

MULTIDIMENSIONAL VERSION OF THE RESULTS OF KOMLÓS, MAJOR AND TUSNÁDY FOR VECTORS WITH FINITE EXPONENTIAL MOMENTS

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ABSTRACT. A multidimensional version of the results of Komlós, Major and Tusnády for the Gaussian approximation of the sequence of successive sums of independent random vectors with finite exponential moments is obtained.

1. INTRODUCTION

The statement of the problem is well known. It is required to construct on a probability space a sequence of independent random vectors X_1, \dots, X_n (with given distributions) and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n (this means that Y_i has the same mean and the same covariance operator as X_i , $i = 1, \dots, n$) so that the quantity

$$\Delta(X, Y) = \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right| \quad (1.1)$$

would be so small as possible with large probability. Here $|x| = \max |x_j|$, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Sometimes we shall use another notation for summands and, by analogy with notation in Sakhanenko (1984), write $\Delta(\tilde{X}, \tilde{Y}) = \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \tilde{X}_i - \sum_{i=1}^k \tilde{Y}_i \right|$ and so on. In some sense this problem is one of the most important in probability approximations because many well-known probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences. This is related to the law of iterated logarithm, to several theorems about large deviations, to the estimates for the rate of convergence of the Prokhorov distance in the invariance principles (see Prokhorov (1956), Skorokhod (1961), Borovkov (1973)) as well as to the Strassen-type (1964) approximations (see, for example, Csörgő and Hall (1984)).

The rate of approximation in the invariance principle was studied by many authors (see, e.g., Prokhorov (1956), Skorokhod (1961), Borovkov (1973), Csörgő and Révész (1975) and the bibliography in Csörgő and Révész (1981),

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Csörgő and Hall (1984), Shao (1995)). Skorokhod (1961) worked out a powerful method known now as Skorokhod embedding. For a long time it seemed that it is impossible to propose a method which could be better. However, in 1975, Komlós, Major and Tusnády (KMT) (1975–76) worked out a new method of dyadic approximation. With the help of this method they obtained an exhaustive solution of the problem mentioned above for sequences of independent identically distributed random variables. Sakhanenko (1984) generalized and essentially improved KMT results for the case of non-identically distributed random variables.

The first attempts to extend the KMT approximations to the multidimensional case (see Berkes and Philipp (1979), Philipp (1979), Berger (1982), Einmahl (1987a,b)) had a partial success only. Comparatively recently U. Einmahl (1989) obtained multidimensional analogs of KMT results which are close to optimal. However, in his results (in general case of vectors with finite exponential moments) there is an unnecessary logarithmic factor. In this paper we improve the result of Einmahl (1989) and obtain multivariate analogs of KMT results.

NOTATION 1.1. Writing $z \in \mathbb{R}^d$ (resp. \mathbb{C}^d), we always keep in mind that $z = (z_1, \dots, z_d) = z_1 e_1 + \dots + z_d e_d$, where $z_j \in \mathbb{R}^1$ (resp. \mathbb{C}^1) and the e_j are the basis vectors. The scalar product of vectors $x, y \in \mathbb{R}^d$ (resp. \mathbb{C}^d) is denoted by $\langle x, y \rangle = x_1 \bar{y}_1 + \dots + x_d \bar{y}_d$. For $z \in \mathbb{R}^d$ (or \mathbb{C}^d) we shall use the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$ and the maximum norm $|z| = \max_{1 \leq j \leq d} |z_j|$.

For $b > 0$ we denote $L(b) = \max \{1, \log b\}$. The distribution and the corresponding covariance operator of a random vector ξ will be denoted by $\mathcal{L}(\xi)$ and $\text{cov } \xi$ (or $\text{cov } F$, if $F = \mathcal{L}(\xi)$). The letter I will be used for the identity operator. The symbols c, c_1, c_2, \dots (without arguments) will be used for absolute positive constants. The letter c can denote different constants when we do not need to fix their numerical values. The numbering of constants starts in each new section again, except Sections 6–9, where we use the common enumeration.

DEFINITION 1.2. (Zaitsev (1986)) Denote by $\mathcal{A}_d(\tau)$ a class of probability distributions depending on a parameter $\tau \geq 0$. The class $\mathcal{A}_d(\tau)$ consists of d -dimensional distributions F for which the function

$$\varphi(z) = \varphi(F, z) = \log \int_{\mathbb{R}^d} e^{\langle z, x \rangle} F\{dx\} \quad (\varphi(0) = 0) \quad (1.2)$$

is defined and analytic for $\|z\| \tau < 1$, $z \in \mathbb{C}^d$, and

$$|d_u d_v^2 \varphi(z)| \leq \|u\| \tau \langle Dv, v \rangle \quad (1.3)$$

for all $u, v \in \mathbb{R}^d$ and $\|z\| \tau < 1$, where $D = \text{cov } F$, and $d_u \varphi$ is the derivative of φ in the direction u .

In Section 2 we consider some properties of classes $\mathcal{A}_d(\tau)$. As examples of distributions from $\mathcal{A}_d(c\tau)$ one can consider the distributions concentrated on the ball $\mathbf{B}_\tau = \{x \in \mathbb{R}^d : \|x\| \leq \tau\}$, more general distributions satisfying Bernstein-type inequality conditions and infinitely divisible distributions

with spectral measures concentrated on \mathbf{B}_τ (see Zaitsev (1986), pp. 205–207).

The following Theorem 1.3 is the main result of the paper. It was announced in Zaitsev (1995a, 1996b, 1997) and, in somewhat weakened form, in Zaitsev (1995b).

THEOREM 1.3. *Suppose that $\tau \geq 1$, $\alpha > 0$ and ξ_1, \dots, ξ_n are random vectors with distributions $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E}\xi_k = 0$, $\text{cov}\xi_k = I$, $k = 1, \dots, n$. Then one can construct on a probability space a sequence of independent random vectors X_1, \dots, X_n and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, and*

$$\mathbb{E} \exp\left(\frac{c_1(\alpha)\Delta(X, Y)}{\tau d^3 L(d)}\right) \leq \exp(c_2(\alpha)d^{9/4+\alpha}L(n/\tau^2)), \quad (1.4)$$

where $c_1(\alpha)$, $c_2(\alpha)$ are positive quantities depending only on α .

In a particular case, when $d = 1$ and all summands have a common variance, Theorem 1.3 is equivalent to the main result of Sakhanenko (1984). One should take into account that, using the Hölder inequality, we can transfer the factors under exponential sign in the right-hand side of (1.4) to the denominator of the fraction in the left-hand side of (1.4). Naturally, this makes the result weaker. Einmahl (1989) considered multidimensional vectors with identical covariance operators, satisfying multidimensional analogs of Sakhanenko's conditions.

One can easily verify that if a vector ξ has finite exponential moments $\mathbb{E}e^{\langle h, \xi \rangle}$, for $h \in V$, where $V \subset \mathbb{R}^d$ is some neighborhood of zero, then $F = \mathcal{L}(\xi) \in \mathcal{A}_d(c(F))$. Therefore, Theorem 1.3 can be considered as a generalization and refinement of the main result of KMT (1975–76). In particular, from Theorem 1.3 one can easily derive the following result, obtained by KMT (1975–76) in the one-dimensional case.

COROLLARY 1.4. *Suppose that a random vector ξ has finite exponential moments $\mathbb{E}e^{\langle h, \xi \rangle}$, for $h \in V$, where $V \subset \mathbb{R}^d$ is some neighborhood of zero. Then one can construct on a probability space a sequence of independent random vectors X_1, X_2, \dots and a corresponding sequence of independent Gaussian random vectors Y_1, Y_2, \dots so that $\mathcal{L}(X_k) = \mathcal{L}(\xi)$, $k = 1, 2, \dots$, and*

$$\sum_{j=1}^n X_j - \sum_{j=1}^n Y_j = O(\log n) \quad a.s. \quad (1.5)$$

An analog of this result was obtained by Einmahl (1989) under additional smoothness-type restrictions on the distributions $\mathcal{L}(\xi)$. Einmahl (1989, Theorem 10), has also proved an analog of Theorem 1.3 but only for sufficiently smooth distributions $\mathcal{L}(\xi)$. As it is noted in KMT (1975–76), from the results of Bártfai (1966) it follows that the accuracy of approximation in (1.5) is the best possible.

The following Theorem 1.5 (by means of its consequence, Theorem 9.1) will be used in the proof of Theorems 1.3 and 1.6. However, it is of independent interest because sometimes it can give a more exact dependence of constants on the characteristics of distributions of summands.

THEOREM 1.5. *Suppose that $\tau \geq 1$, $\lambda \geq 1$, $\alpha > 0$ and $\tilde{\xi}_1, \dots, \tilde{\xi}_n$ are random vectors with distributions $\mathcal{L}(\tilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E} \tilde{\xi}_k = 0$, $\text{cov} \tilde{\xi}_k = I$, $k = 1, \dots, n$. Assume that there exist random vectors $\tilde{\eta}_k$ which have the same moments of the first three orders as the vectors $\tilde{\xi}_k$, $\mathcal{L}(\tilde{\eta}_k) \in \mathcal{A}_d(\tau)$ and $\mathbb{P} \{ |\tilde{\eta}_k| \leq \lambda \} = 1$, $k = 1, \dots, n$. Then one can construct on a probability space a sequence of independent random vectors $\tilde{X}_1, \dots, \tilde{X}_n$ and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n so that $\mathcal{L}(\tilde{\xi}_k) = \mathcal{L}(\tilde{X}_k)$, $k = 1, \dots, n$, and*

$$\mathbb{E} \exp\left(\frac{c_3(\alpha)\Delta(\tilde{X}, Y)}{\lambda d^3 \tau^2}\right) \leq \exp(c_4(\alpha)(\lambda d^{3/2} \tau)^{3/2+\alpha} L(n)),$$

where $c_3(\alpha)$, $c_4(\alpha)$ are positive quantities depending only on α .

The following Theorem 1.6 shows that if all moments of the third order of the vectors ξ_1, \dots, ξ_n in Theorem 1.3 are equal to zero, then the dependence of constants on the dimension can be slightly sharpened.

THEOREM 1.6. *Suppose that $\tau \geq 1$, $\alpha > 0$ and ξ_1, \dots, ξ_n are random vectors with distributions $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E} \xi_k = 0$, $\text{cov} \xi_k = I$, $k = 1, \dots, n$. Assume that for all $u, v, w \in \mathbb{R}^d$ we have*

$$\mathbb{E} \langle \xi_k, u \rangle \langle \xi_k, v \rangle \langle \xi_k, w \rangle = 0, \quad k = 1, \dots, n. \quad (1.6)$$

Then one can construct on a probability space a sequence of independent random vectors X_1, \dots, X_n and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, and

$$\mathbb{E} \exp\left(\frac{c_5(\alpha)\Delta(X, Y)}{d^3 \tau}\right) \leq \exp(c_6(\alpha)d^{9/4+\alpha} L(n/\tau^2)), \quad (1.7)$$

where $c_5(\alpha)$, $c_6(\alpha)$ are positive quantities depending only on α .

REMARK 1.7. It is evident that the condition (1.6) is automatically fulfilled if the vectors ξ_1, \dots, ξ_n have symmetric distributions.

In Theorems 1.3, 1.5 and 1.6 the random vectors ξ_1, \dots, ξ_n and $\tilde{\xi}_1, \dots, \tilde{\xi}_n$ are, generally speaking, non-identically distributed. However, they have the same covariance operator I . Thus, the problem of obtaining an adequate multidimensional generalization of the main result of Sakhnenko (1984) remains open. By the author's opinion, the proof of Theorem 1.3 can be transformed in the proof of an analogous result for vectors with non-identical covariance operators. The conditions $\text{cov} \xi_j = I$, $\tau \geq 1$ can be apparently changed by $B_j^2 \leq \tau^2$, $B_j^2 \leq \lambda \sigma_j^2$, where B_j^2, σ_j^2 are respectively the maximal and the minimal eigenvalues of $\text{cov} \xi_j$, $j = 1, \dots, n$. The constants will depend, naturally, on λ , and n should be changed in the right-hand side of (1.4) by the maximal eigenvalue of the covariance operator of the last sum $\xi_1 + \dots + \xi_n$. The author hopes to consider this situation in a separate paper. Moreover, it is well known that the results similar to Theorems 1.3, 1.5 and 1.6 imply many useful consequences. For example, one can derive estimates in the invariance principle in the case when the summands X_j satisfy less restrictive moment conditions (see, e.g., Shao (1995) and the bibliography in Shao (1995)). The author is going to devote to the consequences of Theorems 1.3, 1.5 and 1.6 also a separate publication.

In Theorems 1.3, 1.5 and 1.6 we consider the case $\tau \geq 1$. Obviously, the assertions of these theorems become stronger for small τ . An analog of Theorem 1.3 in a particular case when the summands have smooth distributions and τ can be arbitrarily small is obtained in Götze and Zaitsev (1997). Note that for small τ the distributions of summands are close to Gaussian ones (see Zaitsev (1986)).

The proof of Theorems 1.3, 1.5 and 1.6 consists of several steps, many of which are close to corresponding steps from the proofs of the main results of KMT (1975–76), Sakhanenko (1984) and Einmahl (1989). In Section 2 we introduce the necessary notation and formulate auxiliary results. In particular (see Definition 2.2), we define the classes $\mathcal{R}_d(\tau, \varepsilon) \subset \mathcal{A}_d(\tau) \times \mathcal{A}_d(\tau)$ of pairs of distributions with close cumulants of the second and third orders and discuss the properties of classes $\mathcal{A}_d(\tau)$ and $\mathcal{R}_d(\tau, 0)$. We define as well the classes $\mathcal{A}_d^*(\tau, \delta, \rho) \subset \mathcal{A}_d(\tau)$ of distributions satisfying some smoothness conditions which guarantee the existence of smooth densities (see Definition 2.12).

The main tool for the proof of Theorems 1.3, 1.5 and 1.6 is the estimates for the closeness of quantiles of one-dimensional conditional distributions contained in Lemmas 3.1 and 4.1. For the proof of Lemmas 3.1 and 4.1 we use Lemmas 2.14 and 2.15 which are completely proved in Zaitsev (1996b), Lemmas 6.1 and 7.1. In Zaitsev (1996b) one can also find a sketch of the proof of Lemmas 3.1 and 4.1. We give the complete proof of these lemmas in Sections 3 and 4. In Lemma 2.14 we give a non-uniform estimate for large deviations in the local CLT for conditional distributions of the last coordinates of vectors with smooth distributions from $\mathcal{A}_d(\tau)$ under condition that the first $d - 1$ coordinates are fixed (see (2.17)). Analogously, Lemma 2.15 contains a non-uniform estimate for the closeness of conditional densities of smooth distributions from $\mathcal{R}_d(\tau, 0)$ (see (2.23)). Moreover, Lemma 2.15 states that if $w > 0$ is separated from zero and from infinity, then the derivative of the logarithm of conditional density at point w is negative. An estimate of this derivative is also presented (see (2.24)). This estimate is essentially used in the proof of Lemma 4.1.

In Lemma 3.1 we give the estimates for the closeness of quantiles in the CLT for conditional distributions of the last coordinates of vectors with smooth distributions from $\mathcal{A}_d(\tau)$ under condition that the first $d - 1$ coordinates are fixed (see (3.1)). Analogously, in Lemma 4.1 we provide an estimate for the closeness of quantiles of conditional distribution functions of smooth distributions from $\mathcal{R}_d(\tau, 0)$ (see (4.1)). The additional information about the closeness of cumulants leads to the better orders of estimates in Lemma 4.1 in comparison with Lemma 3.1.

In Section 5 we describe the KMT (1975–76) scheme of dyadic approximation. This scheme was essentially used by Sakhanenko (1984) and Einmahl (1989). Note, however, that the formulations and the proofs of KMT (1975–76, inequality (2.5)), Sakhanenko (1984, Lemma 1, p. 34) and Einmahl (1989, inequality (4.11)) are close but slightly different. Our approach practically coincides with Einmahl's one. As in KMT (1975–76), Sakhanenko (1984) and Einmahl (1989), we suppose that some independent random vectors with smooth distributions are already constructed and define in-

dependent random vectors with given distributions by successive constructing sums of blocks of summands, containing $2^N, 2^{N-1}, 2^{N-2}, \dots, 4, 2, 1$ summands. For this we apply the so-called Rosenblatt (1952) quantile transformation which was already used by Einmahl (1989). This transformation is a natural generalization of the usual one-dimensional quantile transformation which have been applied by KMT (1975–76) and Sakhanenko (1984).

The scheme of the proof of Theorem 1.3 is very close to that of the main result of Sakhanenko (1984). At first we suppose that the Gaussian vectors Y_1, \dots, Y_n , $n = 2^N$, are already constructed and construct the independent vectors which are bounded with probability one, have sufficiently smooth distributions and the same moments of the first, second and third orders as the needed independent random vectors X_1, \dots, X_n . Then we construct the vectors X_1, \dots, X_n in several steps. After each step the number of X_j which are not constructed becomes smaller in 2^p times, where p is a suitably chosen positive integer. In each step we begin with already constructed vectors which are bounded with probability one and have sufficiently smooth distributions and the needed moments up to the third order. Then we construct the vectors such that, in each block of 2^p summands, only the first vector has the initial bounded smooth distribution. The rest $2^p - 1$ vectors have the needed distributions $\mathcal{L}(\xi_j)$. These $2^p - 1$ vectors from each block will be chosen as X_j and will be not involved in the next steps of the procedure. The coincidence of third moments will allow us to use more precise estimates of the closeness of quantiles of conditional distributions obtained in Section 4.

In Section 6 we realize the first step of the procedure just described. We estimate the rate of approximation in the case when we construct on a probability space the independent Gaussian vectors Y_j and the independent vectors X_j with smooth distributions and the needed moments up to the third order. The main result of Section 6 is formulated in Theorem 6.4.

Sections 7 and 8 are devoted to the estimation of the rate of approximation in the case when we begin with the vectors $\tilde{Y}_1, \dots, \tilde{Y}_n$, $n = 2^N$, having smooth distribution with bounded supports and with the same moments up to the third order as the needed vectors $\tilde{X}_1, \dots, \tilde{X}_n$ and construct the vectors $\tilde{X}_1, \dots, \tilde{X}_n$ in several steps, diminishing after each step the number of \tilde{X}_j which are not constructed in 2^p times. The scheme of the proof is a modified version of that from Sakhanenko (1984). We use the induction with respect to the number of steps of the procedure and prove that this procedure provides the good estimate for

$$\delta_j(N) = \left| \sum_{k=1}^j \tilde{Y}_k - \sum_{k=1}^j \tilde{X}_k \right| \quad (1.8)$$

(see (8.28), (8.94), (8.97)). It is very important that in (1.8) there is no supremum over $j = 1, \dots, 2^N$. This supremum is taken in the last moment only (see (8.98)). At the end of Section 8 we give the proof of Theorem 1.5.

Theorems 1.3 and 1.6 are proved in Section 9. At first we prove an auxiliary result (Theorem 9.1) which is in fact a simple consequence of Theorem 1.5. However, its formulation is more convenient than that of Theorem 1.5 when we investigate the character of dependence of constants of

the parameters. This is taken into account in the proofs of Theorems 1.3 and 1.6. For the proof of Theorem 1.3 we need also Lemma 9.2 which shows that there exist absolute positive constants c_7, c_8 such that for any distribution $F \in \mathcal{A}_d(1)$ with mean zero and $\text{cov } F = I$ there exist a distribution $G \in \mathcal{A}_d(c_7)$ with the same moments up to the third order and such that $G\{\{x : |x| \leq c_8 L(d)\}\} = 1$. It can be verified that this statement is valid without $L(d)$ if all the moments of the third order of the distribution F are equal to zero. Therefore, the factor $L(d)$ from (1.4) is absent in the denominator of the fraction in the left-hand side of the inequality (1.7) of Theorem 1.6.

2. NOTATION AND AUXILIARY RESULTS

NOTATION 2.1. Let \mathcal{F}_d be the set of all d -dimensional probability distributions defined on the σ -field of Borel subsets of the Euclidean space \mathbb{R}^d . Let \mathcal{G}_d be the collection of all Gaussian distributions from \mathcal{F}_d . Below θ symbolizes different quantities not exceeding one in absolute value, E_a is the distribution concentrated at a point a . By $\widehat{F}(t)$, $t \in \mathbb{R}^d$, we denote the characteristic function of a distribution $F \in \mathcal{F}_d$. The product of measures is understood as their convolution: $FG = F * G$. By $\mathcal{N}(0, I) \in \mathcal{G}_d$ we denote the Gaussian distribution with mean zero and covariance operator I . The notation $\Phi(F)$ will be used for the Gaussian distribution whose mean and covariance operator are the same as those of a distribution $F \in \mathcal{F}_d$. In one-dimensional case we denote by $\Phi_\sigma(\cdot)$ the distribution function of the normal law with mean zero and variance σ^2 and by $\varphi_\sigma(\cdot)$ the corresponding density. We shall identify the covariance operators with corresponding covariance matrices.

Let $\tau \geq 0$, $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $\|h\| \tau < 1$, $h \in \mathbb{R}^d$. Then the conjugate distribution $\overline{F} = \overline{F}(h)$ is defined by

$$\overline{F}\{dx\} = (\mathbb{E} e^{\langle h, \xi \rangle})^{-1} e^{\langle h, x \rangle} F\{dx\} \tag{2.1}$$

(sometimes it is called the Cramér transform). Denote by $\overline{\xi}(h)$ a random vector with $\mathcal{L}(\overline{\xi}(h)) = \overline{F}(h)$. It is clear that $\overline{F}(0) = F$ and

$$\text{if } U_1, \dots, U_n \in \mathcal{A}_d(\tau), \quad \|h\| \tau < 1, \quad U = \prod_{j=1}^n U_j, \quad \text{then } \overline{U}(h) = \prod_{j=1}^n \overline{U}_j(h). \tag{2.2}$$

DEFINITION 2.2. For $\tau, \varepsilon \geq 0$ denote by $\mathcal{R}_d(\tau, \varepsilon)$ the collection of pairs of d -dimensional probability distributions (F_1, F_2) such that $F_k \in \mathcal{A}_d(\tau)$, $k = 1, 2$, and

$$\begin{aligned} |d_u d_v^2 g_1(0) - d_u d_v^2 g_2(0)| &\leq \varepsilon \|u\| \tau \langle Dv, v \rangle, \\ |d_v^2 g_1(0) - d_v^2 g_2(0)| &\leq \varepsilon^2 \langle Dv, v \rangle, \end{aligned} \tag{2.3}$$

for all $u, v \in \mathbb{R}^d$, where $D = D_1 + D_2$, $g_k(z) = \varphi(F_k, z)$, $D_k = \text{cov } F_k$.

