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APPROXIMATION OF RELIABILITY FOR A LARGE SYSTEM WITH NON-MARKOVIAN REPAIR-TIMES

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Abstract. Consider a system of many components with constant failure rate and general repair rate. When all components are reliable and easily reparable, the reliability of the system can be evaluated from the probability q of failure before restoration. In [14], authors give an asymptotic approximation by monotone sequences. In the same framework, we propose, here, a bounding for q and apply it in the ageing property case.

Résumé. Le calcul de la fiabilité d'un grand système réparable peut être réduit à celui de la probabilité q de panne avant remise à neuf. Dans [14], les auteurs donnent une approximation asymptotique de q par séquences monotones pour un système k/n. Nous proposons ici un encadrement de q dans un cadre plus général et l'appliquons à des réparations vieillissantes.

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1. Introduction

This paper is motivated by the study of reliability of large and highly reparable systems. It constitutes a continuation of different works originating in Gnedenko and Solovyev [7] and more recently Solovyev and Konstant [14]. These papers were concerned mainly with asymptotic analysis of reliability. Undoubtedly, their approach was profitable to a large area of applications. For more details on asymptotic analysis of reliability, see [6] and references contained therein. We consider here a system with N reparable independent components. The evolution is described by a binary vector $\mathbf{e}(\mathrm{t})=(e_i(t))_{i=1..N}$ where e_i is equal to 0 (resp. 1) according as component number i is working (resp. failed so, in this case, under repair). Each component i has a lifetime X_i with constant failure rate λ_i and repairtime R_i with any distribution.

It is quite natural to define on the system states set E a partial order relation. A state is better than another if it has less failed components:

$$\mathbf{e}(t) \leq \mathbf{e}'(t) \Leftrightarrow \forall i,\ e_i \leq e'_i.$$

Without any risk of confusion, it's possible to identify a state $\mathbf{e}$ and the set of failed components of $\mathbf{e}$. In this way, the order amounts to inclusion.

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The set E can be decomposed in two disjoint parts E_+ and E_- where E_+ is the set of working states and E_- the set of failure states. We assume that this decomposition is compatible with the order relation in the following sense. Let two states be in order, say $\mathbf{e} \leq \mathbf{e}'$. If $\mathbf{e}' \in E_+$ (resp. $\mathbf{e} \in E_-$) then $\mathbf{e} \in E_+$ (resp. $\mathbf{e}' \in E_-$). It seems natural to assume that when all components are functioning ($\mathbf{e} = 0$) the system is functioning too, and the same for repairing $\mathbf{e} = 1$. The system is said to be *coherent* in Barlow-Proschan's sense [1]. In this description, a *minimal cut* is nothing but a minimal subset of $(E^-, \leq)$. Denote by G the set of minimal cuts.

If A is a subset of E, let T_A be the entry time, say

$$T_A = \inf\{t > 0 \mid \exists s < t,\ e(s) \notin A\ e(t) \in A\}\cdot$$

A state is characterized by its associated failed components denoted a. For this state, we note λ_a the exit parameter:

$$\lambda_a = \sum_{i \in a} \lambda_i.$$

The initial state $\mathbf{0}$ plays a particular role. Its entry time and parameter will be denoted by T_0 and λ, and we have

$$\lambda = \sum_{i=1}^{N} \lambda_i.$$

This parameter is the inverse of the mean sojourn time in 0. We need also a global mean repairtime r which is defined by:

$$r = \sum_{i=1..N} \frac{\lambda_i}{\lambda} r_i$$

where r_i is the mean repairtime of component i. This formula has a concrete meaning. If the system is starting from 0, the first failure concerns component i with probability $\frac{\lambda_i}{\lambda}$. In this case, repair starts for a mean time r_i. So r can be seen as the mean repairtime of a component. The product λr measures the stiffness of the system. Generally, for reliable systems, this parameter is very small. Due to the coherence property, the entry time in failure states T_- can be written:

$$T_- = T_{E^-} = \inf_{\gamma \in G} T_\gamma.$$

It is natural to assume the system new at time 0 and to estimate the reliability at time t:

$$R(t) = \mathbf{P}(T_- > t \mid e(0) = 0).$$

This calculation is very important in terms of industrial safety and quality. It's known as a difficult problem. During the last twenty years, many papers were concerned with approximation evaluations. A common feature of essentially all these works (especially [2, 7, 10, 12, 13]) is that they use the fact that the initial state is a regeneration state. It would be too lengthy to mention all these approximations. We choose a representative one in Solovyev [13]:

$$\begin{aligned} R(t) &\geq \exp(-\lambda q t) && (1.1)\\ \text{where } q &= \mathbf{P}(T_- < T_0 \mid e(0) = 0). && (1.2) \end{aligned}$$

With this result, the evaluation of the function R can be reduced to the evaluation of q which is the probability to have a failure system before complete restoration when system goes out from initial state 0.

Our purpose is *to get a good approximation of* q in the most general framework possible. In [14], for a k/n model, authors consider monotone trajectories (a trajectory is monotone if the sequence of visited states is monotone). They prove, in an asymptotic setting, that cumulated probability of all monotone trajectories furnishes a good approximation. It's not surprising that loops "repairs-failures" may have a small probability. For the same model we propose a non-asymptotical approximation of q without any assumption on repair distribution. In addition, when repairtimes verify the ageing property we give even simpler bounds. Failure rates are supposed to be independent of the state system but this restriction can be removed by a bounding of the different rates [7].

The system enters failure states by minimal cuts, so we can decompose q from these minimal cuts, $q = \mathbf{P}(\inf_{\gamma\in G}\{T_\gamma\} < T_0)$, and define a pessimistic version $q_G \geq q$ by summing:

$$q_G = \sum_{\gamma\in G} q_\gamma \quad \text{where} \quad q_\gamma = \mathbf{P}(T_\gamma < T_0 \mid e(0) = 0). \tag{1.3}$$

This bounding is more efficient when minimal cuts have few redundancy. So we need to bound each term q_γ from separate study of each minimal cut.

