by the asymptotics

$$\beta(\alpha, V_\varepsilon(\tau,\rho_\varepsilon)) = \Phi(T_\alpha - u_\varepsilon(\tau,\rho_\varepsilon)) + o(1). \tag{3.1}$$

Here T_α stands for $(1-\alpha)$-quantile of the standard Gaussian distribution: $\Phi(T_\alpha) = 1-\alpha$. The function $u_\varepsilon(\tau,\rho_\varepsilon) = u_\varepsilon$ characterizes the minimax distinguishability.

There are two main types of this function:

$$u_\varepsilon^2(\tau,\rho_\varepsilon) \sim d(\kappa)(\rho_\varepsilon/R)^{A_k(\kappa)}(\varepsilon/R)^{-B_k(\kappa)}, \ k = 1,2; \tag{3.2}$$

where $d(\kappa) > 0$.

For the type G_1 one has:

$$B_1(\kappa) = 4, \quad A_1(\kappa) = \begin{cases} \frac{p(4-q+4sq)}{pq(s-r)+p-q}, & \text{if } q < \infty \\ \frac{p(4s-1)}{p(s-r)-1}, & \text{if } q = \infty \end{cases} \tag{3.3}$$

and for the type G_2 one has:

$$A_2(\kappa) = \begin{cases} \frac{p(1+2sq)}{qs-pr}, & \text{if } q<\infty \\ 2p, & \text{if } q=\infty \end{cases}$$
$$B_2(\kappa) = \begin{cases} \frac{2pq(s-r)+p-q}{qs-pr}, & \text{if } q<\infty \\ \frac{2p(s-r)-1}{s}, & \text{if } q=\infty. \end{cases} \tag{3.4}$$

Put $\Xi^D = \Xi^T \setminus \Xi_D$. Let us define the sets

$$\Xi_{G_1} = \{\kappa \in \Xi^D : r > r_p \ \&\ \{I(\kappa) < 0 \text{ or } p = q = 2\}\}$$

and

$$\Xi_{G_2} = \{\kappa \in \Xi^D : r > r_p \ \&\ I(\kappa) > 0\}\cdot$$

Theorem 4. *For ellipsoidal cases the relations (3.1, 3.2) hold where, if $k = 1,\ \kappa \in \Xi_{G_1}$, then the values $A_k,\ B_k$ are defined by (3.3), and if $k = 2,\ \kappa \in \Xi_{G_2}$, then ones are defined by (3.4). Here $d(\kappa)$ is a positive function on the regions Ξ_{G_1} and Ξ_{G_2} which is bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_k},\ k = 1, 2$.*

Proof of Theorem 4 is given in Sections 5, 6.

Remark 3.1. It follows from the proof later that the function $d(\kappa)$ is continuous Lipschitz function except for, may be, some 3-dimensional sub-manifolds in $\Xi_{G_k},\ k = 1, 2$.

Very cumbersome relations (3.1–3.4) correspond to the solution simple enough equations on the values $z_0 = z_{0,\varepsilon}(\kappa),\ m = m_{0,\varepsilon}(\kappa)$ or $h_0 = h_{0,\varepsilon}(\kappa),\ n = n_{0,\varepsilon}(\kappa)$. For the type G_1 one has:

$$u_\varepsilon^2 \sim c_0(\kappa) m z_0^4, \tag{3.5}$$

where

$$c_1(\kappa) m^{1+pr} z_0^p \sim (\rho_\varepsilon/\varepsilon)^p, \ \begin{cases} c_2(\kappa) m^{1+qs} z_0^q \sim (\varepsilon/R)^{-q}, & \text{if } q<\infty, \\ c_2(\kappa) m^s z_0 \sim (\varepsilon/R)^{-1}, & \text{if } q=\infty \end{cases} \tag{3.6}$$

and for the type G_2 one has:

$$u_\varepsilon^2 \sim c_0(\kappa) n h_0^2, \tag{3.7}$$

where

$$c_1(\kappa) n^{1+pr} h_0 \sim (\rho_\varepsilon/\varepsilon)^p, \ \begin{cases} c_2(\kappa) n^{1+qs} h_0 \sim (\varepsilon/R)^{-q}, & \text{if } q<\infty, \\ c_2(\kappa) n^s \sim (\varepsilon/R)^{-1}, & \text{if } q=\infty. \end{cases} \tag{3.8}$$

Here $c_{0,1,2}(\kappa) > 0$ are functions which are bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_k},\ k = 1, 2$.

These relations are proved in Sections 5, 6. Direct relations for the functions $c_{0,1,2}(\kappa)$ are presented in Section 6 as well. The case $p = q$ is considered later in this section.

Note, that if the values u_ε defined by (3.2) satisfy $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta = \delta(\kappa) > 0$, then the accurate of the relations (3.5–3.8) is $(1 + o(\varepsilon^{\delta_1}))$ where $\delta_1 = \delta_1(\kappa, \delta) > 0$.

The asymptotics of type G_1 arise in Ermakov [6] for $p = q = 2$, in Ingster [11,12] for $p = q \le 2$, in Suslina [26] for $p \le 2, q > p$. The asymptotical minimax families of tests $\psi_{\varepsilon,\alpha} = \mathbf{1}_{\{L_{\varepsilon,\tau} > T_\alpha\}}$ in these cases are based on the statistics

$$L_\varepsilon = L_{\varepsilon,\tau} = u_\varepsilon^{-1} \sum_i z_{\varepsilon,i}^2 (x_i^2 - 1)$$

where $z_\varepsilon = z_\varepsilon(\tau, \rho_\varepsilon)$ are families of sequences,

$$\frac{1}{2}\sum_i z_{\varepsilon,i}^4 = u_\varepsilon^2.$$

The direct description of these families can be given for $p = q \le 2$:

$$z_{\varepsilon,i} = z_0(y^{rp} - y^{sp})_+^{1/(4-p)},\ y = i/m \tag{3.9}$$

and for $r > r_p$ the values $u_\varepsilon,\ z_0,\ m$ are defined by relations (3.5, 3.6) with

$$\begin{aligned} c_0(\kappa) &\sim \frac{1}{2}\int_0^1 (y^{rp} - y^{sp})^{4/(4-p)}dy,\\ c_1(\kappa) &\sim \int_0^1 (y^{rp} - y^{sp})^{p/(4-p)} y^{rp} dy,\\ c_2(\kappa) &\sim \int_0^1 (y^{rp} - y^{sp})^{p/(4-p)} y^{sp} dy. \end{aligned} \tag{3.10}$$

The asymptotics of the type G_2 arise in Ingster [11,12] for $2 < p = q < \infty$. The asymptotical minimax families of tests $\psi_{\varepsilon,\alpha} = \mathbf{1}_{\{L_{\varepsilon,\tau} > T_\alpha\}\cup X_\varepsilon}$ in this case are based on the statistics

$$L_\varepsilon = L_{\varepsilon,\tau} = u_\varepsilon^{-1}\sum_i^{n_\varepsilon} h_{\varepsilon,i}\xi(x_i, z(p))$$

and on the threshold procedure

$$X_\varepsilon = \left\{\max_{1\le i\le n_\varepsilon} |x_i| > \sqrt{2\log n_\varepsilon}\right\}$$

where

$$\xi(x,z) = e^{-z^2/2}\cosh zx - 1. \tag{3.11}$$

Here $\bar h_\varepsilon = \bar h_\varepsilon(\tau)$ for $r > r_p$ are the families of sequences $h_{\varepsilon,i} \in [0,1]$:

$$h_{\varepsilon,i} = h_0(x^{rp} - x^{sp})_+,\ x = i/n, \tag{3.12}$$

the values $u_\varepsilon(\tau),\ h_0,\ n$ are defined by relations (3.7, 3.8) with

$$\begin{aligned} c_0(\kappa) &\sim 2\sinh^2(z^2(p)/2)\int_0^1 (x^{rp} - x^{sp})^2 dx,\\ c_1(\kappa) &\sim z^p(p)\int_0^1 (x^{rp} - x^{sp}) x^{rp} dx,\\ c_2(\kappa) &\sim z^p(p)\int_0^1 (x^{rp} - x^{sp}) x^{sp} dx. \end{aligned} \tag{3.13}$$

Here and later for $p > 2$ we denote by $z(p)$ positive values defined by the relation

$$p\tanh(z^2(p)/2) = z^2(p).$$

If $p < 2$, then we put $z(p) = 0$. These values minimize the functions $f_p(z) = z^{-p}\sinh(z^2/2)$.

The results (3.5–3.13) are presented in Ingster ([11, 12], Ex. 3.1, 3.3 for $s > r = 0$) and are obtained in Section 6 for general case $s > r > r_p$; for $s > r = r_p$ the asymptotics are of different forms (see Sect. 3.2 and Sect. 6 later).

The types G_1 and G_2 arise in Suslina [27] for $r = 0,\ q \neq p < \infty,\quad s > s_{p,q}$.

It is clear that using the relation (3.1) we get the rates which are described by critical radii (rates in Spokoiny [24, 25])

$$\rho^*_\varepsilon(\kappa) = \varepsilon^{B_k(\kappa)/A_k(\kappa)}, \quad k = 1, 2. \tag{3.14}$$

It means

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \to 0 \ \text{ iff } \quad \rho_\varepsilon/\rho^*_\varepsilon(\kappa) \to \infty \tag{3.15}$$

and

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \to 1 - \alpha \ \text{ iff } \quad \rho_\varepsilon/\rho^*_\varepsilon(\kappa) \to 0. \tag{3.16}$$

Using the translation (2.6) we can rewrite the rates for for white Gaussian noise model with $\sigma, \eta \geq 0,\ p, q \geq 1$:

$$\rho^*_\varepsilon(\sigma, \eta, p, q) = \varepsilon^{C_k},\ k = 1, 2,$$

where

$$C_1 = \frac{4(\eta - \sigma)}{4\eta + 1},\ C_2 = \frac{2(\eta - \sigma) + p^{-1} - q^{-1}}{2\eta + 1 - q^{-1}}\cdot$$

These rates for $\sigma = 0$ were obtained in Ingster [9, 10, 12] for Sobolev balls S^η_q with L_p-balls of radii ρ_ε removed (the type G_1, if $p \leq 2,\ q \geq p$ and the type G_2, if $p = q < \infty$); in Lepski and Spokoiny [19], Ingster and Suslina [16] for $q = 2,\ p < 2$ (type G_2); in Spokoiny [25] the rates of the types G_1 and G_2 were obtained also (up to logarithmical factor).

Note that in the regions of main types of Gaussian asymptotics these rates are smaller than the rates in analogous minimax estimation problem that were obtained by Donoho *et al.* [4, 5] an by Lepski *et al.* [18].

It is clear that the rates (3.15, 3.16) with the critical radii (3.14) follow from the inequalities

$$\Phi(T_\alpha - d_1 u_\varepsilon(\kappa, R, \rho_\varepsilon)) + o(1) \leq \beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \leq \Phi(T_\alpha - d_2 u_\varepsilon(\kappa, R, \rho_\varepsilon)) + o(1) \tag{3.17}$$

where $u_\varepsilon(\kappa, R, \rho_\varepsilon)$ are defined by either (3.5, 3.6) or (3.7, 3.8).

Theorem 5. *For Besov bodies case and small enough $\varepsilon > 0$ the relation (3.17) holds where $u_\varepsilon(\kappa, R, \rho_\varepsilon)$ are defined by either (3.5, 3.6), if $\kappa \in \Xi_{G_1}$, or (3.7, 3.8), if $\kappa \in \Xi_{G_2}$ with $c_l(\kappa) = 1, l = 0, 1, 2$. Here $d_1 = d_1(\kappa, R) > 0$ $d_2 = d_2(\kappa, R) > 0$.*

Proof of Theorem 5 is given in Section 7.

3.2. Boundary log-types of Gaussian asymptotics

Let as consider also the "boundary" sets

$$\Xi_{G_3} = \{\kappa \in \Xi^D : r > r_p,\ I(\kappa) = 0 \text{ without } p = q = 2\},$$

$$\Xi_{G_4} = \{\kappa \in \Xi^D : r = r_p,\ s > s^*_{pq},\ p < 2 \text{ or } \ p = 2,\ p \leq q\ \},$$

$$\Xi_{G_5} = \{\kappa \in \Xi^D : r = r_p,\ s > s^*_{pq},\ p > 2 \text{ or } \ p = 2,\ p > q\}\cdot$$

Note, that $A_1(\kappa) = A_2(\kappa),\ B_1(\kappa) = B_2(\kappa)$ for $\kappa \in \Xi_{G_3}$ (these values are defined by (3.3, 3.4)). Observe that the set Ξ_{G_3} is the boundary between Ξ_{G_1} and Ξ_{G_2}, the set Ξ_{G_4} is the boundary between Ξ_{G_1} and Ξ_C, the set Ξ_{G_5} is the boundary between Ξ_{G_2} and Ξ_C.
For $\kappa \in \Xi_{G_3}$ put

$$u_\varepsilon^2 \sim d(\kappa)(\rho_\varepsilon/R)^{A_2(\kappa)}/(\varepsilon/R)^{B_2(\kappa)} \log \varepsilon^{-1}. \tag{3.18}$$

For $\kappa \in \Xi_{G_4}$ put

$$u_\varepsilon^2 \sim d(\kappa)(\rho_\varepsilon/\varepsilon)^4 (\log \varepsilon^{-1})^{(p-4)/p}. \tag{3.19}$$

For $\kappa \in \Xi_{G_5}$ put

$$u_\varepsilon^2 \sim d(\kappa)(\rho_\varepsilon/\varepsilon)^{2p}/\log \varepsilon^{-1}. \tag{3.20}$$

Theorem 6. *For ellipsoidal case the relation (3.1) holds. If $\kappa \in \Xi_{G_3}$, then the values u_ε satisfy (3.18), if $\kappa \in \Xi_{G_4}$, then ones satisfy (3.19), if $\kappa \in \Xi_{G_5}$ then ones satisfy (3.20). Here $d(\kappa)$ are positive functions on the regions $\Xi_{G_3} - \Xi_{G_5}$ which are bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_k},\ k = 3, 4, 5$.*

Proof of Theorem 6 is given in Sections 5, 6.

For the case $r = 0$ the asymptotics type G_3 arise in Suslina [27].

These sharp asymptotics imply the rates (3.15, 3.16) with critical radii of the following form: if $\kappa \in \Xi_{G_3}$, then

$$\rho_\varepsilon^*(\kappa) = \varepsilon^{B_2(\kappa)/A_2(\kappa)} (\log \varepsilon^{-1})^{1/A_2(\kappa)}; \tag{3.21}$$

if $\kappa \in \Xi_{G_4}$, then

$$\rho_\varepsilon^*(\kappa) = \varepsilon (\log \varepsilon^{-1})^{(4-p)/4p}; \tag{3.22}$$

and if $\kappa \in \Xi_{G_5}$, then

$$\rho_\varepsilon^*(\kappa) = \varepsilon (\log \varepsilon^{-1})^{1/2p}. \tag{3.23}$$

Thus the asymptotics are close to classical in the regions $G_4,\ G_5$ (the difference is in log-factor only).

If $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta > 0$, then $\log m_\varepsilon \asymp \log n_\varepsilon \asymp \log h_0^{-1} \asymp \log \varepsilon^{-1}$ and relations (3.18–3.20) correspond to the values $u_\varepsilon(\tau)$ which are defined by the values $z_0 = z_{0,\varepsilon}(\kappa),\ m = m_\varepsilon(\kappa)$ or $h_0 = h_{0,\varepsilon}(\kappa),\ n = n_\varepsilon(\kappa)$ determined by the following relations. If $\kappa \in \Xi_{G_3}$, then

$$u_\varepsilon^2 \sim c_0(\kappa) n h_0^2 \log h_0^{-1} \tag{3.24}$$

where

$$c_1(\kappa) n^{1+pr} h_0 \log h_0^{-1} = (\rho_\varepsilon/\varepsilon)^p, \quad \begin{cases} c_2(\kappa) n^{1+qs} h_0 \log h_0^{-1} = (\varepsilon/R)^{-q}, & \text{if } q < \infty \\ c_2(\kappa) n^s = (\varepsilon/R)^{-1}, & \text{if } q = \infty. \end{cases} \tag{3.25}$$

If $\kappa \in \Xi_{G_4}$, then

$$u_\varepsilon^2 \sim c_0(\kappa) m z_0^4 \log m, \tag{3.26}$$

where

$$c_1(\kappa)m^{p/4}z_0^p\log m = (\rho_\varepsilon/\varepsilon)^p, \quad \begin{cases} c_2(\kappa)m^{1+qs}z_0^q = (\varepsilon/R)^{-q}, & \text{if } q<\infty \\ m^s z_0 = (\varepsilon/R)^{-1}, & \text{if } q=\infty. \end{cases} \tag{3.27}$$

If $\kappa \in \Xi_{G_5}$, then

$$u_\varepsilon^2 \sim c_0(\kappa)nh_0^2\log n \tag{3.28}$$

where

$$c_1(\kappa)n^{1/2}h_0\log n = (\rho_\varepsilon/\varepsilon)^p, \quad \begin{cases} c_2(\kappa)n^{1+qs}h_0 = (\varepsilon/R)^{-q}, & \text{if } q<\infty \\ c_2(\kappa)n^s = (\varepsilon/R)^{-1}, & \text{if } q=\infty. \end{cases} \tag{3.29}$$

These relations are proved in Sections 5 and 6. Direct relations for the functions $c_{0,1,2}(\kappa)$ are presented in Section 6 as well.

Remark 3.2. We assume later in the proofs of upper bounds in the theorems and in some estimations that $u_\varepsilon(\tau,\rho_\varepsilon) = O(\varepsilon^{-\delta})$ for any $\delta>0$, $\kappa\in\Xi_{G_l},\ l=1,2,3$ and $u_\varepsilon(\tau,\rho_\varepsilon)=O(1)$ for $\kappa\in\Xi_{G_l},\ l=4,5$[1].

We consider these relations as the assumptions on the values $\rho_\varepsilon=\rho_\varepsilon(\kappa)$ for the values u_ε defined by the relations of the type (3.2).

These assumptions are not essential. In fact, if $l=1,2,3$ and $u_\varepsilon(\tau,\rho_\varepsilon)\varepsilon^\delta\to\infty$, then, by making ρ_ε smaller, we can get $u_\varepsilon(\tau,\rho_\varepsilon)\asymp\varepsilon^{-\delta}$ and the values $\beta(\alpha,V_\varepsilon(\tau,\rho_\varepsilon))$ are not decrease, but still $\beta(\alpha,V_\varepsilon(\tau,\rho_\varepsilon))\to 0$, as $\varepsilon\to 0$ by the Theorems. The case $l=4,5$ and $u_\varepsilon(\tau,\rho_\varepsilon)\to\infty$ is considered by similar way.

The reader can assume for simplicity $u_\varepsilon(\tau,\rho_\varepsilon)=O(1)$ for $\kappa\in\Xi_{G_l},\ l=1,2,3$ (it is enough to the goals of this paper). We consider more general assumption to make the basis for study of adaptive problems later where we need to consider the case $u_\varepsilon^2(\tau,\rho_\varepsilon)\asymp\log\log\varepsilon^{-1}$.

One can easy check that under assumptions above the following relations hold. If $\kappa\in\Xi_{G_l},\ l=1,3,4$, then

$$z_0m^{-\lambda/(p-q)}\to 0,\ \text{if } p>q;\ z_0m^{-rp/(4-p)}\to 0,\ \text{if } p\le 2;\ z_0\to 0,\ m\to\infty.$$

If $\kappa\in\Xi_{G_l},\ l=1,3$, then

$$z_0=O(\varepsilon^{\delta_1}),\ m^{-1}=O(\varepsilon^{\delta_2}).$$

Also if $\kappa\in\Xi_{G_k},\ k=2,3,5$, then

$$n^{-rp}h_0\to 0,\ \text{if } p>2;\ h_0\to 0,\ n\to\infty.$$

If $\kappa\in\Xi_{G_l},\ l=2,3$, then

$$h_0=O(\varepsilon^{\delta_1}),\ n^{-1}=O(\varepsilon^{\delta_2}).$$

Here $\delta_1=\delta_1(\kappa),\ \delta_2=\delta_2(\kappa)$ are some positive values. We will use these relations in the proofs.

Remark 3.3. Note without proofs that for Besov bodies case the rate asymptotics (3.15, 3.16) hold with critical radii analogous to (3.21–3.23), if $\kappa\in G_3 - G_5$. However the power degree of log-factors depends on the parameters t,h.

[1]This assumption should be extended onto $u_\varepsilon(\tau,\rho_\varepsilon)=O((\log\varepsilon^{-1})^\delta),\ \kappa\in\Xi_{G_l},\ l=4,5$.

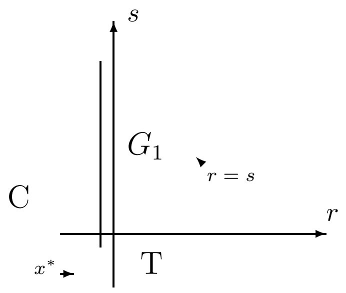

FIGURE 1. $p = q \leq 2$.

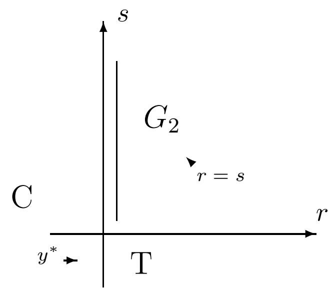

FIGURE 2. $2 < p = q < \infty$.

3.3. Graphical representation

In this section we describe the partition of the planes of the parameters $\{r, s\}$ onto the regions of the asymptotics of different types for fixed values p, q in ellipsoidal case. Remind that the same partition hold for Besov bodies in the sequences space as well. In functional space for the case of Sobolev balls $S_q^\eta(R)$ with Sobolev balls $S_p^\sigma(\rho_\varepsilon)$ removed and for the case of Besov balls $B_{q,t}^\eta(R)$ with Besov balls $B_{p,h}^\sigma(\rho_\varepsilon)$ removed one can get partitions for $\sigma \geq 0,\ \eta \geq 0,\ p \geq 1,\ q \geq 1$ using the translation (2.6) which corresponds to the moving of origin of coordinates to the point $(1/2 - 1/p, 1/2 - 1/q)$ on the pictures. This point is the beginning of vertical half-line (the case $\sigma = 0,\ \eta > 0$) that is presented on the pictures and corresponds to L_p-balls removed.

In Figures 1 and 2 we show the partitions for finite $p = q$. The classical asymptotics C correspond to $r < r_p$ with $r_p = 1/4 - 1/p$, if $p \leq 2$ and $r_p = -1/2p$, if $p > 2$. If $r \geq r_p$, then we have trivial case T for $s \leq r$, of course. Note that regions of the type T are closed on all pictures later.

If $s > r > r_p$, then we have Gaussian asymptotics of the type G_1 for $p \leq 2$, and of the type G_2 for $p > 2$. The boundary $r = r_p,\ s > r_p$ between C and either G_1 or G_2 corresponds to the types either G_4 or G_5. The vertical line on the pictures corresponds to case of functional space: $\sigma = 0$.

The case $q \leq p = \infty$ is presented in Figure 3.

The region C of the classical asymptotics corresponds to $r < 0$ and the Gaussian asymptotics G are replaced onto degenerate D in this case. These results are presented in Ingster [10, 11].

Next pictures correspond to $p < \infty,\ p \neq q$. We denote as $x^* = x^*_{p,q}$ and $y^* = y^*_{p,q}$ the points on the plain $\{r, s\}$ with the coordinates

$$x^* = (1/4 - 1/p,\ 1/4 - 1/q),\ y^* = (-1/2p,\ -1/2q)$$

with evident modification for $q = \infty$.

The case $p \leq 2,\ p \leq q \leq \infty$ (see Fig. 4) is close to $p = q \leq 2$: we have the regions $C,\ T,\ G_1$ with some translation of the boundary between the regions of trivial and Gaussian types; the boundary between C and G_1 corresponds to G_4 as well. Therefore, if $r = 0,\ s > 0$, then we have the interval $(0,\ (q-p)/pq]$ of trivial type. These results are presented in Suslina [26].

The case $2 < p < q \leq \infty$ is presented in Figures 5 and 6. The boundary between the regions G and T is not linear in this case: the break point is x^*. We have the regions G_1 and G_2 of main types of Gaussian asymptotics and we have the type G_3 on the boundary half-line $I = 0$ from the point x^*. The boundary between G_2 and C corresponds to the type G_5. The difference between the cases $p < 4$ and $p > 4$ is the position of the point x^*. These results for $r = 0$ are presented in Suslina [27]. Note that if $r = 1/2 - 1/p$ (vertical half-line), then we have the interval $0 < \eta < (q-p)/2pq$ of the type G_1 and half-line $\eta > (q-p)/2pq$ of the type G_2. These results for functional space and L_p-balls removed are presented in Spokoiny [25] (up to loglog-factor and some additional restrictions).

The most interesting cases seem to be $p > q$ (see Figs. 7 and 8).

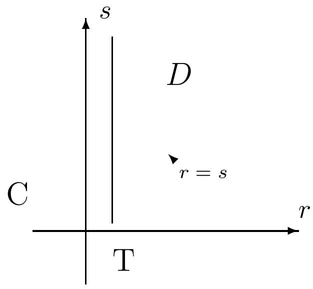

FIGURE 3. $q \le p = \infty$.

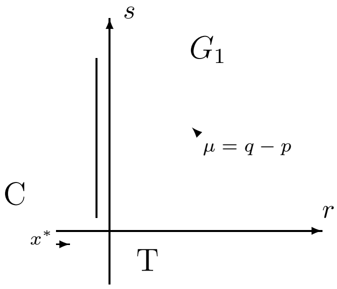

FIGURE 4. $p \le 2,\ p \le q \le \infty$.

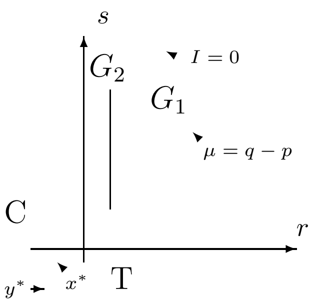

FIGURE 5. $2 < p < q \le \infty;\ p \le 4$.

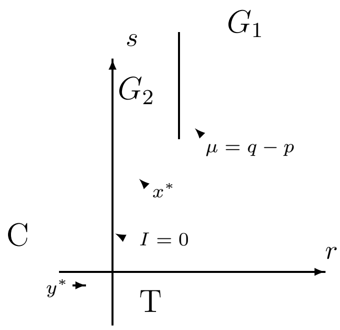

FIGURE 6. $2 < p < q \le \infty;\ p > 4$.

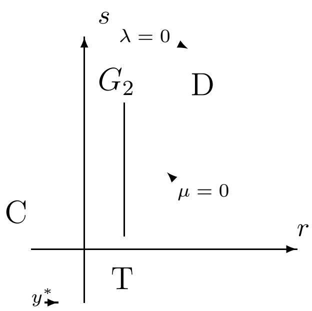

FIGURE 7. $p > q,\ p \ge 2$.

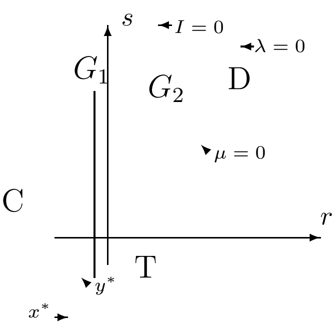

FIGURE 8. $2 > p > q$.

We have regions D of degenerate type here. If $p \ge 2$, then main Gaussian type is G_2; boundary type G_5 (the boundary between G_2 and C for $r = -1/2p$) is presented as well. For $r = 1/2 - 1/p$ (vertical half-line) we have the interval $I_T:\ 0 < \eta < (p-q)/pq$ of the type T, the interval $I_D:\ (p-q)/pq < \eta < (p-q)/2q$ of the type D and half-line $\eta > (p-q)/2q$ of the type G_2. These results for functional space and L_p-balls removed are presented in Spokoiny [25] (up to loglog-factor and some additional restrictions).

If $p < 2$, then all main types of the asymptotics are presented (with the exception of boundary G_5-type). The boundary of the region T has break points x^*, y^* and $(0,0)$. For $r = 1/2 - 1/p$ (vertical half-line) we have the interval $I_T:\ 0 \le \eta < (p-q)/2q$ of the type T, the interval $I_{G_2}:\ (p-q)/2q < \eta < (p-q)/2q(2-p)$

of the type G_2 (for $p > 1$) and half-line $\eta > (p-q)/2q(2-p)$ of the type G_1. These results are presented in Spokoiny [25] (up to the part of the interval I_{G_2}). The case $r = 0$ was considered by Suslina [27].

4. Test procedures

4.1. Degenerate case

We describe common asymptotically minimax test procedures which do not depend on $\kappa \in \Xi_D$ and provide the upper bounds in Theorem 3.

Theorem 7. *Let $\kappa \in \Xi_D$. Then*

1) *For ellipsoidal case let us consider the tests*

$$\psi_{\varepsilon,\alpha} = (1-\alpha)\mathbf{1}_{X_\varepsilon} + \alpha \tag{4.1}$$

which are based on the thresholding

$$X_\varepsilon = \left\{ \max_{1\le i\le N_\varepsilon} |x_i| > \sqrt{2\log N_\varepsilon} \right\} \cup \left\{ \sup_{N_\varepsilon < i < \infty} |x_i|/T_i > 1 \right\} \tag{4.2}$$

with $T_i = \sqrt{2\log i + 2\log\log i}$ and $N_\varepsilon \asymp \log \varepsilon^{-1}$. Then $\alpha(\psi_{\varepsilon,\alpha}) = \alpha + o(1)$ and for any compact $K \subset \Xi_D$ and $B > 1$

$$\sup_{\kappa\in K,\ B^{-1}\le R\le B} (\beta(\psi_{\varepsilon,\alpha}, V_\varepsilon(\tau,\rho_\varepsilon)) - (1-\alpha)\Phi(R_\varepsilon(\tau,\rho_\varepsilon))) \le o(1).$$

Here the values $R_\varepsilon(\tau,\rho_\varepsilon)$ are defined by (2.2).

2) *For Besov bodies case assume $\lambda < 0$, if $hq < pt$. Let us consider the tests (4.1) which are based on the thresholding*

$$X_\varepsilon = \left\{ \max_{1\le j\le J_\varepsilon} \max_{1\le l\le 2^j} |x_{lj}| > \sqrt{2CJ_\varepsilon} \right\} \cup \left\{ \sup_{J_\varepsilon<j<\infty} \max_{1\le l\le 2^j} |x_{lj}|/T_j > 1 \right\}$$

where $C = \log 2$, $T_j = \sqrt{2C(j+\log j)}$, $J_\varepsilon \asymp \log\log\varepsilon^{-1}$. Then $\alpha(\psi_{\varepsilon,\alpha}) = \alpha + o(1)$ and there exists such function $c(\tau) > 0$; $c(\tau) = 1$, if $q \ge t$, that

$$\beta(\psi_{\varepsilon,\alpha}, V_\varepsilon(\tau,\rho_\varepsilon)) \le (1-\alpha)\Phi(\sqrt{2\log n_\varepsilon(\tau)} - c(\tau) n_\varepsilon^{-r}(\tau)\rho_\varepsilon/\varepsilon) + o(1).$$

Here the values $n = n_\varepsilon$ are defined by the relations: $n = 2^{j_0}$, $R/\rho_\varepsilon = c(\tau)2^{j_0(s-r)}$.

The proof of Theorem 7 is given in Section 8.

It is clear that Theorem 7 implies the rates (2.4, 2.5).

4.2. Gaussian case

We describe the test procedures which provide the upper bounds in Theorems 4–6. Note that these test procedures depend essential on $\kappa \in \Xi_{G_l}$, R and on ρ_ε.

Let $\kappa \in \Xi_{G_l}, l = 1,...,5$. Test procedures are defined by two families of sequences: $\bar h_\varepsilon(\tau,\rho_\varepsilon)) = \bar h_\varepsilon = (h_{\varepsilon,1},\dots,h_{\varepsilon,i},\dots)$, $h_{\varepsilon,i} \in [0,1]$ and $\bar z_\varepsilon(\tau,\rho_\varepsilon)) = \bar z_\varepsilon = (z_{\varepsilon,1},\dots,z_{\varepsilon,i},\dots)$, $z_{\varepsilon,i} \ge 0$. In Besov bodies case for $i = 2^j + l$, $l = 1,\dots,2^j$ the values $h_{\varepsilon,i}$, $z_{\varepsilon,i}$ depend on $j \ge 1$ only. The asymptotically minimax families of tests are of the form $\psi_{\varepsilon,\alpha} = \psi_{\varepsilon,\alpha;\tau,\rho_\varepsilon} = \mathbf{1}_{\{L_\varepsilon > T_\alpha\}\cup X_\varepsilon}$. They are based on the statistics

$$L_\varepsilon = u_\varepsilon^{-1} \sum_i h_{\varepsilon,i}\xi(x_i, z_{\varepsilon,i})$$

where the functions $\xi(x,z)$ are defined by (3.11), and on the threshold procedure

$$X_\varepsilon = X_{\varepsilon;\tau,\rho_\varepsilon} = \left\{\sup_i |x_i|/T_{\varepsilon,i} > 1\right\},$$

where for ellipsoidal case thresholds $T_{\varepsilon,i}$ are defined by

$$T_{\varepsilon,i} = \sqrt{(2+\delta)\Delta_{\varepsilon,i}},\ \Delta_{\varepsilon,i} = \log(\|\pi_{\varepsilon,i}\|^{-2}) - z^2_{\varepsilon,i}(1-\delta),$$

and for Besov body case

$$T_{\varepsilon,l,j} = T_{\varepsilon,j} = \sqrt{(2+\delta)\log(\|\pi_{\varepsilon,i}\|^{-2})}.$$

We use the notations

$$\|\pi_{\varepsilon,i}\|^2 = 2h^2_{\varepsilon,i}\sinh^2\frac{z^2_{\varepsilon,i}}{2}$$

which is explained in Section 5.

Here and later we denote by δ small enough positive values (may be, different) which may depend on τ but bounded away from 0 on any compact.

The sequences $\bar h_\varepsilon = \bar h_\varepsilon(\tau,\rho_\varepsilon)$ and $\bar z_\varepsilon = \bar z_\varepsilon(\tau,\rho_\varepsilon)$ for $p=q,\ r>r_p$ were defined in Section 3.1, for general case ones are defined in Sections 6 and 7.

Theorem 8. *The tests $\psi_{\varepsilon,\alpha} = \psi_{\varepsilon,\alpha;\tau,\rho_\varepsilon}$ satisfy the relation: $\alpha(\psi_{\varepsilon,\alpha}) = \alpha + o(1)$ and:*

1) *For ellipsoidal case and for $\kappa \in \Xi_{G_l},\ l=1,\dots,5$*

$$\beta(\psi_{\varepsilon,\alpha}, V_\varepsilon(\tau,\rho_\varepsilon)) \le \Phi(T_\alpha - u_\varepsilon) + o(1)$$

where the values u_ε are defined by Theorems 4 and 6.

2) *For Besov bodies case and for $\kappa \in \Xi_{G_l},\ l=1,2$ there exist such function $c(\tau)>0$ that*

$$\beta(\psi_{\varepsilon,\alpha}, V_\varepsilon(\tau,\rho_\varepsilon)) \le \Phi(T_\alpha - c(\tau)u_\varepsilon) + o(1)$$

where the values u_ε are defined by Theorem 5.

Proof of Theorem 8 follows from the results of Section 6 for ellipsoids and of Section 7 for Besov bodies.

5. Gaussian asymptotics: Reduction to extreme problem

To study sharp and rate asymptotics for Gaussian type we use a generalization of methods of Ingster [12,13] which allows asymptotical reduction of wide enough class of "symmetrical convex separable" minimax hypothesis testing problems to extreme problem: to minimize special Hilbert norm $\|\bar\pi\|$ over the convex set $\Pi_\varepsilon(\tau,\rho_\varepsilon)$ of sequences $\bar\pi = (\pi_1,\dots,\pi_i,\dots)$ where π_i are probability measures on the real line. Under general assumptions (which are formulated in terms of properties of extreme sequences $\bar\pi_\varepsilon$) these extreme sequences (or close to ones) define the asymptotically least favorable priors $\pi^\varepsilon = \pi_{\varepsilon,1}\times\dots\times\pi_{\varepsilon,i}\times\dots$ and asymptotically minimax tests.

The idea of reduction is following. Assume for a moment that a set V_ε is convex and closed. Then (see Burnashev [1], for example) the least favorable prior is Dirac mass δ_{v_ε} at the point $v_\varepsilon \in V_\varepsilon$ nearest to 0:

$$\|v_\varepsilon\| = \inf_{v\in V_\varepsilon}\|v\| > 0.$$

The point v_ε and the norm $u_\varepsilon = \|v_\varepsilon\|$ determine the minimax efficiency and minimax test in the problem (we call this problem as *problem C*):

$$\beta(\alpha, V_\varepsilon) = \Phi(T_\alpha - u_\varepsilon),\ \psi_\alpha(x) = \mathbf{1}_{\{(x,r_\varepsilon)>T_\alpha\}}$$

where $r_\varepsilon = v_\varepsilon/\|v_\varepsilon\|$, $(x,r)=\sum_i x_i r_i$.

In fact,

$$\beta(\alpha, V_\varepsilon) \ge \beta(\alpha, v_\varepsilon) = \Phi(T_\alpha - \|v_\varepsilon\|).$$

From the other hand, by $(x, r_\varepsilon) \sim N((v, r_\varepsilon), 1)$ under P_v-distribution, one has: $\alpha(\psi_\alpha(x)) = \alpha$,

$$\beta(\psi_\alpha(x), V_\varepsilon) = \Phi(T_\alpha - \inf_{v\in V_\varepsilon}(r_\varepsilon, v))$$

and by convexity and minimax theorem

$$\sup_{\|r\|=1}\inf_{v\in V_\varepsilon}(r,v) = \inf_{v\in V_\varepsilon}\sup_{\|r\|=1}(r,v) = \inf_{v\in V_\varepsilon}\|v\| = \inf_{v\in V_\varepsilon}(r_\varepsilon, v).$$

These considerations use only the existence of the points v_ε which minimize the norm over V_ε, the Gaussian structure of the likelihood ratio dP_{v_ε}/dP_0, the $N((v, r_\varepsilon), 1)$-normality of the statistics

$$(x, r_\varepsilon) = \|v_\varepsilon\|^{-1}(\log(dP_{v_\varepsilon}/dP_0) + \|v_\varepsilon\|^2/2)$$

under P_v-distributions and the convexity of the set V_ε.