REMARK 2.3. If $\varepsilon = 0$, then the relation $(F_1, F_2) \in \mathcal{R}_d(\tau, \varepsilon)$ means that the distributions $F_1, F_2 \in \mathcal{A}_d(\tau)$ have identical cumulants of second and third orders.

Classes $\mathcal{A}_d(\tau)$ and $\mathcal{R}_d(\tau, \varepsilon)$ have properties which are very convenient for applying. Some of them were considered in Zaitsev (1986, 1988, 1996a).

It is evident that if $\tau_1 < \tau_2$, then $\mathcal{A}_d(\tau_1) \subset \mathcal{A}_d(\tau_2)$. Moreover, it is easy to see that, for fixed τ , the class $\mathcal{A}_d(\tau)$ is closed with respect to convolution: if $F_1, F_2, \dots, F_n \in \mathcal{A}_d(\tau)$, then $F_1 F_2 \cdots F_n \in \mathcal{A}_d(\tau)$. The class $\mathcal{A}_d(0)$ coincides with the class \mathcal{G}_d of all d -dimensional Gaussian distributions. The following inequality was proved in Zaitsev (1986) and can be considered as an estimate of stability of this characterization: if $F \in \mathcal{A}_d(\tau)$, $\tau > 0$, then $\pi(F, \Phi(F)) \leq c d^2 \tau L(\tau^{-1})$, where $\pi(\cdot, \cdot)$ is the Prokhorov distance. Other useful properties of classes $\mathcal{A}_d(\tau)$ and $\mathcal{R}_d(\tau, 0)$ are contained in the following Lemmas 2.4, 2.5, 2.7, 2.8 and 2.11.

LEMMA 2.4. *Suppose that $\tau \geq 0$, $F \in \mathcal{A}_d(\tau)$, $h \in \mathbb{R}^d$, $\|h\| \tau < 1$, $F = \mathcal{L}(\xi)$, $D = \text{cov } \xi$, $D(h) = \text{cov } \bar{\xi}(h)$. $\mathbb{E} \xi = 0$. Then*

a) *for all $u \in \mathbb{R}^d$ the following relations are valid:*

$$\langle D(h) u, u \rangle = \langle Du, u \rangle (1 + \theta \|h\| \tau), \quad (2.4)$$

$$\log \mathbb{E} e^{\langle h, \xi \rangle} = \frac{1}{2} \langle Dh, h \rangle \left(1 + \frac{1}{3} \theta \|h\| \tau \right), \quad (2.5)$$

$$\log \mathbb{E} e^{i \langle h, \xi \rangle} = -\frac{1}{2} \langle Dh, h \rangle \left(1 + \frac{1}{3} \theta \|h\| \tau \right); \quad (2.6)$$

b) *if $\|h\| \tau \leq 1/2$, then $\bar{F}(h) \in \mathcal{A}_d(2\tau)$.*

Lemma 2.4 is contained in Zaitsev (1986, Lemma 2.1).

LEMMA 2.5. *Suppose that $\tau \geq 0$, $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $y \in \mathbb{R}^m$, $\alpha \in \mathbb{R}^1$, and $A : \mathbb{C}^d \rightarrow \mathbb{C}^m$ is a linear operator such that $A(\mathbb{R}^d) \subset \mathbb{R}^m$. Let $\tilde{\xi} \in \mathbb{R}^k$ be the vector consisting of a subset of coordinates of the vector ξ . Then*

$$\mathcal{L}(A\xi + y) \in \mathcal{A}_m(\|A\| \tau), \quad \text{where } \|A\| = \sup_{x \in \mathbb{R}^d, \|x\| \leq 1} \|Ax\|,$$

$$\mathcal{L}(\alpha \xi) \in \mathcal{A}_d(|\alpha| \tau), \quad \mathcal{L}(\tilde{\xi}) \in \mathcal{A}_k(\tau).$$

Proof. Put

$$\rho(z) = \log \mathbb{E} e^{\langle z, A\xi + y \rangle} = \log \mathbb{E} e^{\langle A^* z, \xi \rangle} + \langle z, y \rangle, \quad (2.7)$$

where $A^* : \mathbb{C}^m \rightarrow \mathbb{C}^d$ is the adjoint operator for A , $z \in \mathbb{C}^m$. Let $D = \text{cov } \xi$, $M = \text{cov } A\xi$. By virtue of (1.2), (1.3), (2.7), for all $u, v \in \mathbb{R}^m$, $z \in \mathbb{C}^m$, $\|A^* z\| \tau < 1$, the following relations are valid:

$$\begin{aligned} |d_u d_v^2 \rho(z)| &= |d_{A^* u} d_{A^* v}^2 \varphi(F, w)|_{w=A^* z} \\ &\leq \|A^* u\| \tau \langle D A^* v, A^* v \rangle \leq \|A^*\| \|u\| \tau \mathbb{E} \langle \xi - \mathbb{E} \xi, A^* v \rangle^2 \\ &= \|A\| \|u\| \tau \mathbb{E} \langle A\xi - \mathbb{E} A\xi, v \rangle^2 = \|u\| \cdot \|A\| \tau \cdot \langle M v, v \rangle. \end{aligned} \quad (2.8)$$

If $\|z\| \cdot \|A\| \tau < 1$ then $\|A^* z\| \tau \leq \|A^*\| \|z\| \tau = \|z\| \cdot \|A\| \tau < 1$ and the inequality (2.8) is also valid. From (2.7), (2.8) and from definition of classes $\mathcal{A}_d(\tau)$ it follows that $\mathcal{L}(A\xi + y) \in \mathcal{A}_m(\|A\| \tau)$. The relations

$\mathcal{L}(\alpha\xi) \in \mathcal{A}_d(|\alpha|\tau)$, $\mathcal{L}(\tilde{\xi}) \in \mathcal{A}_k(\tau)$ follow directly from the statement just proved. \square

COROLLARY 2.6. *Let the conditions of Lemma 2.5 be satisfied, $\alpha \in \mathbb{R}^1$, $(F_1, F_2) \in \mathcal{R}_d(\tau, 0)$, $F_j = \mathcal{L}(\xi_j)$, $y_j \in \mathbb{R}^m$, $j = 1, 2$, and let $\xi_1, \xi_2 \in \mathbb{R}^k$ be the vectors consisting of identical subsets of coordinates of the vectors ξ_1, ξ_2 respectively. Then*

$$\begin{aligned} & (\mathcal{L}(A\xi_1 + y_1), \mathcal{L}(A\xi_2 + y_2)) \in \mathcal{R}_m(\|A\|\tau, 0), \\ & (\mathcal{L}(\alpha\xi_1), \mathcal{L}(\alpha\xi_2)) \in \mathcal{R}_d(|\alpha|\tau, 0), \quad (\mathcal{L}(\tilde{\xi}_1), \mathcal{L}(\tilde{\xi}_2)) \in \mathcal{R}_k(\tau, 0). \end{aligned}$$

Proof. It is sufficient to use Lemma 2.5, the definition of classes $\mathcal{R}_d(\tau, \varepsilon)$ and the corresponding analogs of the relations (2.7), (2.8), taking into account Remark 2.3. \square

LEMMA 2.7. *Suppose that $\tau \geq 0$, $F_k = \mathcal{L}(\xi^{(k)}) \in \mathcal{A}_{d_k}(\tau)$, and the vectors $\xi^{(k)}$, $k = 1, 2$, are independent. Let $\xi \in \mathbb{R}^{d_1+d_2}$ be the vector with the first d_1 coordinates coinciding with those of $\xi^{(1)}$ and with the last d_2 coordinates coinciding with those of $\xi^{(2)}$. Then $F = \mathcal{L}(\xi) \in \mathcal{A}_{d_1+d_2}(\tau)$.*

Proof. Let $D = \text{cov } \xi$, $D^{(k)} = \text{cov } \xi^{(k)}$, $k = 1, 2$. Using the natural analogous notation, we see that for all $u, v \in \mathbb{R}^{d_1+d_2}$ and $z \in \mathbb{C}^{d_1+d_2}$, $\|z\| \tau < 1$,

$$\begin{aligned} |d_u d_v^2 \varphi(F, z)| &= |d_u d_v^2 (\varphi(F_1, z^{(1)}) + \varphi(F_2, z^{(2)}))| \\ &= |d_{u^{(1)}} d_{v^{(1)}}^2 \varphi(F_1, z^{(1)}) + d_{u^{(2)}} d_{v^{(2)}}^2 \varphi(F_2, z^{(2)})| \\ &\leq \|u^{(1)}\| \tau \langle D^{(1)} v^{(1)}, v^{(1)} \rangle + \|u^{(2)}\| \tau \langle D^{(2)} v^{(2)}, v^{(2)} \rangle \\ &\leq \|u\| \tau \langle Dv, v \rangle. \end{aligned}$$

It remains to use the definition of classes $\mathcal{A}_d(\tau)$. \square

LEMMA 2.8. *Suppose that $\tau \geq 0$, $(\mathcal{L}(\xi_1^{(k)}), \mathcal{L}(\xi_2^{(k)})) \in \mathcal{R}_{d_k}(\tau, 0)$, $k = 1, 2$, and the pairs of vectors $\xi_j^{(1)}, \xi_j^{(2)}$, $j = 1, 2$, are the pairs of independent vectors. Let $\xi_j \in \mathbb{R}^{d_1+d_2}$, $j = 1, 2$, be the vectors with the first d_1 coordinates coinciding with those of $\xi_j^{(1)}$ and with the last d_2 coordinates coinciding with those of $\xi_j^{(2)}$. Then $(\mathcal{L}(\xi_1), \mathcal{L}(\xi_2)) \in \mathcal{R}_{d_1+d_2}(\tau, 0)$.*

Proof. It suffices to apply Lemma 2.7, taking into account Remark 2.3. \square

REMARK 2.9. Naturally, Lemmas 2.7 and 2.8 can be evidently extended on the case of any finite number of independent vectors.

REMARK 2.10. Lemma 2.7 can be easily derived from Lemma 2.5, using the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution.

LEMMA 2.11. *Let $\tau \geq 0$ and X_1, \dots, X_n be independent random vectors with $\mathcal{L}(X_i) \in \mathcal{A}_d(\tau)$, $\mathbb{E} X_i = 0$, $S_i = X_1 + \dots + X_i$, for $i = 1, \dots, n$. Let $h, t \in \mathbb{R}^1$, $0 \leq h\tau < 1$, $0 \leq t\tau < 1/2$, $\xi = \max_{1 \leq i \leq n} |S_i|$. Denote by B^2 the maximal eigenvalue of $\text{cov } S_n$. Then*

$$\mathbb{E} e^{h|S_n|} \leq 2de^{h^2B^2}, \quad \mathbb{E} e^{t\xi} \leq 3de^{4t^2B^2}$$

and

$$\mathbb{P} \{ \xi \geq x \} \leq 2d \max \{ \exp(-x^2/4B^2), \exp(-x/4\tau) \}, \quad x \geq 0. \quad (2.9)$$

Proof. By Lemma 2.5, the distributions of the coordinates of the vectors X_i belong to the class $\mathcal{A}_1(\tau)$ and it suffices to prove the statement of the lemma for $d = 1$ (recall that $|\cdot|$ means the maximum norm). Using the completeness of the classes $\mathcal{A}_1(\tau)$ with respect to convolution, we see that by Lemma 2.4, inequality (2.5),

$$\mathbb{E} e^{h|S_n|} \leq \mathbb{E} e^{hS_n} + \mathbb{E} e^{-hS_n} \leq 2 \exp\left(\frac{1}{2} h^2 B^2 \left(1 + \frac{1}{3} h\tau\right)\right) \leq 2e^{h^2 B^2}. \quad (2.10)$$

Note that the random sequence $\exp(h|S_i|)$ (indexed by i) is a positive submartingale. Therefore, by Doob's inequality (see Doob (1953), p. 314) and by (2.10),

$$\mathbb{P}\{\xi \geq x\} = \mathbb{P}\{e^{h\xi} \geq e^{hx}\} \leq \mathbb{E} \exp(h|S_n| - hx) \leq 2 \exp(h^2 B^2 - hx). \quad (2.11)$$

We then deduce (2.9) from (2.11) via the classical Bernstein–Cramér–Chernoff calculation. Integrating by parts and applying (2.11) with $h = 2t$, we get

$$\mathbb{E} e^{t\xi} = 1 + \int_0^\infty t e^{tx} \mathbb{P}\{\xi \geq x\} dx \leq 3e^{4t^2 B^2}.$$

□

DEFINITION 2.12. Let $\tau \geq 0$, $\delta > 0$, $\rho > 0$. Denote by $\mathcal{A}_d^*(\tau, \delta, \rho)$ the class of distributions $F \in \mathcal{A}_d(\tau)$ such that for $h \in \mathbb{R}^d$, $\|h\| \tau < 1$, and for all $v \in \mathbb{R}^d$ the following inequalities hold:

$$(2\pi)^{-d} \int_{\rho\|t\|\tau d \geq 1} |\widehat{F}_h(t)| dt \leq \frac{\tau^2 d^2}{\sigma^2 (2\pi)^{d/2} (\det D)^{1/2}}, \quad (2.12)$$

$$(2\pi)^{-d} \int_{\rho\|t\|\tau d \geq 1} |\langle t, v \rangle \widehat{F}_h(t)| dt \leq \frac{\langle D^{-1}v, v \rangle^{1/2}}{\delta (2\pi)^{d/2} (\det D)^{1/2}}, \quad (2.13)$$

where $F_h = \overline{F}(h)$, $D = \text{cov } F$ and $\sigma^2 > 0$ is the minimal eigenvalue of D .

According to (2.1), we have (if $F = \mathcal{L}(\xi)$)

$$\widehat{F}_h(t) = \mathbb{E} e^{\langle it, \overline{\xi}(h) \rangle} = (\mathbb{E} e^{\langle h, \xi \rangle})^{-1} \mathbb{E} e^{\langle h+it, \xi \rangle}. \quad (2.14)$$

Therefore, the conditions (2.12), (2.13) play in this paper the role which is similar to that of Sakhanenko (1984, inequality (49), p. 9) or Einmahl (1989, inequality (1.5)). Note, however, that the condition (2.13) is needed for estimating the derivatives of densities (see (2.24)). Such estimates are necessary for obtaining more precise bounds for the closeness of conditional distributions when the compared distribution have the coinciding third moments.

NOTATION 2.13. Below for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $d \geq 2$, we shall denote by x' the vector $x' = (x_1, \dots, x_{d-1})$ consisting of the first $d - 1$ coordinates of x . Analogously, we shall use a prime to denote the matrix D' of size

$(d - 1) \times (d - 1)$ consisting of the first $d - 1$ rows and first $d - 1$ columns of a matrix D of size $d \times d$. If $F = \mathcal{L}(\xi)$ and $\xi \in \mathbb{R}^d$ we shall write $F' = \mathcal{L}(\xi')$.

The following Lemmas 2.14 and 2.15 are completely proved in Zaitsev (1996b, Lemmas 6.1 and 7.1).

LEMMA 2.14. *Suppose that $\tau \geq 0$,*

$$F = \mathcal{L}(\xi) \in \mathcal{A}_d^*(\tau, 4, 4) \text{ and, if } d \geq 2, \quad F' = \mathcal{L}(\xi') \in \mathcal{A}_{d-1}^*(\tau, 4, 4). \quad (2.15)$$

Assume that $\mathbb{E}\xi = 0$ and the last coordinate ξ_d of the vector ξ is not correlated with the previous ones. Let $\sigma^2 = \langle De_d, e_d \rangle = \mathbb{E}\xi_d^2 > 0$ be the minimal eigenvalue of $D = \text{cov} F$. Denote by $p(x)$, $x \in \mathbb{R}^d$, the probability density of distribution F . Let $p'(x')$, $x' \in \mathbb{R}^{d-1}$, $d \geq 2$, be the density of the vector ξ' . Denote by $p^{(d)}(x_d|x')$ the conditional density of the d^{th} coordinate ξ_d of the vector $\xi \in \mathbb{R}^d$ for a fixed value of $\xi' = x'$ (for $d = 1$ we define $p^{(1)}(x_1|x') = p(x)$). Denote by G the corresponding (depending on x') one-dimensional conditional distribution.

Then there exist absolute positive constants c_1, \dots, c_6 such that the following statements are true for $\tau d^{3/2}/\sigma \leq c_1$, $|(D')^{-1/2} x'| \leq c_2 \sigma/d^{3/2} \tau$.

For $d \geq 2$ there exists a parameter $h' = h'(x') \in \mathbb{R}^{d-1}$ which gives the solution of the equation $\mathbb{E}\bar{\xi}'(h') = x'$. Define the parameter $h \in \mathbb{R}^d$ which has the first $d - 1$ coordinates coinciding with corresponding coordinates of h' and the last d^{th} coordinate $h_d = 0$. If $d = 1$, we take $h = 0$. Denote $y_0 = \mathbb{E}\langle \bar{\xi}(h), e_d \rangle$. Then

$$|y_0| \leq 2.88\tau \|(D')^{-1/2} x'\|^2 \leq 2.88\tau \frac{\|x'\|^2}{\sigma^2}, \quad (2.16)$$

and, if $|w| \leq \frac{c_3 \sigma^2}{d\tau}$, then for the density $v(w)$ of the distribution GE_{-y_0} the following representation is valid:

$$v(w) = \varphi_\sigma(w) \exp\left(\theta c_4 \frac{\tau}{\sigma} \left(d^{3/2} + d \|(D')^{-1/2} x'\| \left(1 + \frac{w^2}{\sigma^2}\right) + \frac{|w|^3}{\sigma^3}\right)\right), \quad (2.17)$$

and

$$\frac{1}{2\sqrt{2\pi}\sigma} \exp\left(-\frac{w^2}{\sigma^2}\right) \leq v(w) \leq \frac{2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{w^2}{4\sigma^2}\right). \quad (2.18)$$

Moreover, for all $w \in \mathbb{R}^1$,

$$v(w) \leq \frac{c_5}{\sigma} \exp\left(-\min\left\{\frac{w^2}{4\sigma^2}, \frac{c_6|w|}{d\tau}\right\}\right). \quad (2.19)$$

Note that for $d = 1$ we use in the formulations the natural agreement $|(D')^{-1/2} x'| = \|(D')^{-1/2} x'\| = |x'| = \|x'\| = 0$.

LEMMA 2.15. *Suppose that $\tau \geq 0$, $(F_1, F_2) \in \mathcal{R}_d(\tau, 0)$. Let, for $k = 1, 2$,*

$$F_k = \mathcal{L}(\xi_k) \in \mathcal{A}_d^*(\tau, 4, 4) \text{ and, if } d \geq 2, \quad F'_k = \mathcal{L}(\xi'_k) \in \mathcal{A}_{d-1}^*(\tau, 4, 4). \quad (2.20)$$

Assume that $\mathbb{E}\xi_1 = \mathbb{E}\xi_2 = 0$ and the last coordinates $\xi_{k,d}$ of the vectors ξ_k are not correlated with the previous ones. Let $\sigma^2 = \langle De_d, e_d \rangle = \mathbb{E}\xi_{k,d}^2 > 0$ be the minimal eigenvalue of the common covariance operator $D = D_k = \text{cov} F_k$. Let $p_k(z)$, $z \in \mathbb{R}^d$, be the probability densities of distributions F_k . Denote by $p'_k(z')$, $z' \in \mathbb{R}^{d-1}$, the densities of the vectors ξ'_k . Let $p_k^{(d)}(x_{k,d} | x'_k) = p_k(x_k) / p'_k(x'_k)$, $x_k = (x_{k,1}, \dots, x_{k,d})$, be the corresponding conditional densities of the d^{th} coordinates $\xi_{k,d}$ of the vectors $\xi_k \in \mathbb{R}^d$ for fixed values of $\xi'_k = x'_k$ (for $d = 1$ we denote $p_k^{(1)}(x_{k,1} | x'_k) = p_k(x_{k,1})$). Let G_k be the corresponding (depending on x'_k) one-dimensional conditional distributions.

Then there exist absolute positive constants c_7, \dots, c_{11} such that the following statements are true for $\tau d^{3/2} / \sigma \leq c_7$, $|(D')^{-1/2} x'_k| \leq c_8 \sigma / d^{3/2} \tau$, $k = 1, 2$.

For $d \geq 2$ there exist parameters $h'_k = h'_k(x'_k) \in \mathbb{R}^{d-1}$ giving the solutions of the equations $\mathbb{E}\bar{\xi}'_k(h'_k) = x'_k$. Define the parameters $h_k \in \mathbb{R}^d$ which have the first $d - 1$ coordinates coinciding with corresponding coordinates of h'_k and the last d^{th} coordinates $h_{k,d} = 0$. If $d = 1$, we take $h_k = 0$. Let

$$\Lambda = \left\| (D')^{-1/2} x'_1 - (D')^{-1/2} x'_2 \right\|, \quad \Gamma = \max_{k=1,2} \left\| (D')^{-1/2} x'_k \right\|. \quad (2.21)$$

Denote $y_k = \mathbb{E} \langle \bar{\xi}_k(h_k), e_d \rangle$. Then

$$|y_1 - y_2| \leq c_9 (\tau^2 \sigma^{-1} \Gamma^3 + \tau \Lambda \Gamma). \quad (2.22)$$

Denote by $v_k(w)$ the densities of the distributions $G_k E_{-y_k}$. Then the following representation is valid for $|w| \leq \frac{c_{10} \sigma^2}{d \tau}$:

$$v_1(w) = v_2(w) \exp \left(\theta c_{11} \frac{\tau^2}{\sigma^2} \left(d^2 + (\Gamma^2 + \sigma \tau^{-1} \Lambda) \left(d + \frac{w^2}{\sigma^2} \right) + \frac{w^4}{\sigma^4} \right) \right). \quad (2.23)$$

Moreover, for $8\sigma \leq w \leq \frac{c_{10} \sigma^2}{d \tau}$,

$$\frac{d}{dz} \log v_k(z) \Big|_{z=w} \leq -\frac{w}{8\sigma^2}, \quad \frac{d}{dz} \log v_k(z) \Big|_{z=-w} \geq \frac{w}{8\sigma^2}, \quad k = 1, 2. \quad (2.24)$$

3. ESTIMATES FOR THE CLOSENESS OF QUANTILES OF CONDITIONAL DISTRIBUTIONS UNDER THE CONDITIONS OF LEMMA 2.14

LEMMA 3.1. *Let the conditions of Lemma 2.14 be satisfied. Then there exist absolute positive constants c_1, \dots, c_4 such that*

$$\Phi_\sigma(z - \gamma(z)) < \int_{-\infty}^z v(y) dy < \Phi_\sigma(z + \gamma(z)), \quad (3.1)$$

for $\frac{\tau d^{3/2}}{\sigma} \leq c_1$, $|(D')^{-1/2} x'| \leq \frac{c_2 \sigma}{d^{3/2} \tau}$, $|z| \leq \frac{c_3 \sigma^2}{d \tau}$, where

$$\gamma(z) = c_4 \tau \left(d^{3/2} + d \delta \left(1 + \frac{|z|}{\sigma} \right) + \frac{z^2}{\sigma^2} \right), \quad \delta = \left\| (D')^{-1/2} x' \right\|. \quad (3.2)$$

The proof of Lemma 3.1 consists of several steps. For each of them we formulate the corresponding lemma.