The paper is organized as follows. Section 2 is devoted to modelling the system life. Contrary to the Markovian case, the description of the process is not usual. It requires a more precise and complex framework. Main results dealing with bounding of q from monotone sequences are given in Section 3. Section 4 is concerned with applications in the case of ageing repairtimes. In the industrial context, such an assumption is quite reasonable and the simple bounds given at the end of the paper should answer engineer requirements.

2. System modelling

The system is composed of N independent components. For $i = 1..N$, the component i has a lifetime X_i with constant failure rate λ_i. Its repairtime R_i (almost surely positive) has a general distribution defined by the queue function H_i:

$$H_i(t) = \mathbf{P}(R_i > t), \quad \text{so} \quad r_i = \int_0^\infty H_i(u)du.$$

Considering constant failure-rate components is realistic for many industrial components. But for the repairtimes, this assumption is too restrictive. Nevertheless it's reasonable to assume that a current repairing has more chance of success than a starting repair. Many works in the literature deal about the ageing property definition (for more details, see for example [1, 2, 4, 7]). The next definition gives a general version.

Definition 2.1. A random positive variable R_i is said NBUE (New Better than Used in Expectation) if, for any t, we have

$$\mathbf{E}(R_i - t \mid R_i > t) \leq r_i.$$

As a consequence of this relation, the repair time R_i verifies, for any t:

$$\int_t^\infty H_i(u)du \leq r_i e^{-t/r_i}. \tag{2.1}$$

The property defined by this last inequality is usually called HNBUE.

In our context, the NBUE property means that when a component is under repair, mean time to restoration is less than when repair is starting. The more general definition HNBUE is less expressive. Roughly, it amounts to say that the distribution has not an heavy tail (less heavy than the exponential one).

In what follows, we construct independently, for each component i, the process $e_i(t)$ which is Markovian with respect to some filtration which is not the natural one. For clarity of notation, we omit, in this part, the index i of the component.

Let us remark that if the repairtime is instantaneous then the sequence of failures yields a Poisson process with parameter λ. We start from this process and introduce successive repairtimes. Let $(\xi_j)_{j>0}$ be the sequence of jump times of this Poisson process. It represents potential dates of failure. Let $(\rho_j)_{j>0}$ an i.i.d. sequence of random variables with the same distribution as R (repair time of the component) and independent from the previous Poisson process.

Let u be a positive number. We define the process $C(t)$ starting from u as

$$C(t) = (u-t)^+ + \sum_{j=1}^{\infty} (\xi_j + \rho_j - t)^+ .\mathbf{1}_{\xi_j \le t}.$$

As usual, notation $(x)^+$ is used for $\max\{x,0\}$. Let denote $\mathcal{G}_t$ the σ-algebra generated by $\{\xi_j, \rho_j | \xi_j \le t\}$ and $\mathcal{G}$ the corresponding filtration: $\mathcal{G} = (\mathcal{G}_t)_{t\ge 0}$. Due to the Poisson property, the process $C(t)$ is a $\mathcal{G}$-Markov process. It is worthy noting that the corresponding filtration, at time t, allows to anticipate the process.

To construct the process $C(t)$ we have taken into account all the failure-times. To model the evolution of the component we have to cancel the failures which appear when the component is in repair. So we propose to construct a new process $c(t)$. Let fix $u \ge 0$. Set:

$$\sigma_0 = u$$

and define, step by step, for $j \ge 0$:

$$a_{j+1} = \inf_{k>0}(k \mid \xi_k > \sigma_j), \qquad \tau_{j+1} = \xi_{a_{j+1}}, \qquad R_{j+1} = \rho_{a_{j+1}}, \qquad \sigma_{j+1} = \tau_{j+1} + R_{j+1}. \tag{2.2}$$

Component has complete repair at each time σ_j. The next failure after σ_j is the first jump $\xi_{a_{j+1}}$ of the Poisson process. The sequence $(a_j)_j$ is strictly increasing, hence the random variables R_j are i.i.d. with the same distribution as R and are independent of $(\tau_j)_j$. We define:

$$c(t) = (u-t)^+ + \sum_{j=0}^{\infty} (\tau_j + R_j - t)^+ .\mathbf{1}_{\tau_j \le t}. \tag{2.3}$$

The restricted filtration is

$$\mathcal{F} = (\mathcal{F}_t)_{t\ge 0} \qquad \text{with} \qquad \mathcal{F}_t = \sigma\{\tau_j, R_j | \tau_j \le t\} = \sigma\{c(s) | s \le t\}\cdot$$

Remark 1. The process $c(t)$ is Markovian with respect to this filtration.

To prove this, let us suppose $c(t) > 0$, then the end of current repair is known. Because of the loss of memory of the Poisson process, if τ is the next lifetime then $\tau - c(t)$ is independent of $\mathcal{F}_t$. If $c(t) = 0$ the same conclusion holds clearly.

To return to the state evolution of the system we define

$$e(t) = \mathbf{1}_{c(t)>0}.$$

The process $e(t)$ is Markovian with respect to the $(\mathcal{F}_t)_{t\geq 0}$-filtration. We can define, in the same way, a process $E(t)$ which is constructed from $C(t)$:

$$E(t) = \mathbf{1}_{C(t)>0}.$$

Remark 2. For each $t > 0$, $c(t) \leq C(t)$, and $e(t) \leq E(t)$.

Remark 3. The perfect state 0 is a renewal state for the four above processes.

For each component, we make independently the same construction. Now we denote with an adding index i all the variables which were defined above: $\xi_{i,k}, \sigma_{i,j}, \tau_{i,j}, e_i, c_i, E_i, C_i$. We describe the evolution of the system by the new process

$$e(t) = \sum_{i=1}^{N} e_i(t).$$

Similarly, we define the process $c(t) = \sum_{i=1}^{N} c_i(t)$, and the processes $C(t)$ and $E(t)$. For the component i, in the same way as for usual processes, we can define, for $t > 0$, the failure-time and repairtime close to t. Let us denote $\tau_i(t)$ the last failure date before t. Its number is j^* and the end of the associated repairing is $\sigma_i(t)$. The case $j^* = 0$ gives $\sigma_i(t) = u$ otherwise

$$\tau_i(t) = \inf_{j>0; \tau_{i,j} \leq t} (\tau_{i,j}) = \tau_{i,j^*}, \quad \sigma_i(t) = \sigma_{i,j^*} = \tau_i(t) + R_{i,j^*}.$$

At the time t, the component i failed for the last time at the date $\tau_i(t)$ and the residual repairtime is $(\sigma_i(t)-t)^+$.