Of course, sets V_ε are not convex in our problems. However we will try to find asymptotically least favorable priors as product priors $\pi^\varepsilon = \pi_{\varepsilon,1}\times\ldots\times\pi_{\varepsilon,i}\times\ldots$ corresponding to sequences $\bar\pi_\varepsilon$. We will show that under some assumptions the likelihood ratio $dP_{\pi^\varepsilon}/dP_0$ has asymptotically Gaussian structure:

$$\log dP_{\pi^\varepsilon}/dP_0 = -\|\bar\pi_\varepsilon\|^2/2 + L_\varepsilon.$$

Here the statistics $L_\varepsilon \sim N(0, \|\bar\pi_\varepsilon\|)$ under P_0-distribution where $\|\bar\pi\|$ is a norm of Hilbert type on the space of the sequences $\bar\pi$. If we could replace the set V_ε onto some convex set $\Pi_\varepsilon = \{\bar\pi\}$, then we can obtain the problem which is close to the *problem C* above, however not in the sequence space l_2, but in Hilbert space of sequence $\bar\pi$. The results for the *problem C* motivate the consideration of the extreme problem

$$u_\varepsilon = \inf_{\bar\pi\in\Pi_\varepsilon}\|\bar\pi\|. \tag{5.1}$$

We can hope that if a family of sequence $\bar\pi_\varepsilon$ provides infimum in (5.1): $u_\varepsilon = \|\bar\pi_\varepsilon\|$, then the family π^ε provides asymptotically least favorable family of priors.

Simpler variant of this scheme have been used in Ingster [11,12] for ellipsoids with $p = q < \infty$. In this section we realize this scheme following to Ingster [13]. In the next sections we study the extreme problem (5.1). This extreme problem had been studied "on the rate" by Suslina [27] for ellipsoidal case with $r = 0$. We generalize the methods of this paper in Sections 6 and 7.

5.1. Hilbert structure

Let L be a set of sequences $\bar r = (r_1,\ldots,r_i,\ldots)$ of signed measures r_i with finite support on the real line (R^1, B) where B is Borelian σ-algebra. Put

$$(\bar r_1, \bar r_2) = \sum_i (r_{i,1}, r_{i,2}) = \sum_i \int_{R^1}\int_{R^1}(e^{uv}-1)r_{i,1}(du)r_{i,2}(dv). \tag{5.2}$$

Note that

$$(r_1, r_2) = \mathrm{Cov}_{P_{0,1}}\left(\frac{dP_{r_1}}{dP_{0,1}}, \frac{dP_{r_2}}{dP_{0,1}}\right)$$

where $P_r = \int_{R^1} P_{t,1} r(dt)$ is a mixture of one-dimensional Gaussian measures $P_{t,1} = N(t,1)$, Cov is covariation. This yields

$$(r_{i,1}, r_{i,1}) = \|r_{i,1}\|^2 = E\left(\int_{R^1}\left(\exp\left\{-\frac{u^2}{2}+xu\right\}-1\right)r_{i,1}(du)\right)^2 \geq 0,$$

where x is a standard Gaussian variable. Thus the bilinear form $(\bar{r}_1, \bar{r}_2)$ is positive semi-defined. Also one can see that $(r_{i,1}, r_{i,1}) = 0$ if and only if $r_{i,1} = a_i\delta_0$ for any $a_i \in R^1$. Here and later δ_t is Dirac mass at the point t. Put $L' = \{\bar{r} \in L : \|\bar{r}\| < \infty\},\ L_0 = \{a\delta_0,\ a \in R^1\}$. Thus, the bilinear form (5.2) defines Hilbert structure on the set $L^{''} = L'/L_0$ of equivalent classes. We will not use any topological properties of this structure (completeness and so on) and will not consider this properties.

Put

$$\Pi' = \{\bar{\pi} \in L' :\quad \pi_i(dv) \geq 0,\ \pi_i(R^1) \leq 1\ \ \forall i\},$$

$$\Pi = \{\bar{\pi} \in \Pi' :\quad \pi_i \ \text{ are probability measure } \forall i\},$$

and $\Pi^{''} = \Pi'/L_0$. Note that any equivalent class $\bar{\pi}^{''} \in \Pi^{''}$ contains one and only one sequence $\bar{\pi} \in \Pi$ and we can identify the sets $\Pi^{''}$ and Π.

5.2. **Lower bounds**

To obtain asymptotical lower bounds we use asymptotical variant of Bayesian approach. Let us consider Bayesian problems: to test simple hypothesis $H_0 : P = P_0$ *versus* simple Bayesian alternatives $H_{\pi^\varepsilon} : P = P_{\pi^\varepsilon}$, where P_{π^ε} is a mixture:

$$P_{\pi^\varepsilon}(dv_1, \dots, dv_i, \dots) = \int P_u(dv_1, \dots, dv_i, \dots)\pi^\varepsilon(du).$$

First, note (see, for example, Ingster [12], Part II, Sect. 4.1) that if $\pi^\varepsilon(V_\varepsilon) = 1$ or $\pi^\varepsilon(V_\varepsilon) \to 1$, then

$$\beta(\alpha, V_\varepsilon) \geq \beta_{\pi^\varepsilon}(\alpha, V_\varepsilon) \quad \text{or} \quad \beta(\alpha, V_\varepsilon) \geq \beta_{\pi^\varepsilon}(\alpha, V_\varepsilon) + o(1)$$

where $\beta_{\pi^\varepsilon}(\alpha, V_\varepsilon)$ is the minimum second kind errors for tests of level α in Bayesian problems. Also, if

$$E_0\left(\frac{dP_{\pi^\varepsilon}}{dP_0} - 1\right)^2 = E_0\left(\frac{dP_{\pi^\varepsilon}}{dP_0}\right)^2 - 1 \to 0,$$

then $\beta_{\pi^\varepsilon}(\alpha, V_\varepsilon) \to 1 - \alpha$.

Let as consider product prior $\pi^\varepsilon = \pi_{\varepsilon,1} \times \ldots \times \pi_{\varepsilon,i} \times \ldots$ corresponding to a sequence $\bar{\pi}_\varepsilon \in \Pi$. Then

$$P_{\pi^\varepsilon}(dv_1, \dots, dv_i, \dots) = \prod_i \int_{R^1} P_{u_i}(dv_i)\pi_{\varepsilon,i}(du_i)$$

and by the inequality $x \leq \exp(x-1)$ we have:

$$\begin{aligned} E_0\left(\frac{dP_{\pi^\varepsilon}}{dP_0}\right)^2 &= \prod_i E_0\left(\frac{dP_{\pi_{\varepsilon,i}}}{dP_0}\right)^2 \\ &\leq \exp\left(\sum_i E_0\left(\frac{dP_{\pi_{\varepsilon,i}}}{dP_0} - 1\right)^2\right) = \exp\left(\sum_i \|\pi_{\varepsilon,i}\|^2\right) = \exp\left(\|\bar{\pi}_\varepsilon\|^2\right) \to 1 \end{aligned}$$

as $\|\bar{\pi}_\varepsilon\| \to 0$.

This relation motivates to use Hilbert norm $\|\bar{\pi}_\varepsilon\|$ in asymptotical hypotheses testing problems. More over, under some assumptions the norm $\|\bar{\pi}_\varepsilon\|$ defines the asymptotics of error probabilities in this Bayesian hypothesis testing problem (see Ingster [13] and Th. 9 later). To our study it is enough to consider the case when $\pi_{\varepsilon,i}$ are symmetrical three-point measures at the points 0, $z_{\varepsilon,i}$ and $-z_{\varepsilon,i}$:

$$\bar{\pi}_\varepsilon = \bar{\pi}(\bar{h}_\varepsilon,\ \bar{z}_\varepsilon): \quad \pi_{\varepsilon,i} = \pi(z_{\varepsilon,i}, h_{\varepsilon,i}) = (1-h_{\varepsilon,i})\delta_0 + \frac{h_{\varepsilon,i}}{2}(\delta_{z_{\varepsilon,i}} + \delta_{-z_{\varepsilon,i}})$$

(or two-point measures, if $h_{\varepsilon,i} = 1$). Here $\bar{h}_\varepsilon,\ \bar{z}_\varepsilon$ are two sequence, $h_{\varepsilon,i} \in [0,1], \quad z_{\varepsilon,i} \geq 0$. For these measures

$$\|\bar{\pi}_\varepsilon\|^2 = \sum_i \|\pi_{\varepsilon,i}\|^2 = 2\sum_i h_{\varepsilon,i}^2 \sinh^2 \frac{z_{\varepsilon,i}^2}{2}$$

and log-likelihood ratio $l_{\varepsilon,\bar{\pi}_\varepsilon} = \log(dP_{\bar{\pi}_\varepsilon}/dP_0)$ is of the form

$$l_{\varepsilon,\bar{\pi}_\varepsilon} = \sum_i \log(1 + h_{\varepsilon,i}\xi(x_i, z_{\varepsilon,i})).$$

Remind that the function $\xi(x,z)$ is defined by (3.11):

$$\xi(x,z) = e^{-z^2/2}\cosh zx - 1.$$

Note that if x is a standard Gaussian variable, then

$$E\xi(x,z) = 0,\ E(\xi(x,z))^2 = 2\sinh^2\frac{z^2}{2},\ \min_x \xi(x,z) = e^{-z^2/2} - 1 > -1 \tag{5.3}$$

and for any integer $k > 1$ one has

$$E(\xi(x,z))^{2k} \leq C_1(k)\exp(C_2(k)z^2)(E(\xi(x,z))^2)^k \tag{5.4}$$

where $C_1(k) > 0,\ C_2(k) > 0$ are constants (see Lem. 1 in Ingster [13]).

Put the assumptions:

A1. $\sup_i \|\pi_{\varepsilon,i}\| = o(1),\ \|\bar{\pi}_\varepsilon\| \asymp 1.$

A2. As $\varepsilon \to 0$ and $B \to \infty$,

$$\sum_{i: z_{\varepsilon,i} > B} \|\pi_{\varepsilon,i}\|^2 \to 0.$$

B1. For some small enough δ_0 and any δ_1 such that $\delta_0 > \delta_1 > 0$ one has

$$\sup_i \|\pi_{\varepsilon,i}\| = O(\varepsilon^{\delta_0}), \quad \delta_1 < \|\bar{\pi}_\varepsilon\| = O(\varepsilon^{-\delta_1}).$$

B2. For any $\delta > 0$

$$\sum_{i: z_{\varepsilon,i} > \delta\sqrt{\log\varepsilon^{-1}}} \|\pi_{\varepsilon,i}\|^2 = O(\varepsilon^\delta).$$

For ellipsoidal case we use

B3. For any $\eta \in (0,1)$

$$\sum_i \exp(\eta z_{\varepsilon,i}^2)\|\pi_{\varepsilon,i}\|^2 = O(\|\overline{\pi}_\varepsilon\|^2).$$

For Besov body case with $i = 2^j + l$, $l = 1, ...2^j$ we use

B3a. There exist such $\eta \in (0,1)$ that

$$\sum_i \exp(\eta z^2_{\varepsilon,i})\|\pi_{\varepsilon,i}\|^2 = O(\|\overline{\pi}_\varepsilon\|^2).$$

If $z_{\varepsilon,i} = z_{\varepsilon,j},\ h_{\varepsilon,i} = h_{\varepsilon,j}$ do not depend on l, then one can rewrite B3a in the form

$$\sum_j 2^j \exp(\eta z^2_{\varepsilon,j})\|\pi_{\varepsilon,i}\|^2 = O(\|\overline{\pi}_\varepsilon\|^2)\ ,\ \|\overline{\pi}_\varepsilon\|^2 = \sum_j 2^j \|\overline{\pi}_{\varepsilon,j}\|^2.$$

Note that assumptions B1 and either B3 or B3a imply B2 and A2.

Consider the functions

$$L_{\varepsilon,\bar{\pi}_\varepsilon} = \|\bar{\pi}_\varepsilon\|^{-1} \sum_i h_{\varepsilon,i}\xi(x_i, z_{\varepsilon,i}). \tag{5.5}$$

Theorem 9. 1. *Let $\|\bar{\pi}_\varepsilon\| \to 0$. Then $\beta(\alpha, P_{\pi^\varepsilon}) \to 1-\alpha$ for any $\alpha \in (0,1)$.*

2. *Assume A1, A2*

$$\beta(\alpha, P_{\pi^\varepsilon}) = \Phi(T_\alpha - \|\bar{\pi}_\varepsilon\|) + o(1) \tag{5.6}$$

and for any $x \in R^1$

$$P_0(l_{\varepsilon,\bar{\pi}_\varepsilon} < x\|\bar{\pi}_\varepsilon\| + \|\bar{\pi}_\varepsilon\|^2/2) = \Phi(x) + o(1), \tag{5.7}$$

$$P_0(L_{\varepsilon,\bar{\pi}_\varepsilon} < x) = \Phi(x) + o(1). \tag{5.8}$$

3. *Assume B1, B2. Then for small enough $\delta > 0$*

$$\sup_{x\in R^1} |P_0(l_{\varepsilon,\bar{\pi}_\varepsilon} < x\|\bar{\pi}_\varepsilon\| + \|\bar{\pi}_\varepsilon\|^2/2) - \Phi(x)| = O(\varepsilon^\delta) \tag{5.9}$$

and

$$\sup_{x\in R^1} |P_0(L_{\varepsilon,\bar{\pi}_\varepsilon} < x) - \Phi(x)| = O(\varepsilon^\delta). \tag{5.10}$$

Proof. The statements 1, 2 of Theorem 9 are proved in Ingster [13], Theorem 1 and Lemma 1 where wider class of sequences $\bar{\pi}_\varepsilon$ had been considered. The proof of the statement 1 is given in the beginning of this Section. For completeness we give the outline of the proof of the statement 2.

The relation (5.6) follows from (5.7) by

$$\beta(\alpha, P_{\pi^\varepsilon}) = E_{P_0}(\exp(l_{\varepsilon,\bar{\pi}_\varepsilon})\mathbf{1}_{l_{\varepsilon,\bar{\pi}_\varepsilon} < t_{\varepsilon,\alpha}})p33 + o(1) = \int_{-\infty}^{T_\alpha} \exp(-\|\bar{\pi}_\varepsilon\|^2/2 + x\|\bar{\pi}_\varepsilon\|)d\Phi_\varepsilon(x) + o(1)$$

where $t_{\varepsilon,\alpha}$ is $(1-\alpha)$-quantile of the statistic $l_{\varepsilon,\bar{\pi}}$ and Φ_ε is the distribution function of the statistic $(l_{\varepsilon,\bar{\pi}} + \|\bar{\pi}_\varepsilon\|^2/2)/\|\bar{\pi}_\varepsilon\|$ under P_0-distribution.

To proof (5.7) and (5.8) by assumptions A2 and (5.3) it is enough to consider "truncated" statistics $l_{\varepsilon,\bar{\pi}}$ and $L_{\varepsilon,\bar{\pi}}$ with $z_{\varepsilon,i} \le B$ for large enough $B > 0$. The relation (5.8) follows directly from the Central Limit Theorem under Lyapunov conditions: using (5.4) for $k = 2$ and A1 one has

$$\sum_i h^4_{\varepsilon,i} E_{P_0}(\xi(x_i, z_{\varepsilon,i})^4) \le C_1(2)\exp(-C_2(2)B^2) \sum_i \|\pi_{\varepsilon,i}\|^4$$

$$\le C_1(2)\exp(-C_2(2)B^2)\sum_i \|\pi_{\varepsilon,i}\|^2 \sup_i \|\pi_{\varepsilon,i}\|^2 = o(1). \tag{5.11}$$

The relation (5.7) follows from (5.8) and from Taylor expansion (up to second terms) of the function $l_{\varepsilon,\bar{\pi}}$: it is possible by assumptions A1, A2, by the properties (5.3, 5.4); the estimations are analogous to (5.11).

The proof of the statement 3 follows the same outline with the truncation of "tails". It is possible by assumptions B1, B2 which give the accuracy of the rate $o(\varepsilon^\delta)$ for small enough $\delta > 0$: we use the von Bahr-Essen inequality to proof (5.10) and also the Taylor expansion to proof (5.9). The estimations are analogous to (5.11).

Corollary 5.1. *Let $\pi^\varepsilon(V_\varepsilon) \to 1$. Then under assumption A1, A2*

$$\beta(\alpha, V_\varepsilon) \ge \Phi(T_\alpha - \|\bar{\pi}_\varepsilon\|) + o(1).$$

5.3. Upper bounds

To obtains the upper bounds, in what follows we assume that $p < \infty$ and that $p \ge h,\ q \le t$ for Besov bodies case (these assumptions are enough to our study).

For small enough $\delta > 0$ let us consider tests of the form:

$$\psi_{\varepsilon,t_\varepsilon} = \psi_{\varepsilon,t_\varepsilon}(\bar{h}_\varepsilon, \bar{z}_\varepsilon) = \mathbf{1}_{\{L_{\varepsilon,\pi_\varepsilon} > t_\varepsilon\}\cup X_\varepsilon}$$

which are based on the statistics (5.5) and on the threshold procedure

$$X_\varepsilon = \left\{\sup_i |x_i|/T_{\varepsilon,i} > 1\right\}.$$

We consider two different variants of thresholding.

First one is used for ellipsoid case. Put, if $\|\pi_{\varepsilon,i}\| = 0$, then $T_{\varepsilon,i} = \infty$, and if $\|\pi_{\varepsilon,i}\| > 0$, then

$$T_{\varepsilon,i} = \sqrt{(2+2\delta)\Delta_{\varepsilon,i}}\ ,\quad \Delta_{\varepsilon,i} = \log(\|\pi_{\varepsilon,i}\|^{-2}) - z^2_{\varepsilon,i}(1-\delta). \tag{5.12}$$

Second one corresponds to Besov body case with $p > h$ or $q < t$ (the study in Sect. 7 later corresponds to $t = \infty$). We assume $\pi_{\varepsilon,l,j} = \pi_{\varepsilon,j}$ do not depend on l and

$$T_{\varepsilon,l,j} = T_{\varepsilon,j} = \sqrt{(2+2\delta)\log(\|\pi_{\varepsilon,j}\|^{-2})}. \tag{5.13}$$

For $t_\varepsilon = T_\alpha$ these tests are the same that in Section 4.2. Note that $E_0 L_\varepsilon = 0,\ E_0 L^2_\varepsilon = 1$ and

$$E_v L_\varepsilon = (\bar{\pi}_\varepsilon, \bar{\delta}_v)/\|\bar{\pi}_\varepsilon\| = \frac{2}{\|\bar{\pi}_\varepsilon\|}\sum_i h_{\varepsilon,i}\sinh^2\frac{z_{\varepsilon,i}v_i}{2}.$$

Here $\bar{\delta}_v$ is the sequence $(\delta_{v_1}, \dots, \delta_{v_i}, \dots)$ and $(\bar{\pi}_\varepsilon, \bar{\delta}_v)$ is a scalar product in the sense of Section 5.1.

For ellipsoidal case and for thresholding (5.12) put

$$\Re_\varepsilon = \left\{i : \Delta_{\varepsilon,i}/9 \le z^2_{\varepsilon,i} \le 9\Delta_{\varepsilon,i}\right\},\ \aleph_\varepsilon(v) = \left\{i \in \Re_\varepsilon : |v_i| > \left(\sqrt{\Delta_{\varepsilon,i}/(1+3\delta)}\right)/2\right\}$$

and consider the sets

$$\tilde{V}_\varepsilon = \left\{v \in l_2 : \sup_i |v_i|/T_{\varepsilon,i} < 1+\delta,\ \sum_{i\in\aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_0)) < 1\right\} \tag{5.14}$$

where δ_0 is any constant such that $0 < \delta_0 < \delta^*$, $\delta^* = 2((2^{1/2} - 1/2)^{-2} - 1) \approx 0,4$ is an absolute constant. For $v \in l_2$ define the sequence v^*: if $v \notin \tilde{V}_\varepsilon$, then $v^* = v$, and if $v \in \tilde{V}_\varepsilon$, then

$$v_i^* = \begin{cases} v_i, & \text{if } i \notin \aleph_\varepsilon(v) \\ 0, & \text{if } i \in \aleph_\varepsilon(v). \end{cases}$$

Theorem 10. 1. *Assume A1, B3 and $t_\varepsilon = O(1)$. Then for small enough $\delta > 0$ in (5.12) one has:*

$$\alpha(\psi_{\varepsilon,t_\varepsilon}) = \Phi(-t_\varepsilon) + o(1), \quad \beta(\psi_{\varepsilon,t_\varepsilon}, v) = \Phi(t_\varepsilon - (\bar{\pi}_\varepsilon, \bar{\delta}_{v^*})/\|\bar{\pi}_\varepsilon\|) + o(1).$$

2. *Assume B1, B3 and $t_\varepsilon = o(\varepsilon^{-\delta_1})$ for small enough $\delta_1 > 0$. Then for small enough $\delta > 0$ in (5.12) and some $\delta_2 > 0$ one has:*

$$\alpha(\psi_{\varepsilon,t_\varepsilon}) = \Phi(-t_\varepsilon) + o(\varepsilon^{\delta_2})$$

and uniformly on $v \in l_2$

$$\beta(\psi_{\varepsilon,t_\varepsilon}, v) \le \Phi(t_\varepsilon - (\bar{\pi}_\varepsilon, \bar{\delta}_{v^*})/\|\bar{\pi}_\varepsilon\|) + o(\varepsilon^{\delta_2}).$$

Proof of Theorem 10. Statement 1 is contained in the proof of Theorem 2.2 in Ingster [13]. The proof of the statement 2 follows from the analogous considerations. For completeness we give the outline of the proof of the statement 2.

First, note that using B3 one has for some $\delta_2 > 0$:

$$\sum_i \exp(-T_{\varepsilon,i}^2/2) = \sum_i \|\pi_{\varepsilon,i}\|^{2+2\delta} \exp(z_{\varepsilon,i}^2(1-\delta^2))$$
$$\le \sup_i \|\pi_{\varepsilon,i}\|^{2\delta} \sum_i \|\pi_{\varepsilon,i}\|^2 \exp(z_{\varepsilon,i}^2(1-\delta^2)) = o(\varepsilon^{\delta_2})$$

which implies

$$\sup_i \exp(-T_{\varepsilon,i}^2/2) = \sup_i \exp(-\Delta_{\varepsilon,i}(1+\delta)) = o(\varepsilon^{\delta_2}), \tag{5.15}$$

$$P_0(X_\varepsilon) \le 2\sum_i \Phi(-T_{\varepsilon,i}) = o(\varepsilon^{\delta_2}).$$

By

$$P_0(L_{\varepsilon,\bar{\pi}_\varepsilon} > t_\varepsilon) < \alpha(\psi_{\varepsilon,t_\varepsilon}) < P_0(L_{\varepsilon,\bar{\pi}_\varepsilon} > t_\varepsilon) + P_0(X_\varepsilon)$$

these relations and Theorem 9, the statement 2 yield the relation for the first kind error probability $\alpha(\psi_{\varepsilon,t_\varepsilon})$.

To estimate the second kind error probabilities note that

$$\beta(\psi_{\varepsilon,t_\varepsilon}, v) \le \min(P_v(\bar{X}_\varepsilon), P_v(L_{\varepsilon,\bar{\pi}_\varepsilon} \le t_\varepsilon))$$

where $\bar{X}_\varepsilon$ is the complement of the set X_ε. Put

$$V_{\varepsilon,1} = \left\{ v \in l_2 : \sup_i |v_i|/T_{\varepsilon,i} > 1+\delta \right\},$$
$$V_{\varepsilon,2} = \left\{ v \in l_2 \setminus V_{\varepsilon,1} : \max_{i \in \Re_\varepsilon} 2|v_i|/\sqrt{\Delta_{\varepsilon,i}/(1+3\delta)} \le 1 \right\}.$$

Using (5.15) one can see that for some $\delta_3 > 0$ uniformly on $v \in V_{\varepsilon,1}$

$$P_v(\bar{X}_\varepsilon) \le \prod_i (\Phi(-|v_i| + T_{\varepsilon,i})) \le \Phi(-\delta \inf_i T_{\varepsilon,i}) = o(\varepsilon^{\delta_3}).$$

Let $v \in V_{\varepsilon,2}$. Consider the representation

$$L_{\varepsilon,\pi_\varepsilon} = L^v_{\varepsilon,\pi_\varepsilon} + (\bar\pi_\varepsilon, \bar\delta_v)/\|\bar\pi_\varepsilon\| + \Delta L^v_\varepsilon$$

where

$$\begin{aligned} L^v_{\varepsilon,\pi_\varepsilon} &= \|\bar\pi_\varepsilon\|^{-1}\sum_i h_{\varepsilon,i}\xi(x_i - v_i, z_{\varepsilon,i}),\\ \Delta L^v_\varepsilon &= \sum_i \Delta L^v_{\varepsilon,i} = \|\bar\pi_\varepsilon\|^{-1}\sum_i r_{\varepsilon,i}(v_i),\end{aligned}$$

$r_{\varepsilon,i}(v_i) = 2h_{\varepsilon,i}\sinh^2(v_i z_{\varepsilon,i}/2)(\exp(-z^2/2)\sinh(xz_{\varepsilon,i} - v_i z_{\varepsilon,i}/2) - \sinh(v_i z_{\varepsilon,i}/2))$.
Note that P_v-distribution of $L^v_{\varepsilon,\pi_\varepsilon}$ is P_0-distribution of $L_{\varepsilon,\pi_\varepsilon}$ and $E_v(\Delta L^v_\varepsilon) = 0$. By Theorem 9 and Chebyshev inequality it is enough to show that uniformly on $v \in V_{\varepsilon,2}$ for some $\delta_2 > 0$ the following relation holds:

$$E_v(\Delta L^v_\varepsilon)^2 = o(\varepsilon^{\delta_2}(1 + (\bar\pi_\varepsilon, \bar\delta_v)/\|\bar\pi_\varepsilon\|)). \tag{5.16}$$

Using the inequalities $\sinh t \le \exp|t|$, $\cosh t \le \exp|t|$ and $\sinh(t^2/2) > \exp(t^2/2)/4$ for $t > 1$ one can obtain: for $z_{\varepsilon,i} > 1$

$$\begin{aligned} E_v(r_{\varepsilon,i}(v_i))^2 &= 4h^2_{\varepsilon,i}\sinh^2(vz/2)[\sinh z^2 + \sinh^2(vz/2)(\exp z^2 - 1)] \le 4\exp(2z_{\varepsilon,i}|v_i|)\|\pi_{\varepsilon,i}\|^2 && (5.17)\\ E_v(r_{\varepsilon,i}(v_i))^2 &\le 4(\pi_{\varepsilon,i}, \delta_{v_i})\exp(z_{\varepsilon,i}|v_i| + z^2_{\varepsilon,i}/2)\|\pi_{\varepsilon,i}\|. && (5.18)\end{aligned}$$

By inequalities (5.17, 5.18) and the definition of the set $V_{\varepsilon,2}$ the relation (5.16) follows from the inequalities: for some $\delta_3 > 0$

$$\begin{aligned} \sup_{i:\ \Delta_{\varepsilon,i}/9 \le z^2_{\varepsilon,i} \le \Delta_{\varepsilon,i}} \exp\left(z^2_{\varepsilon,i} + z_{\varepsilon,i}\sqrt{\Delta_{\varepsilon,i}/(1+3\delta)}\right)\|\pi_{\varepsilon,i}\|^2 &= o(\varepsilon^{\delta_3}),\\ \sup_{i:\ z^2_{\varepsilon,i} \le \Delta_{\varepsilon,i}/9} \exp\left(z^2_{\varepsilon,i} + 2z_{\varepsilon,i}\sqrt{2\Delta_{\varepsilon,i}(1+\delta)}\right)\|\pi_{\varepsilon,i}\|^2 &= o(\varepsilon^{\delta_3}), \end{aligned} \tag{5.19}$$

$$\begin{aligned} \sum_{i:\ \Delta_{\varepsilon,i} \le z^2_{\varepsilon,i} \le 9\Delta_{\varepsilon,i}} \exp\left(z_{\varepsilon,i}\sqrt{\Delta_{\varepsilon,i}/(1+3\delta)}\right)\|\pi_{\varepsilon,i}\|^2 &= o(\varepsilon^{\delta_3}),\\ \sum_{i:\ z^2_{\varepsilon,i} \ge 9\Delta_{\varepsilon,i}} \exp\left(2z_{\varepsilon,i}\sqrt{2\Delta_{\varepsilon,i}(1+\delta)}\right)\|\pi_{\varepsilon,i}\|^2 &= o(\varepsilon^{\delta_3}). \end{aligned} \tag{5.20}$$

One can easily see that the values under the supremum of (5.19) are of the form $\exp(-\eta\Delta_{\varepsilon,i})$ with some $\eta > 0$. Thus inequalities (5.19) follow from (5.15). Also the values under the sums in the left-hand side of (5.20) are of the form $\exp(\eta z^2_{\varepsilon,i})\|\pi_{\varepsilon,i}\|^2$ with some $\eta \in (0,1)$. Thus inequalities (5.20) follow from B3.

To prove the Theorem we need to consider alternatives from the sets $V_{\varepsilon,3}$:

$$V_{\varepsilon,3} = \left\{ v \in l_2 :\ \sup_i |v_i|/T_{\varepsilon,i} \le 1 + \delta,\ \max_{i \in \Re_\varepsilon} 2|v_i|/\sqrt{\Delta_{\varepsilon,i}/(1+3\delta)} \ge 1 \right\}.$$

Let $v \in V_{\varepsilon,3}$ and

$$\sum_{i \in \aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_0)) \ge 1.$$

This relation and (5.15) imply that for any $\delta_1 \in (\delta_0, \delta^*)$, $B > 0$ and small enough $\varepsilon > 0$ one has

$$\sum_{i \in \aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_1)) > B \log \varepsilon^{-1}.$$

Then

$$\begin{aligned} \beta(\psi_{\varepsilon,t_\varepsilon}, v) &\leq P_v(\bar{X}_\varepsilon) \leq \prod_i (1 - \Phi(|v_i| - T_{\varepsilon,i})) \\ &\leq \exp\left(-\sum_i \Phi\left(-\sqrt{\Delta}_{\varepsilon,i}\left(\sqrt{2(1+\delta)} - 1/2\sqrt{1+3\delta}\right)\right)\right) = o(\varepsilon^{\delta_2}) \end{aligned}$$

for some $\delta_2 > 0$ because for small enough $\delta > 0$, $\delta_1 > 0$ one has

$$\sum_i \Phi\left(-\sqrt{\Delta}_{\varepsilon,i}\left(\sqrt{2(1+\delta)} - 1/2\sqrt{1+3\delta}\right)\right) > \sum_{i \in \aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_1)).$$

Let $v \in V_{\varepsilon,3}$ and

$$\sum_{i \in \aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_0)) < 1$$

(it is the case $v \neq v^*$). Note that $v^* \in V_{\varepsilon,2}$ which implies the inequality of n. 2 of the theorem for $\beta(\psi_{\varepsilon,t_\varepsilon}, v^*)$. Note also that the admissible sets of the tests $\psi_{\varepsilon,t_\varepsilon}$ and all the coordinate cross-sections of these sets are convex and symmetric. Applying Anderson's lemma (see Ibragimov and Khasminskii [8]) to these admissible sets one has the inequality

$$\beta(\psi_{\varepsilon,t_\varepsilon}, v) \leq \beta(\psi_{\varepsilon,t_\varepsilon}, v^*)$$

which implies the inequality of the theorem. Theorem 10 is proved.

For Besov body case and for the thresholding (5.13) put

$$\tilde{V}_\varepsilon = \left\{ v \in l_2 : \sup_i |v_{l,j}|/T_{\varepsilon,j} < 1 + \delta \right\}, \ \tilde{J}_\varepsilon = \{ j : \ z_{\varepsilon,j} > \delta_0 T_{\varepsilon,j} \} \tag{5.21}$$

where δ_0 is small enough absolute constant. Define the sequence v^*: if $v \notin \tilde{V}_\varepsilon$, then $v^* = v$, and if $v \in \tilde{V}_\varepsilon$, then

$$v^*_{l,j} = \begin{cases} v_{l,j}, & \text{if } j \notin \tilde{J}_\varepsilon, \\ 0, & \text{if } i \in \tilde{J}_\varepsilon. \end{cases}$$

For simplicity we formulate next Theorem analogously to Theorem 10, n. 1 only.

Theorem 11. *Assume A1, B3a and $t_\varepsilon = O(1)$. Then for small enough $\delta > 0$ in (5.13) one has:*

$$\alpha(\psi_{\varepsilon,t_\varepsilon}) = \Phi(-t_\varepsilon) + o(1), \quad \beta(\psi_{\varepsilon,t_\varepsilon}, v) = \Phi(t_\varepsilon - (\bar{\pi}_\varepsilon, \bar{\delta}_{v^*})/\|\bar{\pi}_\varepsilon\|) + o(1).$$

Proof of Theorem 11 corresponds to the beginning of the proof of Theorem 10. At first, using B3a, A1 we show that $\sum_j 2^j \Phi(-T_{\varepsilon,j}) = o(1)$ which implies relations for the first kind errors and that we can reject alternatives $v \notin \tilde{V}_\varepsilon$ by the thresholding. Let $v \in \tilde{V}_\varepsilon$. Using Anderson's lemma we get $\beta(\psi_{\varepsilon,t_\varepsilon}, v) \leq \beta(\psi_{\varepsilon,t_\varepsilon}, v^*)$. To estimate $\beta(\psi_{\varepsilon,t_\varepsilon}, v^*)$ we check (5.16) using (5.18), by if $v_i^* = 0$, then $r_{\varepsilon,i} = 0$, and if $v_i^* \neq 0$, then $z_{\varepsilon,j} \leq \delta_0 T_{\varepsilon,j}$ and

$$\exp(2 z_{\varepsilon,i} v_i + z_{\varepsilon,i}^2)\|\pi_{\varepsilon,i}\|^2 \leq \exp(T_{\varepsilon,j}^2((2+2\delta)\delta_0 + \delta_0^2 - 1/(2+\delta))) = o(1)$$

for $\delta_0^2 + 2\delta_0 < 1/2$ and small enough δ in (5.13). Theorem 11 is proved.

Let us discuss the assumptions of Theorems 10 and 11. Note that if

$$\sup_i z_{\varepsilon,i} = O(1), \tag{5.22}$$

then the assumptions B3, B3a are fulfilled, $\Re_\varepsilon = \tilde{J}_\varepsilon = \emptyset$, and $v^* = v$ for any $v \in l_2$. In this case under assumption A1 for any $V_\varepsilon \subset l_2$ Theorems 10, 11 imply the relations

$$\alpha(\psi_{\varepsilon,T_\alpha}) = \alpha + o(\varepsilon^\delta), \quad \beta(\psi_{\varepsilon,T_\alpha}, V_\varepsilon) = \Phi\left(T_\alpha - \inf_{v\in V_\varepsilon}(\bar{\pi}_\varepsilon, \bar{\delta}_{v^*})/\|\bar{\pi}_\varepsilon\|\right) + o(\varepsilon^\delta)$$

and to find the asymptotically best tests we can consider the problem of maximization

$$w_\varepsilon = \sup_{\bar{\pi}} \inf_{v\in V_\varepsilon} (\bar{\pi}_\varepsilon, \bar{\delta}_{v^*})/\|\bar{\pi}_\varepsilon\| \ .$$

If the extreme sequences are the sequences of three-point measures and satisfy A1 and (5.22), then the values w_ε define the upper bounds for minimax asymptotics.

However for considerable problems relation (5.22) does not hold for $p > q,\ \lambda > 0$ (see Sects. 6 and 7 later).

We use the following remark in this case. If alternatives $V_\varepsilon = V_\varepsilon(H_{\varepsilon,1},\ H_{\varepsilon,2})$ are defined by relations

$$V_\varepsilon = \{v \in l_2 :\ f_1(v) > H_{\varepsilon,1},\ f_2(v) < H_{\varepsilon,2}\},$$

then often $v^* \in V_\varepsilon(H'_{\varepsilon,1},\ H_{\varepsilon,2})$ with $H'_{\varepsilon,1} = (1-\delta_\varepsilon)H_{\varepsilon,1}$ and $\delta_\varepsilon > 0,\ \delta_\varepsilon \to 0$. In this case we can obtain the analogues extreme problem for $V'_\varepsilon = V_\varepsilon(H'_{\varepsilon,1},\ H_{\varepsilon,2})$.

More exactly, let us consider ellipsoidal case, when $f_1(v) = \sum_i i^{rp}|v_i|^p$, with $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p$; and Besov bodies case, when

$$f_1(v) = \sum_{j=0}^{\infty}\left(\sum_{l=1}^{2^j} 2^{jpr}|v_{lj}|^p\right)^{h/p}, \ H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^h.$$

Note that $f_2(v)$ are monotone functionals: $f_2(v^*) \le f_2(v)$, if $|v_i^*| \le |v_i|\ \forall i$.

In ellipsoidal case put the assumption:

B4. Either (5.22) holds or for some families $n_\varepsilon \to \infty,\quad N_\varepsilon \to \infty$, $\log n_\varepsilon \asymp \log N_\varepsilon$, for the values $\delta_0,\ \delta^1 \in (0, \delta_0/(2+\delta_0))$, where δ_0 is determined by (5.14), any $i \in \Re_\varepsilon$ and for small enough $\delta' > 0$ one has:

$$|\Delta_{\varepsilon,i} - \log N_\varepsilon| < \delta^1 \Delta_{\varepsilon,i},\ N_\varepsilon^{-\delta'} < i/n_\varepsilon < N_\varepsilon^{\delta'},\ n_\varepsilon^{rp} N_\varepsilon^{1/2} = O(H_{\varepsilon,1} N_\varepsilon^{\delta'}).$$

Proposition 5.1. *In ellipsoidal case under assumption B4 for all $v \in V_\varepsilon(H_{\varepsilon,1},\ H_{\varepsilon,2})$ and for some $\delta_1 > 0$ one has: $v^* \in V_\varepsilon(H'_{\varepsilon,1},\ H_{\varepsilon,2})$ with $H'_{\varepsilon,1} = (1-\delta_\varepsilon)H_{\varepsilon,1}$ and $\delta_\varepsilon = O(N_\varepsilon^{-\delta_1})$.*

Proof of Proposition 5.1. By definition v^* one has: $f_2(v^*) \le f_2(v)$. By

$$1 > \sum_{i\in\aleph_\varepsilon(v)} \exp(-\Delta_{\varepsilon,i}/(2+\delta_0)) \ge N_\varepsilon^{-1/(1-\delta^1)(2+\delta_0)}(\#\aleph_\varepsilon(v))$$

for all $v \in \tilde{V}_\varepsilon$ under assumption B4, we get:

$$\#\aleph_\varepsilon(v) < N_\varepsilon^{1/(1-\delta^1)(2+\delta_0)} = N_\varepsilon^{1/2-3\delta_1},\ \delta_1 > 0.$$

For $\delta' \in (0, \delta_1/(1+|rp|))$ one has:

$$\sum_{i\in\aleph_\varepsilon(v)} |v_i|^p i^{rp} \le B n_\varepsilon^{rp} N_\varepsilon^{\delta'|rp|}(\log N_\varepsilon)^{p/2}(\#\aleph_\varepsilon(v)) = O(N_\varepsilon^{-\delta_1} H_{\varepsilon,1}).$$

Thus $\forall v \in \tilde{V}_\varepsilon(H_{\varepsilon,1},\ H_{\varepsilon,2})$ one has:

$$f_1(v^*) = \sum_i i^{rp}|v_i|^p - \sum_{i\in\aleph_\varepsilon(v)} i^{rp}|v_i|^p \ge f_1(v) - O(N_\varepsilon^{-\delta_1}H_{\varepsilon,1}) \ge H_{\varepsilon,1}(1 - O(N_\varepsilon^{-\delta_1}))$$

which implies the Proposition.