LEMMA 3.2. *Let the conditions of Lemma 3.1 be satisfied. Then there exist absolute positive constants c_5, \dots, c_8 such that if*

$$\frac{\tau d^{3/2}}{\sigma} \leq c_5, \quad |(D')^{-1/2} x'| \leq \frac{c_6 \sigma}{d^{3/2} \tau}, \quad 4\sigma \leq z \leq b = \frac{c_7 \sigma^2}{d\tau}, \quad (3.3)$$

then

$$\int_{z+\lambda(2z)}^b \varphi_\sigma(u) du \leq I = I(z, b) \stackrel{\text{def}}{=} \int_z^b v(y) dy \leq \int_{z-\lambda(z)}^\infty \varphi_\sigma(u) du, \quad (3.4)$$

$$\int_{z+\lambda(2z)}^b \varphi_\sigma(u) du \leq I(-b, -z) = \int_{-b}^{-z} v(y) dy \leq \int_{z-\lambda(z)}^\infty \varphi_\sigma(u) du, \quad (3.5)$$

where

$$\lambda(z) = c_8 \tau (d^{3/2} + d\delta + z(d\delta/\sigma + z/\sigma^2)) \leq z/8. \quad (3.6)$$

Proof. At first we note that, by Lemma 2.14, for $2\sigma \leq |y| \leq c_9 \sigma^2/d\tau$ we have

$$v(y) = \varphi_\sigma(y) \exp\left(\theta c_{10} \frac{\tau}{\sigma} \left(d^{3/2} + \frac{d\delta y^2}{\sigma^2} + \frac{|y|^3}{\sigma^3}\right)\right), \quad (3.7)$$

if c_5, c_6 are small enough. Recall that

$$\varphi_\sigma(u) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right), \quad u \in \mathbb{R}^1. \quad (3.8)$$

We take $c_8 = 16c_{10}$. Thus,

$$\lambda(u) = 16c_{10} \tau (d^{3/2} + d\delta + u(d\delta/\sigma + u/\sigma^2)). \quad (3.9)$$

Consider the following functions:

$$f_-(u) = u - \lambda(u), \quad f_+(u) = u + \lambda(u). \quad (3.10)$$

We shall study some properties of the function $f_-(u)$ for $4\sigma \leq u \leq 2b$ and of the function $f_+(u)$ for $2\sigma \leq u \leq b$. It is clear that choosing c_5, c_6, c_7 to be sufficiently small we can ensure the validity of the relations

$$2b \leq c_9 \sigma^2/d\tau, \quad (3.11)$$

$$0 \leq \lambda(u) \leq u/8 \quad \text{for } 2\sigma \leq u \leq 2b, \quad (3.12)$$

and, consequently,

$$\begin{aligned} 2\sigma \leq u/2 \leq f_-(u) \leq u \leq 2b \leq c_9 \sigma^2/d\tau & \quad \text{for } 4\sigma \leq u \leq 2b, \\ 2\sigma \leq u \leq f_+(u) \leq 2u \leq 2b \leq c_9 \sigma^2/d\tau & \quad \text{for } 2\sigma \leq u \leq b. \end{aligned} \quad (3.13)$$

Moreover, taking again c_5, c_6, c_7 to be small enough, we can ensure the validity of the relation

$$\lambda'(u) = 16c_{10}\tau(d\delta/\sigma + 2u/\sigma^2) \leq 1/2 \quad \text{for } 2\sigma \leq u \leq 2b. \quad (3.14)$$

Since $e^{-2a} \leq 1 - a$ for $0 \leq a \leq 1/2$, we have

$$\begin{aligned} f'_-(u) &= 1 - 16c_{10}\tau(d\delta/\sigma + 2u/\sigma^2) \geq \exp(-32c_{10}\tau(d\delta/\sigma + 2u/\sigma^2)) \\ &\geq \exp(-2c_{10}\tau u^2\sigma^{-2}(d\delta/\sigma + 2u/\sigma^2)), \end{aligned} \quad (3.15)$$

provided that $4\sigma \leq u \leq 2b$. Analogously,

$$\begin{aligned} f'_+(u) &= 1 + 16c_{10}\tau(d\delta/\sigma + 2u/\sigma^2) \leq \exp(16c_{10}\tau(d\delta/\sigma + 2u/\sigma^2)) \\ &\leq \exp(4c_{10}\tau u^2\sigma^{-2}(d\delta/\sigma + 2u/\sigma^2)), \end{aligned} \quad (3.16)$$

provided that $2\sigma \leq u \leq b$. The relations (3.9), (3.10), (3.14) imply that, for $2\sigma \leq u \leq 2b$,

$$0 \leq \lambda'(u) \leq 1/2, \quad 1/2 \leq f'_-(u) \leq 1, \quad 1 \leq f'_+(u) \leq 3/2. \quad (3.17)$$

Hence, there exist the inverse functions $f_+^{-1}(y)$ (for $f_+(2\sigma) \leq y \leq f_+(b)$) and $f_-^{-1}(y)$ (for $f_-(4\sigma) \leq y \leq f_-(2b)$). From (3.13) it follows that

$$y/2 \leq f_+^{-1}(y) \leq y, \quad y \leq f_-^{-1}(y) \leq 2y. \quad (3.18)$$

Using (3.10), (3.17), (3.18), one can easily obtain that

$$\begin{aligned} f_-(f_-^{-1}(y)) &= f_-^{-1}(y) - \lambda(f_-^{-1}(y)), \\ f_+(f_+^{-1}(y)) &= f_+^{-1}(y) + \lambda(f_+^{-1}(y)), \end{aligned} \quad (3.19)$$

and, consequently,

$$\begin{aligned} f_-^{-1}(y) &= y + \lambda(f_-^{-1}(y)) \leq y + \lambda(2y), \\ f_+^{-1}(y) &= y - \lambda(f_+^{-1}(y)) \geq y - \lambda(y). \end{aligned} \quad (3.20)$$

Consider the integral I for $4\sigma \leq z \leq b$. Then, by (3.13), (3.18), we have:

$$z \geq 4\sigma \geq f_+(2\sigma), \quad f_+^{-1}(z) \geq 2\sigma, \quad f_+^{-1}(b) \leq b \quad (3.21)$$

and

$$f_-^{-1}(z) \geq z \geq 4\sigma, \quad b \leq f_-^{-1}(b) \leq 2b. \quad (3.22)$$

By (3.7), (3.8), (3.11), the following estimates hold:

$$I \leq \int_z^b \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{y^2}{2\sigma^2} + c_{10}\frac{\tau}{\sigma}\left(d^{3/2} + \frac{d\delta y^2}{\sigma^2} + \frac{y^3}{\sigma^3}\right)\right) dy \quad (3.23)$$

and

$$I \geq \int_z^b \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{y^2}{2\sigma^2} - c_{10}\frac{\tau}{\sigma}\left(d^{3/2} + \frac{d\delta y^2}{\sigma^2} + \frac{y^3}{\sigma^3}\right)\right) dy. \quad (3.24)$$

In the right-hand side of (3.23) we make the change of variable $y = f_+(u)$. It is clear that $dy = f'_+(u)du$. Moreover, by (3.9), (3.10),

$$y^2 = (u + \lambda(u))^2 \geq u^2 + 2u\lambda(u) = u^2 + 32c_{10}u\tau(d^{3/2} + d\delta + u(d\delta/\sigma + u/\sigma^2)). \tag{3.25}$$

From (3.13) it follows that

$$c_{10} \frac{\tau}{\sigma} \left(d^{3/2} + \frac{d\delta y^2}{\sigma^2} + \frac{y^3}{\sigma^3} \right) \leq 8c_{10} \frac{\tau}{\sigma} \left(\frac{d^{3/2}u}{\sigma} + \frac{d\delta u^2}{\sigma^2} + \frac{u^3}{\sigma^3} \right). \tag{3.26}$$

Using (3.8), (3.16), (3.20), (3.21), (3.25), (3.26), we derive from (3.23) that

$$I \leq \int_{f_+^{-1}(z)}^{f_+^{-1}(b)} \varphi_\sigma(u)du \leq \int_{z-\lambda(z)}^\infty \varphi_\sigma(u)du. \tag{3.27}$$

The upper estimate of $I(z, b)$ in (3.4) is proved. It is clear that the upper estimate of $I(-z, -b)$ in (3.5) can be obtained analogously.

In the right-hand side of (3.24) we make the change of variable $y = f_-(u)$. It is clear that $dy = f'_-(u)du$. Moreover, by virtue of (3.9), (3.10), (3.12), we have

$$y^2 = (u - \lambda(u))^2 \leq u^2 - u\lambda(u) = u^2 - 16c_{10}u\tau(d^{3/2} + d\delta + u(d\delta/\sigma + u/\sigma^2)). \tag{3.28}$$

From (3.13) it follows that

$$c_{10} \frac{\tau}{\sigma} \left(d^{3/2} + \frac{d\delta y^2}{\sigma^2} + \frac{y^3}{\sigma^3} \right) \leq c_{10} \frac{\tau}{\sigma} \left(\frac{d^{3/2}u}{\sigma} + \frac{d\delta u^2}{\sigma^2} + \frac{u^3}{\sigma^3} \right). \tag{3.29}$$

Using (3.8), (3.15), (3.20), (3.22), (3.28), (3.29), we derive from (3.24) that

$$I \geq \int_{f_-^{-1}(z)}^{f_-^{-1}(b)} \varphi_\sigma(u)du \geq \int_{z+\lambda(2z)}^b \varphi_\sigma(u)du. \tag{3.30}$$

The lower estimate of $I(z, b)$ in (3.4) is proved. It is evident that the bound for $I(-z, -b)$ can be obtained in a similar way. The inequality (3.6) follows from (3.12). \square

LEMMA 3.3. *Let the conditions of Lemma 3.1 be satisfied. Then there exist absolute positive constants c_{11}, \dots, c_{14} such that, if*

$$\frac{\tau d^{3/2}}{\sigma} \leq c_{11}, \quad |(D')^{-1/2} x'| \leq \frac{c_{12} \sigma}{d^{3/2} \tau}, \quad 4\sigma \leq z \leq \frac{c_{13} \sigma^2}{d\tau}, \tag{3.31}$$

then

$$\int_{z+\mu(2z)}^{\infty} \varphi_{\sigma}(u)du < I = I(z, \infty) = \int_z^{\infty} v(y)dy < \int_{z-\mu(z)}^{\infty} \varphi_{\sigma}(u)du, \quad (3.32)$$

and

$$\int_{z+\mu(2z)}^{\infty} \varphi_{\sigma}(u)du < I(-\infty, -z) = \int_{-\infty}^{-z} v(y)dy < \int_{z-\mu(z)}^{\infty} \varphi_{\sigma}(u)du, \quad (3.33)$$

where

$$\mu(z) = c_{14} \tau (d^{3/2} + d\delta + z(d\delta/\sigma + z/\sigma^2)) \leq z/4. \quad (3.34)$$

Proof. Choosing c_{11}, c_{12} to be small enough, we ensure the possibility to estimate the integral $I(b, \infty)$ with the help of inequality (2.19) of Lemma 2.14. By this inequality, for $y > b$ we have

$$\begin{aligned} v(y) &\leq \frac{c_{15}}{\sigma} \exp\left(-\min\left\{\frac{y^2}{4\sigma^2}, \frac{c_{16}y}{d\tau}\right\}\right) \\ &\leq \frac{c_{15}}{\sigma} \exp\left(-\min\left\{\frac{c_7y}{4d\tau}, \frac{c_{16}y}{d\tau}\right\}\right) = \frac{c_{15}}{\sigma} \exp\left(-\frac{c_{17}y}{d\tau}\right). \end{aligned} \quad (3.35)$$

Thus, we can estimate $I(b, \infty)$ as follows:

$$\begin{aligned} I(b, \infty) &= \int_b^{\infty} v(y)dy \leq \int_b^{\infty} \frac{c_{15}}{\sigma} \exp\left(-\frac{c_{17}y}{d\tau}\right)dy \\ &= \frac{c_{15}d\tau}{c_{17}\sigma} \exp\left(-\frac{c_{17}c_7\sigma^2}{d^2\tau^2}\right). \end{aligned} \quad (3.36)$$

We shall choose c_{11}, c_{12}, c_{13} so that $c_{13} = \min\{c_7/2, \sqrt{c_7c_{17}}\}$, $c_{11} \leq c_5$, $c_{12} \leq c_6$. Thus, we can use the inequalities (3.4)–(3.6). From (3.4), (3.36) it follows that

$$\begin{aligned} I(z, \infty) &= I(z, b) + I(b, \infty) \\ &\leq \frac{c_{15}d\tau}{c_{17}\sigma} \exp\left(-\frac{c_{17}c_7\sigma^2}{d^2\tau^2}\right) + \int_{z-\mu(z)}^{\infty} \varphi_{\sigma}(u)du - \int_{z-\mu(z)}^{z-\lambda(z)} \varphi_{\sigma}(u)du \end{aligned} \quad (3.37)$$

and

$$I(z, \infty) \geq I(z, b) \geq \int_{z+\mu(2z)}^{\infty} + \int_{z+\lambda(2z)}^{z+\mu(2z)} - \int_b^{\infty} \varphi_{\sigma}(u)du. \quad (3.38)$$

We take $c_{14} = 2c_8$. Taking into account (3.6), (3.31), (3.34), we have:

$$\begin{aligned} \mu(z) &= 2\lambda(z) \leq z/4, & z - \mu(z) &\geq 3z/4, \\ \mu(2z) &= 2\lambda(2z) \leq 8\lambda(z) \leq z, & z + \mu(2z) &\leq 2z. \end{aligned} \quad (3.39)$$

Therefore, taking into account that $c_{13} = \min\{c_7/2, \sqrt{c_7 c_{17}}\}$, choosing c_{11} to be sufficiently small, and using (3.6), (3.8), (3.31), (3.39), we can ensure the validity of the inequalities

$$\begin{aligned} \int_{z-\mu(z)}^{z-\lambda(z)} \varphi_\sigma(u) du &\geq \frac{\lambda(z)}{\sqrt{2\pi\sigma}} \exp\left(-\frac{z^2}{2\sigma^2}\right) \geq \frac{c_8 \tau d^{3/2}}{\sqrt{2\pi\sigma}} \exp\left(-\frac{c_{13}^2 \sigma^2}{2d^2 \tau^2}\right) \\ &> \frac{c_{15} d \tau}{c_{17} \sigma} \exp\left(-\frac{c_{17} c_7 \sigma^2}{d^2 \tau^2}\right) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \int_{z+\lambda(2z)}^{z+\mu(2z)} \varphi_\sigma(u) du &\geq \frac{\lambda(2z)}{\sqrt{2\pi\sigma}} \exp\left(-\frac{2z^2}{\sigma^2}\right) \geq \frac{c_8 \tau d^{3/2}}{\sqrt{2\pi\sigma}} \exp\left(-\frac{2c_{13}^2 \sigma^2}{d^2 \tau^2}\right) \\ &> \exp\left(-\frac{c_7^2 \sigma^2}{d^2 \tau^2}\right) = \exp\left(-\frac{b^2}{4\sigma^2}\right) \geq \int_b^\infty \varphi_\sigma(u) du. \end{aligned} \tag{3.41}$$

The upper bound in (3.32) follows now from (3.37), (3.40). The lower one can be easily derived from (3.38), (3.41). The inequalities (3.33) can be obtained analogously. The inequality (3.34) is contained in (3.39). \square

Proof of Lemma 3.1. We shall choose c_1, c_2, c_3 so small that $c_1 \leq c_{11}$, $c_2 \leq c_{12}$, $c_3 \leq c_{13}$. Thus, we can use the inequalities (3.32)–(3.34).

Let $0 \leq z \leq 4\sigma$. It is evident that, if c_1, c_2 are sufficiently small, then from (3.34) it follows that

$$\mu(8\sigma) \leq 4\mu(4\sigma) \leq c_{18} \tau (d^{3/2} + d\delta) \leq \sigma, \tag{3.42}$$

and, by virtue of (3.2) and Lemma 2.14,

$$v(y) = \varphi_\sigma(y) \exp\left(\theta c_{19} \frac{\tau}{\sigma} (d^{3/2} + d\delta)\right) \quad \text{for } |y| \leq 4\sigma, \tag{3.43}$$

where

$$c_{19} \frac{\tau}{\sigma} (d^{3/2} + d\delta) \leq 1/2. \tag{3.44}$$

From (3.8), (3.43), (3.44) it follows that we can write

$$v(y) = \varphi_\sigma(y) \left(1 + \theta c_{20} \frac{\tau}{\sigma} (d^{3/2} + d\delta)\right) \quad \text{for } |y| \leq 4\sigma. \tag{3.45}$$

Thus, the integral $I(z, 4\sigma)$ can be estimated as follows:

$$I(z, 4\sigma) = \int_z^{4\sigma} v(y) dy \leq \int_z^{4\sigma} \varphi_\sigma(u) du + c_{20} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \tag{3.46}$$

Analogously,

$$I(z, 4\sigma) \geq \int_z^{4\sigma} \varphi_\sigma(u) du - c_{20} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \tag{3.47}$$

Using Lemma 3.3, we see that

$$I(4\sigma, \infty) = \int_{4\sigma}^{\infty} v(y)dy \leq \int_{4\sigma - \mu(4\sigma)}^{\infty} \varphi_{\sigma}(u)du \quad (3.48)$$

and

$$I(4\sigma, \infty) \geq \int_{4\sigma + \mu(8\sigma)}^{\infty} \varphi_{\sigma}(u)du. \quad (3.49)$$

Taking into account (3.8), (3.42), we obtain the inequalities

$$\int_{4\sigma - \mu(4\sigma)}^{4\sigma} \varphi_{\sigma}(u)du \leq \frac{e^{-7}\mu(4\sigma)}{\sqrt{2\pi}\sigma} \leq \frac{c_{18}e^{-7}}{4\sqrt{2\pi}} \frac{\tau}{\sigma} (d^{3/2} + d\delta) \quad (3.50)$$

and

$$\int_{4\sigma}^{4\sigma + \mu(8\sigma)} \varphi_{\sigma}(u)du \leq \frac{e^{-8}\mu(8\sigma)}{\sqrt{2\pi}\sigma} \leq \frac{c_{18}e^{-8}}{\sqrt{2\pi}} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \quad (3.51)$$

Now from (3.46), (3.48), (3.50) it follows that

$$I(z, \infty) = \int_z^{\infty} v(y)dy < \int_z^{\infty} \varphi_{\sigma}(u)du + c_{21} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \quad (3.52)$$

Analogously, from (3.47), (3.49), (3.51) one can easily derive that

$$I(z, \infty) > \int_z^{\infty} \varphi_{\sigma}(u)du - c_{22} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \quad (3.53)$$

Put $c_{23} = \sqrt{2\pi} \max\{c_{21}e^8, c_{22}e^{25/2}\}$. It is clear that choosing c_1, c_2 to be sufficiently small we can ensure the validity of the inequality

$$\phi = c_{23}\tau(d^{3/2} + d\delta) \leq \sigma. \quad (3.54)$$

Hence,

$$z - \phi \geq -\sigma \quad \text{for } z \geq 0. \quad (3.55)$$

Taking into account (3.8), (3.54), (3.55), we see that, for $0 \leq z \leq 4\sigma$,

$$\int_{z-\phi}^z \varphi_{\sigma}(u)du \geq \frac{e^{-8}\phi}{\sqrt{2\pi}\sigma} \geq c_{21} \frac{\tau}{\sigma} (d^{3/2} + d\delta) \quad (3.56)$$

and, analogously,

$$\int_z^{z+\phi} \varphi_{\sigma}(u)du \geq \frac{e^{-25/2}\phi}{\sqrt{2\pi}\sigma} \geq c_{22} \frac{\tau}{\sigma} (d^{3/2} + d\delta). \quad (3.57)$$

Now from (3.52), (3.53), (3.56), (3.57) it follows that

$$\int_{z+\phi}^{\infty} \varphi_{\sigma}(u)du < \int_z^{\infty} v(y)dy < \int_{z-\phi}^{\infty} \varphi_{\sigma}(u)du \quad \text{for } 0 \leq z \leq 4\sigma. \quad (3.58)$$

The statement of Lemma 3.1 for $z \geq 0$ can be easily derived from Lemma 3.3 and from (3.54), (3.58). The case $z \leq 0$ can be considered in a similar way.