In the sequel, we are interested by "chains of failures". We need to bound the probability that a component is under repair between two fixed dates knowing its failure time. The following technical lemma is given for this purpose.

Lemma 2.2. *Let a, b, s, δ positive with $0 \leq s \leq a + \delta \leq b + \delta$. For component i and every initial distribution δ_{u_i}, we have:*

$$\mathbf{P}_{u_i}\left(max\{a, s\} < \tau_i(b) \leq a + \delta;\ b \leq \sigma_i(b)\right) \leq \lambda_i \delta H_i(b - a - \delta) + (\lambda_i \delta)^2/2. \tag{2.4}$$

Proof. If I is an interval, we denote by $\mu(I)$ (resp. $\nu(I)$) the number of failures for masked process (resp. not-masked process):

$$\mu(I) = \text{card}\{j \mid \tau_{i,j} \in I\} \qquad \nu(I) = \text{card}\{j \mid \xi_{i,j} \in I\}.$$

By definition, $\mu(I) \leq \nu(I)$. To establish the result, we have to study the number of jumps in the interval $[\max\{a, s\}, a + \delta]$. Since $\tau_i(s)$ is the last failure-time preceding s, we have $\tau_i(s) \leq s$. And by definition $\tau_i(s) < \sigma_i(s)$. So

$$\mu\left(]\max\{a, s\}, a + \delta]\right) = \mu\left(]\max\{a, \sigma_i(s)\}, a + \delta]\right).$$

The strong Markov property can be used at the renewal point $\sigma_i(s)$. So, for any integer k, we can bound the probability to have more than k jumps in $]a, a + \delta] \cap]s, a + \delta]$:

$$\mathbf{P}_{u_i}(\mu] \max\{a, s\}, a + \delta] \geq k | \mathcal{B}_i(s)) \leq \mathbf{P}_0(\mu](a - \sigma_i(s))^+, (a - \sigma_i(s))^+ + \delta] \geq k).$$

This writing allows to deal with both cases with the same formula. Replacing μ by ν, the problem is comes down to the Poisson process case for which the inequality is well known (see [5]). The probability to have at least one failure in the interval $]x, x+\delta]$ is less than $\lambda_i\delta$ and the probability to have more than two failures is less than $(\lambda_i\delta)^2$. The meaning of the event $\{\max\{a,s\} < \tau_i(b) \le a+\delta;\ b \le \sigma_i(b)\}$ is that there is a jump between $\max\{a,s\}$ and $a+\delta$ with a repairtime ending after $b-a-\delta$. □

Another lemma will be useful in the sequel. It answers the question: *if the process is going out of the perfect state, how long time does it take before complete restoration?* All the components are reparable and no state is absorbing, so such a return happens almost surely. The next lemma gives a bound for this mean return time. The proof is based on a classical result by Klimov [9] for busy period in a queue $M/G/\infty$.

Lemma 2.3. *Let $T_1 = \inf\{t \mid \mathbf{e}(t) \neq \mathbf{0}\}$ be the first failure-time of a component of the system (entry time in $E - \{\mathbf{0}\}$). Recall that T_0 is the entry time to 0. So the mean sojourn time spent out of the perfect state* 0, *starting from 0, is bounded by:*

$$\mathbf{E_0}(T_0 - T_1) \le \frac{e^{\lambda r} - 1}{\lambda}\cdot$$

It follows that, for any $\mathbf{u}$: $E_{\mathbf{u}}(T_0) < \infty$.

Proof. As above, $(\xi_{i,j})_{j=1,2,..}$ is a Poisson process with parameter λ_i and $(\rho_{i,j})_{j=1,2,..}$ is a sequence of random variables with the same distribution as R_i and independent of the Poisson process.

$$C_i(t) = (u_i - t)^+ + \sum_{j=1}^{\infty} (\xi_{i,j} + \rho_{i,j} - t)^+ .\mathbf{1}_{\xi_{i,j}\le t}.$$

We have also:

$$c(t) \le C(t).$$

We want to compare the return times. By analogy to T_0 and T_1 for the process $c(t)$, let denote T_0' and T_1' the return time and the first failure time for the process $V(t)$. Clearly we have $T_1' = T_1$ and

$$\mathbf{E_0}(T_0 - T_1) \le \mathbf{E_0}(T_0' - T_1').$$

It is sufficient to prove the result for the process $C(t)$. This process has the same distribution as the queue process W defined by

$$W(t) = \sum_{j=1}^{\infty} (\zeta_j + U_j - t)^+ .\mathbf{1}_{\zeta_j \le t}$$

where ζ_k is a Poisson process with parameter $\lambda = \sum \lambda_i$ and U_k a sequence of independent random variables with the same distribution as U.

$$P(U > x) = \sum_{i=1}^{N} \frac{\lambda_i}{\lambda} H_i(x).$$

We can easily understand this identity between $C(t)$ and $W(t)$ as follows. Starting from 0, the system has its first failed component at time ζ_1. This failure concerns the component i with probability $\frac{\lambda_i}{\lambda}$. So the repairtime distribution has the same distribution as U. If the process $e(t)$ starts from $(u_i)_{i=1,..,N}$ we use

$$\sum_{i=1,..,N}(u_i-t)^+ \leq \left(\sum_{i=1,..,N} u_i - t\right)^+$$

to prove that

$$c(t) \leq \left(\sum_i u_i - t\right)^+ + W(t).$$

So for this process, the return time $T_0^{''}$ verifies:

$$\mathbf{E}_{(u_i)_i}(T_0) \leq \mathbf{E}_{\sum u_i}(T_0^{''})$$

where $\mathbf{E}_{(u_i)_i}(T_0)$ is the expectation of T_0 starting from u_i for the component i and $\mathbf{E}_{\sum u_i}(T_0'')$ is the mean return time of the queuing process starting from $\sum u_i$. Intuitively, we return at 0 more slowly if we suppose that the repairs are made one after the other. A classical result for $M/G/\infty$ queue (see [9]), gives that expectation is equal to:

$$\lambda.\mathbf{E}_0(T_0^{''} - T_1) = \exp(\lambda \mathbf{E}(U)) - 1$$

$$\mathbf{E}_{\sum u_i}(T_0^{''}) < \infty.$$

And this achieves the proof. □

3. Bounding of failure probability

The aim is to evaluate $q = \mathbf{P}(T_- < T_0)$ which is the probability of failure before restoration, especially its pessimistic version (see Eq. (1.3)), which is built from minimal cuts:

$$q_G = \sum_{\gamma \in G} q_\gamma.$$

The first step consists in evaluating $q_\gamma = P_{\mathbf{0}}(T_\gamma < T_0)$ for a fixed minimal cut γ. Let us fix $\gamma = \{i_1, i_2, .., i_n\}$. Indeed, in the case of highly reliable systems, it is impossible to obtain the failure probability by enumeration of all trajectories ending at γ. Nevertheless a good choice of some trajectories allows to get an efficient approximation.

3.1. **Evaluation of direct failure**

In reliability models, failure probabilities are very dissimilar to repair probabilities. So, when a system failure is obtained with n failure components, it is not very probable to have more than n failures (each pair $repair - failure$ has a small probability). That induces to study trajectories without repairing.

If s is a permutation of γ, we define *direct failure in* γ *via* s by writing that the components of the minimal cut γ fail in the order s and stay under repair until T_γ (time of failure system). When T_γ is given, we can define, as previously, for each component i_j of γ the last failure time before T_γ (which was noted $\tau_{i_j}(T_\gamma)$ above). Recall that $\tau_{s_{i_j},1}$ is the first failure of the component s_{i_j}.

Definition 3.1. The probability p_γ defined by next formula:

$$p_\gamma(s) = \mathbf{P_0}\left(T_1 = \tau_{s_{i_1},1} < ... < \tau_{s_{i_n},1} = T_\gamma \ ; \quad \forall \quad j \leq n \ ,\tau_{s_{i_j},1} = \tau_{s_{i_j}}(T_\gamma); \ \forall k \leq N, \sigma_{k,1} \geq T_\gamma\right)$$

is called probability of direct failure in γ *via* s.

If we consider the set S_γ of all the permutations of γ, we get the *probability of direct failure in* γ:

$$p_\gamma = \sum_{s \in S_\gamma} p_\gamma(s).$$

For these trajectories, each component of γ fails one after the other, without repairing, until failure system.

Lastly, we define the *probability of direct failure* p_G, by considering all minimal cuts:

$$p_G = \sum_{\gamma \in G} p_\gamma.$$

It is noteworthly that, generally, p_G is less than q_G. In what follows, we propose a bounding of q in function of p_G. But before, let us give a lemma which is interesting by itself.

Lemma 3.2. *Let G be the set of minimal cuts, then the probability p_G of direct failure is bounded as follows:*

$$p_G \leq \alpha$$

where α is given by:

$$\alpha = \sum_{\gamma \in G} \pi_\gamma \frac{\mu_\gamma}{\lambda} \qquad \textit{with} \qquad \pi_\gamma = \prod_{i \in \gamma} \lambda_i r_i, \qquad \mu_\gamma = \sum_{i \in \gamma} \frac{1}{r_i}\cdot$$

Proof. First, fix a permutation s of the minimal cut $\gamma = \{i_1, i_2, ...i_n\}$. As previously, $p_\gamma(s)$ is the direct failure probability in order s *via* γ. Regarding the successive failure dates of the direct sequence, we can define a measure $p_\gamma(s, .)$ on R^{+n}. It is absolutely continuous with respect to the corresponding Lebesgue measure dx. Its density can be computed as follows. Let Δ_n be the domain defined by

$$\Delta_n = \{x_i : 0 < x_1 < x_2 < ... < x_n\}\cdot$$

For $j = 1, ..., n$, the times x_j represent the successive failures of the different components s_{i_j} of γ.

$$\frac{dp_\gamma(s,x)}{d\mathbf{x}} = \mathbf{1}_{\Delta_n}\cdot\left(\prod_{j \leq n} \lambda_{s_{i_j}} e^{-\lambda_{s_{i_j}} x_j}\right).e^{-(\lambda-\lambda_\gamma)x_n}.\prod_{j<n} H_{s_{i_j}}(x_n - x_j). \tag{3.1}$$

Recall that λ represents the sum of failure rates of all components and that λ_γ is the sum of failure rates of components of the minimal cut. If we join all exponential terms, using the order of the terms x_j, we obtain:

$$e^{-(\lambda-\lambda_\gamma)x_n}.\prod_{j \leq n} \lambda_{s_{i_j}} e^{-\lambda_{s_{i_j}} x_j} \leq e^{-\lambda x_1} \prod_{i \in \gamma} \lambda_i.$$

So we can write:

$$\frac{dp_\gamma(s,x)}{d\mathbf{x}} \leq \frac{dp_\gamma^+(s,x)}{d\mathbf{x}} \tag{3.2}$$

where the bounding probability p_γ^+ is defined by:

$$\frac{dp_\gamma^+(s,x)}{d\mathbf{x}} = \mathbf{1}_{\Delta_n}.\left(\prod_{i\in\gamma}\lambda_i\right) e^{-\lambda x_1}.\prod_{j<n} H_{s_{i_j}}(x_n - x_j). \tag{3.3}$$

In order to prove the inequality of the lemma, we have to take into account all the permutations.

$$\alpha_\gamma = \sum_{s\in S_\gamma}\int dp_\gamma^+(s,.) = \left(\prod_{i\in\gamma}\lambda_i\right).\sum_{s\in S_\gamma}\int \mathbf{1}_{\Delta_n}.e^{-\lambda x_1}.\prod_{j<n} H_{s_{i_j}}(x_n - x_j)d\mathbf{x}. \tag{3.4}$$