Remark 5.1. It follows from the proof, that δ_1 is bounded away from 0 on any compact $K \subset \Xi$, if assumption B4 holds uniformly on K.

In Besov bodies case put the assumption:
B4a. Either (5.22) holds or

$$\sup_{v\in\tilde{V}_\varepsilon} \sum_{j\in\tilde{J}_\varepsilon} f_{j,1}(v) = o(H_{\varepsilon,1}), \text{where } f_{j,1}(v) = (\sum_{l=1}^{2^j} 2^{jpr}|v_{lj}|^p)^{h/p}.$$

Proposition 5.2. *In Besov body case under assumption B4a for all $v \in V_\varepsilon(H_{\varepsilon,1},\ H_{\varepsilon,2})$ one has: $v^* \in V_\varepsilon(H'_{\varepsilon,1},\ H_{\varepsilon,2})$ with $H'_{\varepsilon,1} = (1-o(1))H_{\varepsilon,1}$.*

Proof of Proposition 5.2. If $v \notin \tilde{V}_\varepsilon$, then $v^* = v$, if $v \in \tilde{V}_\varepsilon$, then

$$f_1(v^*) \ge f_1(v) - \sum_{j\in\tilde{J}_\varepsilon} f_{j,1}(v) \ge H_{\varepsilon,1}(1-o(1)).$$

Thus, we obtain the following

Corollary 5.2. *Let the families $\bar{h}_\varepsilon = \bar{h}_\varepsilon(\tau,\rho_\varepsilon)$, $\bar{z}_\varepsilon = \bar{z}_\varepsilon(\tau,\rho_\varepsilon)$ be given and the tests $\psi_{\varepsilon,\tau,\rho_\varepsilon,t_\varepsilon}$ are considered for alternatives $V_\varepsilon = V_\varepsilon(\tau,\rho_\varepsilon)$ defined by (1.1, 1.2) with $p < \infty$:*

1. *Under assumptions A1 and either B3, B4 in ellipsoidal case or B3a, B4a in Besov bodies case one has: $\alpha(\psi_{\varepsilon,\tau,\rho_\varepsilon,T_\alpha}) = \alpha + o(1)$,*

$$\beta\left(\psi_{\varepsilon,\tau,\rho_\varepsilon,T_\alpha}, V_\varepsilon\right) = \Phi\left(T_\alpha - \inf_{v\in V'_\varepsilon}(\bar{\pi}_\varepsilon, \bar{\delta}_v)/\|\bar{\pi}_\varepsilon\|\right) + o(1).$$

2. *Under assumptions B1, B3, B4 in ellipsoidal case one has: $\alpha(\psi_{\varepsilon,\tau,\rho_\varepsilon,t_\varepsilon}) = \Phi(-t_\varepsilon) + o(\varepsilon^\delta)$,*

$$\beta(\psi_{\varepsilon,\tau,\rho_\varepsilon,t_\varepsilon}, V_\varepsilon) = \Phi\left(t_\varepsilon - \inf_{v\in V'_\varepsilon}(\bar{\pi}_\varepsilon, \bar{\delta}_v)/\|\bar{\pi}_\varepsilon\|\right) + o(\varepsilon^\delta).$$

Here $V'_\varepsilon = V_\varepsilon(\tau,\rho'_\varepsilon)$, $\rho'_\varepsilon = \rho_\varepsilon(1-n_\varepsilon^{-\delta_1})$. If assumptions B1, B3, B4 hold uniformly on K, then the values $\delta,\ \delta_1$ are bounded away from 0 on any compact $K \subset \Xi \times R^1_+$.

Remark 5.2. Assumptions B3, B4 seem to be cumbersome enough. However without assumptions of these type the asymptotics of the likelihood ratio and the asymptotics of error probabilities may be not Gaussian but degenerate or infinite divisible of special type, see Ingster [14].

5.4. **Extreme problem**

Using the results of Section 5.3 for the finding of best tests, we obtain the maximin problem:

$$w'_\varepsilon = \sup_{\bar{\pi}} \inf_{v\in V'_\varepsilon}(\bar{\pi}, \bar{\delta}_v)/\|\bar{\pi}\| = \sup_{\|\bar{r}\|=1} \inf_{v\in V'_\varepsilon}(\bar{r}, \bar{\delta}_v). \tag{5.23}$$

We can replace the set $\Delta_\varepsilon^{'} = \{\bar\delta_v,\ v \in V_\varepsilon^{'}\} \subset \Pi$ in (5.23) onto any wider set $\Pi_\varepsilon^{'} \subset \Pi,\ \Delta_\varepsilon^{'} \subset \Pi_\varepsilon^{'}$ and consider some different maximin problem:

$$u_\varepsilon^{'} = \sup_{\|\bar r\|=1} \inf_{\bar\pi \in \Pi_\varepsilon^{'}} (\bar r, \bar\pi) \le w_\varepsilon^{'}. \tag{5.24}$$

Let the supremum in (5.24) is attained on the family $\bar r_\varepsilon^{'} = \bar\pi_\varepsilon^{'}/\|\bar\pi_\varepsilon^{'}\|,\ \bar\pi_\varepsilon^{'} \in \Pi,$ where $\bar\pi_\varepsilon^{'}$ are sequences of three-points measures satisfying to assumptions A1, B3, B4. Then by Corollary 5.2 we obtain upper bounds

$$\beta(\alpha, V_\varepsilon) \le \Phi(T_\alpha - u_\varepsilon^{'}) + o(1).$$

Let the values u_ε correspond to Π_ε and $\Delta_\varepsilon^{'} \subset \Pi_\varepsilon$. If $u_\varepsilon^{'} = u_\varepsilon + o(1)$ (or $u_\varepsilon^{'} \to \infty$, as $u_\varepsilon \to \infty$), then we can replace $u_\varepsilon^{'}$ onto u_ε in (5.24).

To describe the sets $\Pi_\varepsilon \subset \Pi$ what we use, let us note that if $q < \infty$, then the alternatives in ellipsoidal case (1.1) and in Besov bodies case (1.2) are of the form

$$F_1(\bar\phi_1(v)) \ge H_{\varepsilon,1},\ F_2(\bar\phi_2(v)) \le H_{\varepsilon,2},\quad v \in l_2.$$

Here $\bar\phi_k = (\phi_{k,1}, \dots, \phi_{k,i}, \dots,),\ k = 1,2$ are the sequences of symmetric nonnegative functions on the real line:

$$\phi_{1,i}(x) = |x|^p,\ \phi_{2,i}(x) = |x|^q,\ x \in R^1.$$

For ellipsoidal case (1.1) the functionals $F_1(\bar y)$ and $F_2(\bar y)$ for $q < \infty$ are linear:

$$F_1(\bar y) = \sum_i i^{rp} y_i,\quad F_2(\bar y) = \sum_i i^{sq} y_i$$

and $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p,\ H_{\varepsilon,2} = R^q\varepsilon^{-q}$.

For Besov bodies case (1.2), if $h, p < \infty,\ t, q < \infty$, then

$$F_1(\bar y) = \sum_j 2^{jhr} \left(\sum_{l=1}^{2^j} y_{jl}\right)^{h/p},\quad F_2(\bar y) = \sum_j 2^{jts} \left(\sum_{l=1}^{2^j} y_{jl}\right)^{t/q}$$

with $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^h,\ H_{\varepsilon,2} = R^t\varepsilon^{-t}$; if $q < t = \infty$, then

$$F_2(\bar y) = \sup_j 2^{jsq} \sum_{l=1}^{2^j} y_{jl}$$

with $H_{\varepsilon,2} = R^q\varepsilon^{-q}$. We consider functionals $F_k(\bar y)$ on convex set of nonnegative sequences $\{\bar y = (y_1, \dots, y_i, \dots,)\}$ (of pyramidal structure for Besov bodies case).

We use an approach which is close to Pinsker [22], Donoho and Johnstone [4] in estimation problems. In hypothesis testing problems this approach had been used by Ermakov [6], Ingster [11–14].

Put $\bar\Phi_k(\bar\pi) = (\Phi_{k,1}(\pi_1), \dots, \Phi_{k,i}(\pi_i), \dots,)$ where $\Phi_{k,i}(\pi_i) = E_{\pi_i}\phi_{k,i}$ are π_i-moments. *Define the sets* $\Pi_\varepsilon = \Pi_\varepsilon(\tau, \rho_\varepsilon) \subset \Pi$ *by the moments inequalities:*

$$\Pi_\varepsilon = \{\bar\pi \in \Pi :\ \ G_1(\bar\pi) \ge H_{\varepsilon,1},\ G_2(\bar\pi) \le H_{\varepsilon,2}\ \},$$

where

$$G_1(\bar\pi) = F_1(\bar\Phi_1(\bar\pi)),\ G_2(\bar\pi) = F_2(\bar\Phi_2(\bar\pi)).$$

Denote by $|\pi|$ a half of the length of symmetric convex support of π:

$$|\pi| = \inf\{z > 0 : \ \pi(A) = 0 \ \forall A \subset (R^1 \setminus (-z, z))\}\cdot$$

For $q = \infty$ in ellipsoidal case we consider functionals

$$G_2(\bar{\pi}) = \sup_i i^s |\pi_i|.$$

In Besov bodies case for $q = t = \infty$ we consider functionals

$$G_2(\bar{\pi}) = \sup_j 2^{js} \max_l |\pi_{jl}|.$$

In these cases $H_{\varepsilon,2} = R\varepsilon^{-1}$. It is not difficult to see that functionals G_k may be defined on Π'' (they do not depend on an element of equivalence class). Note that *the functional* G_1 *is concave and the functional* G_2 *is convex* and *the set* Π_ε *is convex* for ellipsoidal case and for Besov bodies case with $p \geq h, \ q \leq t$. Also it is clear that $\Delta_\varepsilon \subset \Pi_\varepsilon$.

Let us consider extreme problem

$$u_\varepsilon = u_\varepsilon(\tau, \rho_\varepsilon) = \inf_{\bar{\pi} \in \Pi_\varepsilon} \|\bar{\pi}\|. \tag{5.25}$$

Lemma 5.1. *Assume:* $\exists \bar{\pi}_\varepsilon = \bar{\pi}_\varepsilon(\tau, \rho_\varepsilon) \in \Pi_\varepsilon$ *such that* $u_\varepsilon = \|\bar{\pi}_\varepsilon\| > 0$. *Then*

$$\sup_{\|\bar{r}\|=1} \inf_{\bar{\pi} \in \Pi_\varepsilon} (\bar{r}, \bar{\pi}) = \inf_{\bar{\pi} \in \Pi_\varepsilon} (\bar{r}_\varepsilon, \bar{\pi}) = u_\varepsilon$$

where $\bar{r}_\varepsilon = \bar{\pi}_\varepsilon / \|\bar{\pi}_\varepsilon\|$.

Proof of the Lemma is contained in Ingster [12], Section 5.3 in some different terms. This simple proof is based on convex properties of the set Π_ε only. One can obtain the Lemma from minimax theorem (see Sion [23], for example), however in this case some topological properties are used.

Remark 5.3. If there exists $\bar{\pi}_\varepsilon$ such that $u_\varepsilon = \|\bar{\pi}_\varepsilon\|$, then it is unique. In fact, if $u_\varepsilon = \|\bar{\pi}^1_\varepsilon\| = \|\bar{\pi}^2_\varepsilon\|$, then it is easy to see that $\|(\bar{\pi}^1_\varepsilon + \bar{\pi}^2_\varepsilon)/2\| < u_\varepsilon$, if $\|(\bar{\pi}^1_\varepsilon - \bar{\pi}^2_\varepsilon)\| > 0$.

Remark 5.4. One can easily obtain the following properties of functions $u_\varepsilon(\tau, \rho_\varepsilon)$ defined by (5.25).

1. $u_\varepsilon(\tau)$ is convex function of variables $(H_{\varepsilon,1}, \ H_{\varepsilon,2})$.
2. Let $u_\varepsilon = \|\bar{\pi}_\varepsilon\|$. Then $G_1(\bar{\pi}_\varepsilon) = H_{\varepsilon,1}$. Assume $\inf\{\|\bar{\pi}\| : G_1(\bar{\pi}) \geq H_{\varepsilon,1}\} = 0$ (it is the infimum without the constraints on G_2, it corresponds to $H_{\varepsilon,2} = \infty$). Then $G_2(\bar{\pi}_\varepsilon) = H_{\varepsilon,2}$.

Define $\rho_{\varepsilon,b} = b\rho_\varepsilon$. Put the assumptions

C1. There exist such $B > 1, \ C > 1$ that $\forall b \in (B^{-1}, B)$ one can find such $\bar{\pi}_\varepsilon(\tau, \rho_{\varepsilon,b}) \in \Pi_\varepsilon(\tau, \rho_{\varepsilon,b})$ that $u_\varepsilon(\tau, \rho_{\varepsilon,b}) = \|\bar{\pi}_\varepsilon(\tau, \rho_{\varepsilon,b})\| > 0$ and $C^{-1} < u_\varepsilon(\tau, \rho_{\varepsilon,b})/u_\varepsilon(\tau, \rho_\varepsilon) < C$.

Remark 5.5. From Remark 5.4, n. 1 we get: under the assumption C1 the function $u_\varepsilon(\tau, \rho_{\varepsilon,b})$ is Lipschitzian function on $b \in (B^{-1}, B)$ with a constant $L = L(B, C) > 0$.

C2. Assumption C1 is fulfilled and $\forall b \in (B^{-1}, B)$ the sequences $\bar{\pi}_\varepsilon(\tau, \rho_{\varepsilon,b})$ are the sequences of three-point measures

$$\pi_{\varepsilon,i}(\tau, \rho_{\varepsilon,b}) = (1 - h_{\varepsilon,i}(\tau, \rho_{\varepsilon,b}))\delta_0 + \frac{h_{\varepsilon,i}(\tau, \rho_{\varepsilon,b})}{2}(\delta_{z_{\varepsilon,i}(\tau,\rho_{\varepsilon,b})} + \delta_{-z_{\varepsilon,i}(\tau,\rho_{\varepsilon,b})})$$

which satisfy: either A1 or B1, and B3, B4 in ellipsoidal case; A1, B3 and B4a in Besov bodies case.

From Lemma 5.1 and Corollaries 5.1, 5.2 we obtain the following

Theorem 12. 1) *Assume C2. Then*

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \le \Phi(T_\alpha - u_\varepsilon(\tau, \rho_\varepsilon)) + o(1)$$

and there exists such family $b_\varepsilon \to 1$ *that this bound is provided by the family of tests* $\psi_{\varepsilon, \bar h_{\varepsilon, b_\varepsilon}, \bar z_{\varepsilon, b_\varepsilon}}$.

2) *Assume that there exists such family* $\bar\pi_{\varepsilon,1}$ *of sequences of three-point measures satisfying to assumptions A1, A2 or B1, B2 that*

$$\|\bar\pi_{\varepsilon,1}\| = u_\varepsilon(\tau, \rho_\varepsilon) + o(1), \ \pi_1^\varepsilon(V_\varepsilon(\tau, \rho_\varepsilon)) \to 1.$$

Then

$$\beta(\alpha, V_\varepsilon(\tau, \rho_\varepsilon)) \ge \Phi(T_\alpha - u_\varepsilon(\tau, \rho_\varepsilon)) + o(1).$$

Theorem 12 translates our problem to the study of extreme problem (5.25) and to checking of assumptions A and B.

6. Extreme problem for ellipsoids

In what follows we assume that $\kappa \in \Xi_G = \cup_{k=1}^5 \Xi_{G_k}$. First, we consider the case $p = q$, where the main properties of the problem are shown and the methods are more simple. The assumption $\kappa \in \Xi_G$ means $p < \infty, \ r_p \le r < s$ in this case.

6.1. The case $p = q < \infty$

It is not difficult to see that extreme problem (5.25) can be separated by the following way. Denote by *Prob* the set of probability measures on the real line.

First, let us consider *one-dimensional problems* of minimization of $\|\pi\|$, $\pi \in Prob$ under the moment constraints:

$$R(\lambda, p) = \inf\{\|\pi\|^2 : \ \pi \in Prob, \ E_\pi |v|^p = \lambda^p\} \cdot \tag{6.1}$$

Then

$$u_\varepsilon^2 = \inf_{\bar\lambda} \sum_i R(\lambda_i, p) : \ \sum_i i^{rp} \lambda_i^p \ge (\rho_\varepsilon/\varepsilon)^p, \sum_i i^{sq} \lambda_i^p \le (R/\varepsilon)^p, \ \lambda_i \ge 0. \tag{6.2}$$

For the case $p = q$ one-dimensional problems have been studied in Ingster [11, 12]. These methods have been generalized in Suslina [27] and we have the following

Lemma 6.1. 1. *If* $p \le 2$, *then the infimum in (6.1) is attained by the two-point measure* $\pi = \frac{1}{2}(\delta_z + \delta_{-z})$ *with* $z = \lambda$.

2. *If* $p > 2$, *then the infimum in (6.1) is attained by three-point measure* $\pi = (1-h)\delta_0 + \frac{h}{2}(\delta_z + \delta_{-z})$ *(or two-point measure, if* $h = 1$*). Let the parameter* $z(p) > 0$ *be defined by the relation:*

$$z^2(p) = p \tanh z^2(p)/2, \ p > 2 \tag{6.3}$$

and $\lambda \le z(p)$. *Then* $z = z(p), \ h = (\lambda/z(p))^p$.

Proof of the lemma is presented in Ingster [11] for $p > 2$ and in Suslina [27] for $p \le 2$ (the case $p \le 2$ also non-directly follows from Ingster [11, 12]).

Using lemma and assuming $\sup_i \lambda_i = o(1)$ (this assumption is equivalent to the first Assumpt. A.1 and is checked later) we obtain from (6.2) the extreme problems: if $p \le 2$, then

$$\inf_{\bar z} 2 \sum_i \sinh^2(z_i^2/2) : \ \sum_i i^{rp} z_i^p \ge (\rho_\varepsilon/\varepsilon)^p, \ \sum_i i^{sp} z_i^p \le (R/\varepsilon)^p, \ z_i \ge 0, \tag{6.4}$$

and if $p > 2$, then

$$\inf_{\bar h} 2\sinh^2 \frac{z^2(p)}{2} \sum_i h_i^2 : \ z^p(p) \sum_i i^{rp} h_i \ge (\rho_\varepsilon/\varepsilon)^p, \ z^p(p) \sum_i i^{sp} h_i \le (R/\varepsilon)^p. \tag{6.5}$$

Using the asymptotics $2\sinh^2(z^2/2) = z^4/2 + O(z^8), \ z \to 0$ and assuming $\sum_i z_i^8 = o(1)$ (we will check this assumption later for the solutions of extreme problem) we can replace the extreme problem (6.4) onto the following:

$$\inf_{\bar z} \frac{1}{2} \sum_i z_i^4 : \ \sum_i i^{rp} z_i^p \ge (\rho_\varepsilon/\varepsilon)^p, \ \sum_i i^{sp} z_i^p \le (R/\varepsilon)^p, \ z_i \ge 0. \tag{6.6}$$

One can easily check that the infimum is 0 in extreme problems analogous to (6.6, 6.5) except for the second constraints:

$$0 = \inf_{\bar z} \frac{1}{2} \sum_i z_i^4 : \ \sum_i i^{rp} z_i^p \ge (\rho_\varepsilon/\varepsilon)^p, \tag{6.7}$$

$$0 = \inf_{\bar h} 2\sinh^2 \frac{z^2(p)}{2} \sum_i h_i^2 : \ z^p(p) \sum_i i^{rp} h_i \ge (\rho_\varepsilon/\varepsilon)^p \tag{6.8}$$

(these estimations are contained in Ingster [12], nn. 4.2, 4.3 in the proof of Th. 2.5). Therefore by Remark 5.4 we can assume the equalities in the constraints. Using the Lagrange multipliers rule it is easy to obtain from (6.6, 6.5) the relations for z_i and h_i. Let us consider differently the cases $p \le 2$ and $p > 2$.

6.1.1. *Proof of Theorems 4, 6 and 8, n. 1 for $p = q \le 2$*

If $p \le 2$, then the values z_i which minimize (6.6) are defined by the some positive parameters $z_0 = z_{\varepsilon,0}, \ m = m_\varepsilon$ by the relations:

$$z_i = z_0((i/m)^{rp} - (i/m)^{sp})_+^{1/(4-p)} \tag{6.9}$$

where the values $m = m_\varepsilon, \ z_0 = z_{0,\varepsilon}$ are defined by the equalities:

$$\begin{aligned} (\rho_\varepsilon/\varepsilon)^p &= z_0^p m^{rp+1} \left(m^{-1} \sum_{1 \le i \le m} ((i/m)^{rp} - (i/m)^{sp})^{p/(4-p)} (i/m)^{rp} \right), \\ (R/\varepsilon)^p &= z_0^p m^{sp+1} \left(m^{-1} \sum_{1 \le i \le m} ((i/m)^{rp} - (i/m)^{sp})^{p/(4-p)} (i/m)^{sp} \right) \end{aligned} \tag{6.10}$$

and

$$2u_\varepsilon^2 \sim \sum_i z_i^4.$$

By (6.9) we have

$$u_\varepsilon^2 \sim \frac{m z_0^4}{2} \left(m^{-1} \sum_{1 \le i \le m} ((i/m)^{rp} - (i/m)^{sp})^{4/(4-p)} \right). \tag{6.11}$$

Let $r > r_p = 1/4 - 1/p$. Assume $z_0 \to 0,\ m \to \infty$. By replacing sums onto integrals we obtain relations (3.5–3.10) (more detailed consideration follows to the scheme of Sect. 6.3.2 later). It is easy to check that if $\rho_\varepsilon \to 0,\ u_\varepsilon = O(\varepsilon^{-\delta})$ for any $\delta > 0$, then for small enough $\delta_1 > 0$

$$z_0 = O(\varepsilon^{\delta_1}),\ m^{-1} = O(\varepsilon^{\delta_1}),\ \sum_i z_i^6 = O\left(u_\varepsilon^2 \sup_i z_i^2\right) = O(\varepsilon^{\delta_1}).$$

Let $r = r_p = 1/4 - 1/p$. Then $s > r_p$, the first sum in (6.10) and the sum in (6.11) are of the rate

$$\begin{aligned} m^{-1} \sum_{1\le i\le m} ((i/m)^{rp} - (i/m)^{sp})^{p/(4-p)} (i/m)^{rp} \quad &\sim \quad \log m, \\ m^{-1} \sum_{1\le i\le m} ((i/m)^{rp} - (i/m)^{sp})^{4/(4-p)} \quad &\sim \quad \log m. \end{aligned} \tag{6.12}$$

Assuming $z_0 \to 0,\ m \to \infty$ we obtain the relations (3.26, 3.27) with $c_0(\kappa) = c_1(\kappa) = 1$ and $c_2(\kappa)$ defined by (3.10). It is easy to check that if $\rho_\varepsilon \to 0,\ u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta = \delta(\kappa) > 0$, then $z_0 \to 0,\ m \to \infty$.

To prove Theorems 4, 6 and 8, n. 1 for the case $r > r_p = 1/4 - 1/p$ it is enough to check the assumptions of Theorem 12. Assumption C.1 follows from asymptotics (3.26, 3.27). If $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta > 0$, then we obtain B.1, by

$$\sup_i z_i \le \max\{z_0, z_0 m^{-rp/(4-p)}\} = o(\varepsilon^{\delta_1}) \tag{6.13}$$

for small enough $\delta_1 = \delta_1(\kappa, \delta) > 0$. Assumptions B.3, B.4 follow from (6.13). Therefore we can use upper bounds of Theorem 12, n. 1.

By $h_i = 1$ and using the relations (6.9), (6.10) we have $\pi_\varepsilon(V_\varepsilon) = 1$. Therefore we can use lower bounds of Theorem 12, n. 2 with original family $\bar\pi_{\varepsilon,1} = \bar\pi_\varepsilon$.

The case $r = r_p = 1/4 - 1/p$ is considered by analogous way. If $u_\varepsilon = O(1)$, then we obtain A.1, B.3, B.4 by

$$\sup_i z_i \le \max\{z_0, z_0 m^{1/4}\} = O((\log m)^{-1/4}) = o(1). \tag{6.14}$$

Theorems 4, 6 and 8, n. 1 are proved for the case $p = q \le 2$.

6.1.2. *Proof of Theorems 4, 6 and 8, n. 1 for $p = q > 2$*

If $p > 2$, then the values h_i which minimize (6.5) are defined by the some positive parameters $h_0 = h_{\varepsilon,0},\ n = n_\varepsilon$ by the relations:

$$h_i = h_0((i/n)^{rp} - (i/n)^{sp})_+$$

where the values $n = n_\varepsilon,,\ h_0 = h_{0,\varepsilon}$ are defined by the equalities:

$$\begin{aligned} (\rho_\varepsilon/\varepsilon)^p \quad &= \quad z^p(p) h_0 n^{rp+1} \left(n^{-1} \sum_{1\le i\le n} ((i/n)^{rp} - (i/n)^{sp})(i/n)^{rp} \right), \\ (R/\varepsilon)^p \quad &= \quad z^p(p) h_0 n^{sp+1} \left(n^{-1} \sum_{1\le i\le n} ((i/n)^{rp} - (i/n)^{sp})(i/n)^{sp} \right) \end{aligned} \tag{6.15}$$

and

$$u_\varepsilon^2 = 2\sinh^2(z^2(p)/2) n h_0^2 \left(n^{-1} \sum_{1\le i\le n} ((i/n)^{rp} - (i/n)^{sp} \right)^2 \tag{6.16}$$

let $r > r_p = -1/2p$. Assume $h_0 \to 0,\ n \to \infty$. By replacing sums onto integrals we obtain relations (3.12, 3.13) (more detailed consideration follows to the scheme of Sect. 6.3.2 later). It is easy to check that if $\rho_\varepsilon \to 0,\ u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta = \delta(\kappa) > 0$, then $h_0 = O(\varepsilon^{\delta_1}),\ n^{-1} = O(\varepsilon^{\delta_1})$ for small enough $\delta_1 > 0$.

Let $r = r_p = -1/2p,\ s > r_p$. Then the first sum in (6.15) and the sum in (6.16) are of the rate

$$\begin{aligned} n^{-1} \sum_{1\le i\le n} ((i/n)^{rp} - (i/n)^{sp})(i/n)^{rp} \quad &\sim \quad \log n, \\ n^{-1} \sum_{1\le i\le n} ((i/n)^{rp} - (i/n)^{sp})^2 \quad &\sim \quad \log n, \end{aligned} \tag{6.17}$$

and assuming $z_0 \to 0,\ n \to \infty$ we obtain the relations (3.26, 3.27) with $c_0(\kappa) = c_1(\kappa) = 1$ and $c_2(\kappa)$ defined by (3.13). It is easy to check that if $\rho_\varepsilon \to 0,\ u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta = \delta(\kappa) > 0$, then $h_0 \to 0,\ n \to \infty$.

To prove Theorems 4, 6 and 8, n. 1 note, that assumptions C.1, C.2 of Theorem 12 follow from asymptotics (3.26, 3.27) and from $\sup_i z_i = z(p)$.

To construct the families $\tilde\pi_\varepsilon$ which provides to the assumptions n. 2 of Theorem 12, it is enough to assume $u_\varepsilon \asymp 1$. This yields: $h_0 \asymp n^{-1/2}$ for $r > r_p$ and $h_0 \asymp (n \log n)^{-1/2}$ for $r = r_p$.

Let us consider the values $\delta_\varepsilon = (\log \varepsilon^{-1})^{-\delta}$ and put

$$\tilde\pi_\varepsilon = \pi_\varepsilon(\kappa, (1+\delta_\varepsilon)\rho_\varepsilon, (1-\delta_\varepsilon)R). \tag{6.18}$$

Using the inequality

$$\begin{aligned} \tilde\pi^\varepsilon(V_\varepsilon) \quad &= \quad \tilde\pi^\varepsilon(F_1(\bar\phi_1(v)) \ge H_{\varepsilon,1},\ F_2(\bar\phi_2(v)) \le H_{\varepsilon,2}) \\ &\ge \quad 1 - \tilde\pi^\varepsilon(F_1(\bar\phi_1(v)) < H_{\varepsilon,1}) - \tilde\pi^\varepsilon(F_2(\bar\phi_2(v)) > H_{\varepsilon,2}), \end{aligned}$$

and Chebyshev inequality, we get the relation $\tilde\pi^\varepsilon(V_\varepsilon) \to 1$ from

$$E_{\tilde\pi^\varepsilon} F_1(\bar\phi_1(v)) = (1+\delta_\varepsilon)^p H_{\varepsilon,1},\ E_{\tilde\pi^\varepsilon} F_2(\bar\phi_2(v)) = (1-\delta_\varepsilon)^p H_{\varepsilon,2} \tag{6.19}$$

and from

$$Var_{\tilde\pi^\varepsilon} F_1(\bar\phi_1(v)) = o(H^2_{\varepsilon,1}\delta^2_\varepsilon),\ Var_{\tilde\pi^\varepsilon} F_2(\bar\phi_2(v)) = o(H^2_{\varepsilon,2}\delta^2_\varepsilon). \tag{6.20}$$

By $H_{\varepsilon,2} \asymp h_0 n^{1+sp},\ H_{\varepsilon,1} \asymp h_0 n^{1+rp}$, if $r > r_p$ and $H_{\varepsilon,1} \asymp h_0 n^{1+rp}\log n$, if $r = r_p$, it is enough to check (6.19). We have:

$$\begin{aligned} Var_{\tilde\pi^\varepsilon} F_2(\bar\phi_2(v)) \quad &= \quad (z(p))^{2p} \sum_{i=1}^n h_i(1-h_i) i^{2sp} \asymp n^{1+2sp} h_0 \int_{1/n}^1 x^{(2s+r)p} dx \\ &= \quad o(H^2_{\varepsilon,2}\delta^2_\varepsilon), \\ Var_{\tilde\pi^\varepsilon} F_1(\bar\phi_1(v)) \quad &= \quad (z(p))^{2p} \sum_{i=1}^n h_i(1-h_i) i^{2rp} \asymp n^{1+2rp} h_0 \int_{1/n}^1 x^{3rp} dx \\ &= \quad o(H^2_{\varepsilon,1}\delta^2_\varepsilon), \end{aligned}$$

for small enough $\delta > 0$ by for $s \ge r$ one has: $2sp + rp + 1 > -1/2$ and

$$\int_{1/n}^1 x^{(2s+r)p} dx \asymp \begin{cases} 1, & \text{if } (2s+r)p+1 > 0, \\ \log n, & \text{if } (2s+r)p+1 = 0, \\ n^{-(2s+r)p-1}, & \text{if } (2s+r)p+1 < 0. \end{cases}$$

Theorems 4, 6 and 8, n. 1 are proved for the case $p = q > 2$.

6.2. The case $p \neq q$: Separation equations system for the extreme problem

It is not difficult to see that the extreme problem (5.23) can be separated by the following way.

6.2.1. *One-dimensional problems*

Let us consider the *one-dimensional problems* of minimization of $\|\pi\|$, $\pi \in Prob$ under the moment constraints.

If $q < \infty$, put for $\lambda \geq 0, \nu \geq 0$

$$R(\lambda, \nu; p, q) = \inf\{ \ \|\pi\|^2 : \ \pi \in Prob, \ E_\pi|v|^p \geq \lambda^p, \ E_\pi|v|^q \leq \nu^q \ \} \cdot \tag{6.21}$$

Then

$$u_\varepsilon^2 = \inf_{\bar\lambda, \bar\nu} \sum_i R(\lambda_i, \nu_i; p, q) : \quad \textstyle\sum_i \quad i^{rp}\lambda_i^p \geq (\rho_\varepsilon/\varepsilon)^p,$$
$$\textstyle\sum_i \quad i^{sq}\nu_i^q \leq R^q\varepsilon^{-q}. \tag{6.22}$$

If $q = \infty$, put for $\lambda \geq 0, \nu \geq 0$

$$R(\lambda, \nu; p, \infty) = \inf\{\|\pi\|^2 : \ \pi \in Prob, \ E_\pi|v|^p \geq \lambda^p, \ |\pi| \leq \nu\} \cdot \tag{6.23}$$

Then

$$u_\varepsilon^2 = \inf_{\bar\lambda, \bar\nu} \sum_i R(\lambda_i, \nu_i; p, \infty) : \quad \textstyle\sum_i \quad i^{rp}\lambda_i^p \geq (\rho_\varepsilon/\varepsilon)^p,$$
$$\sup_i \quad i^s\nu_i \leq R\varepsilon^{-1}. \tag{6.24}$$

In (6.22, 6.24) the infimums are taken over the sets of nonnegative sequences $\bar\lambda, \bar\nu$ under the formulated constraints.

6.2.2. *Solution of one-dimensional problems*

For the case $p \neq q$ one-dimensional problems have been studied in Suslina [27] and we have the following

Lemma 6.2. *If the sets under constraints are not empty, then the infimum in (6.21, 6.23) is attained by three-point measure $\pi = (1-h)\delta_0 + \frac{h}{2}(\delta_z + \delta_{-z})$ (or two-point measure, if $h = 1$) with the parameters $h = h(\lambda, \nu, p, q) \in [0, 1]$ and $z = z(\lambda, \nu, p, q) \geq 0$.*

Proof of the lemma is presented in Suslina [27] The relations for $h = h(\lambda, \nu, p, q) \in [0, 1]$ and $z = z(\lambda, \nu, p, q) \geq 0$ are given in this paper as well, however these relations are not of importance for us at the moment.

Using lemma we can reduce the extreme problems (6.22, 6.24) to the following relations (the infimum is taken under constraints $h_i \in [0, 1], \ z_i \geq 0$):
if $q < \infty$, then

$$u_\varepsilon^2 = \inf_{\bar h, \bar z} 2\sum_i h_i^2 \sinh^2(z_i^2/2) : \quad \textstyle\sum_i \quad i^{rp}h_i z_i^p \geq (\rho_\varepsilon/\varepsilon)^p,$$
$$\textstyle\sum_i \quad i^{sq}h_i z_i^q \leq R^q\varepsilon^{-q}; \tag{6.25}$$

and if $q = \infty$, then

$$u_\varepsilon^2 = \inf_{\bar h, \bar z} 2\sum_i h_i^2 \sinh^2(z_i^2/2) : \quad \textstyle\sum_i \quad i^{rp}h_i z_i^p \geq (\rho_\varepsilon/\varepsilon)^p,$$
$$\sup_i \quad i^s z_i \leq R\varepsilon^{-1}. \tag{6.26}$$

6.2.3. *System of equations for extreme problem, $q < \infty$*

Using the Lagrange multipliers rule we obtain the following system of equations on the variables $h_i,\ z_i$ which attain the infimum in (6.25):

$$\begin{aligned} 4h_i \sinh^2 \frac{z_i^2}{2} &= Ai^{rp} z_i^p - Bi^{sq} z_i^q - C_i, \\ 4h_i \sinh^2 \frac{z_i^2}{2} \left(\frac{z_i^2}{\tanh \frac{z_i^2}{2}} \right) &= Api^{rp} z_i^p - Bqi^{sq} z_i^q. \end{aligned} \tag{6.27}$$

Here

$$A = A_\varepsilon \geq 0,\ B = B_\varepsilon \geq 0,\ C_i = C_{\varepsilon,i} \geq 0$$

and if $C_i > 0$, then $h_i = 1$ (for simplicity we do not consider the Lagrange multipliers which correspond to the constraints $h_i \geq 0,\ z_i \geq 0$ assuming that we will find only the positive solutions).

One can easily check that the infimum is 0 in extreme problems analogous to (6.25, 6.26) except for the second constraints (this follows from the relations (6.7, 6.8)). By Remark 5.4 unknown values A, B, C_i are defined by the equations:

$$\sum_i i^{rp} h_i z_i^p = (\rho_\varepsilon/\varepsilon)^p,\ \sum_i i^{sq} h_i z_i^q = (R/\varepsilon)^q. \tag{6.28}$$

From the Remarks to Lemma 5.1 one can easily see, that any solution of systems (6.27, 6.28) provides the solution of extreme problem (6.25):

$$u_\varepsilon^2 = 2 \sum_i h_i^2 \sinh^2(z_i^2/2). \tag{6.29}$$

In what follows we solve the system (6.27) under some assumptions either on $A_\varepsilon,\ B_\varepsilon$ or on other parameters defined by $A_\varepsilon,\ B_\varepsilon$. Then we find these parameters by solving (6.28) and then we check these assumptions.

First, we try to find the solutions $h_i,\ z_i$ of (6.27) assuming $C_i = 0$. If we obtain $h_i \leq 1$, then the solutions are correct. If we obtain $h_i > 1$, then it is not possible to find such solutions, we put $h_i = 1$ and obtain the equation

$$4 \sinh^2 \frac{z_i^2}{2} \left(\frac{z_i^2}{\tanh \frac{z_i^2}{2}} \right) = Api^{rp} z_i^p - Bqi^{sq} z_i^q \tag{6.30}$$

with the constrain (which corresponds to $C_i \geq 0$)

$$4 \sinh^2 \frac{z_i^2}{2} \leq Ai^{rp} z_i^p - Bi^{sq} z_i^q. \tag{6.31}$$

Next, we solve (6.30, 6.31). Later we realize this outline.