□

4. ESTIMATES FOR THE CLOSENESS OF QUANTILES
OF CONDITIONAL DISTRIBUTIONS
UNDER THE CONDITIONS OF LEMMA 2.15

LEMMA 4.1. *Let the conditions of Lemma 2.15 be satisfied. Then there exist absolute positive constants c_1, \dots, c_4 such that*

$$\int_{-\infty}^{z-\phi(z)} v_1(y)dy < \int_{-\infty}^z v_2(y)dy < \int_{-\infty}^{z+\phi(z)} v_1(y)dy, \tag{4.1}$$

for $\frac{\tau d^{3/2}}{\sigma} \leq c_1$, $|(D')^{-1/2} x'_k| \leq \frac{c_2 \sigma}{d^{3/2} \tau}$, $k = 1, 2$, $|z| \leq \frac{c_3 \sigma^2}{d \tau}$, where

$$\phi(z) = c_4 \frac{\tau^2}{\sigma} \left(d^2 + \delta \left(d + \frac{|z|}{\sigma} \right) + \frac{|z|^3}{\sigma^3} \right), \quad \delta = \Gamma^2 + \sigma \tau^{-1} \Lambda. \tag{4.2}$$

The proof of Lemma 4.1 consists of several steps. For each of them we formulate the corresponding lemma. For $-\infty \leq z_1 \leq z_2 \leq \infty$ we shall denote

$$J_k(z_1, z_2) = \int_{z_1}^{z_2} v_k(y)dy, \quad k = 1, 2.$$

LEMMA 4.2. *Let the conditions of Lemma 4.1 be satisfied. Then there exist absolute positive constants c_5, \dots, c_8 such that, if*

$$\frac{\tau d^{3/2}}{\sigma} \leq c_5, \quad |(D')^{-1/2} x'_k| \leq \frac{c_6 \sigma}{d^{3/2} \tau}, \quad k = 1, 2, \quad 10 \sigma \leq z \leq b = \frac{c_7 \sigma^2}{d \tau}, \tag{4.3}$$

then

$$\int_{z+\beta(5z/4)}^b v_1(u)du \leq J = J_2(z, b) = \int_z^b v_2(y)dy \leq \int_{z-\beta(z)}^\infty v_1(u)du, \tag{4.4}$$

and

$$\int_{-b}^{-z-\beta(5z/4)} v_1(u)du \leq J_2(-b, -z) = \int_{-b}^{-z} v_2(y)dy \leq \int_{-\infty}^{-z+\beta(z)} v_1(u)du, \tag{4.5}$$

where

$$\beta(z) = c_8 \tau^2 \sigma^{-1} (d^2 + d \delta + z (\delta/\sigma + z^2/\sigma^3)) \leq z/8. \tag{4.6}$$

Proof. At first we note that, if c_5, c_6 are small enough, then, by Lemma 2.15 and according to (4.2), for $|y| \leq c_9 \sigma^2/d\tau$ we have

$$v_2(y) = v_1(y) \exp\left(\theta c_{10} \frac{\tau^2}{\sigma^2} \left(d^2 + d \delta + \frac{\delta y^2}{\sigma^2} + \frac{y^4}{\sigma^4} \right)\right) \tag{4.7}$$

and, for $8\sigma \leq y \leq \frac{c_9\sigma^2}{d\tau}$, $k = 1, 2$,

$$\frac{d}{dx} \log v_k(x) \Big|_{x=y} \leq -\frac{y}{8\sigma^2}, \quad \frac{d}{dx} \log v_k(x) \Big|_{x=-y} \geq \frac{y}{8\sigma^2}. \quad (4.8)$$

We take $c_8 = 256c_{10}$. Thus,

$$\beta(u) = 256c_{10}\tau^2\sigma^{-1}(d^2 + d\delta + u(\delta/\sigma + u^2/\sigma^3)). \quad (4.9)$$

Consider the following functions:

$$g_-(u) = u - \beta(u), \quad g_+(u) = u + \beta(u). \quad (4.10)$$

We shall study some properties of the function $g_-(u)$ for $10\sigma \leq u \leq 2b$ and of the function $g_+(u)$ for $8\sigma \leq u \leq b$. It is clear that, choosing c_5, c_6, c_7 to be sufficiently small and taking into account (4.2), (4.9), (4.10), (2.21), we can ensure the validity of the relations

$$2b \leq c_9\sigma^2/d\tau, \quad (4.11)$$

$$0 \leq \beta(u) \leq u/8 \quad \text{for } 8\sigma \leq u \leq 2b, \quad (4.12)$$

and, consequently,

$$\begin{aligned} 8\sigma \leq 4u/5 \leq g_-(u) \leq u \leq 2b \leq c_9\sigma^2/d\tau & \quad \text{for } 10\sigma \leq u \leq 2b, \\ 8\sigma \leq u \leq g_+(u) \leq 5u/4 \leq 2b \leq c_9\sigma^2/d\tau & \quad \text{for } 8\sigma \leq u \leq b. \end{aligned} \quad (4.13)$$

Moreover, taking again c_5, c_6, c_7 to be small enough, we can ensure the validity of the relation

$$\beta'(u) = 256c_{10}\tau^2\sigma^{-1}(\delta/\sigma + 3u^2/\sigma^3) \leq 1/2 \quad \text{for } 8\sigma \leq u \leq 2b. \quad (4.14)$$

Since $e^{-2a} \leq 1 - a$ for $0 \leq a \leq 1/2$, we have

$$\begin{aligned} g'_-(u) &= 1 - 256c_{10}\tau^2\sigma^{-1}(\delta/\sigma + 3u^2/\sigma^3) \\ &\geq \exp(-512c_{10}\tau^2\sigma^{-1}(\delta/\sigma + 3u^2/\sigma^3)) \\ &\geq \exp(-6c_{10}\tau^2u^2\sigma^{-3}(\delta/\sigma + 3u^2/\sigma^3)), \end{aligned} \quad (4.15)$$

provided that $10\sigma \leq u \leq 2b$. Analogously,

$$\begin{aligned} g'_+(u) &= 1 + 256c_{10}\tau^2\sigma^{-1}(\delta/\sigma + 3u^2/\sigma^3) \\ &\leq \exp(256c_{10}\tau^2\sigma^{-1}(\delta/\sigma + 3u^2/\sigma^3)) \\ &\leq \exp(4c_{10}\tau^2u^2\sigma^{-3}(\delta/\sigma + 3u^2/\sigma^3)), \end{aligned} \quad (4.16)$$

provided that $8\sigma \leq u \leq b$. The relations (4.9), (4.10), (4.14) imply that, for $8\sigma \leq u \leq 2b$,

$$0 \leq \beta'(u) \leq 1/2, \quad 1 \leq g'_+(u) \leq 3/2, \quad 1/2 \leq g'_-(u) \leq 1. \quad (4.17)$$

Therefore, there exist the inverse functions $g_+^{-1}(y)$ (for $g_+(8\sigma) \leq y \leq g_+(b)$) and $g_-^{-1}(y)$ (for $g_-(10\sigma) \leq y \leq g_-(2b)$). From (4.13) it follows that

$$4y/5 \leq g_+^{-1}(y) \leq y, \quad y \leq g_-^{-1}(y) \leq 5y/4. \tag{4.18}$$

Using (4.10), (4.17), (4.18), one can easily obtain that

$$\begin{aligned} g_-(g_-^{-1}(y)) &= g_-^{-1}(y) - \beta(g_-^{-1}(y)), \\ g_+(g_+^{-1}(y)) &= g_+^{-1}(y) + \beta(g_+^{-1}(y)), \end{aligned} \tag{4.19}$$

and, consequently,

$$\begin{aligned} g_-^{-1}(y) &= y + \beta(g_-^{-1}(y)) \leq y + \beta(5y/4), \\ g_+^{-1}(y) &= y - \beta(g_+^{-1}(y)) \geq y - \beta(y). \end{aligned} \tag{4.20}$$

Consider the integral J for $10\sigma \leq z \leq b$. Then, by (4.13), (4.18), we have:

$$z \geq 10\sigma \geq g_+(8\sigma), \quad g_+^{-1}(z) \geq 8\sigma, \quad g_+^{-1}(b) \leq b \tag{4.21}$$

and

$$g_-^{-1}(z) \geq z \geq 10\sigma, \quad b \leq g_-^{-1}(b) \leq 5b/4. \tag{4.22}$$

By (4.7), (4.11), the following estimates hold:

$$J \leq \int_z^b v_1(y) \exp\left(c_{10} \frac{\tau^2}{\sigma^2} \left(d^2 + d\delta + \frac{\delta y^2}{\sigma^2} + \frac{y^4}{\sigma^4}\right)\right) dy \tag{4.23}$$

and

$$J \geq \int_z^b v_1(y) \exp\left(-c_{10} \frac{\tau^2}{\sigma^2} \left(d^2 + d\delta + \frac{\delta y^2}{\sigma^2} + \frac{y^4}{\sigma^4}\right)\right) dy. \tag{4.24}$$

In the right-hand side of (4.23) we make the change of variable $y = g_+(u)$. It is clear that $dy = g_+'(u)du$. Moreover, using (4.8)–(4.10), (4.13), we see that

$$\begin{aligned} \log v_1(y) &= \log v_1(u) + \beta(u) \frac{d}{dx} \log v_1(x) \Big|_{x=u+|\theta|\beta(u)} \\ &\leq \log v_1(u) - \frac{u\beta(u)}{8\sigma^2} \\ &= \log v_1(u) - 32c_{10} \frac{\tau^2 u}{\sigma^3} \left(d^2 + d\delta + \frac{u\delta}{\sigma} + \frac{u^3}{\sigma^3}\right). \end{aligned} \tag{4.25}$$

From (4.13) it follows that

$$c_{10} \frac{\tau^2}{\sigma^2} \left(d^2 + d\delta + \frac{\delta y^2}{\sigma^2} + \frac{y^4}{\sigma^4}\right) \leq 3c_{10} \frac{\tau^2}{\sigma^2} \left(\frac{d^2 u}{\sigma} + \frac{d\delta u}{\sigma} + \frac{\delta u^2}{\sigma^2} + \frac{u^4}{\sigma^4}\right). \tag{4.26}$$

Using (4.16), (4.20), (4.21), (4.25), (4.26), we derive from (4.23) that

$$J \leq \int_{g_+^{-1}(z)}^{g_+^{-1}(b)} v_1(u) du \leq \int_{z-\beta(z)}^{\infty} v_1(u) du. \tag{4.27}$$

The upper estimate of $J_2(z, b)$ in (4.4) is proved. It is clear that the upper estimate of $J_2(-z, -b)$ in (4.5) can be obtained analogously.

In the right-hand side of (4.24) we make the change of variable $y = g_-(u)$. It is clear that $dy = g'_-(u) du$. Moreover, using (4.8)–(4.10), (4.13), we see that

$$\begin{aligned} \log v_1(y) &= \log v_1(u) - \beta(u) \frac{d}{dx} \log v_1(x) \Big|_{x=u-|\theta|\beta(u)} \\ &\geq \log v_1(u) + \frac{(u - \beta(u))\beta(u)}{8\sigma^2} \geq \log v_1(u) + \frac{u\beta(u)}{10\sigma^2} \\ &\geq \log v_1(u) + 25c_{10} \frac{\tau^2 u}{\sigma^3} \left(d^2 + d\delta + \frac{u\delta}{\sigma} + \frac{u^3}{\sigma^3} \right). \end{aligned} \tag{4.28}$$

From (4.13) it follows that

$$c_{10} \frac{\tau^2}{\sigma^2} \left(d^2 + d\delta + \frac{\delta y^2}{\sigma^2} + \frac{y^4}{\sigma^4} \right) \leq c_{10} \frac{\tau^2}{\sigma^2} \left(\frac{d^2 u}{\sigma} + \frac{d\delta u}{\sigma} + \frac{\delta u^2}{\sigma^2} + \frac{u^4}{\sigma^4} \right). \tag{4.29}$$

Using (4.15), (4.20), (4.22), (4.28), (4.29), we derive from (4.24) that

$$J \geq \int_{g_-^{-1}(z)}^{g_-^{-1}(b)} v_1(u) du \geq \int_{z+\beta(5z/4)}^b v_1(u) du. \tag{4.30}$$

The lower bound for $J_2(z, b)$ in (4.4) is proved. It is clear that $J_2(-z, -b)$ can be estimated in a similar way. The inequality (4.6) follows from (4.12). \square

LEMMA 4.3. *Let the conditions of Lemma 4.1 be satisfied. Then there exist absolute positive constants c_{11}, \dots, c_{14} such that, if*

$$\frac{\tau d^{3/2}}{\sigma} \leq c_{11}, \quad |(D')^{-1/2} x'_k| \leq \frac{c_{12} \sigma}{d^{3/2} \tau}, \quad k = 1, 2, \quad 10\sigma \leq z \leq \frac{c_{13} \sigma^2}{d\tau}, \tag{4.31}$$

then

$$\int_{z+\nu(5z/4)}^{\infty} v_1(u) du < J = J_2(z, \infty) = \int_z^{\infty} v_2(y) dy < \int_{z-\nu(z)}^{\infty} v_1(u) du, \tag{4.32}$$

and

$$\int_{-\infty}^{-z-\nu(5z/4)} v_1(u) du < J_2(-\infty, -z) = \int_{-\infty}^{-z} v_2(y) dy < \int_{-\infty}^{-z+\nu(z)} v_1(u) du, \tag{4.33}$$

where

$$\nu(z) = c_{14} \tau^2 \sigma^{-1} (d^2 + d\delta + z(\delta/\sigma + z^2/\sigma^3)) \leq z/4. \tag{4.34}$$

Proof. Choosing c_{11}, c_{12} to be small enough, we ensure the possibility to estimate the integrals $J_k(b, \infty)$, $k = 1, 2$, with the help of inequality (2.19) of Lemma 2.14. By virtue of (2.19), we have (for $y > b$, $k = 1, 2$)

$$\begin{aligned} v_k(y) &\leq \frac{c_{15}}{\sigma} \exp\left(-\min\left\{\frac{y^2}{4\sigma^2}, \frac{c_{16}y}{d\tau}\right\}\right) \\ &\leq \frac{c_{15}}{\sigma} \exp\left(-\min\left\{\frac{c_7 y}{4d\tau}, \frac{c_{16} y}{d\tau}\right\}\right) = \frac{c_{15}}{\sigma} \exp\left(-\frac{c_{17} y}{d\tau}\right). \end{aligned} \tag{4.35}$$

Thus, we can estimate $J_k(b, \infty)$ as follows:

$$\begin{aligned} J_k(b, \infty) &= \int_b^\infty v_k(y) dy \leq \int_b^\infty \frac{c_{15}}{\sigma} \exp\left(-\frac{c_{17} y}{d\tau}\right) dy \\ &= \frac{c_{15} d\tau}{c_{17} \sigma} \exp\left(-\frac{c_{17} c_7 \sigma^2}{d^2 \tau^2}\right). \end{aligned} \tag{4.36}$$

We shall choose c_{11}, c_{12}, c_{13} so that

$$c_{11} \leq \min\{c_5, c_{18}\}, \quad c_{12} \leq \min\{c_6, c_{19}\}, \quad c_{13} = \min\left\{c_7, \frac{\sqrt{c_7 c_{17}}}{2}, \frac{2c_{20}}{3}\right\}, \tag{4.37}$$

where c_{18}, c_{19}, c_{20} are the constants c_1, c_2, c_3 from Lemma 2.14. Thus, we can use the inequalities (2.18) for $0 \leq |w| \leq 3c_{13}\sigma^2/2d\tau$ (with the change of $v(\cdot)$ by $v_k(\cdot)$, $k = 1, 2$) and the inequalities (4.4)–(4.6). From (4.4) and (4.36) it follows that

$$\begin{aligned} J_2(z, \infty) &= J_2(z, b) + J_2(b, \infty) \\ &\leq \frac{c_{15} d\tau}{c_{17} \sigma} \exp\left(-\frac{c_{17} c_7 \sigma^2}{d^2 \tau^2}\right) + \int_{z-\nu(z)}^\infty v_1(u) du - \int_{z-\nu(z)}^{z-\beta(z)} v_1(u) du \end{aligned} \tag{4.38}$$

and

$$J_2(z, \infty) \geq J_2(z, b) \geq \int_{z+\nu(5z/4)}^\infty + \int_{z+\beta(5z/4)}^{z+\nu(5z/4)} - \int_b^\infty v_1(u) du. \tag{4.39}$$

We take $c_{14} = 2c_8$. Taking into account the relations (4.3), (4.6), (4.31), (4.34) and (4.37), we have:

$$\begin{aligned} \nu(z) &= 2\beta(z) \leq z/4, & z - \nu(z) &\geq 3z/4, \\ \nu(5z/4) &= 2\beta(5z/4) \leq 4\beta(z) \leq z/2, & z + \nu(5z/4) &\leq 3z/2. \end{aligned} \tag{4.40}$$

Therefore, using (4.6), (4.31), (4.36), (4.37), (4.40), (2.18) and choosing c_{11} to be sufficiently small, we can ensure the validity of the inequalities

$$\begin{aligned} \int_{z-\nu(z)}^{z-\beta(z)} v_1(u) du &\geq \frac{\beta(z)}{2\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{\sigma^2}\right) \geq \frac{c_8 \tau^2 d^2}{2\sqrt{2\pi}\sigma^2} \exp\left(-\frac{c_{13}^2 \sigma^2}{d^2 \tau^2}\right) \\ &> \frac{c_{15} d\tau}{c_{17} \sigma} \exp\left(-\frac{c_{17} c_7 \sigma^2}{d^2 \tau^2}\right) \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} \int_{z+\beta(5z/4)}^{z+\nu(5z/4)} v_1(u) du &\geq \frac{\beta(5z/4)}{2\sqrt{2}\pi\sigma} \exp\left(-\frac{9z^2}{4\sigma^2}\right) \geq \frac{c_8\tau^2 d^2}{2\sqrt{2}\pi\sigma^2} \exp\left(-\frac{9c_{13}^2\sigma^2}{4d^2\tau^2}\right) \\ &> \frac{c_{15}d\tau}{c_{17}\sigma} \exp\left(-\frac{c_{17}c_7\sigma^2}{d^2\tau^2}\right) \geq \int_b^\infty v_1(u) du. \end{aligned} \quad (4.42)$$

The upper bound in (4.32) follows now from (4.38), (4.41). The lower one can be easily derived from (4.39), (4.42). The inequalities (4.33) can be obtained analogously. The inequality (4.34) is contained in (4.40). \square

Proof of Lemma 4.1. We shall choose c_1, c_2, c_3 so small that $c_1 \leq c_{11}$, $c_2 \leq c_{12}$, $c_3 \leq c_{13}$. Thus, we can use the inequalities (4.32)–(4.34).

Let $0 \leq z \leq 10\sigma$. It is evident that, if c_1, c_2 are small enough, then from (4.2), (4.34), (2.21) it follows that

$$\nu(25\sigma/2) \leq 2\nu(10\sigma) \leq c_{21}\tau^2\sigma^{-1}(d^2 + d\delta) \leq \sigma, \quad (4.43)$$

and, by virtue of (4.2) and Lemma 2.15,

$$v_2(y) = v_1(y) \exp\left(\theta c_{22} \frac{\tau^2}{\sigma^2} (d^2 + d\delta)\right) \quad \text{for } |y| \leq 10\sigma, \quad (4.44)$$

where

$$c_{22} \frac{\tau^2}{\sigma^2} (d^2 + d\delta) \leq 1/2. \quad (4.45)$$

From (4.44), (4.45) it follows that we can write

$$v_2(y) = v_1(y) \left(1 + \theta c_{23} \frac{\tau^2}{\sigma^2} (d^2 + d\delta)\right) \quad \text{for } |y| \leq 10\sigma. \quad (4.46)$$

Thus, the integral $J_2(z, 10\sigma)$ can be estimated as follows:

$$J_2(z, 10\sigma) = \int_z^{10\sigma} v_2(y) dy \leq \int_z^{10\sigma} v_1(y) dy + c_{23} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \quad (4.47)$$

Analogously,

$$J_2(z, 10\sigma) \geq \int_z^{10\sigma} v_1(y) dy - c_{23} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \quad (4.48)$$

Using Lemma 4.3, we see that

$$J_2(10\sigma, \infty) = \int_{10\sigma}^\infty v_2(y) dy \leq \int_{10\sigma-\nu(10\sigma)}^\infty v_1(y) dy \quad (4.49)$$

and

$$J_2(10\sigma, \infty) \geq \int_{10\sigma+\nu(25\sigma/2)}^\infty v_1(y) dy. \quad (4.50)$$

Taking into account (4.43), choosing c_1, c_2, c_3 to be small enough and applying the inequality (2.18) of Lemma 2.14, we obtain the inequalities

$$\begin{aligned} \int_{10\sigma-\nu(10\sigma)}^{10\sigma} v_1(y)dy &\leq \int_{10\sigma-\nu(10\sigma)}^{10\sigma} \frac{2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{4\sigma^2}\right) dy \\ &\leq \frac{2e^{-22}\nu(10\sigma)}{\sqrt{2\pi}\sigma} \leq \frac{c_{21}e^{-22}}{\sqrt{2\pi}} \frac{\tau^2}{\sigma^2} (d^2 + d\delta) \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \int_{10\sigma}^{10\sigma+\nu(25\sigma/2)} v_1(y)dy &\leq \int_{10\sigma}^{10\sigma+\nu(25\sigma/2)} \frac{2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{4\sigma^2}\right) dy \\ &\leq \frac{2e^{-25}\nu(25\sigma/2)}{\sqrt{2\pi}\sigma} \leq \frac{2c_{21}e^{-25}}{\sqrt{2\pi}} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \end{aligned} \tag{4.52}$$

Now from (4.47), (4.49), (4.51) it follows that

$$J_2(z, \infty) = \int_z^\infty v_2(y)dy < \int_z^\infty v_1(y)dy + c_{24} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \tag{4.53}$$

Analogously, from (4.48), (4.50), (4.52) one can easily derive that

$$J_2(z, \infty) > \int_z^\infty v_1(y)dy - c_{25} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \tag{4.54}$$

Put $c_{26} = 2\sqrt{2\pi} \max\{c_{24}e^{100}, c_{25}e^{121}\}$. It is clear that choosing c_1, c_2 to be sufficiently small (see (4.2), (2.21)) we can ensure the validity of the inequality

$$r = c_{26}\tau^2\sigma^{-1}(d^2 + d\delta) \leq \sigma. \tag{4.55}$$

Hence,

$$z - r \geq -\sigma \quad \text{for } z \geq 0. \tag{4.56}$$

Taking into account (4.55), (4.56), choosing c_1, c_2, c_3 to be small enough and applying the inequality (2.18) of Lemma 2.14, we see that, for $0 \leq z \leq 10\sigma$,

$$\int_{z-r}^z v_1(y)dy \geq \int_{z-r}^z \frac{1}{2\sqrt{2\pi}\sigma} e^{-y^2/\sigma^2} dy \geq \frac{e^{-100}r}{2\sqrt{2\pi}\sigma} \geq c_{24} \frac{\tau^2}{\sigma^2} (d^2 + d\delta) \tag{4.57}$$

and, analogously,

$$\int_z^{z+r} v_1(y)dy \geq \int_z^{z+r} \frac{1}{2\sqrt{2\pi}\sigma} e^{-y^2/\sigma^2} dy \geq \frac{e^{-121}r}{2\sqrt{2\pi}\sigma} \geq c_{25} \frac{\tau^2}{\sigma^2} (d^2 + d\delta). \tag{4.58}$$

Now from (4.53), (4.54), (4.57), (4.58) it follows that

$$\int_{z+r}^\infty v_1(y)dy < \int_z^\infty v_2(y)dy < \int_{z-r}^\infty v_1(y)dy \tag{4.59}$$

for $0 \leq z \leq 10\sigma$. The statement of Lemma 4.1 for $z \geq 0$ can be easily derived from Lemma 4.3 and from (4.55), (4.59). The case $z \leq 0$ can be considered in a similar way. \square

5. DYADIC SCHEME

Below we describe an extension of the multivariate version of the KMT dyadic scheme due to Einmahl (1989).