With the change of variables: for $j = 1, .., n-1$, $\quad v_j = x_n - x_j$, and $v_n = x_1$, the majorizing probability can be rewritten:

$$\alpha_\gamma = \left(\prod_{i\in\gamma}\lambda_i\right).\sum_{s\in S_\gamma}\int \mathbf{1}_{\Delta_n'}.e^{-\lambda v_n}.\prod_{j<n} H_{s_{i_j}}(v_j)d\mathbf{v}$$

where the domain Δ_n' is associated to the change of variables:

$$\Delta_n' = \{v_i : 0 < v_{n-1} < v_{n-2} < ... < v_1 \ ; \ v_n > 0\}\cdot$$

It's worth remarking that this integral concerns only $n-1$ repair distributions. The sum on all permutations gives the product of integrals of all the different H_j. So we obtain the next simple inequality:

$$p_\gamma \le \alpha_\gamma = \frac{1}{\lambda}\left(\prod_{i\in\gamma}\lambda_i r_i\right).\left(\sum_{i\in\gamma}\frac{1}{r_i}\right).$$

And the lemma is proved by taking into account all the minimal cuts: $\alpha = \sum_{\gamma\in G}\alpha_\gamma$. □

3.2. Upper bound of the failure probability q

Now we establish the main results for the approximation of the failure probability. Some applications are given in the last section. First theorem furnishes an upper bound for the probability q_G (so also for q) from above parameter α. This parameter is easily computed from the mean repairtimes and the mean failure-times. Recall that T_1, (resp. T_0, T_-) denotes the date of first component failure amongst all components (resp. date of the complete restoration after exit from 0, date of system failure). Of course $T_1 < T_0$, and $T_1 < T_-$.

First, let fix γ. With aim of lightening notations, we denote

$$\tau(T_-) = (\tau_1(T_-), \tau_2(T_-), .., \tau_n(T_-))$$

the vector of the last successive failure-times of each component of γ before T_- in a fix order.

Theorem 3.3. *The failure probability q_G obtained by cumulating all the minimal cut probabilities is bounded as follows:*

$$q_G \le (1 + \lambda\mathbf{E_0}(T_0 - T_1))\alpha \le e^{\lambda r}\alpha. \tag{3.5}$$

Proof. With a view to compare p_γ and q_γ, we consider the event

$$A = \{\tau(T_-) \in \Delta_n; T_- < T_0 \; ; \sigma_i(T_-) > T_-\}$$

of successive failures of γ components, in a fix order, before the complete restoration in 0. We want to bound $\mathbf{P}_0(A)$. Let us remark that the event A can be also written:

$$A = \{\tau(T_-) \in \Delta_n; \tau_1(T_-) < T_0 \, ; \sigma_i(T_-) > T_-\}\cdot$$

Two cases are possible, according as the first failure component at time T_1 concerns or not a γ component. This gives a decomposition for A in two parts:

$$A_1 = A \cap \{\tau_1(T_-) = T_1\} \qquad \text{and} \qquad A_2 = A \cap \{\tau_1(T_-) > T_1\}\cdot$$

For A_1, the condition $\{\tau_1(T_-) = T_1\}$ gives clearly $\{\tau_1(T_-) < T_0\}$, hence A_1 is also:

$$A_1 = \{\tau(T_-) \in \Delta_n \, ; \, \tau_1(T_-) = T_1; \, \sigma_i(T_-) > T_-, \, \forall i \in \gamma\}\cdot$$

To evaluate the probability for both cases A_1 and A_2, we use a standard uniform discretisation of space. Let δ a positive real number, $\mathbf{k} = (k_i)_{i=1,..,n}$ an integer vector of Δ_n, and for a fixed $\mathbf{k}$, let note:

$$I_{\mathbf{k}}(\delta) = \{(y_i)_i : i = 1, .., n; \quad k_i\delta < y_i < (k_i + 1)\delta\}\cdot$$

For the special case where $k_1 = 0$, we also use:

$$I'_{\mathbf{k}}(\delta) = \{(y)_i : y_1 = 0; \quad k_i\delta < y_i < (k_i + 1)\delta, \forall i = 2, .., n\}\cdot$$

Remark that $I_{\mathbf{k}}(\delta)$ is included in Δ_n when δ is less than 1.

For A_1, the question is reduced to bound the probability of the event:

$$A_{1,\mathbf{k}} = \{\tau(T_-) \in \Delta_n \cap (T_1 + I'_{\mathbf{k}}(\delta))\} \cap \{\sigma_i(T_-) > T_-\}\cdot$$

For A_2, the corresponding event is:

$$A_{2,\mathbf{k}} = \{\tau(T_-) \in \Delta_n \cap (T_1 + I_{\mathbf{k}}(\delta))\} \cap \{\tau_1(T_-) < T_0\} \cap \{\sigma_i(T_-) > T_-\}\cdot$$

Such an event is contained in:

$$A'_{2,\mathbf{k}} = \{\tau(T_-) \in \Delta_n \cap (T_1 + I_{\mathbf{k}}(\delta))\} \cap \{T_0 > T_1 + k_1\delta\} \cap \{\sigma_i(T_-) > T_-\}\cdot$$

Indeed, the first failure element arrived at time T_1 and, for each component $i \in \gamma$, the last failure $\tau_i(T_-)$ before T_- is after $T_1 + k_1\delta$. So complete restoration cannot be realized before $T_1 + k_1\delta$. If we observe the process at this time, we can use Markov property. The process is in $e(T_1 + k_1\delta)$ and we have:

$$\mathbf{P}_0(A_{2,\mathbf{k}}) \leq \mathbf{P}_0(A'_{2,\mathbf{k}}) = \mathbf{P}_0(T_0 - T_1 \geq k_1.\delta).\mathbf{P}_{e(T_1+k_1\delta)}(A_{2,\mathbf{k}-k_1})$$

where the notation $\mathbf{k} - k_1$ represents vector $(0, k_2 - k_1, ..., k_n - k_1)$. This bounding allows to control what happens when k_n is fixed.