6.3. **Solution of the system (6.27) with $C_i = 0$**

If $C_i = 0$ in (6.27), then we obtain the system

$$\begin{aligned} 4h_i \sinh^2 \frac{z_i^2}{2} \left(\frac{z_i^2 - p \tanh \frac{z_i^2}{2}}{z_i^q \tanh \frac{z_i^2}{2}} \right) &= (p-q) B i^{sq}, \\ 4h_i \sinh^2 \frac{z_i^2}{2} \left(\frac{z_i^2 - q \tanh \frac{z_i^2}{2}}{z_i^p \tanh \frac{z_i^2}{2}} \right) &= (p-q) A i^{rp}. \end{aligned} \tag{6.32}$$

The equations (6.32) imply the solutions of (6.27) with $C_i = 0$:

$$z_i^{p-q}\frac{z_i^2 - p\tanh\frac{z_i^2}{2}}{z_i^2 - q\tanh\frac{z_i^2}{2}} = \frac{B}{A}\, i^{sq-rp},$$
$$h_i = Ai^{rp}\frac{z_i^p}{4\sinh^2\frac{z_i^2}{2}}\left(\frac{(p-q)\tanh\frac{z_i^2}{2}}{z_i^2 - q\tanh\frac{z_i^2}{2}}\right) \tag{6.33}$$

with the constraints

$$z_i^2 > p\tanh\frac{z_i^2}{2}, \quad \text{if } p > q,$$
$$z_i^2 < p\tanh\frac{z_i^2}{2}, \quad \text{if } p < q. \tag{6.34}$$

The constraints (6.34) imply: $z_i \in Z_{p,q}$, where $Z_{p,q} \subset R^1_+$ are the sets:

$$Z_{p,q} = \begin{cases} \emptyset, & \text{if } p \le 2,\ p < q, \\ \{z > z(p)\}, & \text{if } p > 2,\ p > q, \\ \{z < z(p)\}, & \text{if } p > 2,\ p < q, \\ \{z > 0\}, & \text{if } p \le 2,\ p > q. \end{cases}$$

Here the values $z(p)$ are defined in Lemma 6.1.

Introduce the functions

$$\phi_{p,q}(z) = z^{p-q}\frac{z^2 - p\tanh\frac{z^2}{2}}{z^2 - q\tanh\frac{z^2}{2}}; \quad \psi_{p,q}(z) = \frac{z^p}{4\sinh^2\frac{z^2}{2}}\left(\frac{(p-q)\tanh\frac{z^2}{2}}{z^2 - q\tanh\frac{z^2}{2}}\right). \tag{6.35}$$

It is convenient to replace the unknown parameters $A > 0,\ B > 0$ onto another unknown parameters $n > 0,\ h_0 > 0$ for $\lambda \ne 0$, or onto $m > 0,\ z_0 > 0$, for $\Delta \ne 0$:

$$n = n_\varepsilon(\kappa) = (A/B)^{1/\lambda},\ h_0 = h_{0,\varepsilon}(\kappa) = An^{rp};\ x = x_i = i/n,\ i \ge 1 \tag{6.36}$$

or

$$m = m_\varepsilon(\kappa) = \left(\frac{A^{4-q}}{B^{4-p}}\right)^{1/\Delta},\ z_0 = z_{0,\varepsilon}(\kappa) = \left(\frac{A^{sq}}{B^{rp}}\right)^{1/\Delta};\ y = y_i = i/m,\ i \ge 1, \tag{6.37}$$

where we put $\lambda = sq - rp,\quad \Delta = sq(4-p) - rp(4-q)$. It is clear that if $\lambda \ne 0$ and $\Delta \ne 0$, then one has

$$m = nh_0^{(p-q)/\Delta},\ z_0 = h_0^{\lambda/\Delta},\ y = h_0^{-(p-q)/\Delta}x. \tag{6.38}$$

It is convenient to use parameters n, h_0 and variables x for the case when we have solutions with $C_i = 0$. However it is more convenient to use parameters m, z_0 and variables y for the case when we operate with solutions with $C_i > 0$ (this case is considered in the next subsection). As a rule we have the solutions of both types. Therefore we need to consider at the same time both types of parameters.

We can rewrite (6.33) for $\lambda \ne 0$:

$$\phi_{p,q}(z_i) = x^\lambda,\ h_i = h_0x^{rp}\psi_{p,q}(z_i),\quad i \ge 1$$

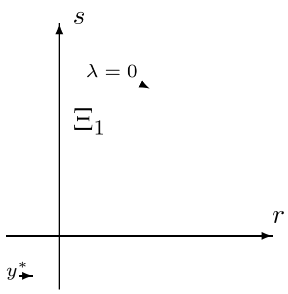

FIGURE 9. $p > q,\ p > 2$.

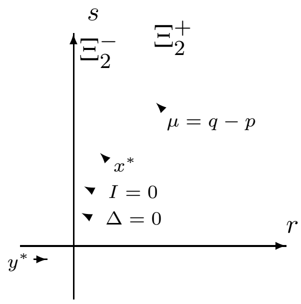

FIGURE 10. $2 < p < q \le \infty$.

or for $\Delta \neq 0$ one has

$$\phi_{p,q}(z_i) = y^\lambda z_0^{p-q},\ h_i = z_0^{4-p} y^{rp} \psi_{p,q}(z_i),\quad i \ge 1.$$

It is possible to check, that if $p > q$, then $\phi_{p,q}(z),\ z \in Z_{p,q}$ is monotone increasing from 0 to ∞, and if $p < q$, then this function is monotone decreasing from ∞ to 0. Therefore it is possible to define the inverse function $\phi_{p,q}^{-1}(x),\ x > 0$ with the values in $Z_{p,q}$. The solutions of (6.27) with $C_i = 0$ are of the form: for $\lambda \neq 0$

$$z_i = \phi_{p,q}^{-1}(x^\lambda) = z(x,\kappa),\ h_i = h_0 x^{rp} \psi_{p,q}(z(x,\kappa)) = h_0 \delta(x,\kappa), \tag{6.39}$$

where

$$z(x,\kappa) = \phi_{p,q}^{-1}(x^\lambda),\ \delta(x,\kappa) = x^{rp} \psi_{p,q}(z(x,\kappa)), \tag{6.40}$$

and if $\Delta \neq 0$, then

$$z_i = z_0(y,\kappa,z_0), h_i = h_0(y,\kappa,z_0) \tag{6.41}$$

where

$$z_0(y,\kappa,z_0) = \phi_{p,q}^{-1}(z_0^{p-q} y^\lambda),\quad h_0(y,\kappa,z_0) = y^{rp} z_0^{4-p} \psi_{p,q}(z_0(y,\kappa,z_0)).$$

Denote, as above, $\Xi_G = \cup_{i=1}^5 \Xi_{G_i}$. Put (see Fig. 9–12)

$$\begin{array}{rcl} \Xi_1 &=& \{\kappa \in \Xi_G :\ p > 2, p > q\}, \\ \Xi_2 &=& \{\kappa \in \Xi_G :\ p > 2, p < q\}, \\ \Xi_3 &=& \{\kappa \in \Xi_G :\ p < 2, p > q\}, \\ \Xi_4 &=& \{\kappa \in \Xi_G :\ p = 2, p > q\}\cdot \end{array} \tag{6.42}$$

Denote $\tilde\Delta = \tilde\Delta(\kappa) = 2(sq - (8-q)r),\ C(p) = z^p(p)/4\sinh^2(z^2(p)/2)$,

$$C_{p,q} = \left(\frac{|q-2|}{|p-2|}\right)^{(p-4)/(p-q)} \frac{|p-q|}{|2-q|},\ C(p,q) = \left(\frac{|q-2|}{|p-2|}\right)^{1/(p-q)},$$

and $C_q = (6(2-q))^{1/(6-q)}$. Note, that if $\kappa \in \Xi_k,\ k \neq 3$, then $\lambda = \lambda(\kappa) > 0$ and if $\kappa \in \Xi_4$, then $I \ge 0$ (it follows from definitions of the sets Ξ_{G_i} in Sect. 3).

The following proposition describes the properties of the functions $z(x,\kappa)$ and $\delta(x,\kappa),\ x > 0,\ \kappa \in \Xi_k,\ k = 1,\dots,4$ (if $\kappa \in \Xi_3$, then we assume $\lambda \neq 0$), and of the functions $z_0(y,\kappa,z_0),\ h_0(y,\kappa,z_0)$, for $\lambda \le 0$ (note that $r < 0,\ s < 0,\ \Delta > 0,\ \kappa \in \Xi_3$ in this cases).

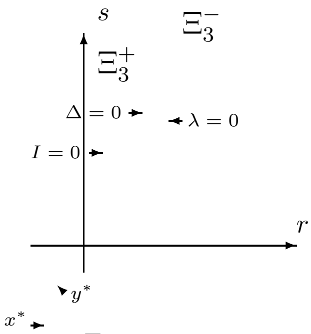

FIGURE 11. $2 > p > q$.

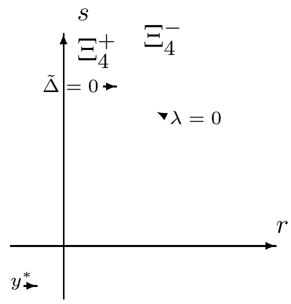

FIGURE 12. $q < p = 2$.

Proposition 6.1.

A). The functions $z(x,\kappa)$ and $\delta(x,\kappa)$ are uniformly continuous positive smooth functions on compacts $K \subset \Xi_k \times R^1_+$ which have no intersection with $\{\lambda = 0\} \times R^1_+$, Uniformly on any compact K of such type the following rate relations 1 – 4 hold:

1. *Let $p > 2,\ p > q\ \ (\kappa \in \Xi_1)$. Then $z(x,\kappa)$ is increasing on x and*

$$z(x,\kappa) \sim \begin{cases} z(p), & \text{if } x \to 0, \\ x^{\lambda/(p-q)}, & \text{if } x \to \infty; \end{cases}$$

$$\delta(x,\kappa) \sim \begin{cases} C(p)x^{rp}, & \text{if } x \to 0, \\ (p-q))x^{rp} z^{p-2}(x,\kappa)\exp(-z^2(x,\kappa)), & \text{if } x \to \infty. \end{cases}$$

2. *Let $p > 2,\ p < q\ \ (\kappa \in \Xi_2)$. Then $z(x,\kappa)$ is decreasing on x and*

$$z(x,\kappa) \sim \begin{cases} z(p), & \text{if } x \to 0, \\ C(p,q)x^{\lambda/(p-q)}, & \text{if } x \to \infty; \end{cases}$$

$$\delta(x,\kappa) \sim \begin{cases} C(p)x^{rp}, & \text{if } x \to 0, \\ C_{p,q}x^{-\Delta/(p-q)}, & \text{if } x \to \infty. \end{cases}$$

3. *Let $p < 2,\ p > q\ \ (\kappa \in \Xi_3)$. If $\lambda < 0$, then $z(x,\kappa)$ is increasing on x, if $\lambda > 0$, then it is decreasing on x;*

$$z(x,\kappa) \sim \begin{cases} C(p,q)x^{\lambda/(p-q)}, & \text{if } x \to 0, \\ x^{\lambda/(p-q)}, & \text{if } x \to \infty\ ; \end{cases}$$

$$\delta(x,\kappa) \sim \begin{cases} C_{p,q}x^{-\Delta/(p-q)}, & \text{if } x \to 0 \text{ for } \lambda > 0 \text{ or } x \to \infty \text{ for } \lambda < 0, \\ (p-q)x^{rp}z^{p-2}e^{-z^2}, & \text{if } x \to \infty \text{ for } \lambda > 0 \text{ or } x \to 0 \text{ for } \lambda < 0; \end{cases}$$

where $z = z(x,\kappa)$.

4. *Let $2 = p > q\ \ (\kappa \in \Xi_4)$. Then $z(x,\kappa)$ is increasing on x and*

$$z(x,\kappa) \sim \begin{cases} C_q x^{\lambda/(6-q)}, & \text{if } x \to 0, \\ x^{\lambda/(2-q)}, & \text{if } x \to \infty; \end{cases}$$

$$\delta(x,\kappa)\sim\begin{cases}C_q^{-2}x^{-\tilde\Delta/(6-q)}, & \text{if } x\to 0,\\ (1-q/2)x^{2r}\exp(-z^2(x,\kappa)), & \text{if } x\to\infty.\end{cases}$$

B). Let $\lambda\le 0$ (remind that $\kappa\in\Xi_3$, $\Delta>0$ in this case). If $\lambda=0$, then $z_0(y,\kappa,z_0)$ is constant. If $\lambda<0$, then it is decreasing continuous on y. If $z_0\to 0$, $z_0m^{-rp/(4-p)}\to 0$, $y\ge m^{-1}$, $m\to\infty$, then $z_0(y,\kappa,z_0)=z_0\tau_0(y,\kappa,z_0)$ where

$$\tau_0(y,\kappa,z_0)\sim\tau_0(y,\kappa)=C(p,q)y^{\lambda/(p-q)},\quad h_0(y,\kappa,z_0)\sim C_{p,q}y^{-\Delta/(p-q)}.$$

Proof of Proposition is based on the standard properties of inverse functions and on the standard asymptotic relations

$$\sinh x\sim\tanh x\sim x; x^2-\tanh x^2/2\sim x^6/12,\quad \text{as } x\to 0.$$

Denote

$$\Xi_\Delta^-=\Xi_1\cup\{\kappa\in\Xi_2\cup\Xi_3:\Delta\le 0\}\cup\{\kappa\in\Xi_4:\tilde\Delta\le 0\};\ \Xi_\Delta^+=\Xi_G\setminus\Xi_\Delta^-.$$

The partitions of the sets Ξ_l onto sets $\Xi_l^-=\Xi_l\cap\Xi_\Delta^-$ and $\Xi_l^+=\Xi_l\cap\Xi_\Delta^+$, $l=2,3,4$ are presented in Figures 10–12. Note that if $\kappa\in\Xi_3$, then the inequality $\Delta\le 0$ yields: $r\ge 0$, $\lambda>0$.

From Proposition 6.1 we obtain:

Corollary 6.1. *1. Assume $\kappa\in\Xi_\Delta^-$ and*

$$n\to\infty,\ h_0\to 0;\ n^{-rp}h_0\to 0 \text{ if } p>2. \tag{6.43}$$

Then for small enough $\varepsilon>0$ the relations

$$z_i=z(i/n,\kappa),\ h_i=h_0\delta(i/n,\kappa) \tag{6.44}$$

define the solutions of the system (6.27) for all i.

2. Assume $\kappa\in\Xi_\Delta^+$ and $\lambda>0$. Then the relations (6.44) define the solutions of the system (6.27) for $i\in I_0$ where the integer set $I_0=I_0(\kappa)$ is defined by the relations:

$$I_0=\begin{cases}i:\ i\le nx_\varepsilon=my_\varepsilon, & \text{if } \kappa\in\Xi_2,\ \Delta>0\ ,\\ i:\ i\ge nx_\varepsilon=my_\varepsilon, & \text{if } \kappa\in\Xi_3,\ \Delta>0,\ \lambda>0 \text{ or } \kappa\in\Xi_4,\ \tilde\Delta>0.\end{cases}$$

Here $x_\varepsilon=(m/n)y_\varepsilon$ is defined by the equality: $h_0\delta(x_\varepsilon,\kappa)=1$. If $p\ne 2$, then we have:

$$y_\varepsilon=y_\varepsilon(\kappa)\sim y_1(\kappa)=C_{p,q}^{(p-q)/\Delta},\quad \text{as}\ \ z_0\to 0.$$

If $p=2$, then (by $(6-q)/\tilde\Delta>(2-q)/\Delta$ for $\kappa\in\Xi_4$) one has

$$y_\varepsilon=y_\varepsilon(\kappa)\sim y_1(\kappa)=(C_q)^{(2q-12)/\tilde\Delta}h_0^{(6-q)/\tilde\Delta-(2-q)/\Delta}\to 0,\ \text{as } h_0\to 0,\ \kappa\in\Xi_4\ .$$

3. Assume $\kappa\in\Xi_\Delta^+$, $z_0\to 0$, $m\to\infty$, $z_0m^{-\lambda/(p-q)}\to 0$ and $\lambda\le 0$ (it means $\kappa\in\Xi_3$). Then the relations $z_i=z_0\tau_0(i/m,\kappa,z_0)$, $h_i=h_0(i/m,\kappa,z_0)$ determine the solutions of the system (6.27) for $i\in I_0=\{i\ge my_\varepsilon\}$, $y_\varepsilon\sim C_{p,q}^{(p-q)/\Delta}$.

Remark 6.1. Note that by Remarks 3.2 the assumptions $n\to\infty$, $h_0\to 0$; $n^{-rp}h_0\to 0$ if $p>2$ and $z_0\to 0$, $m\to\infty$, $z_0m^{-\lambda/(p-q)}\to 0$ for $\kappa\in\Xi_3$, $\lambda\le 0$ follow from the assumption $u_\varepsilon=O(\varepsilon^{-\delta})$ for small enough $\delta>0$ or $u_\varepsilon=O(1)$.

6.3.1. *Solutions of extreme problem for* $\kappa \in \Xi_\Delta^-$

By the Corollary, it is enough to find the values $n,\ h_0$, to obtain u_ε from the relations (6.28) and (6.29) and to check assumptions (6.43). We give the outlines of proofs and omit simple calculations which one can easy restore.

First, assume $r > r_p$. In this case $\kappa \in \Xi_{G_2}$ and we can rewrite the relations (6.28) and (6.29) in the form:

$$\begin{aligned}(\rho_\varepsilon/\varepsilon)^p &= h_0 n^{rp+1} C_{1,\varepsilon}(\kappa,n,h_0) = h_0 n^{rp+1}(c_1(\kappa)+O(n^{-\delta})),\\ (R/\varepsilon)^q &= h_0 n^{sq+1} C_{2,\varepsilon}(\kappa,n,h_0) = h_0 n^{sq+1}(c_2(\kappa)+O(n^{-\delta})),\\ u_\varepsilon^2 &= h_0^2 n C_{0,\varepsilon}(\kappa,n,h_0) = h_0^2 n(c_0(\kappa)+O(n^{-\delta})),\end{aligned} \tag{6.45}$$

for some $\delta = \delta(\kappa) > 0$ where $C_{l,\varepsilon}(\kappa,n,h_0),\ l=0,1,2$ are continuous functions of κ, n, h_0 which are bounded away from 0 and ∞ for small enough n^{-1}, h_0. The relations (6.45) are uniform on all compacts $K \subset \{\kappa \in \Xi_\Delta^- : r > r_p + \delta\}$ for any $\delta > 0$. Here the functions $c_l(\kappa),\ l = 0,1,2$ are defined by the relations:

$$\begin{aligned} c_1(\kappa) &= \int_0^\infty \delta(x,\kappa) z^p(x,\kappa) x^{rp}\,dx\\ c_2(\kappa) &= \int_0^\infty \delta(x,\kappa) z^q(x,\kappa) x^{sq}\,dx\\ c_0(\kappa) &= 2\int_0^\infty \delta^2(x,\kappa)\sinh^2\frac{z^2(x,\kappa)}{2}\,dx.\end{aligned} \tag{6.46}$$

Here the functions $z(x,\kappa)$, $\delta(x,\kappa)$ are defined by (6.35), (6.40).

Using Proposition 6.1 and definition of the set Ξ_{G_2}, one can check that the integrals in (6.46) are finite. In fact, for all integrals $c_l(\kappa),\ l=0,1,2$ one has

$$\int_1^\infty \{...\}dx \asymp \begin{cases} \int_1^\infty x^a \exp(-bx^c)dx,\ b>0,\ c>0 & \text{if } \kappa \notin \Xi_2,\\ \int_1^\infty x^{(I/(p-q))-1}dx,\ I>0, & \text{if } \kappa \in \Xi_2 \end{cases} = O(1). \tag{6.47}$$

If $\kappa \in \Xi_1 \cup \Xi_2 \cup \Xi_4$, then for integrals $c_l(\kappa),\ l=1,0$ one has:

$$\int_0^1 \{...\}dx \asymp \int_0^1 x^{2rp}dx = O(1) \tag{6.48}$$

and for $c_2(\kappa)$

$$\int_0^1 \{...\}dx \asymp \begin{cases} \int_0^1 x^{2rp+\lambda}dx \asymp 1, & \text{if } \kappa \in \Xi_1 \cup \Xi_2,\\ \int_0^1 x^{4r+4\lambda/(6-q)}dx \asymp 1, & \text{if } \kappa \in \Xi_4. \end{cases} \tag{6.49}$$

If $\kappa \in \Xi_3$, then for all integrals $c_l(\kappa),\ l=1,2,0$ one has by $I>0$:

$$\int_0^1 \{...\}dx \asymp \int_0^1 x^{(I/(p-q))-1}dx = O(1). \tag{6.50}$$

These relations imply the existence of the solutions of (6.45):

$$n_\varepsilon = \tilde n_\varepsilon(1+O(\tilde n_\varepsilon^{-\delta})),\ h_{0,\varepsilon} = \tilde h_{0,\varepsilon}(1+O(\tilde n_\varepsilon^{-\delta})),$$

and the relation

$$u_\varepsilon^2 = \tilde h_{0,\varepsilon}^2 \tilde n_\varepsilon(c_0(\kappa)+O(\tilde n_\varepsilon^{-\delta}))$$

where $\tilde{n}_\varepsilon$ and $\tilde{h}_{0,\varepsilon}$ are defined by the relations

$$(\rho_\varepsilon/\varepsilon)^p = \tilde{h}_{0,\varepsilon}\tilde{n}_\varepsilon^{rp+1}c_1(\kappa),\ (R/\varepsilon)^q = \tilde{h}_{0,\varepsilon}\tilde{n}_\varepsilon^{sq+1}c_2(\kappa).$$

In fact, introduce variables

$$z_1 = (n/\tilde{n}_\varepsilon)^{1+rp}h_0/\tilde{h}_{0,\varepsilon} - 1,\ z_2 = (n/\tilde{n}_\varepsilon)^{1+sq}h_0/\tilde{h}_{0,\varepsilon} - 1,$$

and consider the continuous function $f(z) = (f_1(z), f_2(z)) :\ z = (z_1, z_2) \to R^2$:

$$f_1(z_1, z_2) = (\rho_\varepsilon/\varepsilon)^{-p}h_0 n^{rp+1}C_{1,\varepsilon}(\kappa, n, h_0) - 1 = z_1(1 + \delta_1(z_1, z_2)) + \delta_1(z_1, z_2),$$

$$f_2(z_1, z_2) = (R/\varepsilon)^{-q}h_0 n^{sq+1}C_{2,\varepsilon}(\kappa, n, h_0) - 1 = z_2(1 + \delta_2(z_1, z_2)) + \delta_2(z_1, z_2).$$

It follows from (6.45) that $\delta_l(z_1, z_2) = O(\tilde{n}_\varepsilon^{-\delta}),\ l = 1, 2$ for some $\delta > 0$ uniformly on any ball $D^2(a) \subset R^2$ with $a = O(1)$. Thus we have the relation $\|f(z) - z\| = O(\tilde{n}_\varepsilon^{-\delta})$ for any $z \in D^2(a)$. We can rewrite the first and the second equations in (6.45) in the form:

$$\begin{cases} f_1(z_1, z_2) = 0, \\ f_2(z_1, z_2) = 0 \end{cases}$$

and it is enough to use the following simple topological

Lemma 6.3. *Let $f :\ D^k(a) \to R^k,\ k \ge 1$ be such continuous map that $\|f(z) - z\| < b < a$ on the boundary sphere $z \in S^{k-1}(a)$. Then there exists such $z_0 \in D^k(a)$ that $f(z_0) = 0$.*

Proof of the lemma. It follows from assumptions that the families of maps

$$f_t(z) = tz + (1 - t)f(z) : z \to \breve{R}^k = R^k \setminus \{0\}$$

provide the homotopy of the restriction $f = f_0$ on the sphere $S^{k-1}(a)$ to the unit map $f_1(z) = z$ which generates nontrivial homotopy group of $\breve{R}^k$. Therefore it is not possible to continue f_0 to the map $f : D^k(a) \to \breve{R}^k$ which implies the existence of z_0 such that $f(z_0) = 0$.

Thus we have the existence of solutions $h_0 = h_{0,\varepsilon}, n = n_\varepsilon$ of the first and the second equations (6.45) with asymptotics (3.8, 3.7) for $r > r_p$.

Next, let $r = r_p$. In this case we have: $p > 2,\ r_p = -1/2p,\ s > -1/2q$ for $\kappa \in \Xi_\Delta^-$ and $\kappa \in \Xi_{G_5}$. The second relation in (6.46) is of the same form, however the integrals $c_1(\kappa),\ c_0(\kappa)$ diverge in (6.46) and the relations for $(\rho_\varepsilon/\varepsilon)^p,\ u_\varepsilon^2$ in (6.45) could be rewritten in the form

$$\begin{aligned}(\rho_\varepsilon/\varepsilon)^p &= h_0 n^{1/2}\left(\int_{1/n}^1 \delta(x,\kappa)z^p(x,\kappa)x^{-1/2}dx + O(1)\right)\\ &\sim z^{2p}(p)h_0 n^{1/2}\log n\left(4\sinh^2\frac{z^2(p)}{2}\right)^{-1},\end{aligned} \tag{6.51}$$

$$\begin{aligned}u_\varepsilon^2 &= 2nh_0^2\left(\int_{1/n}^1 \delta^2(x,\kappa)\sinh^2\frac{z^2(x,\kappa)}{2}dx + O(1)\right)\\ &\sim h_0^2 n\log n\left(\frac{z^{2p}(p)}{8\sinh^2\frac{z^2(p)}{2}}\right),\end{aligned} \tag{6.52}$$

which provide the relations (3.28, 3.29).

By Remark 3.2 the assumption $u_\varepsilon = O(1)$ implies $h_0 = o(1),\ n^{-1} = o(1)$.

6.3.2. *Proof Theorems 4, 6 and 8, n. 1 for* $\kappa \in \Xi_\Delta^-$

Observe that $\Xi_\Delta^- \subset \Xi_{C_2} \cup \Xi_{C_5}$. Assume $0 < b < u_\varepsilon$ and $u_\varepsilon = O(\varepsilon^{-\delta})$ if $\kappa \in \Xi_{C_2}$ for small enough $b > 0,\ \delta > 0$, $u_\varepsilon = O(1)$ if $\kappa \in \Xi_{C_5}$. To proof theorems it is enough to check the assumptions of Theorem 12.

Assumption C1 follows directly from asymptotics (3.7) and (3.28). Assumptions B3 and either A1 or B1 in C2 can be easily checked using Proposition 6.1, asymptotics (3.7, 3.8) and (3.28, 3.29).

Let us check B4. It is enough to consider the case when $z_i \to \infty$ (it is possible for $p > q,\ \lambda > 0$ only). Put $n_\varepsilon = n,\ N_\varepsilon = h_0^{-2}$. It follows from Proposition 6.1 that $\Delta_{\varepsilon,i} = \log N_\varepsilon + \delta z_i^2 + O(\log z_i)$ as $z_i \to \infty$ which imply $i/n \asymp \log N_\varepsilon$ uniformly for $i \in\asymp (\log N_\varepsilon)^a,\ a > 0$ uniformly on $i \in \Re_\varepsilon$. For small enough δ_1' the relation $n_\varepsilon^{rp} N_\varepsilon^{1/2} = O(H_{\varepsilon,1} N_\varepsilon^{\delta_1'})$ follows from the relations: for small enough $\delta = \delta(\kappa) > 0$ if $r > r_p$, then $0 < b < u_\varepsilon^2 \asymp n N_\varepsilon^{-1} = O(\varepsilon^{-\delta})$, and if $r = r_p$, then $b < u_\varepsilon^2 \asymp n N_\varepsilon^{-1} \log N_\varepsilon = O(1)$, by $\log n_\varepsilon \asymp \log N_\varepsilon \asymp \log \varepsilon^{-1}$ and $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p \asymp n_\varepsilon^{1+rp} N_\varepsilon^{-1/2}$ or $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p \asymp n_\varepsilon^{1+rp} N_\varepsilon^{-1/2} \log n_\varepsilon$.

Let us construct the families $\bar\pi_{\varepsilon,1}$ such that $\|\bar\pi_{\varepsilon,1}\| = u_\varepsilon + o(1),\ \bar\pi_1^\varepsilon(V_\varepsilon) \to 1$. Analogously to Section 6.1.2 let us consider the values $\delta_\varepsilon = (\log \varepsilon^{-1})^{-\delta}$ and put

$$\tilde\pi_\varepsilon = \bar\pi_\varepsilon(\kappa, (1+\delta_\varepsilon)\rho_\varepsilon, (1-\delta_\varepsilon)R).$$

If $r > r_p$, then consider "two-side T_ε-truncated" sequences $\bar\pi_{\varepsilon,1} = \{\pi_{\varepsilon,i,1}\}$ for $T_\varepsilon = \varepsilon^{-a}$ with small enough $a > 0$:

$$\pi_{\varepsilon,i,1} = \begin{cases} \tilde\pi_{\varepsilon,i}, & \text{if } T_\varepsilon^- \le i/n \le T_\varepsilon, \\ \delta_0, & \text{in other cases,} \end{cases}, \ T_\varepsilon^- = T_\varepsilon^{-1}$$

and if $r = r_p$, then consider "one-side T_ε-truncated" sequences:

$$\pi_{\varepsilon,i,1} = \begin{cases} \tilde\pi_{\varepsilon,i}, & \text{if } T_\varepsilon^- \le i/n \le T_\varepsilon, \\ \delta_0, & \text{if } i/n > T_\varepsilon, \end{cases} \quad T_\varepsilon^- = 1/n.$$

It is clear that $\bar\pi_{\varepsilon,1}$ satisfies assumptions B1, B2. The relation $\|\bar\pi_{\varepsilon,1}\| = u_\varepsilon + o(1)$ follows from asymptotics (3.7, 3.28) and (6.46–6.51). The relation $\pi_1^\varepsilon(V_\varepsilon) \to 1$ follows from Chebyshev inequality and relations (6.46–6.51) analogously Section 6.1.2. In fact,

$$E_{\pi_1^\varepsilon} F_1(\bar\phi_1(v)) = (1 + \delta_\varepsilon - O(\varepsilon^{aA_1}))^p H_{\varepsilon,1}, \quad E_{\pi_1^\varepsilon} F_2(\bar\phi_2(v)) \le (1+\delta_\varepsilon)^q H_{\varepsilon,2}$$

for some $A_1 > 0$. One can easily check that

$$\begin{aligned} Var_{\pi_1^\varepsilon} F_2(\bar\phi_2(v)) &= \sum_i z_i^{2q} h_i (1-h_i) i^{2sq} \\ &\asymp n^{1+2sq} h_0 \int_{T_\varepsilon^-}^{T_\varepsilon} \delta(x,\kappa) z^{2q}(x,\kappa) x^{2sq} dx = o(H_{\varepsilon,2}^2 \delta_\varepsilon^2), \end{aligned} \tag{6.53}$$

$$\begin{aligned} Var_{\pi_1^\varepsilon} F_1(\bar\phi_1(v)) &= \sum_i z_i^{2p} h_i (1-h_i) i^{2rp} \\ &\asymp n^{1+2rp} h_0 \int_{T_\varepsilon^-}^{T_\varepsilon} \delta(x,\kappa) z^{2p}(x,\kappa) x^{2rp} dx = o(H_{\varepsilon,1}^2 \delta_\varepsilon^2). \end{aligned} \tag{6.54}$$

To check the last relations in (6.53, 6.54) note that

$$\begin{aligned} H_{\varepsilon,1} &\asymp \begin{cases} n^{1+rp}h_0 & \text{if } r > r_p \ , \\ n^{1/2}h_0 \log h_0^{-1}, \kappa \in \Xi_1 \cup \Xi_2 & \text{if } r = r_p = -1/2p \ , \end{cases} \\ H_{\varepsilon,2} &\asymp n^{1+sq}h_0, \end{aligned} \tag{6.55}$$

and it is enough to show that for both integrals $I_{1,2}$ in (6.53, 6.54) one has: if $r > r_p$, then $I_{1,2} = o(nh_0\delta_\varepsilon^2)$, and if $r = r_p$, then $I_{1,2} = o(nh_0 \log h_0^{-1}\delta_\varepsilon^2)$.

Let $r > r_p$. Then $nh_0 \asymp n^{1/2}u_\varepsilon \asymp \varepsilon^{-\delta}$ for some $\delta > 0$, and these relations follow from estimations:

$$\int_{T_\varepsilon^{-1}}^{T_\varepsilon} \{...\}dx = O(T_\varepsilon^{A_2}) = o(\varepsilon^{-\delta_1})$$

with some $A_2 = A_2(\kappa) > 0$ and one can make $\delta_1 > 0$ arbitrary small by choose small enough $a > 0$.

Let $r = r_p$. Then $p > 2, r_p = -1/2p, \kappa \in \Xi_1 \cup \Xi_2$ and $nh_0 \log h_0^{-1} \asymp (n \log h_0^{-1})^{1/2}u_\varepsilon \asymp \varepsilon^{-\delta}\log \varepsilon^{-1}$ for some $\delta > 0$. Therefore we have:

$$\int_{n^{-1}}^{T_\varepsilon} \{...\}dx \le O(T_\varepsilon^{A_2}) + \int_{n^{-1}}^{1} \{...\}dx$$

and

$$\int_{1/n}^{1} \delta(x,\kappa)z^{2p}(x,\kappa)x^{2rp}dx \asymp \int_{1/n}^{1} x^{3rp}dx \asymp n^{1/2}, \tag{6.56}$$

$$\int_{1/n}^{1} \delta(x,\kappa)z^{2q}(x,\kappa)x^{2sq}dx \asymp \int_{1/n}^{1} x^{3rp+2\lambda}dx = o(n^{1/2}). \tag{6.57}$$

Theorems 4, 6 and 8, n. 1 are proved for the case $\kappa \in \Xi_\Delta^-$.

6.4. **Solution of the system (6.27) with $C_i > 0$**

The inequality $C_i > 0$ means $h_i = 1$ and the equations (6.27) are of the form (6.30) with the constraint (6.31). Using the notations (6.36) we can rewrite (6.30, 6.31):

$$4z_0^{-4}\sinh^2\frac{z^2}{2}\left(\frac{z^2}{\tanh\frac{z^2}{2}}\right) + qy^{sq}(z/z_0)^q = py^{rp}(z/z_0)^p, \tag{6.58}$$

with the constraint

$$4z_0^{-4}\sinh^2\frac{z^2}{2} + y^{sq}(z/z_0)^q \le y^{rp}(z/z_0)^p. \tag{6.59}$$

By Corollary 6.1 we need to solve (6.58, 6.59) for $\kappa \in \Xi_\Delta^+ = \Xi_G \setminus \Xi_\Delta^-$:

$$\Xi_\Delta^+ = \Xi_0 \cup \{\kappa \in \Xi_2 \cup \Xi_3 : \Delta > 0\} \cup \{\kappa \in \Xi_4 : \tilde{\Delta} > 0\}$$

where $\Xi_0 = \{\kappa \in \Xi_G : \ p \le 2, p < q\}$. Observe that if $\kappa \in \Xi_4 : \tilde{\Delta} > 0$, then $\Delta > 0$. Therefore $\Delta > 0$ for any $\kappa \in \Xi_\Delta^+$.

We need to consider $i \in I_1(\kappa),\ y \in Y_\varepsilon(\kappa)$, where integer set $I_1 = I_1(\kappa)$ is the complement of I_0 defined in Corollary 6.1:

$$I_1 = I_1(\kappa) = \begin{cases} \{i = 1, \dots, n, \dots\}, & \text{if } \kappa \in \Xi_0, \\ \{i \ge m y_\varepsilon\}, & \text{if } \kappa \in \Xi_2,\ \Delta > 0\ , \\ \{i \le m y_\varepsilon\}, & \text{if } \kappa \in \Xi_3,\ \Delta > 0 \text{ or } \kappa \in \Xi_4,\ \tilde{\Delta} > 0, \end{cases}$$

and

$$Y_{1,\varepsilon}(\kappa) = Y_1 = \begin{cases} (0, \infty), & \text{if } \kappa \in \Xi_0, \\ [y_\varepsilon, \infty), & \text{if } \kappa \in \Xi_2,\ \Delta > 0, \\ (0, y_\varepsilon], & \text{if } \kappa \in \Xi_3,\ \Delta > 0 \text{ or } \kappa \in \Xi_4,\ \tilde{\Delta} > 0; \end{cases}$$

the values $y_\varepsilon = y_\varepsilon(\kappa) \sim y_1(\kappa)$ are defined in Corollary 6.1. Denote by Y_0 the complement of the set Y_1: $Y_0 = R^1_+ \setminus Y_1$ and consider the set of parameters x corresponding to Y_0:

$$X_\varepsilon(\kappa) = \begin{cases} (0, x_\varepsilon], & \text{if } \kappa \in \Xi_2,\ \Delta > 0\ , \\ [x_\varepsilon, \infty), & \text{if } \kappa \in \Xi_3,\ \Delta > 0 \text{ or } \kappa \in \Xi_4,\ \tilde{\Delta} > 0. \end{cases}$$

Let $h_0 \to 0$. If $\kappa \in \Xi_2,\ \Delta > 0$, then $x_\varepsilon \to \infty$, and if $\kappa \in \Xi_3,\ \Delta > 0,\ \lambda > 0$ or $\kappa \in \Xi_4,\ \tilde{\Delta} > 0$, then $x_\varepsilon \to 0$ (note that $z_0 \to 0$ in these cases). Let $z_0 \to 0$. If $\kappa \in \Xi_3,\ \Delta > 0, \lambda < 0$, then $x_\varepsilon \to \infty$.

Proposition 6.2. 1. *Let $\kappa \in \Xi^+_\Delta,\ i \in I_1(\kappa)$. There exists the unique solution of (6.58, 6.59) $z_i = z_1(y, \kappa, z_0) = z_0\tau(y, z_0, \kappa) > 0$ where $\tau(y, z_0, \kappa)$ is continuous positive smooth function on $z_0,\ y \in Y_{1,\varepsilon},\ \kappa \in \Xi^+_\Delta$ and the boundary continuity condition holds:*

$$z_0\tau(y_\varepsilon, z_0, \kappa) = z_0(y_\varepsilon, \kappa, z_0) = z(x_\varepsilon, \kappa).$$

2. *Assume $z_0 = o(1)$ and if $p \le 2,\ r < 0$, then $z_0 m^{-rp/(4-p)} = o(1)$. Uniformly on $z_0,\ y \in Y_{1,\varepsilon},\ \kappa \in \Xi^+_\Delta$ one has: $z_i \sim z_0\tau$ where $\tau = \tau(y, \kappa)$ is the solution of the equation*

$$2\tau^{4-p} + q y^{sq}\tau^{q-p} = p y^{rp},\ 2\tau^{4-p} + y^{sq}\tau^{q-p} \le y^{rp}. \tag{6.60}$$

The function $\tau(y, \kappa)$ is continuous smooth function on the sets $Y_{1,\varepsilon}(\kappa)$ with the asymptotical properties:

$$\tau(y, \kappa) \sim \begin{cases} (p/2)^{1/(4-p)} y^{rp/(4-p)},\ z_i \to 0, & \text{if } y \to 0,\ \kappa \in \Xi_0 \cup \Xi_3 \cup \Xi_4\ ; \\ (p/q)^{1/(q-p)} y^{-\lambda/(q-p)},\ z_i \to 0, & \text{if } y \to \infty,\ \kappa \in \Xi_0 \cup \Xi_2\ . \end{cases} \tag{6.61}$$

Proof of the Proposition. We give the outline of the proof only. We can rewrite the equation (6.58) in the form: $f(z) = p y^{rp} z_0^{4-p}$ where

$$f(z) = f(z; y, z_0, \kappa) = 2z^{2-p}\sinh(z^2) + q y^{sq} z_0^{4-q} z^{q-p}.$$

Note that $f(z) \to \infty$ as $z \to \infty$. If $p < 4,\ q > p$, then the function $f(z)$ increases on $z > 0,\ f(z) \to 0$ as $z \to 0$. Therefore if $p < 4,\ q > p$, then there exists unique solution of (6.58): $z_i = z(y, \kappa, z_0) > 0$.