Let N be a positive integer and $\{\xi_1, \dots, \xi_{2^N}\}$, $\{Y_1, \dots, Y_{2^N}\}$ are two collections of independent random vectors. We shall always suppose that the random vectors ξ_1, \dots, ξ_{2^N} ; Y_1, \dots, Y_{2^N} have some known distributions from $\mathcal{A}_d(c\tau)$. Assume that the distributions of Y_1, \dots, Y_{2^N} are absolutely continuous. Denote

$$\tilde{S}_0 = 0; \quad \tilde{S}_k = \sum_{l=1}^k \xi_l, \quad 1 \leq k \leq 2^N; \quad (5.1)$$

$$U_{m,k}^* = \tilde{S}_{(k+1) \cdot 2^m} - \tilde{S}_{k \cdot 2^m}, \quad 0 \leq k < 2^{N-m}, \quad 0 \leq m \leq N. \quad (5.2)$$

In particular, $U_{0,k}^* = \xi_{k+1}$, $U_{N,0}^* = \tilde{S}_{2^N} = \xi_1 + \dots + \xi_{2^N}$. In the sequel we shall use the term *block of summands* for a collection of summands with indices of the form $k \cdot 2^m + 1, \dots, (k+1) \cdot 2^m$, where $0 \leq k < 2^{N-m}$, $0 \leq m \leq N$. Thus, $U_{m,k}^*$ is the sum over a block containing 2^m summands. Put also

$$\tilde{U}_{n,k}^* = U_{n-1,2k}^* - U_{n-1,2k+1}^*, \quad 0 \leq k < 2^{N-n}, \quad 1 \leq n \leq N. \quad (5.3)$$

Note that

$$U_{n-1,2k}^* + U_{n-1,2k+1}^* = U_{n,k}^*, \quad 0 \leq k < 2^{N-n}, \quad 1 \leq n \leq N. \quad (5.4)$$

Below we describe a procedure of constructing the random vectors $\{U_{n,k}\}$ with $\mathcal{L}(\{U_{n,k}\}) = \mathcal{L}(\{U_{n,k}^*\})$, provided that the vectors Y_1, \dots, Y_{2^N} are already constructed. For this purpose we shall use the so-called Rosenblatt quantile transformation (see Rosenblatt (1952) and Einmahl (1989)).

Define the coordinate mappings $\pi_i, \hat{\pi}_j$, $i, j = 1, \dots, d$, by the equalities $\pi_i(x_1, \dots, x_d) = x_i$, $\hat{\pi}_j = (\pi_1, \dots, \pi_j)$. Denote by $F_{N,0}^{(1)}(x_1) = \mathbb{P}\{\pi_1(U_{N,0}^*) < x_1\}$, $x_1 \in \mathbb{R}^1$, the distribution function of the first coordinate of the vector $U_{N,0}^*$. Let $F_{N,0}^{(j)}(\cdot | x_1, \dots, x_{j-1})$, $2 \leq j \leq d$, be the regular conditional distribution function (r.c.d.f.) of the random variable $\pi_j(U_{N,0}^*)$, given $\hat{\pi}_{j-1}(U_{N,0}^*) = (x_1, \dots, x_{j-1})$. Furthermore, let $\tilde{F}_{n,k}^{(1)}(\cdot | y_1, \dots, y_d)$ be the r.c.d.f. of $\pi_1(\tilde{U}_{n,k}^*)$, given $U_{n,k}^* = (y_1, \dots, y_d)$. Finally, denote by $\tilde{F}_{n,k}^{(j)}(\cdot | y_1, \dots, y_d; x_1, \dots, x_{j-1})$ the r.c.d.f. of $\pi_j(\tilde{U}_{n,k}^*)$, given $U_{n,k}^* = (y_1, \dots, y_d)$ and $\hat{\pi}_{j-1}(\tilde{U}_{n,k}^*) = (x_1, \dots, x_{j-1})$, $0 \leq k < 2^{N-n}$, $1 \leq n \leq N$, $2 \leq j \leq d$. Put

$$T_0 = 0, \quad T_k = \sum_{l=1}^k Y_l, \quad 1 \leq k \leq 2^N; \quad (5.5)$$

$$V_{m,k} = (V_{m,k}^{(1)}, \dots, V_{m,k}^{(d)}) = T_{(k+1) \cdot 2^m} - T_{k \cdot 2^m}, \quad 0 \leq k < 2^{N-m}, \quad 0 \leq m \leq N; \quad (5.6)$$

$$\tilde{V}_{n,k} = (\tilde{V}_{n,k}^{(1)}, \dots, \tilde{V}_{n,k}^{(d)}) = V_{n-1,2k} - V_{n-1,2k+1}, \quad 0 \leq k < 2^{N-n}, \quad 1 \leq n \leq N.$$

For the r.c.d.f. of these vectors we shall use the similar notation. Denote by $G_{N,0}^{(1)}(x_1) = \mathbb{P}\{\pi_1(V_{N,0}) < x_1\}$, $x_1 \in \mathbb{R}^1$, the distribution function of the first coordinate of the vector $V_{N,0}$. Let $G_{N,0}^{(j)}(\cdot | x_1, \dots, x_{j-1})$ be the r.c.d.f. of $\pi_j(V_{N,0})$, given $\hat{\pi}_{j-1}(V_{N,0}) = (x_1, \dots, x_{j-1})$, $2 \leq j \leq d$. Let $\tilde{G}_{n,k}^{(1)}(\cdot | y_1, \dots, y_d)$ be the r.c.d.f. of $\pi_1(\tilde{V}_{n,k})$, given $V_{n,k} = (y_1, \dots, y_d)$, and $\tilde{G}_{n,k}^{(j)}(\cdot | y_1, \dots, y_d; x_1, \dots, x_{j-1})$ the r.c.d.f. of $\pi_j(\tilde{V}_{n,k})$, given $V_{n,k} = (y_1, \dots, y_d)$ and $\hat{\pi}_{j-1}(\tilde{V}_{n,k}) = (x_1, \dots, x_{j-1})$, $0 \leq k < 2^{N-n}$, $1 \leq n \leq N$, $2 \leq j \leq d$.

Denote now the new collection of random vectors X_k as follows. At first we define

$$U_{N,0}^{(1)} = (F_{N,0}^{(1)})^{-1}(G_{N,0}^{(1)}(V_{N,0}^{(1)})) \quad \text{and, for } 2 \leq j \leq d,$$

$$U_{N,0}^{(j)} = (F_{N,0}^{(j)})^{-1}(G_{N,0}^{(j)}(V_{N,0}^{(j)} | V_{N,0}^{(1)}, \dots, V_{N,0}^{(j-1)}) | U_{N,0}^{(1)}, \dots, U_{N,0}^{(j-1)})$$

(here $(F_{N,0}^{(1)})^{-1}(t) = \sup\{x : F_{N,0}^{(1)}(x) \leq t\}$, $0 < t < 1$, and so on). If the distributions of the random vectors ξ_1, \dots, ξ_{2^N} are absolutely continuous, then this formula can be rewritten in a more natural "symmetric" form (see Sakhanenko (1984), pp. 30–31):

$$F_{N,0}^{(1)}(U_{N,0}^{(1)}) = G_{N,0}^{(1)}(V_{N,0}^{(1)}) \quad \text{and, for } 2 \leq j \leq d,$$

$$F_{N,0}^{(j)}(U_{N,0}^{(j)} | U_{N,0}^{(1)}, \dots, U_{N,0}^{(j-1)}) = G_{N,0}^{(j)}(V_{N,0}^{(j)} | V_{N,0}^{(1)}, \dots, V_{N,0}^{(j-1)}). \tag{5.7}$$

Assume that the random vectors $U_{n,k} = (U_{n,k}^{(1)}, \dots, U_{n,k}^{(d)})$, $0 \leq k < 2^{N-n}$, $1 \leq n \leq N$, corresponding to blocks containing each 2^n summands, are already constructed. Now we want to pass to the sums containing each 2^{n-1} summands. For this for $0 \leq k < 2^{N-n}$ we define

$$\tilde{U}_{n,k}^{(1)} = (\tilde{F}_{n,k}^{(1)})^{-1}(\tilde{G}_{n,k}^{(1)}(\tilde{V}_{n,k}^{(1)} | V_{n,k}) | U_{n,k}), \tag{5.8}$$

and, for $2 \leq j \leq d$,

$$\tilde{U}_{n,k}^{(j)} = (\tilde{F}_{n,k}^{(j)})^{-1}(\tilde{G}_{n,k}^{(j)}(\tilde{V}_{n,k}^{(j)} | V_{n,k}; \tilde{V}_{n,k}^{(1)}, \dots, \tilde{V}_{n,k}^{(j-1)}) | U_{n,k}; \tilde{U}_{n,k}^{(1)}, \dots, \tilde{U}_{n,k}^{(j-1)}). \tag{5.9}$$

It is clear that we can rewrite the relations (5.8), (5.9) in the symmetric form, similar to (5.7). Then we put $\tilde{U}_{n,k} = (\tilde{U}_{n,k}^{(1)}, \dots, \tilde{U}_{n,k}^{(d)})$,

$$U_{n-1,2k} = \frac{1}{2}(U_{n,k} + \tilde{U}_{n,k}),$$

$$U_{n-1,2k+1} = \frac{1}{2}(U_{n,k} - \tilde{U}_{n,k}). \tag{5.10}$$

Thus, we have constructed the random vectors $U_{n-1,k}$, $0 \leq k < 2^{N-n+1}$. After N steps we shall obtain the random vectors $U_{0,k}$, $0 \leq k < 2^N$. Now we set

$$X_k = U_{0,k-1}, \quad S_0 = 0, \quad S_k = \sum_{l=1}^k X_l, \quad 1 \leq k \leq 2^N. \tag{5.11}$$

One can verify that the mutual distribution of the constructed vectors $U_{n,k}$ coincides with the mutual distribution of the vectors $U_{n,k}^*$. In particular, the vectors X_k , $k = 1, \dots, 2^N$, are independent with $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$. Moreover, according to (5.3), (5.4), we have

$$\begin{aligned} \tilde{U}_{n,k} &= U_{n-1,2k} - U_{n-1,2k+1}, & 0 \leq k < 2^{N-n}, \quad 1 \leq n \leq N \\ U_{n,k} &= U_{n-1,2k} + U_{n-1,2k+1}, \end{aligned} \quad (5.12)$$

(it is clear, that (5.12) follows from (5.10)).

The scheme described above is somewhat more general in comparison with that from Einmahl (1989). In Einmahl (1989) it was supposed that the vectors Y_k have independent coordinates. However, the fact that our scheme gives the vectors with needed mutual distributions can be justified with the help of KMT (1975–76, p. 117), Sakhanenko (1984, pp. 30–33) and Einmahl (1989, Theorem 6). For this we note that we can begin with independent Gaussian vectors $\{Z_1, \dots, Z_{2^N}\}$ with $\mathcal{L}(Z_k) = \mathcal{N}(0, I)$ and construct the vectors $\{X_1, \dots, X_{2^N}\}$ and $\{Y_1, \dots, Y_{2^N}\}$ as the functions of vectors $\{Z_1, \dots, Z_{2^N}\}$ using our scheme. The vectors Z_k have independent coordinates and the vectors $\{X_1, \dots, X_{2^N}\}$ and $\{Y_1, \dots, Y_{2^N}\}$ are connected by the relations (5.5)–(5.11). This is especially clear for the relations (5.7). Now we can exclude the vectors $\{Z_1, \dots, Z_{2^N}\}$ from the scheme.

Note that we essentially use here the assumption about the absolute continuity of the distributions of Y_1, \dots, Y_{2^N} . This assumption implies the absolute continuity of the distributions of the vectors $V_{n,k}$ and, hence, the same property of the projections $\hat{\pi}_j(V_{n,k})$. In this case the regular conditional distributions can be defined via conditional densities. Therefore, with probability one, the r.c.d.f. G and \tilde{G} involved in the relations (5.7)–(5.9) are continuous.

LEMMA 5.1. *Let s, r be non-negative integers and $1 \leq m = (2s+1) \cdot 2^r \leq 2^N$. Then*

$$S_m = \frac{m}{2^N} S_{2^N} + \sum_{n=r+1}^N \gamma_n \tilde{U}_{n, l_{n,m}},$$

where $\gamma_n = \gamma_n(m) \in [0, 1]$ and the integers $l_{n,m}$ are defined by

$$l_{n,m} \cdot 2^n < m \leq (l_{n,m} + 1) \cdot 2^n. \quad (5.13)$$

REMARK 5.2. Lemma 5.1 is contained in Einmahl (1986, Lemma 5, p. 47) or (1989, Lemma 5, p. 55). U. Einmahl informed the author that the shortest proof of Lemma 5.1 can be obtained with the help of a geometrical approach due to Massart (1989, p. 275).

COROLLARY 5.3. *Under the conditions of Lemma 5.1*

$$|S_m - T_m| \leq |U_{N,0} - V_{N,0}| + \sum_{n=r+1}^N |\tilde{U}_{n, l_{n,m}} - \tilde{V}_{n, l_{n,m}}|, \quad m = 1, \dots, 2^N.$$

This assertion evidently follows from Lemma 5.1 and from the relations (5.1)–(5.6).

REMARK 5.4. Applying Lemma 5.1 and Corollary 5.3, one should remember that $U_{n,l_n,m}$ is the sum over the block of 2^n summands which contains X_m , the last summand in the sum S_m .

6. ESTIMATION OF THE RATE OF APPROXIMATION
IN CONSTRUCTING THE VECTORS WITH NEEDED MOMENTS
UP TO THE THIRD ORDER

Below we shall need the following useful fact.

LEMMA 6.1. *Assume that $0 < \alpha < 1$ and a random variable μ has values 0 with probability $1 - \alpha^3$ and α^{-1} with probability α^3 . Let $X, K \in \mathbb{R}^d$ be random vectors with $\mathbb{E} X = \mathbb{E} K = 0$, $\text{cov } X = D$, $\text{cov } K = (1 - \alpha)D$, $\mathbb{E} \|X\|^3 < \infty$, $\mathbb{E} \|K\|^3 < \infty$. Suppose also that X, K, μ are mutually independent and the distribution of K is symmetric. Then the random vector $W = \mu X + K$ has mean zero, covariance operator D and, for any $u, v, w \in \mathbb{R}^d$, we have*

$$\mathbb{E} \langle W, u \rangle \langle W, v \rangle \langle W, w \rangle = \mathbb{E} \langle X, u \rangle \langle X, v \rangle \langle X, w \rangle \tag{6.1}$$

(in other words, W and X have the same moments of the first three orders). The proof of Lemma 6.1 is elementary and, therefore, is omitted. One should note that Lemma 6.1 is a simplified (and modified) version of (due to Yurinskii (1978)) Lemma 4 from Sazonov (1981, p. 85).

Consider a one-dimensional symmetric probability distribution $H = \mathcal{L}(\eta)$ concentrated on the interval $[-1, 1]$ and having an everywhere infinitely differentiable density $w(x)$. Suppose that $\mathbb{E} \eta^2 = 1/2$. It is clear that $\mathbb{E} \eta = 0$ and such a distribution can be easily constructed. Below we shall assume that the distribution H with properties indicated above is fixed. Consider a two-dimensional random vector $\eta^* = (\eta_1^*, \eta_2^*)$ having independent coordinates η_j^* , with $\mathcal{L}(\eta_j^*) = H, j = 1, 2$. Define $\zeta = (\eta_1^* + \eta_2^*, \eta_1^* - \eta_2^*)$, $H^* = \mathcal{L}(\zeta)$. Denote by $H_h = \overline{H}(h) \in \mathcal{F}_1, h \in \mathbb{R}^1; H_h^* = \overline{H}^*(h) \in \mathcal{F}_2, h \in \mathbb{R}^2$, the corresponding conjugate distributions.

LEMMA 6.2. *There exist absolute positive constants c_1, \dots, c_{10} such that the following relations are true:*

$$\sup_{|h| \leq c_1} \sup_{|t| \geq c_2} |\widehat{H}_h(t)| \leq c_3 < 1, \tag{6.2}$$

$$\sup_{|h| \leq c_1} |\widehat{H}_h(t)| \leq \exp\left(-\frac{1}{6}t^2\right) \quad \text{for } |t| \leq c_2, \tag{6.3}$$

$$\sup_{|h| \leq 1} \int_{\mathbb{R}^1} |\widehat{H}_h(t)| dt \leq c_4, \quad \sup_{|h| \leq 1} \int_{\mathbb{R}^1} t^2 |\widehat{H}_h(t)| dt \leq c_5, \tag{6.4}$$

$$\sup_{\|h\| \leq c_6} \sup_{|t| \geq c_7} |\widehat{H}_h^*(t)| \leq c_8 < 1, \tag{6.5}$$

$$\sup_{\|h\| \leq c_6} |\widehat{H}_h^*(t)| \leq \exp\left(-\frac{1}{3}\|t\|^2\right) \quad \text{for } |t| \leq c_7, \tag{6.6}$$

$$\sup_{\|h\| \leq 1} \int_{\mathbb{R}^2} |\widehat{H}_h^*(t)| dt \leq c_9, \quad \sup_{\|h\| \leq 1} \int_{\mathbb{R}^2} \|t\|^2 |\widehat{H}_h^*(t)| dt \leq c_{10}. \tag{6.7}$$

Proof. We prove the relations (6.5)–(6.7) only. The inequalities (6.2)–(6.4) can be proved analogously. Denote by $w^*(x)$, $x \in \mathbb{R}^2$, the density of the distribution H^* . Putting $w^*(h, x) = e^{\langle h, x \rangle} w^*(x)$, $h, x \in \mathbb{R}^2$, using (2.14) and integrating by parts, we see that (for $t = (t_1, t_2) \in \mathbb{R}^2$, $t_j \neq 0$, $j = 1, 2$)

$$\begin{aligned} \widehat{H}_h^*(t) &= (\mathbb{E} e^{\langle h, \zeta \rangle})^{-1} \int_{|x| \leq \sqrt{2}} e^{i \langle t, x \rangle} w^*(h, x) dx \\ &= (\mathbb{E} e^{\langle h, \zeta \rangle})^{-1} \int_{|x| \leq \sqrt{2}} t_1^{-4} t_2^{-4} e^{i \langle t, x \rangle} \frac{\partial^8 w^*(h, x)}{\partial x_1^4 \partial x_2^4} dx. \end{aligned} \tag{6.8}$$

Moreover,

$$\sup_{x \in \mathbb{R}^2} \sup_{\|h\| \leq 1} \left| \frac{\partial^8 w^*(h, x)}{\partial x_1^4 \partial x_2^4} \right| dx \leq c_{11} \tag{6.9}$$

(constants here and below are absolute, since the distribution H^* is fixed). From (6.8), (6.9) it follows that

$$\sup_{\|h\| \leq 1} |\widehat{H}_h^*(t)| \leq 2 c_{11} t_1^{-4} t_2^{-4} \tag{6.10}$$

(we take into account that, by the Jensen inequality, $\mathbb{E} e^{\langle h, \zeta \rangle} \geq e^{\mathbb{E} \langle h, \zeta \rangle} = 1$). It is evident that with the help of analogous integrations by parts (different for different values of t) one can prove that

$$\sup_{\|h\| \leq 1} |\widehat{H}_h^*(t)| \leq \frac{c_{12}}{(1 + t_1^4)(1 + t_2^4)}. \tag{6.11}$$

Hence,

$$\sup_{\|h\| \leq 1} \sup_{|t| \geq c_{13}} |\widehat{H}_h^*(t)| \leq 1/2 \tag{6.12}$$

(one can take $c_{13} = (2 c_{12})^{1/4}$). Inequalities (6.7) follow from (6.11).

Obviously, $H^* \in \mathcal{A}_2(c)$. It is easy to see that $\text{cov } H^* = I : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Using Lemma 2.4(b), (2.4) and (2.6), we see that the relation (6.6) is valid, if $c_6 \leq 1$ and $c_7 \leq c_{13}$ are sufficiently small.

It is clear that the distributions H_h^* are absolutely continuous. This implies that the relation $|\widehat{H}_h^*(t)| = 1$ can be valid for $t = 0$ only. Moreover, the function $|\widehat{H}_h^*(t)|$ considered as function of two variables h and t is continuous for all $h, t \in \mathbb{R}^2$. Therefore,

$$\sup_{\|h\| \leq c_6} \sup_{c_7 \leq |t| \leq c_{13}} |\widehat{H}_h^*(t)| \leq c < 1. \tag{6.13}$$

The inequalities (6.12), (6.13) imply (6.5). □

DEFINITION 6.3. Assume that the conditions of Lemma 6.1 are satisfied with $D = I$, $\alpha = 1/2$, and the vector K has independent coordinates K_j with $\mathcal{L}(K_j) = H$, $j = 1, \dots, d$. In this case (it is evident that $\mathbb{E} K = 0$, $\text{cov } K = I/2$) we shall use below the notation $\Psi(\mathcal{L}(X)) = \mathcal{L}(W)$.

Suppose that $\mathcal{L}(X) \in \mathcal{A}_d(\tau)$, $\tau \geq 1$. It is clear that the distribution $\Psi(\mathcal{L}(X))$ is always absolutely continuous with infinitely differentiable density. Using Lemma 2.7, Remark 2.9, the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution and their monotonicity with respect to τ , one can show that there exists c_{14}, c_{15} such that

$$\mathcal{L}(K) \in \mathcal{A}_d(c_{14}), \quad \mathcal{L}(W) = \Psi(\mathcal{L}(X)) \in \mathcal{A}_d(c_{15} \tau), \quad c_{15} \geq 1. \quad (6.14)$$

The simplest way to prove the second relation in (6.14) is to use a connection between classes $\mathcal{A}_d(\tau)$ and classes of distributions satisfying Bernstein-type inequality conditions (see Zaitsev (1986, p. 205; 1987, 1988)).