Now we must control the probability when $k_n\delta$ is close to ∞. Let M a fixed positive real number and B_M the event:

$$B_M = \{T_- > M + T_1\}\cdot$$

We decompose A from discretisation with δ and bounding by M:

$$\mathbf{P}(A) \leq \mathbf{P}(B_M) + \sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(A_{1,\mathbf{k}}) + \sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(A'_{2,\mathbf{k}}). \tag{3.6}$$

Let us give a lemma to bound these different terms.

Lemma 3.4.

$$\begin{aligned}\sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(A'_{2,\mathbf{k}}) &\leq \sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(T_0 - T_1 \geq k_1\delta)\left(\prod_{i\leq n}\lambda_i.\delta.H_i((k_n-k_i-1)\delta) + \left(e^{\frac{\lambda\delta}{2}}-1\right)\prod_{i\leq n}\lambda_i.\delta\right)\\ \sum_{\{k:k_n\delta\leq M,k_1=0\}} \mathbf{P}_0(A_{1,\mathbf{k}}) &\leq \frac{1}{\lambda\delta}\sum_{\{k:k_n\delta\leq M,k_1=0\}}\left(\prod_{i\leq n}\lambda_i.\delta.H_i((k_n-k_i-1)\delta) + \left(e^{\frac{\lambda\delta}{2}}-1\right)\prod_{i\leq n}\lambda_i.\delta\right).\end{aligned}$$

Proof of lemma. We want an upper bound of

$$\mathbf{P}_0(A'_{2,\mathbf{k}}) = \mathbf{P}_0(T_0 - T_1 \geq k_1.\delta).\mathbf{P}_{e(T_1+k_1\delta)}(A_{2,\mathbf{k}-k_1}).$$

To prove the first formula, we have to bound $\mathbf{P}_{e(T_1+k_1\delta)}(A_{2,\mathbf{k}-k_1})$. Beginning with convention $\tau_0(T_-)=0$, we successively write that time $\tau_i(T_-)$ must appear after $\tau_{i-1}(T_-)$ but also between $k_i\delta$ and $(k_i+1)\delta$. Since this time failure is the last one for i, we also have to write that $\sigma_i(T_-)$ is after $k_n\delta$. These conditions allow to apply Lemma 2.2, and we obtain:

$$\mathbf{P}_0(A_{2,\mathbf{k}-k_1}) \leq \prod_{i\leq n}\left(\frac{(\lambda_i.\delta)^2}{2} + \lambda_i\delta H_i((k_n-k_i-1)\delta)\right).$$

This product can be bounded from using expansion in δ and the fact that $H_i \leq 1$ and we obtain:

$$\begin{aligned}\mathbf{P}_0(A_{2,\mathbf{k}-k_1}) &\leq \prod_{i\leq n}\lambda_i.\delta\left(\prod_{i\leq n}H_i((k_n-k_i-1)\delta) + \sum_{i\leq n}\frac{\lambda_i\delta}{2} + \sum_{i,j\leq n}\frac{\lambda_i\lambda_j\delta^2}{4} + ...\right)\\ &\leq \prod_{i\leq n}\lambda_i.\delta\left(\prod_{i\leq n}H_i((k_n-k_i-1)\delta) + \left(e^{\frac{\lambda\delta}{2}}-1\right)\right).\end{aligned}$$

For the case $k_1=0$, reasoning is similar but we must take into account the probability of $\{\tau_1(T_-)=T_1\}$ which is equal to λ_1/λ. That explains the term $\frac{1}{\lambda\delta}$ in the assertion.

Now, we can complete the proof of the theorem. In the decomposition:

$$\mathbf{P}(A) \leq \mathbf{P}(B_M) + \sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(A_{1,\mathbf{k}}) + \sum_{\{k:k_n\delta\leq M\}} \mathbf{P}_0(A'_{2,\mathbf{k}})$$

both last right terms can be bounded. The left one is tending to 0 when M is tending to ∞ because $E_0(T_- - T_1)$ is finite. If δ tends to 0, in the discrete decomposition:

$$\sum_{\{k:k_n\delta\leq M,k_1=0\}}\left(\frac{1}{\lambda\delta}\prod_{i\leq n}\lambda_i.\delta.H_i((k_n-k_i-1)\delta) + \frac{\left(e^{\frac{\lambda\delta}{2}}-1\right)}{\lambda\delta}\prod_{i\leq n}\lambda_i.\delta\right)$$

the second term is $O(\delta)$ and the first term converges to the next integral when we sum on all the different permutations

$$\frac{\prod_{i\leq n}\lambda_i}{\lambda}\int_{v_1>..>v_{n-1}>0}\prod_{i<n}H_i\left(v_i\right)dv_i.$$

In a similar way, in the sum:

$$\sum_{\{k:k_n\delta\leq M\}}\mathbf{P}_0(T_0-T_1\geq k_1\delta)\left(\prod_{i\leq n}\lambda_i.\delta.H_i((k_n-k_i-1)\delta)+\left(e^{\frac{\lambda\delta}{2}}-1\right)\prod_{i\leq n}\lambda_i.\delta\right)$$

the second term is less than $\left(\frac{M}{\delta}+1\right)^n\left(\prod_{i\leq n}\lambda_i.\delta\right)\left(e^{\frac{\lambda\delta}{2}}-1\right)=O(\delta)$ and the first one converges to the next integral when we sum on all the permutations:

$$\frac{\prod_{i\in\gamma}\lambda_i}{\lambda}\left(\int\mathbf{P}_0(T_0-T_1\geq u)du\right)\int_{v_1>..>v_{n-1}>0}\prod_{i<n}H_i\left(v_i\right)dv_i.$$

We recognize the mean return time $\mathbf{E_0}(T_0-T_1)$ (see Lem. 2.3) and the term α which was defined in the previous lemma. To conclude, we cumulate on all the minimal cuts.

□

This result is opening a new approach to evaluate the reliability of highly reliable systems. Using the regenerative approximations which were mentioned at the beginning, we obtain the next practical bound.