Let as show that if $q > p \ge 4$ or $q < p \le 2$, then the equation (6.58) has a positive root $z_i^+ = z(y, \kappa, z_0) > 0$ and $z = z_i^+$ satisfies (6.59) (note that if $p \ne 4$, then there exist two positive roots $z_i^- < z_i^+$, however $z = z_i^-$ does not satisfy (6.59)).

In fact, let $\tilde h_i \ge 1,\ \tilde z_i > 0$ be solutions of (6.32). Then for $z = \tilde z_i > 0$ the following relations hold:

$$f(z) - py^{rp} z_0^{4-p} = 4z^{-p}\sinh^2\frac{z^2}{2}\left(\frac{z^2}{\tanh\frac{z^2}{2}}\right) \quad + \quad h_0 q x^{sq} z^{q-p} - h_0 p x^{rp} \le 0;$$
$$g(z) = x^{sq} z^{q-p}\left(z^2 - q\tanh\frac{z^2}{2}\right) \quad - \quad x^{rp}\left(z^2 - p\tanh\frac{z^2}{2}\right) = 0. \qquad (6.62)$$

By $f(z) \to \infty$ as $z \to \infty$, there exists the solution $z_i^+ \ge \tilde z_i > 0$ of the equation (6.58). Therefore the constraint (6.59) at the point z_i^+ is equivalent to $g(z_i^+) \le 0$. By $g(\tilde z_i) = 0$, using monotone properties of the functions $\phi_{p,q}(z),\ z \in Z_{p,q}$ (see Sect. 6.3) and the sign of the values $z^2 - p\tanh\frac{z^2}{2}$ we have: $g(z) < 0$ for $z > \tilde z_i$ and $g(z) > 0$ for $z < \tilde z_i$.

Using the asymptotics $\sinh x \sim x,\ \tanh x \sim x$, as $x \to 0$ one can easily obtain the asymptotics of n. 2 and (6.60) from (6.58, 6.59). If $q > p > 4$ or $q < p \le 2$, it is possible to check that the point $\tilde y_\varepsilon(\kappa)$ which provides the minimum of the function in left-hand side of the equation (6.60) is bounded away from the set $Y_{1,\varepsilon}(\kappa)$ which implies the smoothness at the point $y_\varepsilon(\kappa)$.

From Propositions 6.1 and 6.2 using the relations (6.38) between variables x, n and y, m, the assumptions: either $u_\varepsilon = O(\varepsilon^{-\delta})$ or $u_\varepsilon = O(1)$ and Remark 3.2 we obtain the following

Corollary 6.2. *Assume $\kappa \in \Xi_\Delta^+$ and the assumptions on z_0, m of Proposition 6.2, n. 2 hold. Then the solutions of (6.27) are defined by the relations:*

1. *If $\lambda > 0$, then we can express the values $z_i,\ h_i$ in terms of variables $x = i/n$:*

$$z_i = \begin{cases} z(i/n, \kappa), & \text{if } i \in I_0, \\ z(i/n, \kappa, h_0) = z_1(i/m, \kappa, z_0), & \text{if } i \in I_1, \end{cases}$$

where $z(x, \kappa, h_0) = z_1(y, \kappa, z_0)$,

$$z(x, \kappa, h_0) \sim \begin{cases} (p/2)^{1/(4-p)} h_0^{1/(4-p)} x^{rp/(4-p)}, & \text{if } x \to 0,\ \kappa \in \Xi_0 \cup \Xi_3, \\ (p/q)^{1/(q-p)} x^{-\lambda/(q-p)}, & \text{if } x \to \infty,\ \kappa \in \Xi_0 \cup \Xi_2, \end{cases}$$

and for $\kappa \in \Xi_4$ one has:

$$z(x, \kappa, h_0) \sim h_0^{1/2} x^r, \ \text{if } x \le x_\varepsilon \sim (C_q^{-2} h_0)^{(6-q)/\tilde\Delta};$$

here

$$h_i = \begin{cases} h_0 \delta(i/n, \kappa), & \text{if } i \in I_0\ , \\ 1, & \text{if } i \in I_1\ . \end{cases}$$

The properties of the function $\delta(x, \kappa),\ z(x, \kappa)$ are determined by Proposition 6.1.

2. *Let $\lambda \le 0$. Then we can express the values $z_i,\ h_i$ in terms of variables $y = i/m$:*

$$z_i = \begin{cases} z_0(i/m, \kappa, z_0), & \text{if } i \in I_0, \\ z_1(i/m, \kappa, z_0), & \text{if } i \in I_1, \end{cases}$$

and

$$h_i = \begin{cases} h_0(i/m, \kappa, z_0), & \text{if } i \in I_0, \\ 1, & \text{if } i \in I_1, \end{cases}$$

where the functions $z_0(y, \kappa, z_0),\ h_0(y, \kappa, z_0)$ are determined by Proposition 6.1, B). The properties of the function $z_1(y, \kappa, z_0)$ are determined by Proposition 6.2.

6.5. **Solution of extreme problem for** $\kappa \in \Xi^+_\Delta$

Propositions 6.1, 6.2 and Corollaries 6.1, 6.2 determine the solutions of equations (6.27) for $\kappa \in \Xi^+_\Delta$ as functions of unknown parameters $z_0,\ m$ or $h_0,\ n,$ if $\lambda \neq 0$. These parameters should be determined from relations (6.28) and (6.29). Let us consider differently cases $I > 0,\ I = 0,\ I < 0$.

Note that the relation $I > 0$ corresponds to the case when solutions with $C_i = 0$ have main part in the sum for u^2_ε (this holds for $\kappa \in \Xi^-_\Delta$); the relations $I = 0$ or $I < 0$ correspond to opposite case: solutions with $C_i > 0$ have essential or main "mass" in the sum. This defines different types of asymptotics in these cases.

As above we give a scheme of investigations and omit elementary calculations which one can easily restore using Propositions 6.1, 6.2 and Corollaries 6.1, 6.2.

6.5.1. *The case* $I > 0$

Note that $\lambda > 0$ and $\kappa \in \Xi_2 \cup \Xi_3 \cup \Xi_4$ in this case. Assume $n \to \infty,\ h_0 \to 0$. Then $z_0 \to 0$. We can rewrite the relations (6.28, 6.29) in the form

$$\begin{aligned} (\rho_\varepsilon/\varepsilon)^p &= h_0 n^{rp+1}(C_{1,\varepsilon}(\kappa) + h_0^{I/\Delta} D_{1,\varepsilon}(\kappa)), \\ (R/\varepsilon)^q &= h_0 n^{sq+1}(C_{2,\varepsilon}(\kappa) + h_0^{I/\Delta} D_{2,\varepsilon}(\kappa)), \\ u^2_\varepsilon &= h_0^2 n (C_{0,\varepsilon}(\kappa) + h_0^{I/\Delta} D_{0,\varepsilon}(\kappa)), \end{aligned} \tag{6.63}$$

where

$$\begin{aligned} C_{1,\varepsilon}(\kappa) &= n^{-1}\sum_{i\in I_0} (i/n)^{rp} z^p(i/n,\kappa)\delta(i/n,\kappa), \\ C_{2,\varepsilon}(\kappa) &= n^{-1}\sum_{i\in I_0} (i/n)^{sq} z^q(i/n,\kappa)\delta(i/n,\kappa), \\ C_{0,\varepsilon}(\kappa) &= 2n^{-1}\sum_{i\in I_0} \delta^2(i/n,\kappa)\sinh^2(z^2(i/n,\kappa)/2); \end{aligned} \tag{6.64}$$

and

$$\begin{aligned} D_{1,\varepsilon}(\kappa) &= m^{-1}\sum_{i\in I_1} (i/m)^{rp}\tau^p(i/m,\kappa)(1+O(z_0^\delta)), \\ D_{2,\varepsilon}(\kappa) &= m^{-1}\sum_{i\in I_1} (i/m)^{sq}\tau^q(i/m,\kappa)(1+O(z_0^\delta)), \\ D_{0,\varepsilon}(\kappa) &= (2m)^{-1}\sum_{i\in I_1} \tau^4(i/m,\kappa)(1+O(z_0^\delta)) \end{aligned} \tag{6.65}$$

(the relation for $D_{0,\varepsilon}(\kappa)$ corresponds to the asymptotics $2\sinh^2(z^2/2) = \frac{1}{2}z^4(1+O(z^2))$, as $z \to 0$.)

Let $r > r_p$ (it means that $\kappa \in \Xi_{G_2}$). Using the asymptotics of Propositions 6.1, 6.2 and of Corollary 6.2, the estimations of Section 6.3.1 and replacing the sums onto integrals, we can check that

$$C_{l,\varepsilon}(\kappa) = c_l(\kappa) + O(n^{-\delta} + h_0^\delta),\ l = 0, 1, 2$$

for some $\delta = \delta(\kappa) > 0$. Here the values $c_l(\kappa)$ are defined by (6.46) and the integrals are finite by the constraints on κ .

In fact, we can replace the sums onto integrals over the sets X_ε with the accuracy $O(n^{-\delta})$. The difference δ_ε with $c_l(\kappa)$ is of the rate

$$\delta_\varepsilon \asymp \begin{cases} \int_{x_\varepsilon}^\infty x^{(I/(p-q))-1}dx, & \text{if } \kappa\in\Xi_2\\ \int_0^{x_\varepsilon} x^{(I/(p-q))-1}dx, & \text{if } \kappa\in\Xi_3\\ \int_0^{x_\varepsilon} x^{4r}dx,\ r>-1/4, & \text{if } \kappa\in\Xi_4,\ l=0,1\\ \int_0^{x_\varepsilon} x^{4r+4\lambda/(6-q)}dx,\ r>-1/4,\ \lambda>0, & \text{if } \kappa\in\Xi_4,\ l=2.\end{cases}$$

Using the properties of the values x_ε one can see that $\delta_\varepsilon = o(\varepsilon^\delta)$ for some $\delta=\delta(\kappa)>0$.

Also one can check that $D_{l,\varepsilon}=O(1)$. In fact, if $\kappa\in\Xi_2$, then, by $\mu>q-p, \Delta>0, y_\varepsilon\asymp 1, m\asymp nh_0^{(p-q)/\Delta}\to\infty$, one has:

$$D_{0,\varepsilon}\asymp\int_{y_\varepsilon}^\infty y^{-(\mu+\Delta)/(q-p)}dy\asymp 1,\quad D_{l,\varepsilon}\asymp\int_{y_\varepsilon}^\infty y^{-\mu/(q-p)}dy\asymp 1,\ l=1,2.$$

Let $\kappa\in\Xi_3\cup\Xi_4$. If $m=o(1)$, then $I_1=\emptyset$ and $D_{l,\varepsilon}=0$; if $m=O(1)$, then $D_{l,\varepsilon}=O(1)$ by $y_\varepsilon\asymp 1$. If $m\to\infty$, then, by $r>r_p=1/4-1/p, \Delta>0$ one has

$$D_{2,\varepsilon}\asymp\int_0^{y_\varepsilon} y^{(4rp+\Delta)/(4-p)}dy\asymp 1,\quad D_{l,\varepsilon}\asymp\int_0^{y_\varepsilon} y^{4rp/(4-p)}dy\asymp 1,\ l=0,1.$$

These relations imply the relations analogous to (6.45). The considerations analogous to Section 6.3.1 show that these relations provide the existence of the solutions $h_0=h_{0,\varepsilon},\ n=n_\varepsilon$ with asymptotics (3.8, 3.7).

By Remark 3.2 the assumption $u_\varepsilon = O(\varepsilon^{-\delta_1})$ implies $h_0=o(\varepsilon^\delta),\ n^{-1}=o(\varepsilon^\delta)$. The accuracy of the asymptotics (3.8, 3.7) is of the rate (ε^{δ_2}) in this case for some $\delta_2=\delta_2(\kappa,\delta_1)>0$ for small enough $\delta_1>0$.

Let $r=r_p=-1/2p$. By $I>0$ this means $p>2,\ \kappa\in\Xi_2,\ \kappa\in\Xi_{G_5}$. By $s>-1/2q$ in this case, one can check that the integrals for $c_0(\kappa), c_1(\kappa)$ diverge. However the relations (6.51) and all estimations for $D_{l,\varepsilon}$ above hold true.

6.5.2. *The case $I<0$*

Assume $r>r_p$ (this corresponds to $\kappa\in\Xi_{G_1}$), $m^{-1}=o(\varepsilon^\delta),\ z_0=o(\varepsilon^\delta)$ and if $p\le 2$, then $m^{-rp/(4-p)}z_0 = o(\varepsilon^\delta)$ for small enough $\delta>0$. Also note that if $u_\varepsilon=O(\varepsilon^{-\delta_1})$ for small enough $\delta_1>0$, then these assumptions hold and $n=mz_0^{-(p-q)/\lambda}>\varepsilon^{-\delta}$ for $\lambda>0$. If $\kappa\in\Xi_3,\ \lambda<0$, then $h_0\to\infty,\ x_\varepsilon\to\infty,\ n\to 0$.

Let $\kappa\in\Xi_0$. The sets I_0 are empty in this case and we can rewrite (6.28, 6.29) in the form

$$\begin{aligned}(\rho_\varepsilon/\varepsilon)^p &= z_0^p m^{rp+1}D_{1,\varepsilon}(\kappa),\\ (R/\varepsilon)^q &= z_0^q m^{sq+1}D_{2,\varepsilon}(\kappa),\\ u_\varepsilon^2 &= z_0^4 m D_{0,\varepsilon}(\kappa)(1+O(z_0^\delta)),\end{aligned}\tag{6.66}$$

where

$$\begin{aligned}D_{1,\varepsilon}(\kappa) &= m^{-1}\sum_i (i/m)^{rp}\tau^p(i/m,\kappa)=\int_0^\infty \tau^p(y,\kappa)y^{rp}dy+O(m^{-\delta}),\\ D_{2,\varepsilon}(\kappa) &= m^{-1}\sum_i (i/m)^{sq}\tau^q(i/m,\kappa)=\int_0^\infty \tau^q(y,\kappa)y^{sq}dy+O(m^{-\delta}),\\ D_{0,\varepsilon}(\kappa) &= (2m)^{-1}\sum_i \tau^4(i/m,\kappa)=\frac{1}{2}\int_0^\infty \tau^4(y,\kappa)dy+O(m^{-\delta}).\end{aligned}\tag{6.67}$$

Therefore

$$\begin{aligned} c_1(\kappa) &= \int_0^\infty \tau^p(y,\kappa)y^{rp}dy, \\ c_2(\kappa) &= \int_0^\infty \tau^q(y,\kappa)y^{sq}dy, \\ c_0(\kappa) &= \frac{1}{2}\int_0^\infty \tau^4(y,\kappa)dy. \end{aligned} \tag{6.68}$$

Let us show that the integrals in (6.68) are finite under the constraints on κ. In fact, by $4rp/(4-p) > -1,\ \mu > q-p,\ \Delta > 0$ for $\kappa \in \Xi_0$, using Proposition 6.2 one has the asymptotics for the integrals in (6.67):

$$\int_0^1 \{...\}dy \asymp \begin{cases} \int_0^1 y^{4rp/(4-p)}dy, & \text{if } l=0,1 \\ \int_0^1 y^{(4rp+\Delta)/(4-p)}dy, & \text{if } l=2 \end{cases} = O(1); \tag{6.69}$$

$$\int_1^\infty \{...\}dy \asymp \begin{cases} \int_1^\infty y^{-\mu/(q-p)}dy, & \text{if } l=1,2 \\ \int_1^\infty y^{-(\mu+\Delta)/(q-p)}dy, & \text{if } l=0. \end{cases} = O(1) \tag{6.70}$$

The relations (6.66–6.68) imply asymptotics (3.5, 3.6).

Let $\kappa \in \Xi_2 \cup \Xi_3$. Then for $\lambda > 0$

$$\begin{aligned} (\rho_\varepsilon/\varepsilon)^p &= z_0^p m^{rp+1}(D_{1,\varepsilon}(\kappa) + z_0^{-I/\lambda}C_{1,\varepsilon}(\kappa)), \\ (R/\varepsilon)^q &= z_0^q m^{sq+1}(D_{2,\varepsilon}(\kappa) + z_0^{-I/\lambda}C_{2,\varepsilon}(\kappa)), \\ u_\varepsilon^2 &= z_0^4 m(D_{0,\varepsilon}(\kappa)(1+O(z_0^\delta)) + z_0^{-I/\lambda}C_{0,\varepsilon}(\kappa)), \end{aligned} \tag{6.71}$$

where $C_{l,\varepsilon}(\kappa), D_{l,\varepsilon}(\kappa)$ are defined in (6.64, 6.65). It is clear that

$$\begin{aligned} D_{1,\varepsilon}(\kappa) &= \int_{Y_1} \tau^p(y,\kappa)y^{rp}dy + O(\varepsilon^\delta), \\ D_{2,\varepsilon}(\kappa) &= \int_{Y_1} \tau^q(y,\kappa)y^{sq}dy + O(\varepsilon^\delta), \\ D_{0,\varepsilon}(\kappa) &= \frac{1}{2}\int_{Y_1} \tau^4(y,\kappa)dy + O(\varepsilon^\delta), \end{aligned} \tag{6.72}$$

for some $\delta > 0$ and the integrals are finite. In fact, if $\kappa \in \Xi_3$, then $4rp/(4-p) > -1$; if $\kappa \in \Xi_2$, then $\mu > q-p$. Therefore

$$\begin{aligned} \int_{Y_1} \tau^p(y,\kappa)y^{rp}dy &\asymp \begin{cases} \int_{y_\varepsilon}^\infty y^{-\mu/(q-p)}dy, & \text{if } \kappa \in \Xi_2 \\ \int_0^{y_\varepsilon} y^{4rp/(4-p)}dy, & \text{if } \kappa \in \Xi_3 \end{cases} = O(1), \\ \int_{Y_1} \tau^q(y,\kappa)y^{sq}dy &\asymp \begin{cases} \int_{y_\varepsilon}^\infty y^{-\mu/(q-p)}dy, & \text{if } \kappa \in \Xi_2 \\ \int_0^{y_\varepsilon} y^{(4rp+\Delta)/(4-p)}dy, & \text{if } \kappa \in \Xi_3 \end{cases} = O(1), \\ \int_{Y_1} \tau^4(y,\kappa)dy &\asymp \begin{cases} \int_{y_\varepsilon}^\infty y^{-(\mu+\Delta)/(q-p)}dy, & \text{if } \kappa \in \Xi_2 \\ \int_0^{y_\varepsilon} y^{4rp/(4-p)}dy, & \text{if } \kappa \in \Xi_3 \end{cases} = O(1). \end{aligned} \tag{6.73}$$

Let us show that

$$\begin{aligned}\tilde{C}_{1,\varepsilon}(\kappa) = z_0^{-I/\lambda} C_{1,\varepsilon}(\kappa) &= \frac{|p-q|}{|I|} C^p(p,q) C_{p,q}^{1+I/\Delta} + O(\varepsilon^\delta);\\ \tilde{C}_{2,\varepsilon}(\kappa) = z_0^{-I/\lambda} C_{2,\varepsilon}(\kappa) &= \frac{|p-q|}{|I|} C^q(p,q) C_{p,q}^{1+I/\Delta} + O(\varepsilon^\delta);\\ \tilde{C}_{0,\varepsilon}(\kappa) = z_0^{-I/\lambda} C_{0,\varepsilon}(\kappa) &= \frac{|p-q|}{|I|} C^4(p,q) C_{p,q}^{2+I/\Delta} + O(\varepsilon^\delta).\end{aligned} \tag{6.74}$$

Let $\kappa \in \Xi_2$. Then $I_0 = \{i \le nx_\varepsilon = my_\varepsilon\},\ x_\varepsilon \to \infty$. Using Proposition 6.1 and Corollary 6.1 one can check

$$\begin{aligned}C_{1,\varepsilon} &= n^{-1} \sum_{i \in I_0} (i/n)^{rp} z^p(i/n,\kappa)\delta(i/n,\kappa)\\ &= \int_{1/n}^1 x^{rp} z^p(x,\kappa)\delta(x,\kappa)dx + \int_1^{x_\varepsilon} x^{rp} z^p(x,\kappa)\delta(x,\kappa)dx + O(n^{-\delta}).\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{1/n}^1 x^{rp} z^p(x,\kappa)\delta(x,\kappa)dx \asymp \int_0^1 x^{rp}dx = O(1);\\ &\int_1^{x_\varepsilon} x^{rp} z^p(x,\kappa)\delta(x,\kappa)dx \asymp \int_1^{x_\varepsilon} x^{I/(p-q)-1}dx \asymp z_0^{I/\lambda} \to \infty.\end{aligned}$$

Consider the family $x_{\varepsilon,1} \to \infty$ such that for small enough $d > 0$

$$x_{\varepsilon,1} = o(x_\varepsilon), \quad \int_1^{x_{\varepsilon,1}} x^{I/(p-q)-1}dx \asymp z_0^{I/\lambda(1+d)}. \tag{6.75}$$

Then using the asymptotics of the functions $z(x,\kappa),\ \delta(x,\kappa)$ as $x \to \infty$ for estimation of the integral on the interval $[x_{\varepsilon,1}, x_\varepsilon]$ we get:

$$\int_1^{x_\varepsilon} x^{rp} z^p(x,\kappa)\delta(x,\kappa)dx = z_0^{I/\lambda} \frac{|p-q|}{|I|} C^p(p,q) C_{p,q}^{1+I/\Delta} + O(z_0^{I/\lambda(1+d)})$$

which imply the first relation in (6.74). The second and the third relation can be proved by similar way.

Let $\kappa \in \Xi_3,\ \lambda > 0$. Then $I_0 = \{i \ge nx_\varepsilon = my_\varepsilon\},\ x_\varepsilon \to 0$. Using Proposition 6.1 and Corollary 6.1 analogously to above consider the family $x_{\varepsilon,1} \to 0$ such that

$$x_{\varepsilon,1}/x_\varepsilon \to \infty, \quad \int_{x_{\varepsilon,1}}^1 x^{I/(p-q)-1}dx \asymp z_0^{I/\lambda(1+d)}. \tag{6.76}$$

Then we use the relations

$$\begin{aligned} C_{1,\varepsilon} &= n^{-1}\sum_{i\in I_0}(i/n)^{rp}z^p(i/n,\kappa)\delta(i/n,\kappa)\\ &= \int_{x_\varepsilon}^1 x^{rp}z^p(x,\kappa)\delta(x,\kappa)dx + \int_1^\infty x^{rp}z^p(x,\kappa)\delta(x,\kappa)dx + O(n^{-\delta});\\ &\int_1^\infty x^{rp}z^p(x,\kappa)\delta(x,\kappa)dx = O(1);\\ &\int_{x_\varepsilon}^1 x^{rp}z^p(x,\kappa)\delta(x,\kappa)dx \sim C^p(p,q)C_{p,q}\int_{x_\varepsilon}^{x_{\varepsilon,1}} x^{I/(p-q)-1}dx \asymp z_0^{I/\lambda}\to\infty. \end{aligned}$$

The estimations analogous to above imply the first relation in (6.74). The second and the third relation can be proved by similar way.

Let $\kappa\in\Xi_3,\ \lambda\le 0$. Then $I_0=\{i\ge my_\varepsilon\},\ y_\varepsilon\sim C_{p,q}^{(p-q)/\Delta}$. Using Proposition 6.1, n. B and Corollary 6.1 we can rewrite the relations (6.28, 6.29) in the form

$$\begin{aligned} (\rho_\varepsilon/\varepsilon)^p &= z_0^p m^{rp+1}(\tilde C_{1,\varepsilon}(\kappa)+D_{1,\varepsilon}(\kappa)),\\ (R/\varepsilon)^q &= z_0^p m^{sq+1}(\tilde C_{2,\varepsilon}(\kappa)+D_{2,\varepsilon}(\kappa)),\\ u_\varepsilon^2 &= z_0^4 m(\tilde C_{0,\varepsilon}(\kappa)+D_{0,\varepsilon}(\kappa)), \end{aligned} \tag{6.77}$$

where

$$\begin{aligned} \tilde C_{1,\varepsilon}(\kappa) &= m^{-1}\sum_{i\in I_0}(i/m)^{rp}\tau_0^p(i/m,\kappa,z_0)h_0(i/m,\kappa,z_0),\\ \tilde C_{2,\varepsilon}(\kappa) &= m^{-1}\sum_{i\in I_0}(i/m)^{sq}\tau_0^q(i/m,\kappa,z_0)h_0(i/m,\kappa,z_0),\\ \tilde C_{0,\varepsilon}(\kappa) &= (2m)^{-1}\sum_{i\in I_0}h_0^2(i/m,\kappa,z_0)\tau_0^4(i/m,\kappa,z_0)(1+O(z_0^\delta)); \end{aligned}$$

(the last asymptotics correspond to $2\sinh^2(z^2/2)=\frac{1}{2}z^4(1+O(z^2)),\ z\to 0.$)

The values $D_{l,\varepsilon}(\kappa)$ are defined in (6.65). The sharp and rate asymptotics of these values are presented in (6.72, 6.73).

The values $\tilde C_{l,\varepsilon},\ l=0,1,2$ satisfy (6.74). In fact,

$$\begin{aligned} \tilde C_{1,\varepsilon} &= m^{-1}\sum_{i\in I_0}(i/m)^{rp}\tau_0^p(i/m,\kappa,z_0)h_0(i/m,\kappa,z_0)\\ &= m^{-1}\sum_{i\in I_0}(i/m)^{rp}C^p(p,q)(i/m)^{p\lambda/(p-q)}C_{p,q}(i/n)^{-\Delta/(p-q)}(1+O(m^{-\delta}))\\ &= \int_{y_\varepsilon}^\infty y^{I/(p-q)-1}dy + O(m^{-\delta}) = \frac{|p-q|}{|I|}C^p(p,q)C_{p,q}^{1+I/\Delta}+O(m^{-\delta}). \end{aligned}$$

The second and the third relations in (6.74) can be proved by similar way.

The relations (6.71) or (6.77) joint with (6.72–6.74) imply the asymptotics (3.5, 3.6) with

$$\begin{aligned} c_1(\kappa) &= \frac{|p-q|}{|I|}C^p(p,q)C_{p,q}^{1+I/\Delta} + \int_{Y_1}\tau^p(y,\kappa)y^{rp}dy, \\ c_2(\kappa) &= \frac{|p-q|}{|I|}C^q(p,q)C_{p,q}^{1+I/\Delta} + \int_{Y_1}\tau^q(y,\kappa)y^{sq}dy, \\ c_0(\kappa) &= \frac{|p-q|}{2|I|}C^4(p,q)C_{p,q}^{2+I/\Delta} + \frac{1}{2}\int_{Y_1}\tau^4(y,\kappa)dy \end{aligned} \tag{6.78}$$

where Y_1 is either $(0, y_1]$ or $[y_1,\infty),\ y_1 = C_{p,q}^{(p-q)/\Delta}$.

Let $r = r_p = 1/4 - 1/p$. This means $\kappa \in \Xi_{G_4},\ p < 2$ or $p = 2,\ q > p;\quad s > 1/4 - 1/q$. The value $c_2(\kappa)$ is defined by (6.78) for $p < 2$ and by (6.68) for $p = 2$. However the integrals for $D_{1,\varepsilon}(\kappa),\ D_{0,\varepsilon}(\kappa)$ diverge and these relations must be replaced onto following:

$$\begin{aligned} D_{1,\varepsilon}(\kappa) &\sim (p/2)^{p/(4-p)}\int_{1/m}^1 y^{-1}dy = (p/2)^{p/(4-p)}\log m, \\ 2D_{0,\varepsilon}(\kappa) &\sim (p/2)^{4/(4-p)}\int_{1/m}^1 y^{-1}dy = (p/2)^{4/(4-p)}\log m, \end{aligned}$$

which also imply asymptotics (3.26), (3.27) with

$$c_1(\kappa) = (p/2)^{p/(4-p)},\ c_0(\kappa) = \frac{1}{2}(p/2)^{4/(4-p)}.$$

6.5.3. *The case* $I = 0$

In this case we have: $\kappa \in \Xi_{G_3},\ \kappa \in \Xi_2 \cup \Xi_3,\ \lambda > 0,\ r > r_p,\ m \to \infty,\ n \to \infty$ or $\kappa \in \Xi_{G_5},\ \kappa \in \Xi_4,\ \lambda > 0,\ r = r_p = -1/4,\ s > 1/4 - 1/q,\ m \to \infty,\ n \to \infty$.

Let $r > r_p$. Then the relations (6.63) - (6.65) hold with $D_{\varepsilon,l} = O(1)$ and

$$\begin{aligned} C_{1,\varepsilon}(\kappa) &= C_{p,q}C^p(p,q)\int_{X_\varepsilon^*} x^{-1}dx + O(1) \\ &\sim C_{p,q}C^p(p,q)\frac{|p-q|}{\Delta}\log h_0^{-1} = c_1(\kappa), \end{aligned} \tag{6.79}$$

$$\begin{aligned} C_{2,\varepsilon}(\kappa) &= C_{p,q}C^q(p,q)\int_{X_\varepsilon^*} x^{-1}dx + O(1) \\ &\sim C_{p,q}C^q(p,q)\frac{|p-q|}{\Delta}\log h_0^{-1} = c_2(\kappa), \end{aligned} \tag{6.80}$$

$$\begin{aligned} C_{0,\varepsilon}(\kappa) &= \frac{1}{2}C_{p,q}^2C^4(p,q)\int_{X_\varepsilon^*} x^{-1}dx + O(1) \\ &\sim \frac{1}{2}C_{p,q}^2C^4(p,q)\frac{|p-q|}{\Delta}\log h_0^{-1} = c_0(\kappa), \end{aligned} \tag{6.81}$$

where

$$X_\varepsilon^* = \begin{cases} [x_\varepsilon(\kappa), 1], & \text{if } \kappa \in \Xi_3, \\ [1, x_\varepsilon(\kappa)), & \text{if } \kappa \in \Xi_2. \end{cases}$$

Let $r = r_p = -1/4$. Then $x_\varepsilon \to 0,\ y_\varepsilon \to 0$.

The considerations analogous to Section 6.3.1 show that the relations (6.79–6.81) provide the existence of the solutions $h_0 = h_{0,\varepsilon},\ n = n_\varepsilon$ with asymptotics (3.24, 3.25) where

$$(\rho_\varepsilon/\varepsilon)^2 \sim h_0 n^{1/2}\left(\int_{x_\varepsilon}^1 x^{-1}dx + \int_{1/m}^{y_\varepsilon} y^{-1}dy + O(1)\right) \sim \frac{2-q}{\Delta} h_0 n^{1/2}\log h_0^{-1},$$

$$u_\varepsilon^2 \sim \frac{1}{2}h_0^2 n\left(\int_{x_\varepsilon}^1 x^{-1}dx + \int_{1/m}^{y_\varepsilon} y^{-1}dy + O(1)\right) \sim \frac{2-q}{2\Delta} h_0^2 n\log h_0^{-1}.$$

At last observe that by

$$h_0 n^{sq+1} D_{2,\varepsilon}(\kappa) = z_0^q m^{sq+1} D_{2,\varepsilon}(\kappa) \asymp z_0^q m^{sq+1}\int_{1/m}^{y_\varepsilon} y^\Delta dy = o(h_0 n^{sq+1}).$$

we get $(R/\varepsilon)^q \sim h_0 n^{sq+1} c_2(\kappa)$ where $c_2(\kappa)$ is defined by (6.46).

These relations imply the asymptotics (3.28, 3.29).

6.5.4. *Proof of Theorems 4, 6 and 8, n. 1 for* $\kappa \in \Xi_\Delta^+$

To proof the Theorems it is enough to check the assumptions of Theorem 12 assuming $0 < b < u_\varepsilon^2 = O(\varepsilon^{-\delta})$. Assumption C1 follows directly from the asymptotics (3.5, 3.7) and (3.28). Assumptions B1, B3 in C2 can be easily checked using Propositions 6.1, 6.2, Corollaries 6.1 – 6.2 and asymptotics (3.5, 3.6) or (3.28, 3.29).

To check B4, analogously to Section 6.3.2, it is enough to consider the cases of unbounded z_i. It is possible for $p > q,\ \lambda > 0$. As in Section 6.3.2, put $n_\varepsilon = n = m z_0^{-(p-q)/\lambda},\ N_\varepsilon = h_0^{-2} = z_0^{-2\Delta/\lambda}$. It follows from Proposition 6.1 that $\Delta_{\varepsilon,i} = \log N_\varepsilon + \delta z_i + O(\log z_i)$, as $z_i \to \infty$, which imply $i/n \asymp \log N_\varepsilon$ uniformly for $i \in \Re_\varepsilon$.

Let $I > 0$. Then for small enough $\delta_1' > 0$ the relation $n_\varepsilon^{rp} N_\varepsilon^{1/2} = O(H_{\varepsilon,1} N_\varepsilon^{\delta_1'})$ follows from relations: for small enough $\delta = \delta(\kappa) > 0$, if $r > r_p$, then $0 < b < u_\varepsilon^2 \asymp nN_\varepsilon^{-1} = O(\varepsilon^{-\delta})$, and if $r = r_p$, then $0 < b < u_\varepsilon^2 \asymp nN_\varepsilon^{-1}\log N_\varepsilon = O(1)$ and $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p \asymp n_\varepsilon^{1+rp} N_\varepsilon^{-1/2}\log n_\varepsilon$. The case $I = 0$ can be considered by similar way.

Let $I < 0$. Then for small enough $\delta_1' > 0$ the relation $n_\varepsilon^{rp} N_\varepsilon^{1/2} = O(H_{\varepsilon,1} N_\varepsilon^{\delta_1'})$ follows from relations: for small enough $\delta = \delta(\kappa) > 0$ if $r > r_p$, then $0 < b < u_\varepsilon^2 \asymp m z_0^4 = O(\varepsilon^{-\delta})$, and if $r = r_p$, then $0 < b < u_\varepsilon^2 \asymp m z_0^4 \log m_\varepsilon = O(1)$ and $H_{\varepsilon,1} = (\rho_\varepsilon/\varepsilon)^p \asymp m_\varepsilon^{1+rp} z_0^p \log m$ and $\log m \asymp \log \varepsilon^{-1}$.

Let us construct the families $\bar\pi_{\varepsilon,1} = \{\pi_{\varepsilon,i,1}\}$ such that $\|\bar\pi_{\varepsilon,1}\| = u_\varepsilon + o(1),\ \bar\pi_1^\varepsilon(V_\varepsilon) \to 1$. It is enough to assume $u_\varepsilon = O(1)$. First, note that if $\kappa \in \Xi_0$, then $\pi_\kappa^\varepsilon(V_\varepsilon) = 1$ by $h_i = 1$ for all i in this case and we put $\bar\pi_{\varepsilon,1} = \bar\pi_\varepsilon$.

Let $\kappa \notin \Xi_0$. Analogously to Sections 6.1.2 and 6.3.2 let us consider the values $\delta_\varepsilon = (\log\varepsilon^{-1})^{-\delta}$ and put $\tilde\pi_\varepsilon = \bar\pi_\varepsilon(\kappa, (1+\delta_\varepsilon)\rho_\varepsilon, (1-\delta_\varepsilon)R)$.

If $I = 0,\ \kappa \in \Xi_4,\ r = r_p = -1/4$, then we put $\bar\pi_{\varepsilon,1} = \tilde\pi_\varepsilon$.

In other cases let us consider families $T_\varepsilon = \varepsilon^{-a}$ with small enough $a = a(\kappa) > 0$.

Let $r > r_p,\ I > 0$. Then we consider "two-side T_ε-truncated" sequences $\bar\pi_{\varepsilon,1}$:

$$\pi_{\varepsilon,i,1} = \begin{cases} \tilde\pi_{\varepsilon,i}, & \text{if } T_\varepsilon^{-1} \le i/n \le T_\varepsilon, \\ \delta_0, & \text{in other cases.} \end{cases}$$

Let either $r = r_p$ or $I \le 0$ ($\kappa \notin \Xi_4$). Then we consider "one-side T_ε-truncated" sequences of following type. If $I > 0,\ r = r_p$ or $I \le 0,\ \kappa \notin \Xi_2$, then

$$\pi_{\varepsilon,i,1} = \begin{cases} \tilde\pi_{\varepsilon,i}, & \text{if } 1/n \le i/n \le T_\varepsilon \text{ for } I > 0,\ r = r_p \text{ or } 1/m \le i/m \le T_\varepsilon \text{ for } I \le 0, \\ \delta_0, & \text{in other cases.} \end{cases}$$

If $I \le 0,\ \kappa \in \Xi_2$, then

$$\pi_{\varepsilon,i,1} = \begin{cases} \tilde{\pi}_{\varepsilon,i}, & \text{if } i/m \ge T_\varepsilon^{-1} \\ \delta_0, & \text{in other cases.} \end{cases}$$

We assume that for $I \le 0,\ \lambda > 0$ the values T_ε and T_ε^{-1} satisfy conditions (here the values $x_{\varepsilon,1}$ are determined by (6.75, 6.76)):

1) Let $\kappa \in \Xi_2$. Then $T_\varepsilon^{-1} = y_\varepsilon x_{\varepsilon,1}/x_\varepsilon$; also if $I+\mu \ne 0$, then $T_\varepsilon^{-(I+\mu)/(p-q)} = o(m\delta_\varepsilon^2)$, and if $I+\mu = 0$, then $\log(T_\varepsilon) = o(m\delta_\varepsilon^2)$;
2) Let $\kappa \in \Xi_3$. Then $T_\varepsilon = y_\varepsilon x_{\varepsilon,1}/x_\varepsilon$; also if $I+\mu \ne 0$, then $T_\varepsilon^{(I+\mu)/(p-q)} = o(m\delta_\varepsilon^2)$, and if $I+\mu = 0$, then $\log(T_\varepsilon) = o(m\delta_\varepsilon^2)$.