Let $\tau \geq 1$ and $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^N}$, be independent random vectors with known distributions such that $\mathcal{L}(\tilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E}\tilde{\xi}_k = 0$, $\text{cov}\tilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Assume that we have the independent random vectors ξ_1, \dots, ξ_{2^N} with $\mathcal{L}(\xi_k) = \Psi(\mathcal{L}(\tilde{\xi}_k))$, $k = 1, \dots, 2^N$. According to Lemma 6.1, ξ_k and $\tilde{\xi}_k$ have the same moments of the first three orders. Let Y_1, \dots, Y_{2^N} be independent Gaussian random vectors with $\mathcal{L}(Y_k) = \Phi(\mathcal{L}(\tilde{\xi}_k))$. Assume that the random vectors X_k , $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$; $V_{n,k}, \tilde{V}_{n,k}$; $U_{n,k}, \tilde{U}_{n,k}$, S_k, T_k are constructed as functions of the vectors Y_1, \dots, Y_{2^N} as in Section 5. Taking into account (6.14) and using the completeness of classes $\mathcal{A}_d(c_{15} \tau)$ with respect to convolution, we have

$$\begin{aligned} \mathcal{L}(X_k) &= \mathcal{L}(\xi_k) \in \mathcal{A}_d(c_{15} \tau), \quad k = 1, \dots, 2^N; \\ \mathcal{L}(U_{n,k}) &\in \mathcal{A}_d(c_{15} \tau), \quad 0 \leq k < 2^{N-n}, \quad 0 \leq n \leq N. \end{aligned} \quad (6.15)$$

THEOREM 6.4. *Let the conditions described above be satisfied with $N \geq 1$. Then there exist absolute positive constants c_{16}, c_{17} such that*

$$\mathbb{E} \exp\left(\frac{c_{16} \Delta}{d^{3/2} \tau}\right) \leq \exp(c_{17} N L(d)),$$

where

$$\Delta = \Delta(X, Y) = \max_{1 \leq i \leq 2^N} |S_i - T_i|. \quad (6.16)$$

The rest of this section is devoted to the proof of Theorem 6.4. Therefore, below we assume that its conditions are fulfilled. The following Lemma 6.5 can be considered as an analog of Lemma 4 from Einmahl's (1989, p. 55).

LEMMA 6.5. *There exist absolute positive constants c_{18}, \dots, c_{23} such that*

a) *if $\tau d^{3/2}/2^{N/2} \leq c_{18}$, then*

$$|U_{N,0} - V_{N,0}| \leq c_{19} d^{3/2} \tau (1 + 2^{-N} |U_{N,0}|^2) \quad (6.17)$$

provided that $|U_{N,0}| \leq \frac{c_{20} \cdot 2^N}{d^{3/2} \tau}$;

b) *if $1 \leq n \leq N$, $0 \leq k < 2^{N-n}$, $\tau d^{3/2}/2^{n/2} \leq c_{21}$, then*

$$|\tilde{U}_{n,k} - \tilde{V}_{n,k}| \leq c_{22} d^{3/2} \tau \left(1 + 2^{-n} \max\{|U_{n,k}|^2, |\tilde{U}_{n,k}|^2\}\right) \quad (6.18)$$

provided that $\max\{|U_{n,k}|, |\tilde{U}_{n,k}|\} \leq \frac{c_{23} \cdot 2^n}{d^{3/2} \tau}$.

Proof. At first we prove assertion a). From the properties of classes $\mathcal{A}_d(\tau)$ and from (6.15) it follows that $\mathcal{L}(U_{N,0}) \in \mathcal{A}_d(c_{15}\tau)$, $\mathcal{L}(V_{N,0}) = \Phi(\mathcal{L}(U_{N,0}))$. Below we prove that if c_{18} is small enough, then these distributions satisfy the conditions of Lemmas 2.14 and 3.1 with $D = 2^N I$, $\sigma^2 = 2^N$ and with the change of τ by $c_{24}\tau$, where $c_{24} = \max\{c_{15}, c_1^{-1}, 1\}$. For this we shall verify that for $h \in \mathbb{R}^d$, $\|h\| c_{24}\tau < 1$, and for all $v \in \mathbb{R}^d$ the following inequalities hold (see (2.15) and Definition 2.12):

$$(2\pi)^{-d} \int_T |\widehat{F}_h(t)| dt \leq \frac{c_{24}^2 \tau^2 d^2}{2^N (2\pi)^{d/2} \cdot 2^{Nd/2}}, \quad (6.19)$$

$$(2\pi)^{-d} \int_T |\langle t, v \rangle \widehat{F}_h(t)| dt \leq \frac{\|v\|}{4 \cdot 2^{N/2} (2\pi)^{d/2} \cdot 2^{Nd/2}}, \quad (6.20)$$

where $F_h = \overline{F}(h)$, $F = \mathcal{L}(U_{N,0})$,

$$T = \{t \in \mathbb{R}^d : 4 \|t\| c_{24}\tau d \geq 1\}. \quad (6.21)$$

Besides, we need the validity of analogous relations in the $(d-1)$ -dimensional situation for the distributions $F^l = \mathcal{L}(U'_{N,0})$ (see Notation 2.13).

From the definition of distributions $\Psi(\cdot)$ and from (2.2) it follows that

$$|\widehat{F}_h(t)| \leq \prod_{j=1}^d |\widehat{H}_{h_j}(t_j)|^{2^N}. \quad (6.22)$$

Since $c_{24} \geq c_1^{-1}$, $\tau \geq 1$, we can use for estimating $|\widehat{H}_{h_j}(t_j)|$ the inequalities (6.2)–(6.4). In particular, from (6.2), (6.3), (6.21) one can easily derive that

$$\sup_{t \in T} \prod_{j=1}^d |\widehat{H}_{h_j}(t_j)| \leq \max\{\exp(-(96c_{24}^2 \tau^2 d^2)^{-1}), c_3\} \leq \exp\left(-\frac{c_{25}}{\tau^2 d^2}\right). \quad (6.23)$$

From (6.22), (6.23) it follows that

$$|\widehat{F}_h(t)| \leq \exp\left(-\frac{2^N c_{25}}{2\tau^2 d^2}\right) \prod_{j=1}^d |\widehat{H}_{h_j}(t_j)|^{2^{N-1}}, \quad t \in T. \quad (6.24)$$

Choosing c_{18} to be sufficiently small and taking into account that $\tau \geq 1$, we ensure the validity of the inequality

$$2^{N-1} \geq 2. \quad (6.25)$$

Using (6.3), we obtain

$$\int_{|t_j| \leq c_2} |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j \leq \int_{|t| \leq c_2} \exp\left(-\frac{2^N t^2}{12}\right) dt \leq \frac{c}{2^{N/2}}. \quad (6.26)$$

Analogously, using (6.2), (6.4), (6.25), we have

$$\int_{|t_j| > c_2} |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j \leq \int_{|t| > c_2} c_3^{2^{N-2}} |\widehat{H}_{h_j}(t)| dt \leq c \exp(-c \cdot 2^N) \leq \frac{c}{2^{N/2}}. \tag{6.27}$$

From (6.26), (6.27) it follows that

$$\int_{\mathbb{R}^1} |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j \leq \frac{c}{2^{N/2}}. \tag{6.28}$$

Using now (6.24), (6.28), we see that

$$(2\pi)^{-d} \int_T |\widehat{F}_h(t)| dt \leq \frac{1}{2^{Nd/2} (2\pi)^{d/2}} \exp\left(c_{26} d - \frac{c_{27} \cdot 2^N}{\tau^2 d^2}\right). \tag{6.29}$$

It is evident that, choosing c_{18} to be small enough, we can ensure the validity of the inequality

$$\exp\left(c_{26} d - \frac{c_{27} \cdot 2^N}{\tau^2 d^2}\right) \leq \frac{c_{24}^2 \tau^2 d^2}{2^N} \leq \frac{1}{4}. \tag{6.30}$$

The inequalities (6.29), (6.30) imply (6.19).

For the proof of (6.20) we apply the Hölder inequality and obtain that

$$\int_T |\langle t, v \rangle \widehat{F}_h(t)| dt \leq \|v\| \left(\int_T |\widehat{F}_h(t)| dt\right)^{1/2} \left(\int_T \|t\|^2 |\widehat{F}_h(t)| dt\right)^{1/2}. \tag{6.31}$$

For estimating the first integral in the right-hand side of (6.31) we note that, according to (6.29), (6.30),

$$(2\pi)^{-d} \int_T |\widehat{F}_h(t)| dt \leq \frac{1}{4(2\pi)^{d/2} \cdot 2^{Nd/2}}. \tag{6.32}$$

Let us estimate the second integral. For this, using (6.3), we obtain

$$\int_{|t_j| \leq c_2} t_j^2 |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j \leq \int_{|t| \leq c_2} t^2 \exp\left(-\frac{2^N t^2}{12}\right) dt \leq \frac{c}{2^{3N/2}}. \tag{6.33}$$

Moreover, using (6.2), (6.4), (6.25), we have

$$\begin{aligned} \int_{|t_j| > c_2} t_j^2 |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j &\leq \int_{|t| > c_2} c_3^{2^{N-2}} t^2 |\widehat{H}_{h_j}(t)| dt \\ &\leq c \exp(-c \cdot 2^N) \leq \frac{c}{2^{3N/2}}. \end{aligned} \tag{6.34}$$

From (6.33), (6.34) it follows that

$$\int_{\mathbb{R}^1} t_j^2 |\widehat{H}_{h_j}(t_j)|^{2^{N-1}} dt_j \leq \frac{c}{2^{3N/2}}. \tag{6.35}$$

Using now (6.24), (6.28), (6.35), we see that

$$(2\pi)^{-d} \int_T \|t\|^2 |\widehat{F}_h(t)| dt \leq \frac{1}{2^N (2\pi)^{d/2} \cdot 2^{Nd/2}} \exp\left(c_{28} d - \frac{c_{29} \cdot 2^N}{\tau^2 d^2}\right). \tag{6.36}$$

It is evident that, choosing c_{18} to be small enough, we can ensure the validity of the inequality

$$\exp\left(c_{28} d - \frac{c_{29} \cdot 2^N}{\tau^2 d^2}\right) \leq \frac{1}{4}. \tag{6.37}$$

The inequalities (6.31), (6.32), (6.36), (6.37) imply (6.20).

The analogs of the relations (6.19), (6.20) for the distributions F' follow, in fact, from (6.19), (6.20) applied in the $(d - 1)$ -dimensional case.

Using Lemma 2.5, we can analogously prove that the conditions of Lemmas 2.14 and 3.1 with $D = 2^N I$, $\sigma^2 = 2^N$ and with the change of τ by $c_{24} \tau$ are satisfied in the j -dimensional situation for the distributions $\mathcal{L}(U_{N,0}^j), \mathcal{L}(V_{N,0}^j), j = 1, \dots, d$, where the vectors

$$U_{N,0}^j = (U_{N,0}^{(1)}, \dots, U_{N,0}^{(j)}), \quad V_{N,0}^j = (V_{N,0}^{(1)}, \dots, V_{N,0}^{(j)}) \tag{6.38}$$

consist of the first j coordinates of the vectors $U_{N,0}, V_{N,0}$ respectively.

Taking into account (5.7), noting that all distribution functions $G_{N,0}^{(j)}$ coincide with $\Phi_\sigma(\cdot)$ and applying Lemma 3.1 and the inequality (2.16) of Lemma 2.14 in the j -dimensional case, $j = 1, \dots, d$, we obtain that

$$|U_{N,0}^{(1)} - V_{N,0}^{(1)}| \leq c_{30} \tau \left(1 + \frac{|U_{N,0}^{(1)}|^2}{2^N}\right), \tag{6.39}$$

if $\frac{\tau}{2^{N/2}} \leq c_{31}, |U_{N,0}^{(1)}| \leq \frac{c_{32} \cdot 2^N}{\tau}$;

$$|U_{N,0}^{(j)} - V_{N,0}^{(j)}| \leq c_{30} \tau \left(j^{3/2} + j^{3/2} \frac{|U_{N,0}^{j-1}|}{2^{N/2}} \left(1 + \frac{|U_{N,0}^{(j)}|}{2^{N/2}}\right) + \frac{|U_{N,0}^{(j)}|^2}{2^N} + j \frac{|U_{N,0}^{j-1}|^2}{2^N} \right), \tag{6.40}$$

if

$$\frac{\tau j^{3/2}}{2^{N/2}} \leq c_{31}, \quad |U_{N,0}^{(j)}| \leq \frac{c_{32} \cdot 2^N}{j \tau}, \quad \frac{|U_{N,0}^{j-1}|}{2^{N/2}} \leq \frac{c_{33} \cdot 2^{N/2}}{j^{3/2} \tau}, \quad 2 \leq j \leq d. \tag{6.41}$$

Checking the inequality (6.40), one should take into account that in Lemmas 2.14 and 3.1 the densities $v(\cdot)$ are the densities of shifted conditional distributions. The possibility to write the inequality (6.40) and the conditions (6.41) without shift is ensured by a suitable choice of sufficiently small c_{32} and c_{33} with the use of inequality (2.16). Inequality (6.17) with $c_{18} \leq c_{31}, c_{19} = 4c_{30}, c_{20} = \min\{c_{32}, c_{33}\}$ follows from (6.39), (6.40) and from the definitions of $U_{N,0}, U_{N,0}^j$.

Let us prove assertion b). Let $L_{n,k}^* = (U_{n-1,2k}, U_{n-1,2k+1}) \in \mathbb{R}^{2d}$ be the vector with first d coordinates coinciding with those of $U_{n-1,2k}$ and with last d coordinates coinciding with those of $U_{n-1,2k+1}$. Denote by $L_{n,k} = (U_{n,k}, \tilde{U}_{n,k}) \in \mathbb{R}^{2d}$ the vector constructed analogously from the coordinates of the vectors $U_{n,k}, \tilde{U}_{n,k}$. It is evident that the vectors $U_{n-1,2k}$ and $U_{n-1,2k+1}$ are independent. Moreover, according to (6.15), the distributions of these vectors belong to $\mathcal{A}_d(c_{15}\tau)$. From Lemma 2.7 it follows that $\mathcal{L}(L_{n,k}^*) \in \mathcal{A}_{2d}(c_{15}\tau)$. The relations (5.12) show that $L_{n,k} = AL_{n,k}^*$, where $A : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear operator with $\|A\| = \sqrt{2}$. Therefore, using Lemma 2.5, we see that $\mathcal{L}(L_{n,k}) \in \mathcal{A}_{2d}(c_{15}\sqrt{2}\tau)$.

Consider as well the Gaussian random vectors $R_{n,k} = (V_{n,k}, \tilde{V}_{n,k}) \in \mathbb{R}^{2d}$. It can be easily verified that (see (5.1)–(5.6)) $\mathcal{L}(R_{n,k}) = \Phi(\mathcal{L}(L_{n,k}))$. Define also the vectors

$$L_{n,k}^j = (L_{n,k}^{(1)}, \dots, L_{n,k}^{(j)}), \quad R_{n,k}^j = (R_{n,k}^{(1)}, \dots, R_{n,k}^{(j)}), \quad j = d, \dots, 2d, \tag{6.42}$$

consisting of the first j coordinates of the vectors $L_{n,k}, R_{n,k}$ respectively.

One can show that if c_{21} is small enough, then the distributions $\mathcal{L}(L_{n,k}^j)$ and $\mathcal{L}(R_{n,k}^j)$, $j = d + 1, \dots, 2d$, satisfy in j -dimensional situation the conditions of Lemmas 2.14, 3.1 with $D = 2^n I$, $\sigma^2 = 2^n$ and with the change of d by j and τ by $c_{34}\tau$, where $c_{34} = \max\{c_{15}\sqrt{2}, c_1^{-1}, c_6^{-1}, 1\}$. For this one should fix j and verify that for $h^* \in \mathbb{R}^j$, $\|h^*\| c_{34}\tau < 1$, and for all $v^* \in \mathbb{R}^j$, the following inequalities hold (see (2.15) and Definition 2.12):

$$(2\pi)^{-j} \int_T |\widehat{F}_{h^*}(t^*)| dt^* \leq \frac{c_{34}^2 \tau^2 j^2}{2^n (2\pi)^{j/2} \cdot 2^{nj/2}}, \tag{6.43}$$

$$(2\pi)^{-j} \int_T |\langle t^*, v^* \rangle \widehat{F}_{h^*}(t^*)| dt^* \leq \frac{\|v^*\|}{4 \cdot 2^{n/2} (2\pi)^{j/2} \cdot 2^{nj/2}}, \tag{6.44}$$

where $F_{h^*} = \overline{F}(h^*)$, $F = \mathcal{L}(L_{n,k}^j)$, $h^* = (h_1^*, \dots, h_d^*) \in \mathbb{R}^j$,

$$\begin{aligned} T &= \{t^* = (t_1^*, \dots, t_d^*) \in \mathbb{R}^j : 4 \|t^*\| c_{34}\tau j \geq 1\}; \\ h_m^* &= (h_m, h_{d+m}) \in \mathbb{R}^2, \quad t_m^* = (t_m, t_{d+m}) \in \mathbb{R}^2, \quad m = 1, \dots, j-d; \\ h_m^* &= h_m \in \mathbb{R}^1, \quad t_m^* = t_m \in \mathbb{R}^1, \quad m = j-d+1, \dots, d. \end{aligned} \tag{6.45}$$

Besides, we need the validity of analogous relations in the $(j-1)$ -dimensional situation for the distributions $F^l = \mathcal{L}(L_{n,k}^{j-1})$. Thus, one should verify the fulfilment of (6.43), (6.44) for $j = d, \dots, 2d$.

Using the definition of distributions $\Psi(\cdot)$ and of random vectors $L_{n,k}^j$ and the relations (2.2), (5.1), (5.2), (5.12), we obtain that

$$|\widehat{F}_{h^*}(t^*)| \leq \prod_{m=1}^{j-d} |\widehat{H}_{h_m^*}(t_m^*)|^{2^{n-1}} \prod_{m=j-d+1}^d |\widehat{H}_{h_m}(t_m)|^{2^n}.$$

By the definition of c_{34} , we can use for estimating $|\widehat{H}_{h_m}(t_m)|$ the inequalities (6.2)–(6.4) and for estimating $|\widehat{H}_{h_m^*}(t_m^*)|$ the inequalities (6.5)–(6.7).

The rest of the proof of inequalities (6.43), (6.44) does not differ from the proof of (6.19), (6.20) and therefore is omitted. The only difference is in the use of (6.5)–(6.7) instead of (6.2)–(6.4) and in the change of constants and notation (see also the proof of (6.43), (6.44) in somewhat more complicated situation in Section 7).

Note that all distribution functions $\tilde{G}_{n,k}^{(j-d)}(\cdot|\cdot)$ coincide with $\Phi_\sigma(\cdot)$. Therefore, using (5.8), (5.9) and applying Lemma 3.1 and the inequality (2.16) of Lemma 2.14 in the j -dimensional case, $j = d + 1, \dots, 2d$, we obtain that

$$|L_{n,k}^{(j)} - R_{n,k}^{(j)}| \leq c_{35} \tau \left(j^{3/2} + j^{3/2} \frac{|L_{n,k}^{j-1}|}{2^{n/2}} \left(1 + \frac{|L_{n,k}^{(j)}|}{2^{n/2}} \right) + \frac{|L_{n,k}^{(j)}|^2}{2^n} + j \frac{|L_{n,k}^{j-1}|^2}{2^n} \right), \quad (6.46)$$

if

$$\frac{\tau j^{3/2}}{2^{n/2}} \leq c_{36}, \quad |L_{n,k}^{(j)}| \leq \frac{c_{37} \cdot 2^n}{j \tau}, \quad \frac{|L_{n,k}^{j-1}|}{2^{n/2}} \leq \frac{c_{38} \cdot 2^{n/2}}{j^{3/2} \tau}. \quad (6.47)$$

Checking the inequality (6.46) should be carried out analogously to the proof of (6.40). The possibility to write the inequality (6.46) and the conditions (6.47) without shift is also ensured by a suitable choice of sufficiently small constants (c_{37} and c_{38}) with the use of inequality (2.16).

Inequality (6.18) follows from (6.42), (6.46) and from the definition of $L_{n,k}, R_{n,k}$ provided that we choose $c_{21} \leq 2^{-3/2} c_{36}$, $c_{22} = 2^{3/2} \cdot 4 c_{35}$ and $c_{23} = 2^{-3/2} \min \{ c_{37}, c_{38} \}$. \square

Proof of Theorem 6.4. Let us prove that there exists a c_{39} such that

$$\mathbb{P} \{ \Delta \geq x \} \leq (15d)^N \exp \left(- \frac{x}{c_{39} d^{3/2} \tau} \right), \quad x \geq 0. \quad (6.48)$$

Denote

$$c_{40} = \min \{ c_{20}, c_{23}, c_{15}^{-1} \}, \quad c_{41} = \frac{4c_{40}}{\min \{ c_{18}^2, c_{21}^2, 2^{-1} \}}, \quad y = \frac{c_{40}}{d^{3/2} \tau}. \quad (6.49)$$

Assume (without loss of generality) that

$$x \geq c_{41} d^{3/2} \tau \quad (6.50)$$

and choose the integer M from the condition

$$x < 4y \cdot 2^M \leq 2x. \quad (6.51)$$

Note that $M \geq 1$ since (6.49), (6.50) and $\tau \geq 1$ imply the inequality $x \geq 8y$. Denote $L = \min \{ M, N \}$,

$$\Delta_1 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^L} |S_{k \cdot 2^M + s} - S_{k \cdot 2^M}|, \quad (6.52)$$

$$\Delta_2 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^L} |T_{k \cdot 2^M + s} - T_{k \cdot 2^M}|, \quad (6.53)$$

$$\Delta_3 = \max_{0 \leq k < 2^{N-M}} |S_{k \cdot 2^M} - T_{k \cdot 2^M}|, \quad (6.54)$$

where the maximums are taken over integer values of k and s . It is evident that

$$\Delta \leq \Delta_1 + \Delta_2 + \Delta_3. \tag{6.55}$$

Define the event

$$A = \{\omega : \max\{\Delta_1, \Delta_2\} < x/4\} \tag{6.56}$$

(we shall always assume that all considered random vectors are measurable mappings of elementary event $\omega \in \Omega$). Recall that all d -dimensional Gaussian distributions belong to classes $\mathcal{A}_d(\tau)$ for all $\tau \geq 0$. Applying Lemma 2.11 with parameters $\tau^* = c_{15} \tau$, $B^2 = 2^L \leq 2^M$, and using (6.49), (6.51)–(6.53) and (6.56), we obtain that

$$\begin{aligned} 1 - \mathbb{P}\{A\} &\leq 2^N \cdot 4d \max\{\exp(-x^2/64B^2), \exp(-x/16\tau^*)\} \\ &\leq 2^N \cdot 4d \exp\left(-\frac{cx}{d^{3/2}\tau}\right). \end{aligned} \tag{6.57}$$

If $M \geq N$ then $\Delta_3 = 0$, and, for $\omega \in A$, we have $\Delta < x/2$ (see (6.55) and (6.56)). Hence, $\mathbb{P}\{\Delta \geq x\} \leq 1 - \mathbb{P}\{A\}$ and one can conclude via (6.57).