Corollary 3.5. *A system is composed with N components. For $i=1,...,N$, each component i has a constant failure-rate λ_i and a general repair time distribution (mean time r_i). Let denote G the set of the minimal cuts. The reliability at time t is bounded, in a pessimistic way, by:*

$$R(t)\geq exp(-at)$$

where a is defined by

$$a=exp\left(\sum_{i=1}^{N}\lambda_i r_i\right).\sum_{\gamma\in G}\pi_\gamma\mu_\gamma\qquad with\qquad \pi_\gamma=\prod_{i\in\gamma}\lambda_i r_i,\qquad \mu_\gamma=\sum_{i\in\gamma}\frac{1}{r_i}\cdot$$

It is a direct consequence of previous theorem because $q\leq q_G\leq e^{\lambda r}\alpha$.

3.3. **Lower bound**

In this paragraph, we give a lower bound for the failure probability. It is pleasant to see that the same parameter α is found again in the calculation of this bound. It would be useful to measure the accuracy of the approximation.

Theorem 3.6. *With the same notations as above, the probability of failure before restoration (which cumulates all minimal cuts) verifies:*

$$p_G\geq\alpha(1-\rho)$$

where ρ is the parameter:

$$\rho = \sup_{\gamma \in G} \int_{\mathbf{R}^{+\mathbf{n}}} \left(1 - e^{\lambda \max\{v_i\}}\right) \prod_{i \in \gamma} H_i(v_i) \frac{dv_i}{r_i} \cdot$$

Proof. First, fix a minimal cut γ . Using expression (3.3), we write $x_j < x_n$ in the exponential term of (3.1). So we get:

$$\alpha_\gamma - p_\gamma(s, \infty) \leq \int_{\mathbf{R}^{+\mathbf{n}}} \left(1 - e^{\lambda(x_n - x_1)}\right) dp_\gamma^+(x).$$

With the same change of variables as above, $v_n = x_1$ and $v_j = x_n - x_j$ for $j < n$, we obtain:

$$\alpha_\gamma - p_\gamma(s, \infty) \leq \rho \alpha_\gamma$$

where α_γ is the upper bound when γ is fixed. Lastly we take into account all minimal cuts. □

Now it would be interesting to sum up all these results. Taking the same notations as in the above sections, we obtain successive inequalities which can be used in concrete situations.

Corollary 3.7. *The failure probability q, the cumulated failure probability q_G on minimal cuts, the direct failure probability p_G can be ordered.*

$$\alpha(\mathbf{1} - \rho) \leq \mathbf{p}_G \leq \mathbf{q} \leq \mathbf{q}_G \leq \alpha \mathbf{e}^{\lambda r} \tag{3.7}$$

where α is given by:

$$\alpha = \sum_{\gamma \in G} \pi_\gamma \frac{\mu_\gamma}{\lambda} \qquad \text{with} \qquad \pi_\gamma = \prod_{i \in \gamma} \lambda_i r_i, \qquad \mu_\gamma = \sum_{i \in \gamma} \frac{1}{r_i} \cdot$$

That is to say that the failure probability q can be approximated with α. It is worth noting that this parameter can be computed without any information on the detail of the system architecture. Only minimal cuts must be known.

A special case engages our attention. It concerns the n-out-of-N systems (n/N systems). Failure arrives when more than n components among N are failed. This case is the source of a lot of works (see, for example [11]). It has been studied in an asymptotic approach by Solovyev and Konstant [14]. We recall their result in order to compare with our boundings.

Theorem 3.8. *Here we assume that failure rates $\lambda_i(e)$ are depending from state system e. Failure system is characterized by more than n failed components. Let denote*

$$\bar{\lambda}_i = \max_{e \in E^+} \lambda_i(e) \quad \text{and} \quad \bar{\lambda} = \max_{e \in E} \sum_i \lambda_i(e).$$

If the parameters $\lambda_i(e)$ and the functions $H_i(x)$ are varying in such a manner that:

$$\frac{\bar{\lambda}_i}{\lambda_i} \leq c < \infty \quad \text{and} \quad \frac{1}{r_i} \int_0^\infty \left(1 - e^{\bar{\lambda} x}\right) H_i(x) dx \to 0 \tag{3.8}$$

then, with the same notations as above, the next approximations are valid:

$$(q - p_G) = o(p_G) = o(\alpha).$$

So for n/N systems, with some general asymptotic point of view, it is proved that direct sequences give approximately the failure distribution. Here, studying this case thoroughly, we have built a similar result in a non asymptotic context and we have given a measure of efficiency of this approximation.

For example, let us consider the particular case of a parallel system with N identical exponential components. Let denote (λ, μ), the failure and repair common rate. From Corollary 3.5, we get an approximation of reliability:

$$R(t) \geq \exp(-N\mu e^{N\epsilon}\epsilon^N t)$$

where $\epsilon = \frac{\lambda}{\mu}$ is the stiffness coefficient of the system.

4. Ageing repairtimes

4.1. Refined bounds

Previous boundings give a method to evaluate, *a priori*, failure probability $q \simeq \alpha$. Only expectations of lifetime and repairtime are used. Efficiency of this approximation is measured by λr which is an easily known parameter and a small parameter ρ defined by:

$$\rho = \max_{\gamma \in G} \int_{\mathbf{R}^{+\mathbf{n}}} \left(1 - e^{\lambda \max\{v_i\}}\right) \prod_{i \in \gamma} H_i(v_i) \frac{dv_i}{r_i} \cdot$$

But this writing is not very expressive. From each component i and repairtime R_i we define a new random variable R'_i whose density is given by queue function H_i. With this variable, ρ can be more conveniently written:

$$\rho = \max_{\gamma \in G} \left\{ \mathbf{E} \left(1 - \exp\left(-\lambda \max_{j \in \gamma} R'_j \right) \right) \right\} \cdot$$

It is clearly bounded by:

$$\rho \leq \lambda \max_{\gamma \in G} \left(\mathbf{E} \left(\max_{j \in \gamma} R'_j \right) \right).$$

That induces us to seek a bounding of the expectation of the maximum of the random variables R'_j. In the popular case where all components have both lifetime and repairtime exponentially distributed, random variable R'_j has the same exponential distribution as R_j and the bound for ρ follows immediately:

$$\rho \leq \frac{\lambda}{\mu}$$

where $\lambda = \sum \lambda_i$ and $\mu = \min \mu_i$. This parameter can be seen as a *stiffness coefficient* of the Markov model. Consider now some systems involving ageing properties for repairtimes.