Observe that the relation $\|\bar{\pi}_{\varepsilon,1}\| = u_\varepsilon + o(1)$ follows from the estimations which are given in Sections 6.5.1–6.5.3.

Analogously to Sections 6.1.2, 6.3.2 the relation $\pi_1^\varepsilon(V_\varepsilon) \to 1$ follows from Chebyshev inequality, from the relations (6.19, 6.20) and from relations:

$$E_{\pi_1^\varepsilon} F_1(\bar{\phi}_1(v)) = (1+\delta_\varepsilon - O(\varepsilon^{aA_1}))^p H_{\varepsilon,1}, \quad E_{\pi_1^\varepsilon} F_2(\bar{\phi}_2(v)) \le (1+\delta_\varepsilon)^q H_{\varepsilon,2}$$

for some $A_1 > 0$. Also we use the following estimations of variances:

$$\begin{aligned} Var_{\pi_1^\varepsilon} F_2(\bar{\phi}_2(v)) = \sum_{i\in I_0} z_i^{2q} i^{2sq} h_i(1-h_i) &\asymp \begin{cases} n^{1+2sq} h_0 I_1, & \text{if } \lambda > 0 \\ m^{1+2sq} z_0^{2q} I_2, & \text{if } \lambda \le 0 \end{cases} \\ &= o(H_{\varepsilon,2}^2 \delta_\varepsilon^2), \end{aligned} \tag{6.82}$$

$$I_1 = \int_{x\in X_0^*} \delta(x,\kappa) z^{2q}(x,\kappa) x^{2sq} dx,\ I_2 = \int_{y\in Y_0^*} h_0(y,\kappa,z_0)\tau_0^{2q}(y,\kappa) y^{2sq} dy; \tag{6.83}$$

$$\begin{aligned} Var_{\pi_1^\varepsilon} F_1(\bar{\phi}_1(v)) = \sum_{i\in I_0} z_i^{2p} i^{2rp} h_i(1-h_i) &\asymp \begin{cases} n^{1+2rp} h_0 I_1, & \text{if } \lambda > 0 \\ m^{1+2rp} z_0^{2p} I_2, & \text{if } \lambda \le 0 \end{cases} \\ &= o(H_{\varepsilon,1}^2 \delta_\varepsilon^2), \end{aligned} \tag{6.84}$$

$$I_1 = \int_{x\in X_0^*} \delta(x,\kappa) z^{2p}(x,\kappa) x^{2rp} dx,\ I_2 = \int_{y\in Y_0^*} h_0(y,\kappa,z_0)\tau_0^{2q}(y,\kappa) y^{2sq} dy, \tag{6.85}$$

where

$$X_0^* = \begin{cases} X_\varepsilon \cap [T_\varepsilon^{-1}, T_\varepsilon], & \text{if } I > 0,\ r > r_p, \\ X_\varepsilon \cap [n^{-1}, T_\varepsilon], & \text{if } I > 0,\ r = r_p, \\ X_\varepsilon, & \text{if } I = 0,\ \kappa \in \Xi_4, \\ X_\varepsilon \cap [x_{\varepsilon,1}, \infty), & \text{if } I \le 0,\ \kappa \in \Xi_2, \\ X_\varepsilon \cap [n^{-1}, x_{\varepsilon,1}], & \text{if } I \le 0,\ \kappa \in \Xi_3,\ \lambda > 0; \end{cases}$$

$$Y_0^* = Y_0 \cap [m^{-1}, T_\varepsilon].$$

To show the equalities in (6.82, 6.84), first, assume $I \ge 0$. Then $\lambda > 0,\ \log h_0^{-1} \asymp \log n$ and

$$H_{\varepsilon,1} \asymp \begin{cases} n^{1+rp} h_0, & \text{if } r > r_p \text{ and } I > 0\ , \\ n^{1+rp} h_0 \log h_0^{-1}, & \text{if } r = r_p \text{ or } I = 0\ , \end{cases}$$

$$H_{\varepsilon,2} \asymp \begin{cases} n^{1+sq} h_0, & \text{if } I > 0 \text{ or } I = 0,\ r = r_p,\ \kappa \in \Xi_4, \\ n^{1+sq} h_0 \log h_0^{-1}. & \text{if } I = 0,\ \kappa \notin \Xi_4\ . \end{cases}$$

It is enough to show that for both integrals $I_{1,2}$ in (6.83, 6.85) one has: if $r > r_p$, then $I_{1,2} = o(nh_0\delta_\varepsilon^2)$, and if $r = r_p$ or $I = 0$, then $I_{1,2} = o(nh_0 \log h_0^{-1} \delta_\varepsilon^2)$.

Let $r > r_p, I > 0$. Then $nh_0 \asymp n^{1/2}u_\varepsilon > \varepsilon^{-\delta_1}$ for some $\delta_1 > 0$ and these relations follow from estimations:

$$\int_{X_\varepsilon \cap [T_\varepsilon^{-1}, T_\varepsilon]} \{...\}dx = O(T_\varepsilon^{A_2}) = o(\varepsilon^{-\delta})$$

where $A_2 = A_2(\kappa) > 0$ and one can make $\delta > 0$ arbitrary small by choose $a > 0$ small enough.

Let $r = r_p, I > 0$. Then $p > 2, r_p = -1/2p, \kappa \in \Xi_2$ and $nh_0 \log h_0^{-1} \asymp (n \log h_0^{-1})^{1/2} u_\varepsilon \asymp \varepsilon^{-\delta} \log \varepsilon^{-1}$ for some $\delta > 0$. We have:

$$\int_{X_\varepsilon \cap [n^{-1}, T_\varepsilon]} \{...\}dx \le O(T_\varepsilon^{A_2}) + \int_{n^{-1}}^1 \{...\}dx$$

and analogously to (6.56, 6.57) one has:

$$\int_{1/n}^1 \delta(x,\kappa) z^{2p}(x,\kappa) x^{2rp} dx \asymp \int_{1/n}^1 x^{3rp} dx \asymp n^{1/2}, \tag{6.86}$$

$$\int_{1/n}^1 \delta(x,\kappa) z^{2q}(x,\kappa) x^{2sq} dx \asymp \int_{1/n}^1 x^{3rp+2\lambda} dx = o(n^{1/2}). \tag{6.87}$$

Let $I = 0,\ r > r_p$ (this means $\kappa \in \Xi_2 \cup \Xi_3,\ \lambda > 0$). If $\kappa \in \Xi_2$, then $X_0^* = [x_{\varepsilon,1}, x_\varepsilon],\ \mu > 0$ and we calculate the variances directly:

$$Var_{\pi_1^\varepsilon} F_1(\bar\phi_1(v)) \asymp h_0 n^{2rp+1} \int_{X_0^*} x^{2rp} z^{2p}(x,\kappa)\delta(x,\kappa)dx \tag{6.88}$$

$$\asymp \quad h_0 n^{2rp+1} \int_{x_{\varepsilon,1}}^{x_\varepsilon} x^{\mu/(p-q)-1} dx = h_0 n^{2rp+1} o(1) = o(n^{2rp+1} \log^{1-2\delta} h_0^{-1}); \tag{6.89}$$

$$Var_{\pi_1^\varepsilon} F_2(\bar\phi_2(v)) \asymp h_0 n^{2sq+1} \int_{X_0^*} x^{2sq} z^{2q}(x,\kappa)\delta(x,\kappa)dx \tag{6.90}$$

$$\asymp \quad h_0 n^{2sq+1} \int_{x_{\varepsilon,1}}^{x_\varepsilon} x^{\mu/(p-q)-1} dx = h_0 n^{2sq+1} o(1) = o(n^{2sq+1} \log^{1-2\delta} h_0^{-1}). \tag{6.91}$$

If $\kappa \in \Xi_3$, then $X_0^* = [x_\varepsilon, x_{\varepsilon,1}]$ and it is possible that $\mu = 0$. By repeating the estimations (6.89, 6.91) we obtain the same results (small difference is at the point $s = r = -1/4$ where $\mu = 0$, which implies unessential additional log-factor). Remind that $H_{\varepsilon,1}^2 \asymp n^{2rp+1} \log h_0^{-1},\ H_{\varepsilon,2}^2 \asymp n^{2sq+1} \log h_0^{-1}$ in these cases.

Let $I = 0,\ r = r_p = -1/4$ (this means $\kappa \in \Xi_4$). Then

$$Var_{\pi_1^\varepsilon} F_1(\bar\phi_1(v)) = \sum_{i \in I_0} z_i^{2p} i^{2rp} h_i(1-h_i) \asymp h_0 \int_{x_\varepsilon}^1 x^{(4\lambda - \tilde\Delta)/(6-q)-1} dx$$

$$\asymp \begin{cases} h_0, & \text{if } 4\lambda - \tilde\Delta > 0 \\ h_0 \log h_0^{-1}, & \text{if } 4\lambda - \tilde\Delta = 0 \\ h_0^{4\lambda/\tilde\Delta}, & \text{if } 4\lambda - \tilde\Delta < 0 \end{cases} = o(\log n);$$

$$Var_{\pi_1^\varepsilon} F_2(\bar\phi_2(v)) = \sum_{i \in I_0} z_i^{2q} i^{2sq} h_i(1-h_i) \asymp h_0 n^{2sq+1} \int_{x_\varepsilon}^1 x^{(8\lambda + \tilde\Delta)/(6-q)-2} dx$$

$$\asymp \begin{cases} h_0 n^{2sq+1}, & \text{if } \tilde\Delta + 8\lambda - 6 + q > 0 \\ h_0 n^{2sq+1} \log h_0^{-1}, & \text{if } \tilde\Delta + 8\lambda - 6 + q = 0 \\ h_0^{12\lambda/\tilde\Delta} n^{2sq+1}, & \text{if } \tilde\Delta + 8\lambda - 6 + q < 0 \end{cases} = o(n^{2sq+1}/\log n);$$

(remind that $H^2_{\varepsilon,1} \asymp \log n,\ H^2_{\varepsilon,2} \asymp n^{2sq+1}/\log n$ in this case).

Let $I < 0$ (this means $\kappa \in \Xi_2 \cup \Xi_3$). Then

$$\begin{aligned} H_{\varepsilon,1} &\asymp \begin{cases} m^{1+rp} z_0^p, & \text{if } r > r_p, \\ m^{1+rp} z_0^p \log m, & \text{if } r = r_p\ , \end{cases} \\ H_{\varepsilon,2} &\asymp m^{1+sq} z_0^q, \end{aligned} \tag{6.92}$$

and the relations (6.82), (6.84) follow from estimations: if $\lambda > 0$, then:

$$\begin{aligned} h_0 n^{2rp+1} \int_{X_\varepsilon^*} x^{2rp} z^{2p}(x,\kappa)\delta(x,\kappa)dx &\asymp m^{2rp+1} z_0^{2p} \int_{Y_0^*} y^{((I+\mu)/(p-q))-1} dy \\ &= o(m^{2rp+2} z_0^{2p} \delta_\varepsilon^2), \end{aligned} \tag{6.93}$$

$$\begin{aligned} h_0 n^{2sq+1} \int_{X_\varepsilon^*} x^{2sq} z^{2q}(x,\kappa)\delta(x,\kappa)dx &\asymp m^{2sq+1} z_0^{2q} \int_{Y_0^*} y^{((I+\mu)/(p-q))-1} dy \\ &= o(m^{2sq+2} z_0^{2q} \delta_\varepsilon^2); \end{aligned} \tag{6.94}$$

if $\lambda \le 0$, then using Proposition 6.1, B we directly obtain the relation analogous to (6.93, 6.94). These estimations hold for $I < 0, r = r_p$ (this means that $\kappa \in \Xi_3$) as well.

Theorems 4, 6 and 8, n. 1 are proved for the case $\kappa \in \Xi_\Delta^+$.

6.6. **Extreme problem for** $q = \infty$

For the case $p < q = \infty$ from (6.26) we obtain to the following system of equations:

$$\begin{aligned} 4h_i \sinh^2 \frac{z_i^2}{2} &= A i^{rp} z_i^p - C_i, \\ 4h_i^2 \sinh^2 \frac{z_i^2}{2} \left(\frac{z_i^2}{\tanh \frac{z_i^2}{2}} \right) &= A p i^{rp} h_i z_i^p - B_i i^s z_i \end{aligned} \tag{6.95}$$

and the constraints are

$$(\rho_\varepsilon/\varepsilon)^p = \sum_i h_i z_i^p i^{rp}, \quad \sup_i z_i i^s \le R/\varepsilon. \tag{6.96}$$

Here

$$A = A_\varepsilon \ge 0,\ B_i = B_{\varepsilon,i} \ge 0,\ C_i = C_{\varepsilon,i} \ge 0;$$

and if $C_i > 0$, then $h_i = 1$, if $B_i > 0$, then $z_i = i^{-s} R \varepsilon^{-1}$.

First, we try to find the solutions $h_i,\ z_i$ of (6.95) assuming $B_i = C_i = 0$. It is possible for $p > 2$ and we obtain the relations:

$$z_i = z(p),\ h_i = AC(p) i^{rp}; \quad C(p) = \psi_p(z(p)), \tag{6.97}$$

where

$$\psi_p(z) = z^p / 4 \sinh^2(z^2/2). \tag{6.98}$$

If $0 \le h_i \le 1,\ i^{-s}R\varepsilon^{-1} \ge z(p)$, then these relations determine the solutions of (6.95). Let either $p \le 2$ or $p > 2$ and (6.97) do not satisfied. If $p \le 2$, then we put $h_i = 1,\ B_i = 0$, and, assuming $z_i = o(1)$ and using the relations $\sinh x \sim \tanh x \sim x$, as $x \to 0$, we obtain the equation and the constraint:

$$z_i \sim (Api^{rp}/2)^{1/(4-p)} \le i^{-s}R\varepsilon^{-1} \tag{6.99}$$

(the inequality $C_i \ge 0$ holds by $p \le 2$). If $p > 2$ and (6.97) do not satisfy, then we put $z_i = i^{-s}R\varepsilon^{-1}, C_i = 0$ and we obtain the equation and the constraint:

$$h_i = Ai^{rp}\psi(z_i) \le 1 \tag{6.100}$$

(the inequality $B_i > 0$ holds by $p > 2$).

The realization of this outline gives the following results.

6.6.1. *Proof of Theorems 4, 6 and 8, n. 1 for* $p \le 2$

In this case we have $s > r + 1/p,\ r \ge 1/4 - 1/p,\ s > 1/4$. Introduce variables $m = m_\varepsilon(\kappa),\ z_0 = z_{0,\varepsilon}(\kappa)$ by the relations

$$z_0 m^s = R/\varepsilon,\ z_0 m^{-rp/(4-p)} = (Ap/2)^{1/(4-p)} \tag{6.101}$$

and assume $z_0 \to 0,\ m \to \infty$ and $z_0 m^{-rp/(4-p)} \to 0$. Then we have: $h_i = 1$,

$$z_i \sim \begin{cases} z_0(i/m)^{rp/(4-p)}, & \text{if } i \le m \\ z_0(i/m)^{-s}, & \text{if } i \ge m. \end{cases}$$

The constraints (6.96) imply the relations

$$(\rho_\varepsilon/\varepsilon)^p \sim z_0^p m^{rp+1} \begin{cases} c_1(\kappa), & \text{if } r > r_p = 1/4 - 1/p, \\ \log m, & \text{if } r = r_p = 1/4 - 1/p, \end{cases}$$

and we have:

$$u_\varepsilon^2 \sim m z_0^4 \begin{cases} c_0(\kappa), & \text{if } r > r_p = 1/4 - 1/p, \\ (\log m)/2, & \text{if } r = r_p = 1/4 - 1/p, \end{cases}$$

where

$$\begin{aligned} c_1(\kappa) &= \int_0^1 y^{4rp/(4-p)}dy + \int_1^\infty y^{-p(s-r)}dy = \frac{4-p}{4rp+4-p} + \frac{1}{p(s-r)-1}, \\ c_0(\kappa) &= \frac{1}{2}\int_0^1 y^{4rp/(4-p)}dy + \frac{1}{2}\int_1^\infty y^{-4s}dy = \frac{4-p}{2(4rp+4-p)} + \frac{1}{2(4s-1)} \end{aligned}$$

which imply asymptotics (3.5, 3.6) for $r > r_p$ and (3.26, 3.27) for $r = r_p$. By Remark 3.2 the relations $z_0 \to 0,\ m \to \infty$ follow from the assumptions $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta > 0$.

It is not difficult to check the assumptions of Theorem 12 (note that $\pi^\varepsilon(V_\varepsilon) = 1$ by $h_i = 1$ for all i). Theorems 4, 6 and 8, n. 1 are proved for $q = \infty, p \le 2$.

6.6.2. *Proof of Theorems 4, 6 and 8, n. 1 for* $p > 2$

Note that $r \ge -1/2p,\ s > 0$ in this case. Put $\Delta = \Delta(\kappa) = s(4-p) + rp,\ I = 2s(p-2) - 2rp - 1$,

$$n = n_\varepsilon(\kappa) = (R/z(p)\varepsilon)^{1/s},\ h_0 = h_{0,\varepsilon}(\kappa) = An^{rp}$$

and assume $h_0 \to 0,\ n \to \infty$.

First, consider the case $\Delta \le 0$ and note that $I > 0$ in this case (see Fig. 10). For $x = i/n$ we have:

$$z_i = \begin{cases} z(p), & \text{if } x \le 1 \\ z(p)x^{-s}, & \text{if } x \ge 1 \end{cases} \tag{6.102}$$

$$h_i = h_0 x^{rp}\psi_p(z_i) = h_0\delta(x,\kappa); \tag{6.103}$$

here and later $\delta(x,\kappa) = x^{rp}\psi_p(z_i)$, where the function $\psi_p(z)$ is determined by (6.98); $\delta(x,\kappa) \sim z(p)^{p-4}x^{\Delta}$ as $x \to \infty$.

The constraints (6.96) imply the relations

$$(\rho_\varepsilon/\varepsilon)^p \sim h_0 n^{rp+1}\begin{cases} c_1(\kappa), & \text{if } r > r_p = -1/2p, \\ z^p(p)\psi_p(z(p))\log n, & \text{if } r = r_p = -1/2p, \end{cases}$$

$$u_\varepsilon^2 \sim nh_0^2\begin{cases} c_0(\kappa), & \text{if } r > r_p = -1/2p, \\ \psi_p(z(p))\frac{z^p(p)}{2}\log n, & \text{if } r = r_p = -1/2p, \end{cases}$$

where

$$\begin{aligned} c_1(\kappa) &= z^p(p)\psi_p(z(p))\int_0^1 x^{2rp}dx + \int_1^\infty x^{(r-s)p}\delta(x,\kappa)dx \\ &= z^p(p)\psi_p(z(p))/(2rp+1) + I_1, \\ c_0(\kappa) &= \psi_p(z(p))\frac{z^p(p)}{2}\int_0^1 x^{2rp}dx + 2\int_1^\infty \delta^2(x,\kappa)\sinh^2\left(\frac{z^2(p)}{2x^{2s}}\right)dx \\ &= \psi_p(z(p))\frac{z^p(p)}{2}/(2rp+1) + I_0 \end{aligned}$$

and the integrals are finite: $I_l \le B\int_1^\infty x^{-I-1}dx = B/I,\ l = 0,1$.

These imply asymptotics (3.7), (3.8) for $r > r_p$ and (3.28), (3.29) for $r = r_p$; by Remark 3.2 the relations $h_0 \to 0,\ n \to \infty$ follow from the assumptions $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta > 0$.

Let $\Delta > 0$. Then we put $m = nh_0^{-1/\Delta},\ z_0 = z(p)h_0^{s/\Delta}$. We have for $x = i/n,\ y = i/m$:

$$z_i = \begin{cases} z(p), & \text{if } x \le 1 \\ z(p)x^{-s} = z_0 y^{-s}, & \text{if } x \ge 1 \end{cases}, \tag{6.104}$$

$$h_i = \begin{cases} h_0 C(p)x^{rp}, & \text{if } x \le 1 \\ h_0\delta(x,\kappa), & \text{if } 1 \le x \le x_\varepsilon \\ 1, & \text{if } x \ge x_\varepsilon \end{cases}, \tag{6.105}$$

where x_ε is defined by the relation: $h_0\delta(x_\varepsilon,\kappa) = 1$. Using the asymptotics of $\delta(x,\kappa)$ as $x \to \infty$ we get:

$$x_\varepsilon = (m/n)y_\varepsilon \sim h_0^{-1/\Delta}(z(p))^{(4-p)/\Delta} \to \infty,\ y_\varepsilon \sim (z(p))^{(4-p)/\Delta}.$$

As in Section 6.5, we need to consider differently cases $I > 0,\ I = 0,\ I < 0$. Using the considerations analogous to ones used in Section 6.5 for $\kappa \in \Xi_2$ and the relations (6.104, 6.105) we also obtain the required asymptotics (3.5, 3.6–3.28, 3.29). Checking of the assumptions of Theorem 12 can to be carried out analogously to Section 6.5.4 as well (note that we need to estimate π_1^ε-variation of the functional F_1 only).

Theorems 4, 6 and 8, n. 1 are proved for $q = \infty, p > 2$.

6.7. Some additional properties of the solution of (6.27)

In this section we formulate two propositions which will be used to study the adaptive problems. For simplicity assume $q < \infty$.

6.7.1. *Continuous properties of the solutions of extreme problem*

Denote

$$\Xi_{G_0} = \{\kappa \in \Xi_{G_1} \cup \Xi_{G_2} :\ p \neq q,\ p \neq 2,\ \lambda \neq 0,\ \Delta \neq 0\}$$

and

$$\Xi_{G_{01}} = \Xi_{G_0} \cap \Xi_{G_1},\ \Xi_{G_{02}} = \Xi_{G_0} \cap \Xi_{G_2}.$$

For $L_1 > 0$, $L_2 > 0$ and $\kappa_0 \in \Xi_{G_{02}}$ (or $\kappa_0 \in \Xi_{G_{01}}$ respectively) let $\Delta(\kappa_0, L)$ be the set of such $\kappa \in \Xi_{G_0}$ and $\tilde{n} > 0,\ \tilde{h}_0 > 0$ (or $\tilde{m} > 0,\ \tilde{z}_0 > 0$) that

$$\|\kappa - \kappa_0\| = |r - r_0| + |s - s_0| + |p - p_0| + |q^{-1} - q_0^{-1}| < L_1$$

and

$$|\tilde{n}/n - 1| + |\tilde{h}_0/h_0 - 1| < L \text{ or } |\tilde{m}/m - 1| + |\tilde{z}_0/z_0 - 1| < L_2.$$

Here $n = n(\varepsilon),\ h_0 = h_0(\varepsilon)$ (or $m = m(\varepsilon),\ z_0 = z_0(\varepsilon)$) are the values that correspond to the solutions of (6.27, 6.28) for $\kappa = \kappa_0$. Let $\bar{h}_0 = \bar{h}_\varepsilon(\kappa_0),\ \bar{z}_0 = \bar{z}_\varepsilon(\kappa_0)$ be the sequences $h_i(\kappa_0, n, h_0),\ z_i(\kappa_0, n, h_0)$ or, respectively, $h_i(\kappa_0, m, z_0),\ z_i(\kappa_0, m, z_0)$ for these solutions. Let also $(\bar{h}^*,\ \bar{z}^*) = (\bar{h}^*_L(\kappa_0),\ \bar{z}^*_L(\kappa_0))$ be the sequences

$$h_i^* = \sup_{(\kappa,\tilde{n},\tilde{h}_0)\in\Delta(\kappa_0,L)} h_i(\kappa, \tilde{n}, \tilde{h}_0),\quad \widehat{z}_i = \sup_{(\kappa,\tilde{n},\tilde{h}_0)\in\Delta(\kappa_0,L)} z_i(\kappa, \tilde{n}, \tilde{h}_0)$$

or, respectively,

$$h_i^* = \sup_{(\kappa,\tilde{m},\tilde{z}_0)\in\Delta(\kappa_0,L)} h_i(\kappa, \tilde{m}, \tilde{z}_0),\quad \widehat{z}_i = \sup_{(\kappa,\tilde{m},\tilde{z}_0)\in\Delta(\kappa_0,L)} z_i(\kappa, \tilde{m}, \tilde{z}_0)$$

and $z_i^* = \hat{z}_i \mathbf{1}_{\hat{z}_i < B\sqrt{\log\log\varepsilon^{-1}}}$, $B > 0$, where the sequences $\bar{h}(\kappa, \tilde{n}, \tilde{h}_0)$ and $\bar{z}(\kappa, \tilde{n}, \tilde{h}_0)$ are the solutions of (6.27) for the values A, B which correspond to $\kappa, \tilde{n}, \tilde{h}_0$ according to (6.36) or the sequences $\bar{h}(\kappa, \tilde{m}, \tilde{z}_0)$ and $\bar{z}(\kappa, \tilde{m}, \tilde{z}_0)$ are the solutions of (6.27) for the values A, B which correspond to $\kappa, \tilde{m}, \tilde{z}_0$ according to (6.37). Remind that the rate properties of these sequences are defined by Propositions 6.1, 6.2 and by Corollaries 6.1, 6.2. Put

$$u_L = u(\bar{h}^*_L(\kappa_0), \bar{z}^*_L(\kappa_0)),\ u_0 = u_{0,\varepsilon} = u(\bar{h}_0, \bar{z}_0)$$

where

$$u^2(\bar{h}, \bar{z}) = \sum_i u^2(h_i, z_i) = 2\sum_i h_i^2 \sinh^2(z_i^2/2).$$

Let $K \subset \Xi_{G_0}$ be a compact.

Proposition 6.3. *Let $1 \le u_0 = o(\varepsilon^{-\delta})$ for any $\delta > 0$. There exist such positive value δ_0 (which depends on a compact K) that for any $B > 0$, $L_1 = o(1/\log\log\varepsilon^{-1})$, $L_2 = o(1)$, $\kappa_0 \in K$ one has: $u_L/u_0 \le 1 + O(L_1^{\delta_0} + L_2)$.*

Proof of Proposition 6.3 is based on simple estimations. The scheme of the estimations is following. We estimate the difference $u^2(\bar{h}, \bar{z}) - u^2(\bar{h}_0, \bar{z}_0)$ between the sums over "the middle" $cn < i < Cn$ (or $cm < i < Cm$) and between the sums over "the tails" $i \le cn,\ i \ge Cn$ (or $i \le cm,\ i \ge Cm$) for small enough c and large enough C. By Propositions 6.1, 6.2 and by Corollaries 6.1, 6.2 all items over "the middle" are uniformly Lipschitzian on κ and on $\tilde{n}/n,\ \tilde{h}_0/h_0$ (or on $\tilde{m}/m,\ \tilde{z}_0/z_0$). Also one can construct the uniform majorantes for the difference between the sums over "the tails". These estimations imply the proposition.

6.7.2. *Correlation properties*

Let

$$\rho(\kappa_1,\kappa_2;\varepsilon)=\frac{(\bar\pi_{\kappa_1,\varepsilon},\bar\pi_{\kappa_2,\varepsilon})}{\|\bar\pi_{\kappa_1,\varepsilon}\|\|\bar\pi_{\kappa_2,\varepsilon}\|}=\frac{\sum_i h_{i,1}h_{i,2}\sinh^2(z_{i,1}z_{i,2}/2)}{\sqrt{\sum_i h_{i,1}^2\sinh^2(z_{i,1}^2/2)}\sqrt{\sum_i h_{i,2}^2\sinh^2(z_{i,2}^2/2)}}$$

where $\bar\pi_{\kappa_l,\varepsilon}$ is the sequence of the three-point measures corresponding to the sequences $\bar h_\varepsilon(\kappa_l)=\{h_{i,l}\},\ \bar z_\varepsilon(\kappa_l)=\{z_{i,l}\},\ l=1,2$ which are the solutions of (6.27, 6.28) for $\kappa=\kappa_l,\ l=1,2$. Let $n_l=n_{l,\varepsilon}$ or $m_l=m_{l,\varepsilon}$ be the values which correspond to these sequences. Let $K_1\subset\Xi_{G_{0,1}}$ or $K_2\subset\Xi_{G_{0,2}}$ be a compact.

Proposition 6.4. *Let $\kappa_l\in K_1$ (or $\kappa_l\in K_2$), $1\le u_l=\|\bar\pi_{\kappa_l,\varepsilon}\|\le\varepsilon^{-\delta},\ l=1,2$. Then there exist such positive values $\varepsilon_0,\ \delta_0,\ \delta_1,\ \delta_2,\ L_0>0,\ B$ (that may depend on a compact K_1 or K_2) that for any $\varepsilon<\varepsilon_0,\ \delta<\delta_0,\ L<L_0,\ \|\kappa_1-\kappa_2\|<L$ one has: if $\kappa_l\in K_2,\ l=1,2,$ and $n_1\le n_2$, then*

$$\rho(\kappa_1,\kappa_2;\varepsilon)\le B\left(\left(\frac{n_1}{n_2}\right)^{\delta_1}+\varepsilon^{\delta_2}\right),$$

if $\kappa_l\in K_1,\ l=1,2$ and $m_1\le m_2$, then

$$\rho(\kappa_1,\kappa_2;\varepsilon)\le B\left(\left(\frac{m_1}{m_2}\right)^{\delta_1}+\varepsilon^{\delta_2}\right).$$

Proof. Let $\kappa_l\in K_2=K$. Note that

$$I(\kappa_l)>0,\ \|\bar\pi_{\kappa_l,\varepsilon}\|=u_\varepsilon(\kappa_l)\asymp h_{0,l}n_l^{1/2}.$$

Using the estimations of Section 6.3.1 for some $\delta_2>0,\ C_1>0$ uniformly on K for small enough ε one has

$$\sum_{i\in I_1(\kappa_l)}h_{i,l}^2\sinh^2(z_{i,l}^2/2)\le C_1u_\varepsilon^2(\kappa_l)\varepsilon^{\delta_2}$$

where the sets $I_0=I_0(\kappa),\ I_1=I_1(\kappa)$ are determined in the Corollary 6.1 and in Section 6.2. By this relation and Cauchy inequality we can consider the items in the numerator with $i\in I_0(\kappa_1)\cap I_0(\kappa_2)$ only. Also one can choose such $C_2>0$ that for every $\kappa_l\in K,\ l=1,2$ and $i\in I_0(\kappa_l)$

$$h_{i,l}\sinh(z_{i,l}^2/2)\le C_2h_{0,l}\begin{cases}(i/n_l)^{a_l-1/2} & \text{if } i<n_l\\ (i/n_l)^{-b_l-1/2} & \text{if } i\ge n_l\end{cases}$$

where $a_l>0,\ b_l>0$ are bounded away from 0 uniformly on $\kappa_l\in K$. In fact, by Proposition 6.1 and by Corollary 6.1 we can put

$$a_l=\begin{cases}r_lp_l+1/2, & \text{if } \kappa_l\in\Xi_1\cup\Xi_2\cup\Xi_4\\ I(\kappa_l)/2(p_l-q_l), & \text{if } \kappa_l\in\Xi_3,\end{cases}$$

$$b_l=\begin{cases}I(\kappa_l)/2(p_l-q_l), & \text{if } \kappa_l\in\Xi_2\\ D, & \text{if } \kappa_l\in\Xi_1\cup\Xi_3\cup\Xi_4\end{cases}$$

with any $D>0$ by the exponential decrease of the items for $\kappa_l\in\Xi_1\cup\Xi_3\cup\Xi_4$.

It is enough to assume $n_1/n_2<c,\ n_1>C$ for small enough c and large enough C. Let $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ be the sums $\sum_i h_{i,1}h_{i,2}\sinh^2(z_{i,1}z_{i,2}/2)$ over $i\in I_0(\kappa_1)\cap I_0(\kappa_2)$ with, respectively, $i<n_1,\ n_1\le i\le n_2,\ i>n_2$. Using the estimations above we have uniformly on $\kappa_l\in K$:

$$\Sigma_1\le C_2^2h_{0,1}h_{0,2}(n_1/n_2)^{a_2-1/2}\sum_{i<n_1}(i/n_1)^{a_1+a_2-1}\asymp h_{0,1}h_{0,2}(n_1n_2)^{1/2}(n_1/n_2)^{a_2},$$

$$\Sigma_3 \le C_2^2 h_{0,1}h_{0,2}(n_2/n_1)^{-b_1-1/2}\sum_{i>n_2}(i/n_2)^{-b_1-b_2-1} \asymp h_{0,1}h_{0,2}(n_1n_2)^{1/2}(n_1/n_2)^{b_2}.$$

Also if $a_2 < b_1$, then

$$\Sigma_2 \le C_2^2 h_{0,1}h_{0,2}(n_1/n_2)^{a_2-1/2}\sum_{i\ge n_1}(i/n_1)^{a_2-b_1-1} \asymp h_{0,1}h_{0,2}(n_1n_2)^{1/2}(n_1/n_2)^{a_2},$$

if $a_2 > b_1$, then

$$\Sigma_2 \le C_2^2 h_{0,1}h_{0,2}(n_2/n_1)^{-b_1-1/2}\sum_{i\le n_2}(i/n_2)^{a_2-b_1-1} \asymp h_{0,1}h_{0,2}(n_1n_2)^{1/2}(n_1/n_2)^{b_1}$$

and if $a_2 = b_1$, then

$$\Sigma_2 \le C_2^2 h_{0,1}h_{0,2}(n_1/n_2)^{a_2-1/2}\sum_{i\ge n_1}(i/n_1)^{-1} \asymp h_{0,1}h_{0,2}(n_1n_2)^{1/2}(n_1/n_2)^{a_2}\log(n_2/n_1).$$

These relations imply the statement of the Proposition for $\kappa_l \in K_2$.

Let $\kappa_l \in K_1 = K$. In this case we have:

$$I(\kappa_l) < 0,\ \|\bar\pi_{\kappa_l,\varepsilon}\| = u_\varepsilon(\kappa_l) \asymp z_{0,l}^2 m_l^{1/2}.$$

Put $I_\varepsilon(\delta,l) = \{i:\ \ z_{i,l} > \delta\}$. Using the estimations analogous to Section 6.3.3, Propositions 6.1, 6.2 and Corollaries 6.1, 6.2 one can choose such positive $\delta_2,\ C_1,\ \delta_\varepsilon \to 0$ that for small enough ε uniformly on K

$$\sum_{i\in I_\varepsilon(\delta_\varepsilon,l)} h_{i,l}^2 \sinh^2(z_{i,l}^2/2) \le C_1 u_\varepsilon^2(\kappa_l)\varepsilon^{\delta_2}.$$

By this relation and Cauchy inequality we can consider the items in the numerator with $i \in I_\varepsilon(\delta_\varepsilon,1)\cap I_\varepsilon(\delta_\varepsilon,2)$ only.

Then one can choose such $C_2 > 0$ that for every $\kappa_l \in K,\ l = 1,2$ and $i \in I_\varepsilon(\delta_\varepsilon,l)$

$$h_{i,l}\sinh(z_{i,l}^2/2) \le C_2 z_{0,l}^2 \begin{cases}(i/m_l)^{a_l-1/2} & \text{if } \ i < m_l\\ (i/m_l)^{-b_l-1/2} & \text{if } \ i \ge m_l\end{cases}$$

where $a_l > 0,\ b_l > 0$ are bounded away from 0 uniformly on $\kappa_l \in K$. In fact, by Propositions 6.1, 6.2 and by Corollaries 6.1, 6.2 we can put

$$a_l = \begin{cases}(2r_lp_l/(4-p_l)) + 1/2, & \text{if } \ \kappa_l \in \Xi_0\cup\Xi_3\\ I(\kappa_l)/2(p_l-q_l), & \text{if } \ \kappa_l \in \Xi_2,\end{cases}$$

$$b_l = \begin{cases}I_l/2(p_l-q_l), & \text{if } \ \kappa_l \in \Xi_2\cup\Xi_3\\ (2\lambda(\kappa_l)/(q_l-p_l)) + 1/2, & \text{if } \ \kappa_l \in \Xi_0.\end{cases}$$

Then the estimations are analogous to above. The Proposition is proved.

7. Extreme problem for Besov bodies

We give the proofs of Theorems 5 and 8 in this section. It is clear that we need to prove Theorem 8, n. 2 which implies upper bounds of Theorem 5, and to obtain lower bounds of Theorem 5 in this section. We

consider the Besov bodies with $p<\infty$, $h\le p$, $q\le t$ by the required convex properties of Section 5.4 assuming $\kappa\in\Xi_G=\Xi_{G_1}\cup\Xi_{G_2}$. By the symmetry on $l=1,\dots,2^j$ for all $j>0$ the extreme problem is of the form:
if $q<\infty$, $t<\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar\lambda,\bar\nu}\sum_j 2^jR(\lambda_j,\nu_j;p,q):\quad \textstyle\sum_j\ 2^{j(rh+h/p)}\lambda_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\textstyle\sum_j\ 2^{j(st+t/q)}\nu_j^t\le(R/\varepsilon)^t; \tag{7.1}$$

if $q<t=\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar\lambda,\bar\nu}\sum_j 2^jR(\lambda_j,\nu_j;p,q):\quad \textstyle\sum_j\ 2^{j(rh+h/p)}\lambda_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\sup_j\ 2^{j(sq+1)}\nu_j^q\le(R/\varepsilon)^q; \tag{7.2}$$

and if $q=t=\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar\lambda,\bar\nu}\sum_j 2^jR(\lambda_j,\nu_j;p,\infty):\quad \textstyle\sum_j\ 2^{j(rh+h/p)}\lambda_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\sup_j\ 2^{js}\nu_j\le R\varepsilon^{-1}. \tag{7.3}$$

Here the values $R(\lambda,\nu;p,q)$ are determined by the relations (6.21, 6.23).