Let now $M < N$. Fix some integer k , $1 \leq k \leq 2^{N-M}$. According to Corollary 5.3, we have

$$|S_{k \cdot 2^M} - T_{k \cdot 2^M}| \leq |U_{N,0} - V_{N,0}| + \sum_{n=M+1}^N |\tilde{U}_{n,l_{n,j}} - \tilde{V}_{n,l_{n,j}}|, \tag{6.58}$$

where $l_{n,j}$ are integers, defined by $l_{n,j} \cdot 2^n < j \stackrel{\text{def}}{=} k \cdot 2^M \leq (l_{n,j} + 1) \cdot 2^n$ (see (5.13)).

By virtue of (6.49)–(6.51), we have $c_{41} d^{3/2} \tau \leq x \leq \frac{4c_{40} \cdot 2^M}{d^{3/2} \tau}$ and, consequently,

$$\tau \leq \frac{\min\{c_{18}, c_{21}\} \cdot 2^{M/2}}{d^{3/2}}, \quad \frac{\tau d^{3/2}}{2^{M/2}} \leq \min\{c_{18}, c_{21}\}. \tag{6.59}$$

Recall that $U_{n,l} = S_{(l+1) \cdot 2^n} - S_{l \cdot 2^n}$. According to (6.49), (6.51) and (6.56), for $\omega \in A$ we have

$$|U_{M,l}| \leq \Delta_1 < y \cdot 2^M \leq \frac{\min\{c_{20}, c_{23}\} \cdot 2^M}{d^{3/2} \tau}, \quad 0 \leq l < 2^{N-M}. \tag{6.60}$$

The relations (6.59), (6.60) allow us to apply Lemma 6.5 for estimating the right-hand side of (6.58). Using exactly the same arguments as in KMT (1975–76, pp. 120–121), we obtain (for $\omega \in A$) the bound

$$|S_{k \cdot 2^M} - T_{k \cdot 2^M}| \leq c_{42} d^{3/2} \tau \left(N + W_{M,l_{M,j}} + \sum_{n=M}^{N-1} W_{n,\tilde{l}_{n,j}} \right), \tag{6.61}$$

where (for $0 \leq n \leq N, 0 \leq l < 2^{N-n}$)

$$W_{n,l} = \begin{cases} 2^{-n} |U_{n,l}|^2, & \text{if } |U_{n,l}| \leq y \cdot 2^n, \\ 0, & \text{otherwise.} \end{cases} \tag{6.62}$$

and $U_{n,l_{n,j}}$, $M \leq n \leq N$, is the sum over the block of 2^n summands which contains the summand X_j (see Remark 5.4). The sum $U_{n,\tilde{l}_{n,j}}$ does not contain X_j and $U_{n+1,l_{n+1,j}} = U_{n,l_{n,j}} + U_{n,\tilde{l}_{n,j}}$, $M \leq n < N$ (see (5.12)). Thus, all summands in the right-hand side of (6.61) are functions of sums of disjoint blocks of independent summands. Therefore, they are themselves independent.

Let us show that

$$\mathbb{E} \exp(tW_{n,l}) \leq 2d + 1, \quad \text{for } 0 \leq t \leq \frac{1}{8}. \tag{6.63}$$

Indeed, integrating by parts, we obtain

$$\begin{aligned} \mathbb{E} \exp(tW_{n,l}) &= 1 + \int_0^{y^2 \cdot 2^n} t e^{tu} \mathbb{P} \{W_{n,l} \geq u\} du \\ &\leq 1 + \frac{1}{8} \int_0^{y^2 \cdot 2^n} e^{u/8} \mathbb{P} \{|U_{n,l}| \geq 2^{n/2} \sqrt{u}\} du. \end{aligned} \tag{6.64}$$

Recall that the sum $U_{n,l}$ contains 2^n independent summands X_i . Hence, using Lemma 2.11, (6.15) and (6.49), we get

$$\mathbb{P} \{|U_{n,l}| \geq 2^{n/2} \sqrt{u}\} \leq 2d \exp\left(-\min\left\{\frac{2^n u}{4 \cdot 2^n}, \frac{2^{n/2} \sqrt{u}}{4c_{15} \tau}\right\}\right) = 2de^{-u/4} \tag{6.65}$$

provided that $0 \leq u \leq y^2 \cdot 2^n$. The relation (6.63) immediately follows from (6.64) and (6.65).

The relations (6.61), (6.63) imply that

$$\mathbb{E} e^{h\Delta_3} \mathbf{1}_A \leq 2^N \cdot (2d + 1)^N e^{N/8}, \quad \text{where } h = (8c_{42} d^{3/2} \tau)^{-1} \tag{6.66}$$

and $\mathbf{1}_A$ is the indicator function of the event A . By (6.55) and (6.56), we have $\Delta < x/2 + \Delta_3$, for $\omega \in A$. Therefore,

$$\mathbb{P} \{\Delta \geq x\} \leq 1 - \mathbb{P} \{A\} + e^{-hx/2} \mathbb{E} e^{h\Delta_3} \mathbf{1}_A. \tag{6.67}$$

Now (6.48) follows via (6.57), (6.66) and (6.67).

Let the quantities $\varepsilon, x_0 > 0$ be defined by the equalities $\varepsilon = \frac{1}{2c_{39} d^{3/2} \tau}$, $e^{\varepsilon x_0} = (15d)^N$. Integrating by parts and using (6.48), we obtain

$$\begin{aligned} \mathbb{E} e^{\varepsilon \Delta} &= 1 + \int_0^\infty \varepsilon e^{\varepsilon x} \mathbb{P} \{\Delta \geq x\} dx, \\ \int_0^{x_0} \varepsilon e^{\varepsilon x} \mathbb{P} \{\Delta \geq x\} dx &\leq \int_0^{x_0} \varepsilon e^{\varepsilon x} dx = e^{\varepsilon x_0} - 1 = (15d)^N - 1, \\ \int_{x_0}^\infty \varepsilon e^{\varepsilon x} \mathbb{P} \{\Delta \geq x\} dx &\leq \int_{x_0}^\infty \varepsilon e^{-\varepsilon(x-x_0)} dx = 1, \end{aligned}$$

and, hence, $\mathbb{E} e^{\varepsilon \Delta} \leq (15d)^N + 1 \leq (16d)^N$. Together with (6.16) this completes the proof of Theorem 6.4. \square

7. ESTIMATES IN THE DYADIC SCHEME
 UNDER THE CONDITION OF COINCIDENCE OF MOMENTS
 OF THE FIRST THREE ORDERS

Suppose that p, N are natural numbers, $p \leq N$, $\tau \geq 1$, and $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^N}$ are random vectors with known distributions $\mathcal{L}(\tilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E} \tilde{\xi}_k = 0$, $\text{cov} \tilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Let $\tilde{\eta}_k$ be random vectors which have the same moments of the first three orders as the vectors $\tilde{\xi}_k$, $\mathcal{L}(\tilde{\eta}_k) \in \mathcal{A}_d(\tau)$, $k = 1, \dots, 2^N$. Let Y_1, \dots, Y_{2^N} be independent random vectors with distributions $\mathcal{L}(Y_k) = \Psi(\mathcal{L}(\tilde{\eta}_k))$, $k = 1, \dots, 2^N$, where the distributions $\Psi(\cdot)$ are defined in Section 6. Consider as well the random vectors ξ_1, \dots, ξ_{2^N} with $\mathcal{L}(\xi_l) = \mathcal{L}(Y_l)$ for $l = (m - 1) \cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$, and with $\mathcal{L}(\xi_l) = \mathcal{L}(\tilde{\xi}_l)$ otherwise. Let the random vectors X_k , $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$; $V_{n,k}, \tilde{V}_{n,k}$, and $U_{n,k}, \tilde{U}_{n,k}$ be constructed as functions of Y_1, \dots, Y_{2^N} as in Section 5.

REMARK 7.1. From Lemma 6.1 and from the definition of the distributions $\Psi(\cdot)$ it follows that the vectors ξ_k, X_k, Y_k have the same moments of the first three orders for each $k = 1, \dots, 2^N$.

Let c_1, \dots, c_{42} be absolute positive constants from Section 6. By virtue of (6.14) we have

$$\mathcal{L}(X_k), \mathcal{L}(Y_k) \in \mathcal{A}_d(c_{15} \tau), \quad k = 1, \dots, 2^N. \tag{7.1}$$

LEMMA 7.2. *There exist absolute positive constants c_{43}, \dots, c_{48} such that*

a) *if $1 \leq n \leq N$, $0 \leq k < 2^{N-n}$, $\tau^2 d^3 p / 2^{n-p} \leq c_{43}$, then*

$$\begin{aligned} |\tilde{U}_{n,k} - \tilde{V}_{n,k}| \leq & \frac{c_{44} d^2 \tau^2}{2^{n/2}} \left(1 + 2^{-3n/2} \max\{|U_{n,k}|^3, |\tilde{U}_{n,k}|^3\} \right) \\ & + \frac{1}{8} |U_{n,k} - V_{n,k}| \end{aligned} \tag{7.2}$$

provided that $\max\{|U_{n,k}|, |V_{n,k}|, |\tilde{U}_{n,k}|\} \leq \frac{c_{45} \cdot 2^n}{d^{3/2} \tau}$;

b) *if $\tau^2 d^3 p / 2^{N-p} \leq c_{46}$, then*

$$|U_{N,0} - V_{N,0}| \leq c_{47} d^2 \tau^2 \cdot 2^{-N/2} (1 + 2^{-3N/2} |U_{N,0}|^3) \tag{7.3}$$

provided that $|U_{N,0}| \leq \frac{c_{48} \cdot 2^N}{d^{3/2} \tau}$.

Proof. At first we prove assertion a). Suppose that the vectors $L_{n,k}^*, L_{n,k}$ are constructed from coordinates of vectors $U_{n-1,2k}, U_{n-1,2k+1}$; $U_{n,k}, \tilde{U}_{n,k}$ as in the proof of Lemma 6.5. Let the vectors $R_{n,k}^*, R_{n,k}$ be analogously constructed from coordinates of vectors $V_{n-1,2k}, V_{n-1,2k+1}$; $V_{n,k}, \tilde{V}_{n,k}$. It is evident that the vectors $U_{n-1,2k}$ and $U_{n-1,2k+1}$ are as before independent. The same may be said about the vectors $V_{n-1,2k}, V_{n-1,2k+1}$. Using the relations (5.2), (7.1), the properties of classes $\mathcal{A}_d(\tau)$, the definition of classes $\mathcal{R}_d(\tau, \varepsilon)$, Lemma 2.8 and Remarks 2.3, 7.1, we can easily verify that $(\mathcal{L}(L_{n,k}^*), \mathcal{L}(R_{n,k}^*)) \in \mathcal{R}_{2d}(c_{15} \tau, 0)$. The relations (5.6), (5.12) show that $L_{n,k} = AL_{n,k}^*$, $R_{n,k} = AR_{n,k}^*$, where $A : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear operator with $\|A\| = \sqrt{2}$. Therefore, using Corollary 2.6, we see that

$$(\mathcal{L}(L_{n,k}), \mathcal{L}(R_{n,k})) \in \mathcal{R}_{2d}(c_{15} \sqrt{2} \tau, 0). \tag{7.4}$$

Now we define the vectors $L_{n,k}^j, R_{n,k}^j$, $j = d, \dots, 2d$, consisting of the first j coordinates of the vectors $L_{n,k}, R_{n,k}$ respectively (see (6.42)). Below we show that the distributions $\mathcal{L}(L_{n,k}^j), \mathcal{L}(R_{n,k}^j)$, $j = d+1, \dots, 2d$, satisfy in j -dimensional situation the conditions of Lemmas 2.15, 4.1 with $D = 2^n I$, $\sigma^2 = 2^n$ and with the change of d by j and τ by $c_{34} \tau$, where $c_{34} = \max\{c_{15} \sqrt{2}, c_1^{-1}, c_6^{-1}, 1\}$ was already defined above. For this one should fix j and verify that, if $\tau^2 d^3 p / 2^{n-p} \leq c_{49}$ with sufficiently small c_{49} , then for $h^* \in \mathbb{R}^j$, $\|h^*\| c_{34} \tau < 1$, and for all $v^* \in \mathbb{R}^j$ the inequalities (6.43), (6.44) hold, where $F_{h^*} = \overline{F}(h^*)$, $F = \mathcal{L}(L_{n,k}^j)$ or $F = \mathcal{L}(R_{n,k}^j)$, $h^* = (h_1^*, \dots, h_d^*)$ and the relations (6.45) are satisfied (see (2.20) and Definition 2.12). Besides, we need the validity of analogous relations in the $(j-1)$ -dimensional situation for the distributions F' . Finally, one should take into account the relation (7.4). It is clear that it suffices to check the conditions (6.43), (6.44) for $F = \mathcal{L}(L_{n,k}^j)$, $j = d, \dots, 2d$, if c_{49} is small enough. For $F = \mathcal{L}(R_{n,k}^j)$ these conditions can be verified analogously.

Choosing c_{49} to be sufficiently small and taking into account that $\tau \geq 1$, we ensure the validity of the inequality

$$2^{n-p-2} \geq 2. \quad (7.5)$$

From the definition of distributions $\Psi(\cdot)$ and random vectors $L_{n,k}^j$ and from (2.2), (5.1), (5.2), (5.12), (7.5) it is easy to deduce that (in the conditions of Section 7)

$$|\widehat{F}_{h^*}(t^*)| \leq \prod_{m=1}^{j-d} |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-1}} \prod_{m=j-d+1}^d |\widehat{H}_{h_m}(t_m)|^{2^{n-p}}. \quad (7.6)$$

By definition of c_{34} , we can use for estimating $|\widehat{H}_{h_m}(t_m)|$ the inequalities (6.2)–(6.4) and for estimating $|\widehat{H}_{h_m^*}^*(t_m^*)|$ the inequalities (6.5)–(6.7). In particular, from (6.2), (6.3), (6.5), (6.6), (6.45) one can easily derive that

$$\sup_{t \in T} \prod_{m=1}^{j-d} |\widehat{H}_{h_m^*}^*(t_m^*)| \prod_{m=j-d+1}^d |\widehat{H}_{h_m}(t_m)|^2 \leq \max\{\exp(-(48c_{34}^2 \tau^2 d^2)^{-1}), c_3^2, c_8\} \\ \leq \exp\left(-\frac{c_{50}}{\tau^2 d^2}\right). \quad (7.7)$$

From (7.6), (7.7) it follows that, for $t^* \in T$,

$$|\widehat{F}_{h^*}(t^*)| \leq \exp\left(-\frac{2^{n-p-2} c_{50}}{\tau^2 d^2}\right) \prod_{m=1}^{j-d} |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} \prod_{m=j-d+1}^d |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}}. \quad (7.8)$$

Using (6.3), (6.6), we obtain

$$\int_{|t_m^*| \leq c_7} |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* \leq \int_{|t| \leq c_7, t \in \mathbb{R}^2} \exp\left(-\frac{2^{n-p} \|t\|^2}{12}\right) dt \leq \frac{c}{2^{n-p}}, \\ \int_{|t_m| \leq c_2} |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m \leq \int_{|t| \leq c_2, t \in \mathbb{R}^1} \exp\left(-\frac{2^{n-p} t^2}{12}\right) dt \leq \frac{c}{2^{(n-p)/2}}. \quad (7.9)$$

Analogously, using (6.2), (6.4), (6.5), (6.7), (7.5), we have

$$\begin{aligned} \int_{|t_m^*| > c_7} |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* &\leq \int_{|t| > c_7} c_3^{2^{n-p-3}} |\widehat{H}_{h_m^*}^*(t)| dt \\ &\leq c \exp(-c \cdot 2^{n-p}) \leq \frac{c}{2^{n-p}}, \\ \int_{|t_m| > c_2} |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m &\leq \int_{|t| > c_2} c_3^{2^{n-p-2}} |\widehat{H}_{h_m}(t)| dt \\ &\leq c \exp(-c \cdot 2^{n-p}) \leq \frac{c}{2^{(n-p)/2}}. \end{aligned} \tag{7.10}$$

From (7.9), (7.10) it follows that

$$\int_{\mathbb{R}^2} |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* \leq \frac{c}{2^{n-p}}, \quad \int_{\mathbb{R}^1} |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m \leq \frac{c}{2^{(n-p)/2}}. \tag{7.11}$$

The inequalities (7.8), (7.11) imply that

$$(2\pi)^{-j} \int_T |\widehat{F}_{h^*}(t^*)| dt^* \leq \frac{1}{2^p \cdot 2^{nj/2} (2\pi)^{j/2}} \exp\left(c_{51} dp - \frac{c_{52} \cdot 2^{n-p}}{\tau^2 d^2}\right). \tag{7.12}$$

Choosing c_{49} to be small enough, we can ensure the validity of the inequalities

$$\exp\left(c_{51} dp - \frac{c_{52} \cdot 2^{n-p}}{\tau^2 d^2}\right) \leq \frac{c_{34}^2 \tau^2 d^2}{2^{n-p}} \leq \frac{c_{34}^2 \tau^2 j^2}{2^{n-p}} \leq \frac{4c_{34}^2 \tau^2 d^2}{2^{n-p}} \leq \frac{1}{4}. \tag{7.13}$$

The inequalities (7.12), (7.13) imply (6.43).

For the proof of (6.44) we apply the Hölder inequality and obtain that

$$\int_T |\langle t^*, v^* \rangle \widehat{F}_{h^*}(t^*)| dt^* \leq \|v^*\| \left(\int_T |\widehat{F}_{h^*}(t^*)| dt^* \int_T \|t^*\|^2 |\widehat{F}_{h^*}(t^*)| dt^* \right)^{1/2}. \tag{7.14}$$

For estimating the first integral in the right-hand side of (7.14) we note that, according to (7.12), (7.13),

$$(2\pi)^{-j} \int_T |\widehat{F}_{h^*}(t^*)| dt^* \leq \frac{1}{4(2\pi)^{j/2} \cdot 2^{nj/2}}. \tag{7.15}$$

Let us estimate the second integral. By (6.3), (6.6), we have

$$\begin{aligned} \int_{|t_m^*| \leq c_7} \|t_m^*\|^2 |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* &\leq \int_{|t| \leq c_7, t \in \mathbb{R}^2} \|t\|^2 \exp\left(-\frac{2^{n-p} \|t\|^2}{12}\right) dt \\ &\leq \frac{c}{2^{2(n-p)}}, \\ \int_{|t_m| \leq c_2} t_m^2 |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m &\leq \int_{|t| \leq c_2, t \in \mathbb{R}^1} t^2 \exp\left(-\frac{2^{n-p} t^2}{12}\right) dt \\ &\leq \frac{c}{2^{3(n-p)/2}}. \end{aligned} \tag{7.16}$$

Moreover, using (6.2), (6.4), (6.5), (6.7), (7.5), we obtain

$$\begin{aligned} \int_{|t_m^*| > c_7} \|t_m^*\|^2 |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* &\leq \int_{|t| > c_7} c_8^{2^{n-p-3}} \|t\|^2 |\widehat{H}_{h_m^*}^*(t)| dt \\ &\leq c \exp(-c \cdot 2^{n-p}) \leq \frac{c}{2^{2(n-p)}}, \\ \int_{|t_m| > c_2} t_m^2 |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m &\leq \int_{|t| > c_2} c_3^{2^{n-p-2}} t^2 |\widehat{H}_{h_m}(t)| dt \\ &\leq c \exp(-c \cdot 2^{n-p}) \leq \frac{c}{2^{3(n-p)/2}}. \end{aligned} \tag{7.17}$$

From (7.16), (7.17) it follows that

$$\begin{aligned} \int_{\mathbb{R}^2} \|t_m^*\|^2 |\widehat{H}_{h_m^*}^*(t_m^*)|^{2^{n-p-2}} dt_m^* &\leq \frac{c}{2^{2(n-p)}}, \\ \int_{\mathbb{R}^1} t_m^2 |\widehat{H}_{h_m}(t_m)|^{2^{n-p-1}} dt_m &\leq \frac{c}{2^{3(n-p)/2}}. \end{aligned} \tag{7.18}$$

Using now (7.8), (7.11), (7.18), we see that

$$(2\pi)^{-j} \int_T \|t^*\|^2 |\widehat{F}_{h^*}(t^*)| dt^* \leq \frac{1}{2^n (2\pi)^{j/2} \cdot 2^{nj/2}} \exp\left(c_{53} dp - \frac{c_{54} \cdot 2^{n-p}}{\tau^2 d^2}\right). \tag{7.19}$$

Choosing c_{49} to be small enough, we can ensure the validity of the inequality

$$\exp\left(c_{53} dp - \frac{c_{54} \cdot 2^{n-p}}{\tau^2 d^2}\right) \leq \frac{1}{4}. \tag{7.20}$$

The inequalities (7.14), (7.15), (7.19), (7.20) imply (6.44).

Using (2.21), (4.2), (5.8), (5.9) and applying Lemma 4.1 and the inequality (2.22) of Lemma 2.15 (for the parameter $\tilde{\tau} = c_{34} \tau$) in the j -dimensional case, $j = d + 1, \dots, 2d$, we obtain that

$$\begin{aligned} |L_{n,k}^{(j)} - R_{n,k}^{(j)}| &\leq \frac{c\tau^2}{2^{n/2}} \left(j^2 + (\Gamma_{j-1}^2 + 2^{n/2} \tau^{-1} \Lambda_{j-1}) \left(j + \frac{|L_{n,k}^{(j)}|}{2^{n/2}} \right) + \frac{|L_{n,k}^{(j)}|^3}{2^{3n/2}} \right. \\ &\quad \left. + \Gamma_{j-1}^3 + 2^{n/2} \tau^{-1} \Lambda_{j-1} \Gamma_{j-1} \right), \end{aligned} \tag{7.21}$$

where

$$\Lambda_{j-1} = 2^{-n/2} j^{1/2} |L_{n,k}^{j-1} - R_{n,k}^{j-1}|, \quad \Gamma_{j-1} = 2^{-n/2} j^{1/2} \max\{|L_{n,k}^{j-1}|, |R_{n,k}^{j-1}|\}, \tag{7.22}$$

provided that

$$\frac{\tau^2 j^3 p}{2^{n-p}} \leq c_{55}, \quad |L_{n,k}^{(j)}| \leq \frac{c_{56} \cdot 2^n}{j\tau}, \quad \frac{\max\{|L_{n,k}^{j-1}|, |R_{n,k}^{j-1}|\}}{2^{n/2}} \leq \frac{c_{57} \cdot 2^{n/2}}{j^{3/2} \tau} \tag{7.23}$$

and c_{55}, c_{56}, c_{57} are small enough. Verifying the inequality (7.21), one should take into account that in Lemmas 2.15 and 4.1 the densities $v_m(\cdot)$, $m = 1, 2$, are the densities of shifted conditional distributions. The possibility to write the inequality (7.21) and the conditions (7.23) without shift is ensured by choosing c_{56}, c_{57} to be sufficiently small and by using the inequality (2.16) for y_m , $m = 1, 2$, involved in Lemma 2.15.