Proposition 4.1. *If each component of the system is HNBUE, then:*

$$\rho \leq \lambda \max_{\gamma \in G} \left\{ (1 + \log |\gamma|) \max_{j \in \gamma} r_j \right\}$$

where $|\gamma|$ represents here the number of elements of γ.

Proof. The expectation of the maximum of R'_i can be written

$$\mathbf{E}\left(\max_{j \in \gamma} R'_j\right) = \int_0^\infty \mathbf{P}\left(\max_{j \in \gamma} R'_j > u\right) du.$$

Fix $a > 0$ and decompose the integral:

$$\mathbf{E}\left(\max_{j\in\gamma} R'_j\right) \leq \int_0^a 1du + \int_a^\infty \mathbf{P}\left(\max_{j\in\gamma} R'_j > u\right) du \tag{4.1}$$

$$\leq a + \int_a^\infty \sum_{j\in\gamma} \mathbf{P}(R'_j > u)du. \tag{4.2}$$

Using property of Definition 2.1, we get, for any a,

$$\mathbf{E}\left(\max_{j\in\gamma} R'_j\right) \leq a + \int_a^\infty \sum_{j\in\gamma} r_j e^{-u/r_j} du.$$

So the next relation is available for any a:

$$\mathbf{E}\left(\max_{j\in\gamma} R'_j\right) \leq a + \sum_{j\in\gamma} r_j.e^{-a/r_j}.$$

The right term is a function of a which has a minimum for the solution of the equation

$$\sum_{j\in\gamma} e^{-a/r_j} = 1.$$

Thus, we get a first upper bound for ρ but not an explicit form. When all the expectations r_j are less than r the right part of the inequality is less than $a + |\gamma| r e^{-a/r}$. We can choose $a = r.\log(n)$, then the inequality of the proposition is following. □

Corollary 4.2. *With the same notations as above, assume that all repairtimes have the HNBUE property. If moreover, for any* i*,* $r_i \leq r_{max}$ *then*

$$\rho \leq \lambda(1 + log|\gamma_{max}|) r_{max}$$

where $|\gamma_{max}|$ *is the dimension of the largest minimal cut.*

For a n/N system, when the number of components are increasing, ρ is increasing as $\log(n)$ in comparison to an increasing in N. We can refine the approximation of the failure probability by direct failure.

Corollary 4.3. *Let denote* $\rho^+ = \lambda(1 + log(N)) r_{max} \leq 1$*. With the same assumptions as previously, then*

$$1 \leq \frac{q}{p_G} \leq \frac{e^{\rho^+}}{1-\rho^+} \cdot \tag{4.3}$$

The parameter ρ^+ bounds the coefficient ρ and the stiffness λr of the system. It can be evaluated without knowledge about architecture. With regard to monotone sequences, it is easy to enumerate or simulate them.

4.2. **Some applications**

In this paragraph we want to investigate two special cases. The first one would can be used in concrete situations when we know approximatively the repairtime. It concerns the uniform distribution for repairtimes. It is clear that those distributions have NBUE property and we can apply above results. But it's also possible to bound directly by writing that this random variable has a bounded support.

Corollary 4.4. *If all repairtimes have a support in* $[0,a]$*, then* ρ *is bounded with:*

$$\rho \leq \lambda a.$$

In the special case where each repairtime component has an uniform distribution with mean r_i *this formula can be expressed as:*

$$\rho \leq 2\lambda \max_{i=1..n} r_i.$$

Proof. Using expression of ρ from random variable R'_j, it's sufficient to prove that R'_j has also its support bounded by a. In the case of uniform distributions, we have $\mathbf{E}(\max_{j\in\gamma} R'_j) \leq 2\max_{i\in\gamma} r_i$. □

This corollary gives a rough general bound for ρ. We remark that the size of minimal cuts doesn't appear. It confirms that when repairtime is small in front of the lifetime, the probability of failure is very small.

With a view to compare to classical results, we propose to take again the case of n/N systems. Let assume that all N components are identical and exponentially distributed with failure and repair rates respectively equal to λ, μ. Let consider the case where the failure system arrives when more than n components are failed. By symmetry, there is exactly $\binom{N}{n}$ identical minimal cuts. If we apply inequalities (3.7) then we obtain:

$$\binom{N-1}{n-1}\epsilon^{n-1}(1-\rho^+) \leq q \leq \binom{N-1}{n-1} e^{N\epsilon}.\epsilon^{n-1}.$$

Here, ϵ is as above, a stiffness coefficient, that is: $\epsilon = \frac{\lambda}{\mu}$. This expression can be rewritten as follows:

$$\binom{N-1}{n-1}(1-N\epsilon(1+\log(n)))\epsilon^{n-1} \leq q \leq \binom{N-1}{n-1}(1+N\epsilon)\epsilon^{n-1}.$$

Out of curiosity, let us compare to exact formulas that we can found, for example, in [3]:

$$q = \left(\sum_{k=0}^{n-1} \frac{1}{\binom{N-1}{k}}\epsilon^{-k}\right)^{-1}. \tag{4.4}$$

Usually ϵ is very small for reliability applications and we can appreciate our general formula as a good approximation even in this Markov case.

In conclusion when all components of a system are reparable and highly reliable, it's natural to use approximations built from regenerative processes. In practical cases, the efficiency of this kind of approximations is very good. To justify this use, different authors have given asymptotic reasons when stiffness vanishes. Here, after modeling from Poisson Process, we have furnished exact bounds. These bounds confirm the approximation of order one with respect to stiffness. Moreover these bounds can be applied when special property about repairtime distributions is known. In this way, for concrete situations, it is possible to use more realistic distributions than exponential one (for example, uniform distributions).

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