Using Lemma 6.2 we can reduce the extreme problems (7.1–7.3) to the following ones (the infimum is considered under constraints $h_j\in[0,1]$, $z_j\ge 0$): if $q<\infty$, $t<\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar h,\bar z}2\sum_j 2^jh_j^2\sinh^2\frac{z_j^2}{2}:\quad \textstyle\sum_j\ 2^{j(rh+h/p)}h_j^{h/p}z_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\textstyle\sum_j\ 2^{j(st+t/q)}h_j^{t/q}z_j^t\le R^t\varepsilon^{-t}; \tag{7.4}$$

if $q<t=\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar h,\bar z}2\sum_j 2^jh_j^2\sinh^2\frac{z_j^2}{2}:\quad \textstyle\sum_j\ 2^{j(rh+h/p)}h_j^{h/p}z_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\sup_j\ 2^{j(sq+1)}h_jz_j^q\le(R/\varepsilon)^q; \tag{7.5}$$

and if $q=t=\infty$, then

$$u_\varepsilon^2(\tau)=\inf_{\bar h,\bar z}2\sum_j 2^jh_j^2\sinh^2\frac{z_j^2}{2}:\quad \textstyle\sum_j\ 2^{j(rh+h/p)}h_j^{h/p}z_j^h\ge(\rho_\varepsilon(\tau)/\varepsilon)^h,$$
$$\sup_j\ 2^{js}z_j\le R\varepsilon^{-1}. \tag{7.6}$$

The outline of the proof of Theorem 8, n. 2 is following. We consider the "widest" sets $t=\infty$ and assume $0<h\le p$ (it is enough to assume h is small enough). We show that the analogous to either (3.5, 3.6) or (3.7, 3.8) rates hold in this extreme problem. These imply the inequality: $u_\varepsilon(\tau)\ge c(\tau)u_\varepsilon(\kappa,R,\rho_\varepsilon)$ for small enough $\varepsilon>0$, where the values $u_\varepsilon(\kappa,R,\rho_\varepsilon)$ are determined by (3.2) with $d(\kappa)=1$ and either (3.3) or (3.4).

We study the extreme problem for $p\ne q<\infty$ only (the considerations for $p=q<\infty$ or for $q=\infty$ are more simple). Using Lagrange multipliers rule we obtain from (7.5) the following system of equations on the variables $h_j,\ z_j$:

$$\begin{aligned} 2^{j+2}h_j\sinh^2\frac{z_j^2}{2} &= (h/p)A2^{j(rh+h/p)}h_j^{(h/p)-1}z_j^h - B_j2^{j(sq+1)}z_j^q - C_j,\\ 2^{j+2}h_j\sinh^2\frac{z_j^2}{2}\left(\frac{z_j^2}{\tanh\frac{z_j^2}{2}}\right) &= hA2^{j(rh+h/p)}h_j^{(h/p)-1}z_j^h - qB_j2^{j(sq+1)}z_j^q. \end{aligned} \tag{7.7}$$

Here $A = A_\varepsilon \ge 0,\ B_j = B_{\varepsilon,j} \ge 0,\ C_j = C_{\varepsilon,j} \ge 0$; if $C_j > 0$, then $h_j = 1$ and if $B_j > 0$, then $2^{j(sq+1)}h_jz_j^q = (R/\varepsilon)^q$ (for simplicity we do not consider the Lagrange multipliers corresponding to the constraints $h_j \ge 0$, $z_j \ge 0$ assuming that we consider positive solutions only). The values $A = A(\kappa, h)$ are determined by the relation

$$\sum_j 2^{j(rp+h/p)}h_j^{h/p}z_j^h = (\rho_\varepsilon(\tau)/\varepsilon)^h, \tag{7.8}$$

(this follows from Rem. 5.4 to Lem. 5.1, Sect. 5.4). In Sections 7.1.1–7.1.4 we describe the solutions of (7.7) using some different parameters $h_0, n = 2^{j_0}$ or $z_0, m = 2^{j_1}$ (as in Sect. 6). Using (7.8) we obtain the asymptotics of these parameters and the asymptotics of $u_\varepsilon(\tau)$. Then we use Theorem 12, n. 1. which imply the statement of Theorem 8.

It follows from convex properties of extreme problem and by the solution is unique (see Sect. 5.4) that it is enough to find *any* solution of the system (7.7) under constraints above.

To obtain the lower bounds we construct such families $\bar\pi_\varepsilon = \bar\pi_\varepsilon(\kappa)$ that $\|\bar\pi_\varepsilon\| \asymp u_\varepsilon(\kappa)$ and $\pi^\varepsilon(V_\varepsilon(\kappa, t, h)) \to 1$ for all positive t, h. Then we use Corollary 5.1 and obtain the lower bounds of Theorem 5.

7.1. Study of the system (7.7)

We consider differently the cases of zero and of positive values $C_j,\ B_j$ in (7.7).

7.1.1. *The case* $C_j = B_j = 0$

In this case we have from (7.7) the equation

$$z_j^2 = p\tanh(z_j^2/2)$$

which have the solution $z_j = z(p)$ for $p > 2$ only. Define the values $h_0,\ n = 2^{j_0}$ by the relations

$$h_0^{2-h/p} = (h/p)AC(p)z^{h-p}(p)2^{j_0(rh+h/p-1)},\ z^q(p)h_02^{j_0(sq+1)} = (R/\varepsilon)^q \tag{7.9}$$

where $C(p) = z^p(p)/4\sinh^2(z^2(p)/2)$. Assume

$$j_0 \to \infty, h_02^{j_0/2}j_0^{-\delta} \to 0 \text{ for small enough } \delta = \delta(\kappa, h) > 0. \tag{7.10}$$

We have the equations:

$$z_j = z(p),\ h_j = h_02^{a(j-j_0)};\ p > 2 \tag{7.11}$$

where

$$a = (rh + h/p - 1)/(2 - h/p). \tag{7.12}$$

Note that $a > 1/2$ by $2rp > -1$ for $\kappa \in \Xi_G,\ p > 2$. The constraints $h_j \le 1,\ z_j^q h_j 2^{j(sq+1)} \le (R/\varepsilon)^q$ for $0 \le j \le j_0$ are of the form

$$h_0 2^{a(j-j_0)} \le 1,\ 2^{b(j-j_0)} \le 1;\ b = a + sq + 1,\ 0 \le j \le j_0. \tag{7.13}$$

These constraints hold under assumptions above for small enough $\delta > 0$ by $sq + 1/2 > 0,\ rp + 1/2 > 0$ for $\kappa \in \Xi_G,\ p > 2$ and $b > 0$ for $h/p \in (0, 1]$.

Note that

$$2^{2+j-j_0}(h_j/h_0)^2 \sinh^2 \frac{z_j^2}{2} = C(p,h)\left(2^{(j-j_0)(1+rp)}(h_j/h_0) z_j^p\right)^{h/p}$$

where $C(p,h) = 4\sinh^2(z^2(p)/2)(z(p))^{-h}$.

7.1.2. *The case* $C_j = 0,\ B_j > 0$

In this case we have from (7.7) the equations and the constraints:

$$\begin{array}{rcl} h_j &=& (R/\varepsilon)^q z_j^{-q} 2^{-j(sq+1)},\ \phi_{p,q,h}(z_j) = (p-q)(h/p)A(R/\varepsilon)^{-q(2-h/p)}2^{jc} \\ h_j &\le& 1,\ \begin{cases} z_j \le z(p), & \text{if } q > p \\ z_j \ge z(p), & \text{if } q < p, \end{cases} \end{array} \tag{7.14}$$

where

$$\phi_{p,q,h}(z) = 4z^{-2q+(h/p)(q-p)}\left(\frac{z^2}{\tanh\frac{z^2}{2}} - q\right)\sinh^2\frac{z^2}{2}$$

and $c = 2sq + 1 + (rp - sq)h/p > 0$ for $p \ge 2, \kappa \in \Xi_G$. The constraints on z_j in (7.14) follow from the assumption $B_j \ge 0$. By the constraints on z_j in (7.14) these solutions are not possible for $p \le 2,\ q > p$.

It is easy to check that if $z \ge z(p), p > q$ (we assume $z(p) = 0$ for $p \le 2$), then the function $\phi_{p,q,h}(z)$ increases on z from $\phi_{p,q,h}(z(p)) > 0$ to ∞, and if $0 < z \le z(p),\ 2 < p < q$, then it increases from $-\infty$ to $\phi_{p,q,h}(z(p)) < 0$. If $p > 2$ and the values h_0, j_0 are defined by (7.9), then the values $h_0, z(p)$ are the solutions of (7.14) for $j = j_0$ and $c > 0$ in the right-hand side of (7.14) (note that we can consider (7.14) for all real j).

Therefore there exist the solutions of the equation in (7.14) with the constraints on z_j for $j \ge j_0$, when $p > 2$ or for $j \ge 0$, when $q < p \le 2$.

To define the asymptotics of the values $z_j,\ h_j$, introduce the values:

$$b_1 = c/d,\ d = 2(q-2) - h(q/p - 1),\ a_1 = qb_1 - sq - 1. \tag{7.15}$$

It is easy to check that $d > 0, b_1 > 0$ for $q > p > 2$ and $d < 0, a_1 < 0$ for $q < p \le 2$.

Also let $\kappa \in \Xi_G$. One can check, that if $a_1 \le 0$ for $q > p > 2$, then $I > 0$, and if $b_1 \ge 0$ for $q < p \le 2$, then $I < 0$.

Let $p > 2$. Then for $j > j_0$

$$\begin{array}{rcl} z_j &=& \begin{cases} c_j 2^{-b_1(j-j_0)}, & \text{if } q > p \\ c_j(1 + j - j_0)^{1/2}, & \text{if } q < p \end{cases}, \\ h_j &=& h_0 \begin{cases} d_j 2^{a_1(j-j_0)}, & \text{if } q > p \\ d_j 2^{-(sq+1)(j-j_0)}(1 + j - j_0)^{-q/2}, & \text{if } q < p, \end{cases} \end{array} \tag{7.16}$$

and

$$2^{2+j-j_0}(h_j/h_0)^2 \sinh^2\frac{z_j^2}{2} = C(p,q,h)\left(2^{(j-j_0)(1+rp)}(h_j/h_0) z_j^p\right)^{h/p}\left(\frac{z_j^2}{\tanh\frac{z_j^2}{2}} - q\right)^{-1}$$

where

$$C(p,q,h) = 4(p-q)\sinh^2(z^2(p)/2)(z(p))^{-h}.$$

One can check that c_j increases and d_j decreases on j, if $z_j^2/\tanh\frac{z_j^2}{2} < q-1$ for $q > p$. Here and later we denote $c_j = c_j(\tau), d_j = d_j(\tau), \tau = (\kappa, h),\ h = \xi p$, to be positive values (may be, different in different relations) which are bounded away from 0 and ∞ uniformly on $\kappa \in K,\ \xi \in [\delta, 1],\ j,\ \varepsilon$ for all compacts $K \in G,\ \delta \in (0,1)$ and small enough $\varepsilon > 0$.

We need to check the constraints $h_j \le 1$. It is clear that if $p > q, p > 2$ or $q > p > 2,\ a_1 \le 0$, then these constraints hold for $j > j_0$ under assumptions (7.10). Thus, joint with Section 7.1.1 we have obtained the solutions of (7.7) for $2 < p,\ q < p$.

Let $2 \ge p$ or $2 < p < q,\ a_1 > 0$. Introduce the values $z_0,\ m = 2^{j_1}$ by the relations

$$2^{j_1+2}\sinh^2\frac{z_0^2}{2}\left(\frac{z_0^2}{\tanh\frac{z_0^2}{2}} - q\right) = (h/p)(p-q)A2^{j_1(rh+h/p)}z_0^h,\ 2^{j_1(sq+1)}z_0^q = (R/\varepsilon)^q \tag{7.17}$$

(note that the values $z_j = z_0,\ h_j = 1$ are the solutions of (7.14) for $j = j_1$).

Assume

$$z_0 2^{-j_1((rh+h/p-1)/(4-h))}j_1^\delta \to 0,\ p \le 2,\ j_1 \to \infty \text{ for small enough } \delta = \delta(\kappa) > 0. \tag{7.18}$$

Also for $2 \ge p > q$ introduce the values $h_0,\ n = 2^{j_0}$ by the relations analogous to (7.9):

$$h_0^{2-h/p} \asymp A2^{j_0(rh+h/p-1)},\ h_0 2^{j_0(sq+1)} \asymp (R/\varepsilon)^q \tag{7.19}$$

and note that

$$2^{-b_1(j_1-j_0)} \asymp z_0,\ h_0 2^{a_1(j_1-j_0)} \asymp 1.$$

Let $2 < p < q,\ a_1 > 0$. Then the values h_j increase on j. Therefore the constraints $h_j \le 1$ hold for $j \le j_1$. Also uniformly for $j_1 > j,\ j \asymp j_1$ we have:

$$z_j \sim z_0 2^{-b_1(j-j_1)} \asymp 2^{-b_1(j-j_0)},\ h_j \sim 2^{a_1(j-j_1)} \asymp h_0 2^{a_1(j-j_0)}. \tag{7.20}$$

Thus, if $2 < p < q,\ a_1 > 0$, then the constraints $h_j \le 1$ hold for $j_0 \le j \le j_1$.

Let $q < p \le 2$. Then $a_1 < 0$ and the values h_j decrease on j. Therefore the constraints $h_j \le 1$ hold for $j \ge j_1$. If $b_1 \ge 0$, then for $j_1(1-o(1)) > j$ we have the relations (7.20) as well. If $b_1 < 0$, then the values z_j increase on j. These relations imply $h_j \asymp h_0, z_j \asymp 1$ for $j = j_0 + O(1)$. Note that $j_0 > j_1(1+\delta) \to \infty$ under assumptions (7.18). Thus, we have:

$$\begin{aligned} z_j &= \begin{cases} c_j 2^{-b_1(j-j_0)}, & \text{if } j < j_0 \\ c_j(1+j-j_0)^{1/2}, & \text{if } j > j_0 \end{cases}, \\ h_j &= h_0 \begin{cases} d_j 2^{a_1(j-j_0)}, & \text{if } j < j_0 \\ d_j 2^{-(sq+1)(j-j_0)}(1+j-j_0)^{-q/2}, & \text{if } j > j_0 \end{cases} \end{aligned} \tag{7.21}$$

and if $j \ge j_0$

$$2^{j-j_0}(h_j/h_0)^2\sinh^2\frac{z_j^2}{2} \asymp \left(2^{(j-j_0)(1+rp)}(h_j/h_0)z_j^p\right)^{h/p} z_j^{-2}.$$

7.1.3. *The case* $C_j > 0, B_j > 0$

We consider this case for $p \le 2$ or for $2 < p < q$ and $a_1 > 0$. Let $p \le 2$. Introduce the values $z_{00}, \tilde m = 2^{j_{11}}$ by the relations

$$2^{j_{11}+2}\sinh^2\frac{z_{00}^2}{2}\left(\frac{z_{00}^2}{\tanh\frac{z_{00}^2}{2}}\right) = hA2^{j_{11}(rh+h/p)}z_{00}^h,\ 2^{j_{11}(sq+1)}z_{00}^q = (R/\varepsilon)^q. \tag{7.22}$$

Assume

$$z_{00}2^{-j_{11}((rh+h/p-1)/(4-h))}j_{11}^{\delta} \to 0,\ p \le 2,\ j_{11} \to \infty \text{ for small enough } \delta = \delta(\kappa) > 0. \tag{7.23}$$

Note that $z_{00} \asymp z_0, j_1 > j_{11} = j_1 + O(1)$ for $2 \ge p > q$ where the values z_0, j_1 are defined by (7.17). The assumptions (7.23) equivalent to (7.18) in these cases.

In this case we have the relations:

$$h_j = 1,\ z_j = z_0 2^{-(s+1/q)(j-j_1)} = z_{00}2^{-(s+1/q)(j-j_{11})}$$

and the following constraints:

$$\psi_j(z_j)\phi(z_j) < hA,\ \psi_j(z_j)(q-\phi(z_j)) < h(q-p)A/p, \tag{7.24}$$

where j_1, z_0 are defined by (7.17) and

$$\psi_j(z_j) = 2^{2+j(1-rh-h/p)}z_j^{-h}\sinh^2\frac{z_j^2}{2} \sim 2^{-h_1 j}\left(\frac{R}{\varepsilon}\right)^{4-h}, \phi(z_j) = \frac{z_j^2}{\tanh\frac{z_j^2}{2}} \sim 2 \text{ as } z_j \to 0.$$

Here

$$h_1 = (4-h)(s+1/q) - 1 + rh + h/p > 0$$

for $\kappa \in \Xi_G$, $p \le 2$ or $2 < p < q, a_1 > 0$; $0 < h \le p$ by $h_1 = da_1/q$. This implies that $\psi_j(z_j)$ increases on j for $z_j = o(1)$.

The equality in the second relation in (7.24) holds for $j = j_1, z_j = z_0$. If $p \le 2$, then the equality in the first relation in (7.24) holds for $j = j_{11}, z_j = z_{00}$. If $2 < p < q$ and $a_1 > 0$, then this equality is not possible, the second inequality in (7.24) implies the first one and holds for $j > j_1$,. If $p < q$, $p \le 2$, then the first inequality in (7.24) implies the second inequality and holds for $j > j_{11}$. If $2 \ge p > q$, then these inequalities hold for $j_{11} < j < j_1$ only.

Thus, joint with Sections 7.1.2, 7.1.3 we have obtained the solutions of (7.7) for $2 < p < q$.

7.1.4. *The case* $C_j > 0, B_j = 0$

This case means $h_j = 1$.

$$2^{j+2}\sinh^2\frac{z_j^2}{2}\left(\frac{z_j^2}{\tanh\frac{z_j^2}{2}}\right) = hA2^{j(rh+h/p)}z_j^h,\ 2^{j(sq+1)}z_j^q \le (R/\varepsilon)^q.$$

We need consider this case for $p \le 2$ which imply the constraints $z_j > z(p) = 0$. Using the values z_{00}, j_{11} defined by (7.22) and assuming $z_j = o(1)$ we can rewrite these equations and constraints in the form

$$z_j \sim z_{00}2^{b_2(j-j_{11})},\ z_j \le z_{00}2^{-(s+1/q)(j-j_{11})} \tag{7.25}$$

where

$$b_2 = (rh + h/p - 1)/(4 - h). \tag{7.26}$$

Note that $b_2 + s + 1/q = h_1/(4-h) > 0$ for $\kappa \in \Xi_G$ which imply that the constraint in (7.25) holds for $j \le j_{11}$.
Thus, joint with Sections 7.1.1, 7.1.2 we have obtained the solutions of (7.7) for $2 \ge p > q$ and for $2 \ge p, q > p$.

7.1.5. *The solutions of (7.7)*

The following proposition is combination of the results of Sections 7.1.1–7.1.4. We use here the constants a, a_1, b_1, b_2 defined by (7.12, 7.15, 7.26) and the values $h_0, j_0;\ z_0, j_1$.

Proposition 7.1. *For $p > 2$ or for $2 \ge p > q, b_1 < 0$ define the values j_0, h_0 by the relations (7.9) or by (7.19) and assume (7.10) for $p > 2$. For $q > p > 2, a_1 > 0$ or for $p \le 2$ define the values j_1, z_0 by the relations (7.17) and assume (7.18). Then there exist the solutions $z_j > 0,\ h_j \in (0,1],\ j > 0$ of (7.7) and the following asymptotics hold*

1. *Let $p > 2, p > q$ (note that $I > 0$ in this case). Then*

$$z_j \asymp \begin{cases} z(p), & \text{if } j \le j_0 \\ (1 + j - j_0)^{1/2}, & \text{if } j \ge j_0, \end{cases}$$

$$2^{j-j_0}(h_j/h_0)^2 \sinh^2 \frac{z_j^2}{2} \asymp \left(2^{(j-j_0)(1+rp)}(h_j/h_0) z_j^p\right)^{h/p} z_j^{-2} \text{ for } j \ge j_0;$$

and

$$h_j \asymp h_0 \begin{cases} 2^{a(j-j_0)}, & \text{if } j \le j_0 \\ 2^{-(sq+1)(j-j_0)}(1 + j - j_0)^{-q/2}, & \text{if } j \ge j_0 \end{cases};\ \sup_j h_j = o(1).$$

2. *Let $2 < p < q$. If $a_1 \le 0$ (remind that $I > 0$ in this case), then*

$$z_j \asymp \begin{cases} z(p), & \text{if } j \le j_0 \\ 2^{-b_1(j-j_0)}, & \text{if } j_0 < j, \end{cases}$$

and

$$h_j \asymp h_0 \begin{cases} 2^{a(j-j_0)}, & \text{if } j \le j_0 \\ 2^{a_1(j-j_0)}, & \text{if } j_0 < j. \end{cases}$$

If $a_1 > 0$, then

$$z_j \asymp \begin{cases} z(p), & \text{if } j \le j_0 \\ 2^{-b_1(j-j_0)}, & \text{if } j_0 < j < j_1 \\ z_0 2^{-(s+1/q)(j-j_1)}, & \text{if } j \ge j_1 \end{cases},\ h_j \asymp \begin{cases} h_0 2^{a(j-j_0)}, & \text{if } j \le j_0 \\ h_0 2^{a_1(j-j_0)}, & \text{if } j_0 < j < j_1 \\ 1, & \text{if } j_1 \ge j. \end{cases}$$

3. *Let $2 \ge p > q$. If $b_1 < 0$, then*

$$z_j \asymp \begin{cases} z_0 2^{b_2(j-j_1)}, & \text{if } j \le j_1 \\ 2^{-b_1(j-j_0)} \asymp z_0 2^{-b_1(j-j_1)}, & \text{if } j_1 < j < j_0 \\ (1 + j - j_0)^{1/2}, & \text{if } j \ge j_0, \end{cases}$$

$$2^{j-j_0}(h_j/h_0)^2 \sinh^2 \frac{z_j^2}{2} \asymp \left(2^{(j-j_0)(1+rp)}(h_j/h_0) z_j^p\right)^{h/p} z_j^{-2} \text{ for } j \ge j_0;$$

$$h_j \asymp \begin{cases} 1, & \textit{if } j \le j_1 \\ h_0 2^{a_1(j-j_0)} \asymp 2^{a_1(j-j_1)}, & \textit{if } j_1 < j < j_0 \\ h_0 2^{-(sq+1)(j-j_0)}(1+j-j_0)^{-q/2}, & \textit{if } j \ge j_0; \end{cases}$$

if $b_1 \ge 0$ *(remind that* $I < 0$ *in this case), then*

$$z_j \asymp z_0 \begin{cases} 2^{b_2(j-j_1)}, & \textit{if } j \le j_1 \\ 2^{-b_1(j-j_1)}, & \textit{if } j_1 < j \end{cases}, \ \sup_j z_j = o(1)$$

and

$$h_j \asymp \begin{cases} 1, & \textit{if } j \le j_1 \\ 2^{a_1(j-j_1)}, & \textit{if } j_1 < j. \end{cases}$$

4. *Let* $2 \ge p, q > p$ *(note that* $I < 0$ *in this case). Then*

$$z_j \asymp \begin{cases} z_0 2^{b_2(j-j_1)}, & \textit{if } j \le j_1 \\ z_0 2^{-(s+1/q)(j-j_1)}, & \textit{if } j \ge j_1 \end{cases}, \ \sup_j z_j = o(1); \quad h_j = 1.$$

7.2. Solutions of extreme problems and upper bounds

We need to estimate the values h_0, j_0 or z_0, j_1 from the relation (7.8) and Proposition 7.1. By Remark 3.2 assumptions (7.10) and (7.18) follow from the assumption $u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $\delta = \delta(\kappa) > 0$ by $2^{j_0} \asymp n,\ 2^{j_1} \asymp m$.

7.2.1. *The cases* $p > 2,\ p > q$ *or* $q > p > 2$ *and* $a_1 \le 0$

These cases correspond to $I > 0,\ \kappa \in \Xi_{G_2}$ and we need to obtain the rates (3.7, 3.8). Note that

$$\sum_j 2^{j(rh+h/p)} h_j^{h/p} z_j^h = \sum_j (2^{j(rp+1)} h_j z_j^p)^{h/p} \asymp (h_0 2^{j_0(rp+1)})^{h/p}(\Sigma_1 + \Sigma_2),$$

where

$$\Sigma_1 \asymp \sum_{j=1}^{j_0} 2^{(j-j_0)(a+rp+1)(h/p)} \asymp 1 \tag{7.27}$$

by $a + rp + 1 > 0$ and

$$\Sigma_2 \asymp \begin{cases} \sum_{j=j_0}^{\infty} 2^{-(j-j_0)(sq-rp)(h/p)}(1+j-j_0)^{d_j}, & \text{if } p > 2,\ p > q \\ \sum_{j=j_0}^{\infty} 2^{-(j-j_0)(pb_1-a_1-rp-1)(h/p)}, & \text{if } q > p > 2, a_1 \le 0 \end{cases} \asymp 1 \tag{7.28}$$

by $sq > rp$ for $p > 2$ and

$$pb_1 - a_1 - rp - 1 = I/d; \tag{7.29}$$

(remind that $d = 2(q-2) + (h/p)(p-q),\ I = 2(p-2)sq - 2(q-2)rp + p - q$) which implies $pb_1 - a_1 - rp - 1 > 0$ for $q > p > 2,\ I > 0$ by $d > 0$ in this case. Thus from (7.8) we have the rate relation for $n = 2^{j_0},\ h_0$:

$$h_0 n^{rp+1} \asymp (\rho_\varepsilon/\varepsilon)^p \tag{7.30}$$

(we omit the considerations which show the existence the solutions j_0, h_0).

Let us obtain the asymptotics of the values u_ε. We have:

$$u_\varepsilon^2 = 2\sum_j 2^j h_j^2 \sinh^2 \frac{z_j^2}{2} \asymp h_0^2 2^{j_0}(\Sigma_1' + \Sigma_2')$$

where

$$\Sigma_1' = \sum_{j=1}^{j_0} 2^{(j-j_0)(2a+1)} \asymp 1 \tag{7.31}$$

by $2a+1>0$ and

$$\Sigma_2' \asymp \begin{cases} \Sigma_2, & \text{if } p>2,\ p>q \\ \sum_{j=j_0}^\infty 2^{-(j-j_0)(4b_1-2a_1-1)}, & \text{if } q>p>2,\ a_1\le 0 \end{cases} \asymp 1$$

by Proposition 7.1 and by

$$4b_1 - 2a_1 - 1 = hI/pd \tag{7.32}$$

which implies $4b_1-2a_1-1>0$ for $q>p>2$ by $I>0,\ d>0$ in this case. Thus we have the relation

$$h_0^2 n \asymp u_\varepsilon^2 \tag{7.33}$$

which joint with (7.30) and (7.9) imply the rates (3.7, 3.8).

7.2.2. *The cases $p\le 2,\ q>p$ or $2\ge p>q$ and $b_1\ge 0$*

These cases correspond to $I<0,\ \kappa\in\Xi_{G_1}$ and we need to obtain the rates (3.5, 3.6). Analogously to above

$$\sum_j 2^{j(rh+h/p)} h_j^{h/p} z_j^h = \sum_j (2^{j(rp+1)} h_j z_j^p)^{h/p} \asymp (z_0^p 2^{j_1(rp+1)})^{h/p}(\Sigma_1+\Sigma_2),$$

where

$$\Sigma_1 \asymp \sum_{j=1}^{j_1} 2^{(j-j_1)(pb_2+rp+1)(h/p)} \asymp 1 \tag{7.34}$$

by $pb_2+rp+1>0$ and

$$\eta_2 \asymp \begin{cases} \sum_{j=j_1}^\infty 2^{-(j-j_1)((s-r)p-1+p/q)(h/p)}, & \text{if } p\le 2,\ q>p \\ \sum_{j=j_1}^\infty 2^{-(j-j_1)(pb_1-a_1-rp-1)(h/p)}, & \text{if } 2\ge p>q,\ b_1\ge 0 \end{cases} \asymp 1 \tag{7.35}$$

by $(s-r)pq>q-p,\ a_1>0$ for $p\le 2,\ p<q$ and by (7.29) which implies $pb_1-a_1-rp-1>0$ for $2\ge p>q$ by $I<0,\ d<0$ in this case. Thus from (7.8) we have the rate relation for $m=2^{j_1},\ z_0$:

$$z_0^p m^{rp+1} \asymp (\rho_\varepsilon/\varepsilon)^p \tag{7.36}$$

(we also omit the considerations which show the existence of the solutions z_0, j_1).

To obtain the asymptotics of the values u_ε note that $\sinh^2(z^2/2)\asymp z^4$ for $z=O(1)$ and

$$u_\varepsilon^2 = 2\sum_j 2^j h_j^2 \sinh^2 \frac{z_j^2}{2} \asymp z_0^4 2^{j_1}(\Sigma_1' + \Sigma_2')$$

where

$$\Sigma_1' = \sum_{j=1}^{j_1} 2^{(j-j_1)(4b_2+1)} \asymp 1 \tag{7.37}$$

by $4b_2+1>0$ and

$$\Sigma_2' \asymp \begin{cases} \sum_{j=j_1}^{\infty} 2^{-(j-j_1)(4s+4/q-1)} & \text{if } p \le 2,\ p<q \\ \sum_{j=j_0}^{\infty} 2^{-(j-j_1)(4b_1-2a_1-1)}, & \text{if } 2 \ge p > q,\ b_1 \ge 0 \end{cases} \asymp 1 \tag{7.38}$$

by $s > 1/4 - 1/q$ for $p \le 2,\ p<q$ and by (7.32) which implies $4b_1 - 2a_1 - 1 > 0$ for $2 \ge p > q$ by $I<0,\ d<0$ in this case.

Thus we have the relation

$$z_0^4 m \asymp u_\varepsilon^2 \tag{7.39}$$

which joint with (7.36) and (7.17) imply the rates (3.5, 3.6).

7.2.3. *The case* $2<p<q,\ a_1>0$

This case corresponds to $I>0,\ \kappa \in \Xi_{G_2}$ or $I<0,\ \kappa \in \Xi_{G_1}$ and we need to obtain the rates (3.5, 3.6) for $I<0$ and (3.7, 3.8) for $I>0$. By $j_0<j_1$ we have

$$\sum_j 2^{j(rh+h/p)} h_j^{h/p} z_j^h = \sum_j (2^{j(rp+1)} h_j z_j^p)^{h/p}$$

$$\asymp \sum_{j<j_0} + \sum_{j>j_1} + \sum_{j_0<j<j_1} \asymp d_1\Sigma_1 + d_2(\Sigma_2+\Sigma_3) \asymp d_2\Sigma_2 + d_1(\Sigma_1+\Sigma_4)$$

where the value Σ_1 is defined by (7.27), the value Σ_2 is defined by (7.35) for $q>p$.

$$d_1 = (h_0 2^{j_0(rp+1)})^{h/p},\ d_2 = (z_0^p 2^{j_1(rp+1)})^{h/p}$$

and

$$\Sigma_3 \asymp \sum_{j=j_0}^{j_1} 2^{(j-j_1)(a_1-pb_1+rp+1)(h/p)}, \Sigma_4 \asymp \sum_{j=j_0}^{j_1} 2^{(j-j_0)(a_1-pb_1+rp+1)(h/p)}.$$

Remind that $h_0/z_0^p \asymp 2^{(pb_1-a_1)(j_1-j_0)}$ and by (7.29)

$$d_1/d_2 \asymp 2^{(j_1-j_0)(pb_1-a_1-rp-1)h/p} = 2^{(j_1-j_0)hI/pd}. \tag{7.40}$$

The estimations above show that $\Sigma_1 \asymp 1$ by $a+rp+1>0$ and $\Sigma_2 \asymp 1$ by $\mu = pq(s-r) > q-p$ for $q>p>2$.

Let $I>0$. Then $d_2 = o(d_1)$ and $\Sigma_4 \asymp 1$ which imply asymptotics (7.30). Let $I<0$. Then $d_1 = o(d_2)$ and $\Sigma_3 \asymp 1$ which imply asymptotics (7.36).

To obtain the asymptotics of the values u_ε note that

$$u_\varepsilon^2 = 2\sum_j 2^j h_j^2 \sinh^2 \frac{z_j^2}{2} \asymp \sum_{j<j_0} + \sum_{j>j_1} + \sum_{j_0<j<j_1}$$

$$\asymp c_1\Sigma_1' + c_2(\Sigma_2' + \Sigma_3') \asymp c_2\Sigma_2' + c_1(\Sigma_1' + \Sigma_4')$$

where $c_1 = h_0^2 2^{j_0},\ c_2 = z_0^4 2^{j_1}$, the value Σ_1' is defined by (7.31), the value Σ_2' is defined by (7.38) for $q>p$.

$$\Sigma_3^{'} \asymp \sum_{j=j_0}^{j_1} 2^{(j-j_1)(2a_1-4b_1+1)}, \Sigma_4^{'} \asymp \sum_{j=j_0}^{j_1} 2^{(j-j_0)(2a_1-4b_1+1)}.$$

Note that

$$c_1/c_2 \asymp 2^{(j_1-j_0)(4b_1-2a_1-1)} = 2^{hI(j_1-j_0)/pd}. \tag{7.41}$$

The estimations above show that $\Sigma_1^{'} \asymp 1$ by $2a+1>0$ and $\Sigma_2^{'} \asymp 1$ by $4sq > q-4$ for $a_1 > 0$.

Let $I>0$. Then $c_2 = o(c_1)$ and $\Sigma_4^{'} \asymp 1$ which imply asymptotics (7.33). Let $I<0$. Then $c_1 = o(c_2)$ and $\Sigma_3^{'} \asymp 1$ which imply asymptotics (7.39).

These relations imply the rates (3.7, 3.8) for $I>0$ and (3.5, 3.6) for $I<0$.

7.2.4. *The case* $2 \geq p > q,\ b_1 < 0$

This case corresponds to $I>0,\ \kappa \in \Xi_{G_2}$ or $I<0,\ \kappa \in \Xi_{G_1}$ and we need to obtain the rates (3.5, 3.6) for $I<0$ and (3.7, 3.8) for $I>0$. By $j_0 > j_1$ we have similarly to above

$$\sum_j 2^{j(rh+h/p)} h_j^{h/p} z_j^h \asymp \sum_{j<j_1} + \sum_{j>j_0} + \sum_{j_1<j<j_0} \asymp d_2\Sigma_1 + d_1(\Sigma_2+\Sigma_4) \asymp d_1\Sigma_2 + d_2(\Sigma_1+\Sigma_3)$$

where the value Σ_1 is defined by (7.34), the value Σ_2 is defined by (7.28) for $p>q$, $d_1,\ d_2$ are the same as above and

$$\Sigma_3 \asymp \sum_{j=j_1}^{j_0} 2^{(j-j_1)(a_1-pb_1+rp+1)(h/p)}, \Sigma_4 \asymp \sum_{j=j_1}^{j_0} 2^{(j-j_0)(a_1-pb_1+rp+1)(h/p)}.$$

The estimations above show that $\Sigma_1 \asymp 1$ by $pb_2+rp+1>0,\ \Sigma_2 \asymp 1$ by $sq > rp$ for $b_1 < 0$.

Let $I>0$. Then by (7.40) where $d<0,\ j_1<j_0$ we have $d_2 = o(d_1)$ and $\Sigma_4 \asymp 1$ which imply asymptotics (7.30). Let $I<0$. Then $d_1 = o(d_2)$ and $\Sigma_3 \asymp 1$ which imply asymptotics (7.36).

To obtain the asymptotics of the values u_ε note that

$$u_\varepsilon^2 \asymp \sum_{j<j_1} + \sum_{j>j_0} + \sum_{j_1<j<j_0} \asymp c_2\Sigma_1^{'} + c_1(\Sigma_2^{'}+\Sigma_4^{'}) \asymp c_1\Sigma_2^{'} + c_2(\Sigma_1^{'}+\Sigma_3^{'})$$

where the values $c_1,\ c_2$ are defined as above, the value $\Sigma_1^{'}$ is defined by (7.37), the value $\Sigma_2^{'}$ is defined by (7.28) for $p>q$, and

$$\Sigma_3^{'} \asymp \sum_{j=j_1}^{j_0} 2^{(j-j_1)(2a_1-4b_1+1)}, \Sigma_4^{'} \asymp \sum_{j=j_1}^{j_0} 2^{(j-j_0)(2a_1-4b_1+1)}.$$

The estimations above show that $\Sigma_1^{'} \asymp 1$ by $4b_2+1>0$ and $\Sigma_2^{'} \asymp 1$ by $sq-rp>0$ for $b_1<0$.

Let $I>0$. Then by (7.41) where $d<0,\ j_1<j_0$ we have $c_2 = o(c_1)$ and $\Sigma_4^{'} \asymp 1$ which imply asymptotics (7.33). Let $I<0$. Then $c_1 = o(c_2)$ and $\Sigma_3^{'} \asymp 1$ which imply asymptotics (7.39).

These relations imply the rates (3.7, 3.8) for $I>0$ and (3.5, 3.6) for $I<0$.

7.2.5. *Upper bounds*

To obtain the statement of Theorem 8, n. 2 it is enough to check the assumptions of Theorem 12, n. 1. Assumption C1 follows directly from the asymptotics (3.5) and (3.7). One can easily check assumptions B1, B3a in C2 using Propositions 7.1 and the rates type of (3.5, 3.6) or (3.7, 3.8).

We need to check assumption B4a for $p>q,\ \lambda>0$ which correspond to $sq>rp,\ I>0$ and to the asymptotics type of G_2 by $z_{\varepsilon,j} = O(1)$ in other cases. It follows from Propositions 7.1 and from estimations above, that

$z_{\varepsilon,j} = O(1)$ for $j \le j_0$, and if $j > j_0$, then

$$T^2_{\varepsilon,j} \sim (\log 2)(2+\delta)j + B(j-j_0),\ z^2_{\varepsilon,j} \asymp j - j_0,\ \tilde{J} \subset \{j \ge (1+\delta_1)j_0\},\ \delta_1 > 0,\ B = B(\tau) > 0$$

for δ_0 small enough in (5.21). Let $v \in \tilde{V}_\varepsilon$. Then using the inequality

$$2^{jrp}\sum_l |v_{lj}|^p \le \max_l |v_{lj}|^{p-q} 2^{j(rp-sq)} 2^{jsq} \sum_l |v_{lj}|^q \le \max_l |v_{lj}|^{p-q} 2^{j(rp-sq)} (R/\varepsilon)^q$$

and relations

$$(\rho_\varepsilon/\varepsilon)^p \asymp h_0 2^{j_0(rp+1)} = 2^{j_0(rp-sq)} h_0 2^{j_0(sq+1)} \asymp 2^{j_0(rp-sq)}(R/\varepsilon)^q$$

we get:

$$f_{j,1}(v) = (\sum_{l=1}^{2^j} 2^{jpr}|v_{lj}|^p)^{h/p} \le B_1 T^{h(p-q)/p}_{\varepsilon,j} 2^{(j-j_0)(rp-sq)h/p}(\rho_\varepsilon/\varepsilon)^h.$$

Therefore

$$\sup_{v\in\tilde{V}_\varepsilon}\sum_{j\in\tilde{J}_\varepsilon} f_{j,1}(v) \le B_2(\rho_\varepsilon/\varepsilon)^h \sum_{j\ge(1+\delta_1)j_0} j^{h(p-q)/2p} 2^{(j-j_0)(rp-sq)h/p} = o(H_{\varepsilon,1}).$$

Thus Theorem 8, n. 2 and the upper bounds of Theorem 5 for Besov bodies case are proved.

7.3. **Lower bounds**

To obtain the upper bounds of Theorem 5 by Corollary 5.1 it is enough to construct such sequences of three-point measures $\bar{\pi}_\varepsilon = \{\pi_{\varepsilon,i,j}\} = \bar{\pi}_\varepsilon(\tau),\ \tau = (\kappa,t,h)$ that $\|\bar{\pi}_\varepsilon\| \asymp u_\varepsilon,\ \pi^\varepsilon(V_\varepsilon(\tau)) \to 1$ and assumptions A1, A2 hold. We can assume that $b < u_\varepsilon = O(\varepsilon^{-\delta})$ for small enough $b > 0,\ \delta = \delta(\tau) > 0$.

Put

$$\pi_{\varepsilon,i,j} = \begin{cases} \delta_0, & \text{if } j \ne j^* \\ (1-h_{j^*})\delta_0 + \frac{h_{j^*}}{2}(\delta_{z_{j^*}} + \delta_{-z_{j^*}}), & \text{if } j = j^* \end{cases},\ 1 \le i \le 2^j$$

where δ_z is Dirac mass at the point $z \in R^1$,

$$j^* = j_0,\ h_{j^*} = h_0,\ z_{j^*} = 1,\ \text{if } I > 0; \quad j^* = j_1,\ h_{j^*} = 1,\ z_{j^*} = z_0,\ \text{if } I < 0,$$

and the values $n = 2^{j_0}, h_0,\ m = 2^{j_1}, z_0$ are determined by the relations analogous to (3.8, 3.6) with different $\rho'_\varepsilon = B\rho_\varepsilon,\ R' = R/B$ for any $B > 1$:

$$n^{rp+1}h_0 = (\rho'_\varepsilon/\varepsilon)^p,\ n^{sq+1}h_0 = (R'/\varepsilon)^q$$

or

$$m^{rp+1}z_0^p = (\rho'_\varepsilon/\varepsilon)^p,\ m^{sq+1}z_0^q = (R'/\varepsilon)^q.$$

It is clear that $\|\bar{\pi}_\varepsilon\| \asymp u_\varepsilon$ where u_ε is defined by (3.7, 3.5). By the measures π^ε are supported on one level j^*, the relation $\pi^\varepsilon(V_\varepsilon(\tau)) \to 1$ follows from the relations

$$\pi^\varepsilon\{2^{rpj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^p > (\rho_\varepsilon/\varepsilon)^p\} \to 1,\ \pi^\varepsilon\{2^{sqj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^q < (R/\varepsilon)^q\} \to 1. \tag{7.42}$$

If $I < 0$, then one can easy check that these relations hold with π^ε-probability 1. If $I > 0$, then one can easy check these relations using Chebyshev inequality by

$$E_{\pi^\varepsilon}\left(2^{rpj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^p\right) = n^{rp+1}h_0 = (B\rho_\varepsilon/\varepsilon)^p,$$

$$E_{\pi^\varepsilon}\left(2^{sqj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^q\right) = n^{sq+1}h_0 = (R/B\varepsilon)^q$$

and

$$Var_{\pi^\varepsilon}\left(2^{rpj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^p\right) < n^{2rp+1}h_0 = o((\rho_\varepsilon/\varepsilon)^{2p}),$$

$$Var_{\pi^\varepsilon}\left(2^{sqj^*}\sum_{i=1}^{2^{j^*}}|v_{ij^*}|^q\right) < n^{2sq+1}h_0 = o((R/\varepsilon)^{2q}),$$

if $nh_0 \to \infty$ which holds for $u_\varepsilon = O(\varepsilon^{-\delta})$ and small enough $\delta > 0$.

Theorems 5 and 8 are proved.

8. Degenerate type: Proof of Theorems 3, 7

8.1. Upper bounds: Ellipsoids

Let us consider the tests $\psi_{\varepsilon,\alpha}$ from Theorem 7. By

$$\alpha(\psi_{\varepsilon,\alpha}) = \alpha + (1-\alpha)P_0(X_\varepsilon),\ \ \beta(\psi_{\varepsilon,\alpha}, v) = (1-\alpha)P_v(\bar{X}_\varepsilon),$$

to prove n. 1 of Theorem 7 we need to show that uniformly on $\kappa \in K \subset \Xi_D,\ B^{-1} < R < B$

$$\begin{aligned} P_0(X_\varepsilon) &\to 0, \\ \sup_{v\in V_\varepsilon} P_v(\bar{X}_\varepsilon) &\leq \Phi\left(\sqrt{2\log n_\varepsilon(\tau)} - n_\varepsilon^{-r}(\tau)\rho_\varepsilon/\varepsilon\right) + o(1) \end{aligned} \qquad (8.1)$$

where $\bar{X}_\varepsilon$ is a complement of X_ε,

$$\tau = (\kappa, R),\ V_\varepsilon = V_\varepsilon(\tau,\rho_\varepsilon),\ n = n_\varepsilon(\tau,\rho_\varepsilon) = (R/\rho_\varepsilon)^{1/(s-r)} \to \infty$$

by $s > r \geq 0$ for $\kappa \in \Xi_D$ ($r > 0$ for $p < \infty$). We can assume that $n \geq N_\varepsilon$.

Let us consider the properties of the thresholding (4.2). Using the standard relation:

$$\Phi(-x) \sim \frac{1}{\sqrt{2\pi}x}\exp(-x^2/2),\ \text{as } x \to \infty \qquad (8.2)$$

we have the first relation in (8.1):

$$P_0(X_\varepsilon) \leq 2N_\varepsilon\Phi(-\sqrt{2\log N_\varepsilon}) + 2\sum_{i=N_\varepsilon}^{\infty}\Phi(-T_i) \asymp \frac{1}{\sqrt{\log N_\varepsilon}} + \sum_{i=N_\varepsilon}^{\infty}\frac{1}{i(\log i)^{3/2}} \to 0.$$

Let $v \in V_\varepsilon(\tau,\rho_\varepsilon)$. Then

$$\begin{aligned} P_v(\bar{X}_\varepsilon) \leq \min\Bigg\{ &\min_{i\leq N_\varepsilon}\left(\Phi\left(\sqrt{2\log N_\varepsilon} - |v_i|\right) - \Phi\left(-\sqrt{2\log N_\varepsilon} - |v_i|\right)\right), \\ &\inf_{N_\varepsilon < i}(\Phi(T_i - |v_i|) - \Phi(-T_i - |v_i|))\Bigg\} \leq \min_{i\leq n}(\Phi(T_n - |v_i|) - \Phi(-T_n - |v_i|)). \end{aligned}$$

By $T_n = \sqrt{2\log n} + o(1), \Phi(-T_n - |v_i|) \to 0$, the second relation in (8.1) follows from the

Lemma 8.1. *Let* $n = n_\varepsilon(\tau) = (R/\rho_\varepsilon)^{1/(s-r)}$ *and* $s > r > 0$, $p \geq q$, $\lambda = sq - rp \leq 0$ *(note that these assumptions hold for* $\kappa \in \Xi_D$*). Then*

$$\inf_{v \in V_\varepsilon} \max_{i \leq n} |v_i| \geq \rho_\varepsilon/\varepsilon n^r.$$

Proof of the lemma. For simplicity assume $q < p < \infty$ (the case $\infty = p \geq q$ is simpler).

First, note that

$$(\rho_\varepsilon/\varepsilon)^p \leq \sum_i (i^r|v_i|)^p \leq \sup_i \{i^{-\lambda}|v_i|^{p-q}\} \sum_i (i^s|v_i|)^q \leq \sup_i i^{-\lambda}|v_i|^{p-q}(R/\varepsilon)^q$$

which imply

$$\sup_i i^{-\lambda}|v_i|^{p-q} \geq \rho_\varepsilon^p / R^q \varepsilon^{p-q}.$$

Next, by $|v_i| \leq R/\varepsilon i^s$ and by definition of n we have for any $i_0 > n$ and $i \geq i_0$:

$$i^{-\lambda}|v_i|^{p-q} \leq i_0^{p(r-s)}(R/\varepsilon)^{p-q} < \rho_\varepsilon^p/R^q\varepsilon^{p-q}.$$

Therefore the supremum is attained at $i \leq n$ and these relations imply

$$n^{-\lambda} \max_{i \leq n} |v_i|^{p-q} \geq \max_{i \leq n} i^{-\lambda}|v_i|^{p-q} \geq \rho_\varepsilon^p/R^q\varepsilon^{p-q}.$$

Thus we have the inequality of the lemma:

$$\max_{i \leq n} |v_i| \geq n^{\lambda/(p-q)}(\rho_\varepsilon/R)^{q/(p-q)}\rho_\varepsilon/\varepsilon = n^{-r}\rho_\varepsilon/\varepsilon.$$

The lemma and Theorem 7, n. 1 are proved.

8.2. **Upper bounds: Besov bodies**

The consideration of this case is analogous to above: we need the relations

$$\begin{aligned} P_0(X_\varepsilon) &\rightarrow 0, \\ \sup_{v \in V_\varepsilon} P_v(\bar{X}_\varepsilon) &\leq \Phi(\sqrt{2\log n_\varepsilon(\tau)} - c(\tau)n_\varepsilon^{-r}(\tau)\rho_\varepsilon/\varepsilon) + o(1) \end{aligned} \tag{8.3}$$

for some $c(\tau) > 0$ and $n = n_\varepsilon(r, s, R) = 2^{j_0}$, $R/\rho_\varepsilon = c(\tau)2^{j_0(s-r)}$.

The first relation in (8.3) is obtained as above. To obtain the second relation we use the considerations analogous to above and the following

Lemma 8.2. *Let* $v \in V_\varepsilon = V_\varepsilon(\tau, \rho_\varepsilon)$, $\tau = (\kappa, R, t, h)$ *and* $s > r \geq 0$, $p > q$, $\lambda = sq - rp \leq 0$ *and* $\lambda < 0$, *if* $hq < pt$. *Then there exist such constant* $c(\tau) > 0$ *that*

$$\inf_{v \in V_\varepsilon} \max_{j \leq j_0} \max_{1 \leq i \leq 2^j} |v_{ij}| \geq c(\tau)n^{-r}\rho_\varepsilon/\varepsilon.$$

Proof of the lemma. To simplicity assume $p, q, t, h < \infty$. Let $v \in V_\varepsilon$. For a positive sequence $\{d_j\}$ (which is determined concretely later) we have:

$$(\rho_\varepsilon/\varepsilon)^h \leq \sum_j 2^{jrh}\left(\sum_i |v_{ij}|^p\right)^{h/p} \leq \left(\sup_j 2^{-\lambda j} d_j^{-1} \max_i |v_{ij}|^{p-q}(R/\varepsilon)^q\right)^{h/p} \sum_j \left(d_j x_j^{q/t}\right)^{h/p}$$

where

$$x_j = \left(2^{jsq}\sum_i |v_{ij}|^q(\varepsilon/R)^q\right)^{t/q}\ ;\ \sum_j x_j \le 1.$$

This implies

$$\sup_j 2^{-\lambda(j-j_0)} d_j^{-1}\max_i |v_{ij}|^{p-q} \ge 2^{\lambda j_0}\rho_\varepsilon^p R^{-q}\varepsilon^{q-p}\left(\sum_j \left(d_j x_j^{hq/pt}\right)^{hq/pt}\right)^{-p/h}.$$

If $a = hq/pt \ge 1$, then we put $d_j = 1,\ c(\tau) = 1$ and by $\sum_j x_j^{hq/pt} \le 1$ we have

$$\sup_j 2^{-\lambda(j-j_0)}\max_i |v_{ij}|^{p-q} \ge 2^{\lambda j_0}\rho_\varepsilon^p/R^q\varepsilon^{p-q} = (2^{-rj_0}\rho_\varepsilon/\varepsilon)^{p-q}$$

and by

$$\sup_j 2^{sj}\max_i |v_{ij}| \le R/\varepsilon \tag{8.4}$$

analogously to the proof of Lemma 8.1 we have:

$$\sup_{j>j_0} 2^{-\lambda(j-j_0)/(p-q)}\max_i |v_{ij}| < 2^{-rj_0}\rho_\varepsilon/\varepsilon.$$

These relations imply the necessary inequality with $c(\tau) = 1$.

Let $a = hq/pt < 1,\ \lambda < 0$. Put

$$d_j = \begin{cases} 2^{-\lambda(j-j_0)}, & \text{if } j \le j_0\\ 2^{p(r-s)(j-j_0)}, & \text{if } j > j_0.\end{cases}$$

By Holder inequality

$$\sum_j d_j^{h/p}x_j^a \ \le\ \left(\sum_j x_j\right)^a\left(\sum_j d_j^{h/p(1-a)}\right)^{1-a} < \left(\sum_{j\le 0} 2^{-jh\lambda/p(1-a)} + \sum_{j>0} 2^{jh(r-s)/(1-a)}\right)^{1-a} = b(\tau).$$

Put $c(\tau) = (b(\tau))^{-p/h(p-q)}$. Then

$$\max\left\{\max_{j\le j_0}\max_i |v_{ij}|,\ \sup_{j>j_0} 2^{(j-j_0)s}\max_i |v_{ij}|\right\} > c(\tau)2^{-j_0 r}\rho_\varepsilon/\varepsilon$$

and by (8.4)

$$\sup_{j>j_0} 2^{(j-j_0)s}\max_i |v_{ij}| \le 2^{-sj_0}R/\varepsilon = c(\tau)2^{-rj_0}\rho_\varepsilon/\varepsilon.$$

Thus we get:

$$\max_{j\le j_0}\max_i |v_{ij}| > c(\tau)2^{-rj_0}\rho_\varepsilon/\varepsilon.$$

The lemma and Theorem 7 are proved.

8.3. **Lower bounds: Ellipsoids**

The lower bounds of Theorem 3 follow from the relation: if $\kappa \in \Xi_D$ and $n_\varepsilon = (R/\rho_\varepsilon)^{1/(s-r)}$, then

$$\beta(\alpha, V_\varepsilon(\kappa, R, \rho_\varepsilon)) \geq (1-\alpha)\Phi(\sqrt{2\log n_\varepsilon} - n_\varepsilon^{-r}\rho_\varepsilon/\varepsilon) + o(1). \tag{8.5}$$

To prove (8.5) we can assume

$$n_\varepsilon^{-r}\rho_\varepsilon/\varepsilon = O\left(\sqrt{2\log n_\varepsilon}\right). \tag{8.6}$$

Put

$$V_{1,\varepsilon}(\bar{x}) = \{v_k = \{v_{ki}\} \in l_2,\ n_1 \leq k \leq n\}$$

where

$$n = n_\varepsilon,\ n_1 = n_{1,\varepsilon} = n(1 - 1/\log n),\ \bar{x} = \{x_i,\ n_1 \leq i \leq n\},\ x_i = i^{-r}\rho_\varepsilon/\varepsilon$$

and

$$v_{ki} = \begin{cases} 0, & \text{if } k \neq i \\ x_i, & \text{if } k = i. \end{cases}$$

It is clear that $V_{1,\varepsilon}(\bar{x}) \subset V_\varepsilon$ which implies the inequality

$$\beta(\alpha, V_\varepsilon) \geq \beta(\alpha, V_{1,\varepsilon}(\bar{x})). \tag{8.7}$$

Using Theorem 4.2 in Ingster [12], Part II, n. 4.4 with $u_i = x_i$ we obtain the inequality

$$\beta(\alpha, V_{1,\varepsilon}(\bar{x})) \geq (1-\alpha)\Phi(R_\varepsilon) + o(1) \tag{8.8}$$

where R_ε are such values that

$$\sum_{i=n_1}^{n} \Phi(-x_i - R_\varepsilon) \asymp 1. \tag{8.9}$$

Put $R_\varepsilon = \sqrt{2\log n_\varepsilon} - n_\varepsilon^{-r}\rho_\varepsilon/\varepsilon + \delta_\varepsilon$. Then the relation (8.6–8.8) imply (8.5), if we could choose such $\delta_\varepsilon \to 0$ that (8.9) holds. It is clear that this possibility follows from the relations: for any $\delta > 0$

$$\sum_{n_1 \leq i \leq n} \Phi(-x_i - R_\varepsilon + \delta) \to \infty, \quad \sum_{n_1 \leq i \leq n} \Phi(-x_i - R_\varepsilon - \delta) \to 0. \tag{8.10}$$

By $x_i + R_\varepsilon = \sqrt{2\log n} + o(1)$ uniformly on $n_1 \leq i \leq n$, using (8.2) one can easy check the relations (8.10).

The relation (8.5) and Theorem 3 for ellipsoidal case are proved.

8.4. **Lower bounds: Besov bodies**

The lower bounds of Theorem 3 follow from the relation: if $\kappa \in \Xi_D$ and $n_\varepsilon = 2^{j_0}$, $j_0 = j_{0,\varepsilon} = [\ (s-r)^{-1}\log_2(R/\rho_\varepsilon)\ ]$, where $[\ t\ ]$ is an integral part of $t > 0$, then

$$\beta(\alpha, V_\varepsilon(\kappa, R, \rho_\varepsilon)) \geq (1-\alpha)\Phi\left(\sqrt{2\log n_\varepsilon} - n_\varepsilon^{-r}\rho_\varepsilon/\varepsilon\right) + o(1). \tag{8.11}$$

To prove (8.11) let us consider the level j_0 and the set

$$V_{1,\varepsilon} = \{v_k = \{v_{kij}\} \in l_2,\ 1 \leq k \leq 2^{j_0}\}$$

with

$$v_{kij} = \begin{cases} 0, & \text{if } k \neq i,\ j \neq j_0 \\ 2^{-j_0 r}\rho_\varepsilon/\varepsilon, & \text{if } k = i,\ j = j_0. \end{cases}$$

It is clear that $V_{1,\varepsilon} \subset V_\varepsilon$ which implies the inequality

$$\beta(\alpha, V_\varepsilon) \geq \beta(\alpha, V_{1,\varepsilon}) \tag{8.12}$$

and (8.11) follows from (8.12) and the inequality of Ingster [12], Part II, n. 4.4 for $u_\varepsilon = 2^{-j_0 r}\rho_\varepsilon/\varepsilon$:

$$\beta(\alpha, V_{1,\varepsilon}) \geq (1-\alpha)\Phi(\sqrt{2\log n} - 2^{-j_0 r}\rho_\varepsilon/\varepsilon) + o(1).$$

The relation (8.11) and Theorem 3 are proved.

9. Trivial type: Proof of Theorem 2

9.1. **Ellipsoidal case**

Let

$$\kappa \in \Xi_T,\ \infty \geq p \geq q,\ r \geq 0$$

(note that $s \leq r$ in this case) and $R > \rho_\varepsilon$, if $s = r$. If $r > 0$, then the set V_ε contains the points $v_n \in l_2$ with only one nonzero coordinate $v_{n,i} = i^{-r}\rho_\varepsilon/\varepsilon \to 0$, where $i = i(n) \to \infty$ as $n \to \infty$ which implies the theorem on this case. If $r = 0,\ \infty \geq p \geq q$, consider the points $v_i \in l_2$:

$$v_{i,j} = \begin{cases} \rho_\varepsilon/\varepsilon, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and the set $V_{\varepsilon,n} = \{v_i,\ m+1 \leq i \leq m+n\} \subset V_\varepsilon$ for large enough m. Using the inequality in Ingster [12], Part II, p. 181 with $u_\varepsilon = \rho_\varepsilon/\varepsilon$ we have, as $n \to \infty$:

$$\beta(\alpha, V_\varepsilon) \geq \beta(\alpha, V_{\varepsilon,n}) \geq (1-\alpha)\Phi(\sqrt{2\log n} - \rho_\varepsilon/\varepsilon) + o(1) \to 1-\alpha.$$

To obtain Theorem 2 for other cases for a fixed $\varepsilon > 0,\ \rho_\varepsilon > 0,\ \kappa \in \Xi_T$ and $R > \rho_\varepsilon$, if $\mu = 0$ it is enough to construct such sequences $\bar\pi_n = \{\pi_{n,i}\} = \bar\pi_{n,\varepsilon,\rho_\varepsilon,R,\kappa}$ that

$$\|\bar\pi_n\| \to 0,\ \pi^n(V_\varepsilon) \to 1 \tag{9.1}$$

where $V_\varepsilon = V_{\varepsilon,\rho_\varepsilon,R}(\kappa)$ and π^n is product measure corresponding to $\bar\pi_n$. As above we use the sequences of three-point measures

$$\pi_{n,i} = (1-h_{n,i})\delta_0 + \frac{h_{n,i}}{2}(\delta_{z_{n,i}} + \delta_{-z_{n,i}}).$$

Let $p = q < \infty,\ s \leq r < 0$. Put for $p \leq 2$

$$h_{n,i} = 1,\ z_{n,i} = \begin{cases} 0, & \text{if } i > n \\ b_n i^{rp/(4-p)}, & \text{if } i \leq n, \end{cases}$$

and for $p > 2$

$$z_{n,i} = 1,\ h_{n,i} = \begin{cases} 0, & \text{if } i > n \\ a_n i^{rp}, & \text{if } i \leq n, \end{cases}$$

where $a_n,\ b_n$ are such values that

$$\sum_{i=1}^n i^{rp} z_{n,i}^p = b_n^p \sum_{i=1}^n i^{4rp/(4-p)} = (\rho_\varepsilon/\varepsilon)^p,\ (R/\varepsilon)^p > \sum_{i=1}^n i^{rp} h_{n,i} = a_n \sum_{i=1}^n i^{2rp} > (\rho_\varepsilon/\varepsilon)^p.$$

Then we can obtain the relations (9.1) by the estimations similar to the proof of Theorem 2.5 in Ingster [12], Sections 4.2 and 4.3.

Thus, we need to consider the cases $\kappa \in \Xi_T$ with $\infty > p > q,\ r < 0$ and $p < q$. For simplicity we assume $q < \infty$ (for $p < q = \infty$ one can use similar consideration). Remind the notations:

$$\lambda = sq - rp,\ \mu = pq(s-r),\ \Delta = 4\lambda - \mu = sq(4-p) - rp(4-q),$$
$$I = 2\mu - 4\lambda + p - q = 2(p-2)sq - 2(q-2)rp + p - q.$$

Lemma 9.1. *Let*

$$p \neq q,\ \Delta/(q-p) > 0,\ 0 \le \mu/(q-p) \le 1,\ \lambda/(q-p) \ge 0.$$

If $\mu/(q-p) = 0$, then we assume $\rho_\varepsilon \le R$. Put

$$h_{n,i} = 1,\ z_{n,i} = \begin{cases} \delta_n i^{-\lambda/(q-p)}, & \text{if } m_{n,1} \le i \le m_{n,2} \\ 0, & \text{in other cases} \end{cases}$$

where $m_1 = m_{n,1} \to \infty,\ m_2 = m_{n,2} \to \infty,\ \delta_n \asymp 1$, as $n \to \infty$ are such values that

$$A_n = \sum_{i=m_1}^{m_2} i^{-\mu/(q-p)} \asymp 1, \quad (\rho_\varepsilon/\varepsilon)^p \le \delta_n^p A_n,\ (R/\varepsilon)^q \ge \delta_n^q A_n$$

(if $\mu/(q-p) > 0$, then one can easy chooses such values. If $\mu/(q-p) = 0$, put $m_1 = m_2 = n,\ \delta_n = R/\varepsilon$). Then the relations (9.1) hold.

Proof of the lemma. By the assumptions

$$\sum_{i=m_1}^{m_2} z_{n,i}^p i^{rp} = \delta_n^p \sum_{i=m_1}^{m_2} i^{-\mu/(q-p)} = \delta_n^p A_n \ge (\rho_\varepsilon/\varepsilon)^p,$$

$$\sum_{i=m_1}^{m_2} z_{n,i}^q i^{sq} = \delta_n^q \sum_{i=m_1}^{m_2} i^{-\mu/(q-p)} = \delta_n^q A_n \le (R/\varepsilon)^q,$$

which imply $\pi^n(V_{\varepsilon,\rho_\varepsilon}(\kappa)) = 1$. Also

$$\|\bar\pi_n\|^2 \asymp \sum_{i=m_1}^{m_2} z_{n,i}^4 = \delta_n^4 \sum_{i=m_1}^{m_2} i^{-(\mu+\Delta)/(q-p)} < m_1^{-\Delta/(q-p)} \delta_n^4 \sum_{i=m_1}^{m_2} i^{-\mu/(q-p)} = O(m_1^{-\Delta/(q-p)}) \to 0$$

The lemma is proved.

Lemma 9.2. *Let*

$$p \neq q,\ \Delta/(q-p) \le 0,\ I/(q-p) \le 0,\ \lambda/(q-p) \ge 0,\ 0 < \mu/(q-p).$$

Put

$$h_{n,i} = \begin{cases} a_n i^{\Delta/(q-p)}, & \text{if } m_{n,1} \le i \le m_{n,2} \\ 0, & \text{in other cases} \end{cases},$$

$$z_{n,i} = \begin{cases} \delta_n i^{-\lambda/(q-p)}, & \textit{if } \ m_{n,1} \le i \le m_{n,2} \\ 0, & \textit{in other cases} \end{cases}$$

where $m_1 = m_{1,n} \to \infty, \ m_2 = m_{2,n} \to \infty, \ \delta_n \asymp 1, \ a_n \to 0$ *as* $n \to \infty$ *are such values that*

$$A_n = a_n \sum_{i=m_1}^{m_2} i^{-1-I/(q-p)} \asymp 1, \quad (\rho_\varepsilon/\varepsilon)^p < \delta_n^p A_n, \ (R/\varepsilon)^q > \delta_n^q A_n.$$

(one can easy chooses such values). Then the relations (9.1) hold.

Proof of the lemma. By the assumptions

$$E_{\pi^n}\left(\sum_i i^{rp}|v_i|^p\right) = \sum_{i=m_1}^{m_2} h_{n,i} z_{n,i}^p i^{rp} = a_n \delta_n^p \sum_{i=m_1}^{m_2} i^{-1-I/(q-p)} = \delta_n^p A_n > (\rho_\varepsilon/\varepsilon)^p,$$

$$E_{\pi^n}\left(\sum_i i^{sq}|v_i|^q\right) = \sum_{i=m_1}^{m_2} h_{n,i} z_{n,i}^q i^{sq} = a_n \delta_n^q \sum_{i=m_1}^{m_2} i^{-1-I/(q-p)} = \delta_n^q A_n < (R/\varepsilon)^q,$$

$$Var_{\pi^n}\left(\sum_i i^{rp}|v_i|^p\right) \asymp Var_{\pi^n}\left(\sum_i i^{sq}|v_i|^q\right)$$
$$\asymp a_n \sum_{i=m_1}^{m_2} i^{-1-(I+\mu)/(q-p)} < m_1^{-\mu/(q-p)} A_n \to 0$$

which by Chebyshev inequality imply $\pi^n(V_{\varepsilon,\rho_\varepsilon}(\kappa)) \to 1$. Also

$$\|\bar{\pi}_n\|^2 \asymp \sum_{i=m_1}^{m_2} h_{n,i}^2 z_{n,i}^4 = a_n^2 \delta_n^4 \sum_{i=m_1}^{m_2} i^{-1-I/(q-p)} = O(a_n) \to 0.$$

The lemma is proved.

Theorem 2 for $\infty > p > q, \ r < 0$ and $p < q$ follows directly from Lemmas 9.1, 9.2 and from following monotone property. Let $\kappa = (p,q,r,s), \ \kappa' = (p,q,r,s'), \ s' < s$. Then $V = V_\varepsilon(\kappa, R, \rho) \subset V_\varepsilon(\kappa', R, \rho) = V'$. This yields: $\beta_\varepsilon(\alpha, V) \le \beta_\varepsilon(\alpha, V')$. Therefore it is enough to check the triviality for large enough s from the region Ξ_T. In fact, let $\infty > p > q, \ r < 0$. If $0 > r \ge -1/2p$, then we can use Lemma 9.2 by $\lambda \le 0$ and $\mu < 0, \ I \ge 0, \Delta \ge 0$ for large enough s in this case. If $1/4 - 1/p \le r \le -1/2p$ (it is possible for $p < 2$), then also we can use Lemma 9.2 by $I \ge 0$ and $\lambda \le 0, \ \mu < 0, \Delta \ge 0$ for large enough s in this case.

Let $p < q$. If $r > 1/4 - 1/p$, then we can use Lemma 9.1 by $\mu \le q - p$ and $\lambda > 0, \ \Delta > 0$ for large enough s in this case. If $r \le 1/4 - 1/p$, then we can use Lemma 9.2 by $I \le 0$ and $\mu > 0, \ \lambda \ge 0, \ \Delta \le 0$ for large enough s in this case.

Theorem 2 is proved for ellipsoidal case.

9.2. Besov bodies case

Let $\kappa \in \Xi_T$. First, assume $I \ne 0$ and $R > \rho_\varepsilon$, if $s = r$. Then the considerations of this case are analogous to above. We consider only one level $j_0 = j_{n,0} \to \infty$. Let $\kappa \in \Xi_T, \ \infty \ge p \ge q, \ r \ge 0$ and $R > \rho_\varepsilon$, if $s = r$. Then the set V_ε contains 2^{j_0} points $v_n \in l_2$ with only one nonzero coordinate $v_{n,ij_0} = 2^{-rj_0}\rho_\varepsilon/\varepsilon, \ i = 1, \dots, 2^{j_0}$ which implies the theorem on this case.

Let $p = q < \infty, \ s \le r < 0$. Put for $p \le 2$ and $r > 1/4 - 1/p$.

$$h_{n,ij} = 1, \ z_{n,ij} = \begin{cases} 0, & \text{if } j \ne j_0 \\ 2^{-j_0(r+1/p)}(\rho_\varepsilon/\varepsilon), & \text{if } j = j_0, \ i = 1, \dots, 2^{j_0}, \end{cases}$$

and for $p > 2$ and $r > -1/2p$, if $s = r$

$$z_{n,ij} = \rho_\varepsilon/\varepsilon,\ h_{n,ij} = \begin{cases} 0, & \text{if } j \neq j_0 \\ 2^{-j_0(rp+1)}, & \text{if } j = j_0,\ i = 1, \dots, 2^{j_0}. \end{cases}$$

Then we can easily obtain the relations (9.1). Note that the cases $r = 1/4 - 1/p$, $s = r$ and $r = -1/2p$, $s = r$ correspond to $I = 0$. The proof of Theorem 2 is analogous to the proof of Lemma 9.5 later; if $s < r$, then we use monotone property.

Let $\infty > p \neq q$. Analogously to Lemma 9.1 and Lemma 9.2 we have

Lemma 9.3. *Let*

$$p \neq q,\ \Delta/(q-p) > 0,\ 0 \leq \mu/(q-p) \leq 1,\ \lambda/(q-p) \geq 0.$$

If $\mu = 0$ or $\mu = q - p$, then assume $\rho_\varepsilon \leq R$. Put

$$h_{n,ij} = 1,\ z_{n,ij} = \begin{cases} b_0 z_{j_0}, & \textit{if } j = j_0,\ 1 \leq i \leq m \\ 0, & \textit{in other cases} \end{cases}$$

where $z_{j_0} = 2^{-j_0\lambda/(q-p)}$, $m = a_0 2^{j_0\mu/(q-p)}$, a_0 and b_0 are such values that $\rho_\varepsilon/\varepsilon \leq b_0 a_0^{1/p}$, $R/\varepsilon \geq b_0 a_0^{1/q}$ and $a_0 \geq 1$ if $\mu = 0$, $a_0 \leq 1$ if $\mu = q - p$ Then the relations (9.1) hold.

Proof of the lemma. We have

$$\left(\sum_j \left(2^{jrp} \sum_i |z_{n,ij}|^p \right)^{h/p} \right)^{p/h} \sim m 2^{j_0 rp} (b_0 z_{j_0})^p = a_0 b_0^p,$$

$$\left(\sum_j \left(2^{jsq} \sum_i |z_{n,ij}|^q \right)^{t/q} \right)^{q/t} \sim m 2^{j_0 sq} (b_0 z_{j_0})^q = a_0 b_0^q$$

which imply $\pi^n(V_\varepsilon) = 1$. Also

$$\|\bar{\pi}_n\|^2 \asymp m z_{j_0}^4 = a_0 b_0^4 2^{-j_0\Delta/(q-p)} \to 0.$$

The lemma is proved.

Lemma 9.4. *Let*

$$p \neq q,\ \Delta/(q-p) \leq 0,\ I/(q-p) < 0,\ \lambda/(q-p) \geq 0,\ 0 < \mu/(q-p).$$

Put

$$h_{n,ij} = \begin{cases} a_0 h_{j_0}, & \textit{if } \ j = j_0,\ 1 \leq i \leq m \\ 0, & \textit{in other cases,} \end{cases}$$

$$z_{n,i} = \begin{cases} b_0 z_{j_0} & \textit{if } \ j = j_0,\ 1 \leq i \leq m \\ 0, & \textit{in other cases} \end{cases}$$

where a_0 and b_0 are such values that $\rho_\varepsilon/\varepsilon < b_0 a_0^{1/p}$, $R/\varepsilon > b_0 a_0^{1/q}$ and

$$h_{j_0} = h_0 2^{j_0\Delta/(q-p)},\ z_{j_0} = 2^{-j_0\lambda/(q-p)},\ m = 2^{j_0(1+I/(q-p))}/h_0$$

where $h_0 \to 0$, $h_0 > 2^{j_0 I/(q-p)}$. Then the relations (9.1) hold.

Proof of the lemma. We have

$$E_{\pi^n}\left(\sum_j\left(2^{jrp}\sum_i|v_{ij}|^p\right)^{h/p}\right)^{p/h} \sim m2^{j_0rp}h_{j_0}z_{j_0}^p a_0b_0^p = a_0b_0^p,$$

$$E_{\pi^n}\left(\sum_j\left(2^{jsq}\sum_i|v_{ij}|^q\right)^{t/q}\right)^{q/t} \sim m2^{j_0sq}h_{j_0}z_{j_0}^q a_0b_0^q = a_0b_0^q,$$

$$Var_{\pi^n}\left(\sum_j\left(2^{jrp}\sum_i|v_{ij}|^p\right)^{h/p}\right)^{p/h} \asymp m2^{2j_0rp}h_{j_0}z_{j_0}^{2p} \asymp 2^{-j_0\mu/(q-p)} \to 0$$

$$Var_{\pi^n}\left(\sum_j\left(2^{jsq}\sum_i|v_{ij}|^q\right)^{t/q}\right)^{q/t} \asymp m2^{2j_0sq}h_{j_0}z_{j_0}^{2q} \asymp 2^{-j_0\mu/(q-p)} \to 0$$

which imply $\pi^n(V_\varepsilon) \to 1$ as $n \to \infty$. Also

$$\|\bar\pi_n\|^2 \asymp mh_{j_0}^2z_{j_0}^4 = h_0 \to 0.$$

The lemma is proved.

Theorem 2 for $I \neq 0$ and $\infty > p > q,\ r < 0$ or $p < q$ follows directly from Lemmas 9.3 and 9.4 and from monotone property noted above.

Let $\kappa \in \Xi_T,\ I = 0$. Note (see Sect. 5.2 above or Ingster [12], Part II, Sect. 4.1) that it is enough to construct such measures π^n on l_2 that

$$\pi^n(V_\varepsilon) \to 1,\ E_0\left(\frac{dP_{\pi^n}}{dP_0} - 1\right)^2 = E_0\left(\frac{dP_{\pi^n}}{dP_0}\right)^2 - 1 \to 0 \tag{9.2}$$

where $P_{\pi^n}(A) = \int P_v(A)\pi^n(dv)$ is a mixture. For simplicity we consider the case $p \neq q$ only.

Lemma 9.5. *Assume*

$$I/(q-p) = 0,\ p \neq q,\ \Delta/(q-p) \leq 0,\ \lambda/(q-p) \geq 0,\ 0 < \mu/(q-p).$$

Let us consider the product measures $\bar\pi_k$ corresponding to the sequences $\bar h_k,\ \bar z_k$ where

$$h_{k,ij} = \begin{cases} a_0h_k, & \text{if } j = k,\ 1 \leq i \leq 2^k \\ 0, & \text{in other cases,}\end{cases}$$

$$z_{k,ij} = \begin{cases} b_0z_k, & \text{if } j = k,\ 1 \leq i \leq 2^k \\ 0, & \text{in other cases.}\end{cases}$$

Here b_0 and a_0 are such values that $\rho_\varepsilon/\varepsilon < b_0a_0^{1/p},\ R/\varepsilon > b_0a_0^{1/q}$ and

$$h_k = 2^{j\Delta/(q-p)},\ z_k = 2^{-j\lambda/(q-p)}.$$

Put

$$\pi^n = j_0^{-1}\sum_{k=j_0+1}^{2j_0}\bar\pi_k,\quad j_0 = j_{n,0} \to \infty.$$

Then the relations (9.2) hold.

Proof of the lemma. Let us consider the variables

$$x_j = 2^{jrp} \sum_{i=1}^{2^j} |v_{ij}|^p, \ y_j = 2^{jsq} \sum_{i=1}^{2^j} |v_{ij}|^q.$$

We have: $P_{\bar{\pi}_k}\{x_j = 0, \ y_j = 0\} = 1$, if $j \neq k$, and if $j = k$, then

$$E_{\bar{\pi}_j}(x_j) = a_0 2^{j(rp+1)} h_j (b_0 z_j)^p = a_0 b_0^p, \ E_{\bar{\pi}_j}(y_j) = a_0 2^{j(sq+1)} h_j (b_0 z_j)^q = a_0 b_0^q,$$

$$Var_{\bar{\pi}_j}(x_j) \asymp Var_{\bar{\pi}_j}(y_j) \asymp 2^{-j\mu/(q-p)}.$$

By Chebyshev inequality these relations imply $\bar{\pi}_k(V_\varepsilon) \to 1$ as $k \to \infty$ which imply the first relation in (9.2).

To obtain the second relation note that

$$\|\bar{\pi}_k\|^2 \asymp 2^k h_k^2 z_k^4 \asymp 1$$

and

$$E_0 \left(\frac{dP_{\pi^n}}{dP_0} - 1 \right)^2 = j_0^{-2} \sum_{j,k=j_0+1}^{2j_0} E_0 \left(\frac{dP_{\bar{\pi}_j}}{dP_0} - 1 \right) \left(\frac{dP_{\bar{\pi}_k}}{dP_0} - 1 \right) = j_0^{-2} \sum_{k=j_0+1}^{2j_0} E_0 \left(\frac{dP_{\bar{\pi}_k}}{dP_0} - 1 \right)^2 \to 0$$

by

$$E_0 \left(\frac{dP_{\bar{\pi}_k}}{dP_0} - 1 \right)^2 \leq \exp(\|\bar{\pi}_k\|^2) - 1 = O(1)$$

The lemma and Theorem 2 are proved.

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