Using again the possibility to choose c_{55}, c_{56}, c_{57} to be small enough, we can easily derive from (7.21)–(7.23) that if the conditions (7.23) are satisfied, then

$$|L_{n,k}^{(j)} - R_{n,k}^{(j)}| \leq \frac{c_{58} \tau^2 d^2}{2^{n/2}} \left(1 + \frac{|L_{n,k}^j|^3}{2^{3n/2}}\right) + \frac{1}{16} |L_{n,k}^{j-1} - R_{n,k}^{j-1}|. \quad (7.24)$$

For the proof of (7.24) one should estimate each summand in the right-hand side of the inequality (7.21) separately. We use, in particular, that $\Gamma_{j-1} \leq \Lambda_{j-1} + 2^{-n/2} j^{1/2} |L_{n,k}^{j-1}|$.

Let us prove that, for $j = d + 1, \dots, 2d$,

$$|L_{n,k}^{(j)} - R_{n,k}^{(j)}| \leq \frac{c_{44} \tau^2 d^2}{2 \cdot 2^{n/2}} \left(1 + \frac{|L_{n,k}^j|^3}{2^{3n/2}}\right) + \frac{1}{16} |L_{n,k}^{j-1} - R_{n,k}^{j-1}|, \quad (7.25)$$

$$|R_{n,k}^{(j)}| \leq \frac{c_{57} \cdot 2^n}{4d^{3/2}\tau}, \quad (7.26)$$

provided that

$$\frac{\tau^2 d^3 p}{2^{n-p}} \leq c_{43}, \quad \max\{|U_{n,k}|, |V_{n,k}|, |\tilde{U}_{n,k}|\} \leq \frac{c_{45} \cdot 2^n}{d^{3/2}\tau}, \quad (7.27)$$

where

$$c_{44} = 2c_{58}, \quad c_{45} = \min\left\{\frac{c_{56}}{2}, \frac{c_{57}}{16}, \frac{1}{\sqrt{c_{44}}}\right\}, \quad c_{43} = \min\left\{\frac{c_{55}}{8}, \left(\frac{c_{45}}{c_{44}}\right)^{2/3}\right\}. \quad (7.28)$$

It is clear that (7.28) implies

$$c_{59} \stackrel{\text{def}}{=} c_{45} + \frac{c_{44}}{2} (c_{43}^{3/2} + c_{45}^3) + \frac{c_{45}}{8} + \frac{c_{57}}{64} \leq \frac{c_{57}}{4}. \quad (7.29)$$

We shall use the induction with respect to j . Inequality (7.25) in the case $j = d + 1$ follows directly from (7.24), (7.28) and from the definition of vectors $L_{n,k}, R_{n,k}$, since the conditions (7.23) are in this case the consequences of (7.27), (7.28). Moreover, using (7.25), (7.27)–(7.29), we see that

$$|R_{n,k}^{(d+1)}| \leq |L_{n,k}^{(d+1)}| + |L_{n,k}^{(d+1)} - R_{n,k}^{(d+1)}| \leq \frac{c_{59} \cdot 2^n}{d^{3/2}\tau} \leq \frac{c_{57} \cdot 2^n}{4d^{3/2}\tau}. \quad (7.30)$$

This completes the proof of (7.25), (7.26) for $j = d + 1$.

Suppose now that the inequalities (7.25) and (7.26) are already proved for $j = d + 1, \dots, s - 1$, $d + 2 \leq s \leq 2d$. We shall prove that these inequalities are valid for $j = s$. Inequality (7.25) for $j = s$ follows from (7.24), (7.28)

since the validity of (7.23) can be easily derived in this case from (7.26) (with $j = d + 1, \dots, s - 1$), (7.27), (7.28). Moreover, using (7.25) (with $j = s$), (7.26) (with $j = d + 1, \dots, s - 1$), (7.27)–(7.29), we obtain the relation (7.30) with the change of the superscript $(d + 1)$ by (s) . This completes the proof of (7.25), (7.26) for $j = d + 1, \dots, 2d$.

Inequality (7.2) follows from (7.25), (6.42) and from the definition of $L_{n,k}$ and $R_{n,k}$.

Let us prove assertion b). Consider the vectors $U_{N,0}^j, V_{N,0}^j, j = 1, \dots, d$, consisting of the first j coordinates of the vectors $U_{N,0}, V_{N,0}$ respectively (see (6.38)). From the properties of classes $\mathcal{R}_d(\tau, \varepsilon)$, Corollary 2.6, Remarks 2.3, 7.1 and the relations (5.2), (7.1) it follows that

$$(\mathcal{L}(U_{N,0}^j), \mathcal{L}(V_{N,0}^j)) \in \mathcal{R}_j(c_{15} \tau, 0). \tag{7.31}$$

One can verify that, if $\tau^2 j^3 p / 2^{N-p} \leq c_{60}$ with sufficiently small c_{60} , then the pairs of distributions $(\mathcal{L}(U_{N,0}^j), \mathcal{L}(V_{N,0}^j))$ satisfy in j -dimensional situation the conditions of Lemmas 2.15 and 4.1 with $D = 2^N I, \sigma^2 = 2^N$ and with the change of τ by $c_{24} \tau$, where $c_{24} = \max\{c_{15}, c_1^{-1}, 1\}$ was defined in Section 6. For this one should verify that, for $h \in \mathbb{R}^j, \|h\| c_{24} \tau < 1$ and for all $v \in \mathbb{R}^j$, the following inequalities hold, if c_{60} is small enough (see (2.20) and Definition 2.12):

$$(2\pi)^{-j} \int_T |\widehat{F}_h(t)| dt \leq \frac{c_{24}^2 \tau^2 j^2}{2^N (2\pi)^{j/2} \cdot 2^{Nj/2}}, \tag{7.32}$$

$$(2\pi)^{-j} \int_T |\langle t, v \rangle \widehat{F}_h(t)| dt \leq \frac{\|v\|}{4 \cdot 2^{N/2} (2\pi)^{j/2} \cdot 2^{Nj/2}}, \tag{7.33}$$

where $F_h = \overline{F}(h), F = \mathcal{L}(U_{N,0}^j)$ or $F = \mathcal{L}(V_{N,0}^j)$,

$$T = \{t \in \mathbb{R}^j : 4 \|t\| c_{24} \tau j \geq 1\}. \tag{7.34}$$

The inequalities (7.32), (7.33) automatically imply the validity of analogous relations in the $(j - 1)$ -dimensional situation, $j = 2, \dots, d$, for the distributions F' . It is clear that it suffices to check the conditions (7.32), (7.33) for $F = \mathcal{L}(U_{N,0}^j), j = 1, \dots, d$, if c_{60} is small enough. For $F = \mathcal{L}(V_{N,0}^j)$ these conditions can be verified analogously.

From the definition of distributions $\Psi(\cdot)$ and random vectors $U_{N,0}^j$ and from (2.2), (5.1), (5.2) one can easily derive that

$$|\widehat{F}_h(t)| \leq \prod_{m=1}^j |\widehat{H}_{h_m}(t_m)|^{2^{N-p}}. \tag{7.35}$$

By the definition of c_{24} , we can use for estimating $|\widehat{H}_{h_m}(t_m)|$ the inequalities (6.2)–(6.4). The rest of the proof of inequalities (7.32), (7.33) does not practically differ from the proof of (6.19), (6.20) and is simpler than the proof of (6.43), (6.44) in Section 7. Therefore it is omitted. The only difference is in the change of constants and notation.

Using the relations (2.21), (4.2), (5.7) and applying Lemma 4.1 and the inequality (2.22) of Lemma 2.15 in the j -dimensional case, $j = 1, \dots, d$, we obtain that

$$|U_{N,0}^{(1)} - V_{N,0}^{(1)}| \leq \frac{c_{61} \tau^2}{2^{N/2}} \left(1 + \frac{|U_{N,0}^{(1)}|^3}{2^{3N/2}}\right), \tag{7.36}$$

if

$$\frac{\tau^2 p}{2^{N-p}} \leq c_{62}, \quad |U_{N,0}^{(1)}| \leq \frac{c_{63} \cdot 2^N}{\tau}, \tag{7.37}$$

and, for $2 \leq j \leq d$,

$$|U_{N,0}^{(j)} - V_{N,0}^{(j)}| \leq \frac{c_{61} \tau^2 d^2}{2^{N/2}} \left(1 + \frac{|U_{N,0}^{(j)}|^3}{2^{3N/2}}\right) + \frac{1}{2} |U_{N,0}^{(j-1)} - V_{N,0}^{(j-1)}|, \tag{7.38}$$

if

$$\frac{\tau^2 j^3 p}{2^{N-p}} \leq c_{62}, \quad |U_{N,0}^{(j)}| \leq \frac{c_{63} \cdot 2^N}{j \tau}, \quad \frac{\max\{|U_{N,0}^{(j-1)}|, |V_{N,0}^{(j-1)}|\}}{2^{N/2}} \leq \frac{c_{64} \cdot 2^{N/2}}{j^{3/2} \tau} \tag{7.39}$$

and c_{62}, c_{63}, c_{64} are small enough. Checking inequalities (7.36), (7.38) should be carried out analogously to the proof of (7.24).

Let us prove that, for $j = 1, \dots, d$,

$$|V_{N,0}^{(j-1)}| \leq \frac{c_{64} \cdot 2^N}{d^{3/2} \tau}, \tag{7.40}$$

$$|U_{N,0}^{(j)} - V_{N,0}^{(j)}| \leq \frac{c_{47} \tau^2 d^2}{2 \cdot 2^{N/2}} \left(1 + \frac{|U_{N,0}^{(j)}|^3}{2^{3N/2}}\right) + \frac{1}{2} |U_{N,0}^{(j-1)} - V_{N,0}^{(j-1)}|, \tag{7.41}$$

if

$$\frac{\tau^2 d^3 p}{2^{N-p}} \leq c_{46}, \quad |U_{N,0}| \leq \frac{c_{48} \cdot 2^N}{d^{3/2} \tau}, \tag{7.42}$$

where

$$c_{47} = 2c_{61}, \quad c_{48} = \min\left\{c_{63}, \frac{c_{64}}{8}, \frac{1}{\sqrt{c_{47}}}\right\}, \quad c_{46} = \min\left\{c_{62}, \left(\frac{c_{48}}{c_{47}}\right)^{2/3}\right\},$$

$$|V_{N,0}^{(0)}| = |U_{N,0}^0 - V_{N,0}^0| = 0. \tag{7.43}$$

It is clear that (7.43) implies

$$c_{65} \stackrel{\text{def}}{=} c_{48} + \frac{c_{47}}{2} (c_{46}^{3/2} + c_{48}^3) + \frac{c_{48}}{2} + \frac{c_{64}}{2} \leq c_{64}. \tag{7.44}$$

We shall use the induction with respect to j . Inequality (7.41) for $j = 1$ follows directly from (7.36), (7.43), since the conditions (7.37) are in this case the consequences of (7.42), (7.43). Inequality (7.40) for $j = 1$ is valid automatically (see (7.43)).

Suppose now that the inequalities (7.40) and (7.41) are already proved for $j = 1, \dots, s-1$, $2 \leq s \leq d$. We shall prove that these inequalities are valid for $j = s$. Using (7.40) (with $j = 1, \dots, s-1$), (7.41) (with $j = s-1$), (7.42)–(7.44), we obtain

$$|V_{N,0}^{(s-1)}| \leq |U_{N,0}^{(s-1)}| + |U_{N,0}^{(s-1)} - V_{N,0}^{(s-1)}| \leq \frac{c_{65} \cdot 2^N}{d^{3/2}\tau} \leq \frac{c_{64} \cdot 2^N}{d^{3/2}\tau}. \quad (7.45)$$

Moreover, it is clear that the inequality (7.41) for $j = s$ can be easily derived from (7.38), (7.43) since the validity of (7.39) follows in this case from (7.40) (with $j = 1, \dots, s-1$), (7.42), (7.43), (7.45). This completes the proof of (7.40), (7.41) for $j = 1, \dots, d$.

Inequality (7.3) follows now from (7.41)–(7.43) and from the definition of $U_{N,0}$, $U_{N,0}^j$, $U_{N,0}^{(j)}$, $V_{N,0}$, $V_{N,0}^j$, $V_{N,0}^{(j)}$. \square

8. PROOF OF THEOREM 1.5

In this section it is convenient to assume that the constants c_1, \dots, c_{65} are the same as in Sections 6 and 7. Let u, v, w be arbitrary numbers satisfying the relations

$$\frac{5}{8} \leq v < \frac{2}{3}, \quad u = 1 - v, \quad w = 2 - 3v. \quad (8.1)$$

It is clear that

$$\frac{1}{3} < u \leq \frac{3}{8}, \quad 0 < w \leq \frac{1}{8}, \quad 3u - w = 1. \quad (8.2)$$

Below the symbols a_1, \dots, a_7 will be used for denoting positive numbers depending on v only. The letter a can denote different positive quantities depending only on v when we do not need to fix their numerical values. It is clear that, if v is fixed, a, a_1, \dots, a_7 are absolute constants. Denote

$$\begin{aligned} c_{66} &= \max\{c_{44}, c_{47}\}, & c_{67} &= \min\{c_{45}, c_{48}, 1\}, & c_{68} &= \min\{c_{43}, c_{46}\}, \\ a_1 &= \min\left\{\frac{c_{67}}{5}, \frac{1}{(126c_{66})^{1/2}}, \frac{1-2^{-w}}{2}\right\}, & a_2 &= \min\{c_{68}, a_1^2, 1\}. \end{aligned} \quad (8.3)$$

Let $\tau \geq 1$, $\lambda \geq 1$. Define p as the minimal integer satisfying the inequalities

$$p \geq 4, \quad \frac{a_1 \cdot 2^{vp}}{d^{3/2}\tau} \geq 4\lambda. \quad (8.4)$$

Denote by M the maximal integer such that

$$\frac{\tau^2 d^3 p}{2^{M-p}} \geq a_2. \quad (8.5)$$

Let N be a positive integer such that

$$\frac{\tau^2 d^3 p}{2^{N-p}} < a_2, \quad \frac{N-M}{p} \text{ is integer.} \quad (8.6)$$

Obviously, the relations (8.3)–(8.6) imply

$$N > M \geq p, \quad N \geq 2p. \tag{8.7}$$

Let $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^N}$ be random vectors with distributions $\mathcal{L}(\tilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbb{E} \tilde{\xi}_k = 0$, $\text{cov} \tilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Let $\tilde{\eta}_k$ be random vectors which have the same moments of the first three orders as the vectors $\tilde{\xi}_k$, $\mathcal{L}(\tilde{\eta}_k) \in \mathcal{A}_d(\tau)$ and $\mathbb{P} \{ |\tilde{\eta}_k| \leq \lambda \} = 1$, $k = 1, \dots, 2^N$. Suppose that the independent random vectors \tilde{Y}_k , $k = 1, \dots, 2^N$, with distributions $\mathcal{L}(\tilde{Y}_k) = \Psi(\mathcal{L}(\tilde{\eta}_k))$ are already constructed (the distributions $\Psi(\cdot)$ are defined in Section 6). According to Lemma 6.1, \tilde{Y}_k and $\tilde{\xi}_k$ have the same moments of the first three orders. Our aim is to construct independent random vectors \tilde{X}_k , $k = 1, \dots, 2^N$, with $\mathcal{L}(\tilde{X}_k) = \mathcal{L}(\tilde{\xi}_k)$ so that the sequential sums of \tilde{X}_k and \tilde{Y}_k would be close with large probability. For this we define the random vectors

$$\{ Y_k^{[N-lp]}, Z_{(k-1) \cdot 2^{lp+1}}^{[N-lp]}, 1 \leq k \leq 2^{N-lp}, 0 \leq l \leq (N - M - p)/p \} \tag{8.8}$$

with the help of a recurrent procedure described below. Note that in one-dimensional case a very similar procedure was used by Sakhanenko (1984).

At first, we set $Y_k^{[N]} = \tilde{Y}_k$, $k = 1, \dots, 2^N$. After this, using the procedure described in Sections 5 and 7, we define the independent random vectors $Z_k^{[N]}$, $k = 1, \dots, 2^N$, with $\mathcal{L}(Z_r^{[N]}) = \Psi(\mathcal{L}(\tilde{\eta}_r))$ for $r = (m - 1) \cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$. For $r = 1, \dots, 2^N$, $r \neq (m - 1) \cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$, we take $\mathcal{L}(Z_r^{[N]}) = \mathcal{L}(\tilde{\xi}_r)$ and define $\tilde{X}_r = Z_r^{[N]}$. Thus, the amount of vectors \tilde{X}_k , which are not constructed, is decreased in 2^p times. In the case $N = M + p$, the first step of applying this procedure is simultaneously the last one, since the random vectors (8.8) with $l = (N - M - p)/p = 0$ are already constructed.

Suppose now that $N > M + p$. Then we repeat the same procedure, working only with vectors which have the distributions $\Psi(\mathcal{L}(\tilde{\eta}_r))$ and diminishing their number after each step in 2^p times. Namely, if the vectors $\{ Y_k^{[N-lp+p]}, Z_{(k-1) \cdot 2^{lp-p+1}}^{[N-lp+p]}, 1 \leq k \leq 2^{N-lp+p} \}$ are already defined for an integer l and $1 \leq l \leq (N - M - p)/p$, we set $Y_k^{[N-lp]} = Z_{(k-1) \cdot 2^{lp+1}}^{[N-lp+p]}$, $k = 1, \dots, 2^{N-lp}$, and then, using again the dyadic scheme described in Sections 5 and 7, construct the independent random vectors $Z_{(k-1) \cdot 2^{lp+1}}^{[N-lp]}$, $k = 1, \dots, 2^{N-lp}$, distributed as follows. In the case $r = (m - 1) \cdot 2^{lp+p} + 1$, $m = 1, \dots, 2^{N-lp-p}$, we choose $\mathcal{L}(Z_r^{[N-lp]}) = \Psi(\mathcal{L}(\tilde{\eta}_r))$. In the opposite case, when $r = (s - 1) \cdot 2^{lp} + 1$, $s = 1, \dots, 2^{N-lp}$, and when r cannot be represented in the form $r = (m - 1) \cdot 2^{lp+p} + 1$, for some $m = 1, \dots, 2^{N-lp-p}$, we take $\mathcal{L}(Z_r^{[N-lp]}) = \mathcal{L}(\tilde{\xi}_r)$ and define $\tilde{X}_r = Z_r^{[N-lp]}$.

After the last step (with $l = (N - M - p)/p$) all independent random vectors \tilde{X}_i , $i = 1, \dots, 2^N$, are constructed except \tilde{X}_r having indices of the form $r = (m - 1) \cdot 2^{N-M} + 1$, $m = 1, \dots, 2^M$. Instead of this we have constructed the vectors with distributions $\mathcal{L}(Z_r^{[M+p]}) = \Psi(\mathcal{L}(\tilde{\eta}_r))$ for $r = (m - 1) \cdot 2^{N-M} + 1$, $m = 1, \dots, 2^M$. Together with \tilde{X}_i these vectors form a collection of mutually independent vectors. This can be easily verified

by induction, if we take into account that, in each step, the new collection of independent vectors is constructed as a function of the vectors which are independent of all already constructed vectors \tilde{X}_i .

In order to add the missing elements to the sequence of already constructed \tilde{X}_i , we define the random vectors $Y_k^* = Y_k^{[M]} = Z_{(k-1) \cdot 2^{N-M} + 1}^{[M+p]}$, $k = 1, \dots, 2^M$, and, using again the procedure described in Section 5, construct the independent random vectors X_k^* , $k = 1, \dots, 2^M$, with distributions $\mathcal{L}(X_k^*) = \mathcal{L}(\tilde{\xi}_{(k-1) \cdot 2^{N-M} + 1})$. Then we set $\tilde{X}_{(k-1) \cdot 2^{N-M} + 1} = X_k^*$. It is clear that the vectors X_k^* are mutually independent and they do not depend on \tilde{X}_i constructed earlier, since, by construction, X_k^* are functions of $Y_1^*, \dots, Y_{2^M}^*$. This completes the procedure of constructing \tilde{X}_i , $i = 1, \dots, 2^N$.

By analogy with Sakhanenko (1984), we shall prove that the procedure described above provides good approximation, using an induction with respect to N . It is clear that the procedure of constructing the independent random vectors $X_{*,k} = \tilde{X}_{(k-1) \cdot 2^p + 1}$, $k = 1, \dots, 2^{N-p}$, coincides with the procedure of constructing the vectors \tilde{X}_k , $k = 1, \dots, 2^N$. The only difference is in the change of N by $N - p$. Thus, below we shall assume that we have the good approximation in the case when we construct 2^{N-p} summands and prove the analogous statement in the case of 2^N summands.

We shall use the notation which was already used in Sections 5 and 7 with $Y_k = Y_k^{[N]}$, $X_k = Z_k^{[N]}$, $k = 1, \dots, 2^N$. Denote $S = S_j$, $T = T_j$ (see (5.5), (5.11)). Sometimes we shall omit for simplicity the index j , $1 \leq j \leq 2^N$, keeping in mind that it is fixed. By virtue of (6.14), by construction and by the definition of distributions $\Psi(\cdot)$ we have

$$\mathbb{P} \{ |Y_k| \leq 4\lambda \} = 1, \quad k = 1, \dots, 2^N. \tag{8.9}$$

It is clear that from (5.5), (8.9) it follows that

$$\mathbb{P} \{ |T| \leq 4\lambda \cdot 2^N \} = 1. \tag{8.10}$$

For fixed integers j , $1 \leq j \leq 2^N$, and s , $0 \leq s \leq N$, define the integer $k_s = k_s(j)$ by the relation $(k_s - 1) \cdot 2^s < j \leq k_s \cdot 2^s$. Now we can represent the sums S and T in the form

$$S = S^{(s)} - S^{\{s\}}, \quad T = T^{(s)} - T^{\{s\}}, \tag{8.11}$$

where

$$\begin{aligned} S^{(s)} &= S_{k_s \cdot 2^s}, & S^{\{s\}} &= S^{(s)} - S, \\ T^{(s)} &= T_{k_s \cdot 2^s}, & T^{\{s\}} &= T^{(s)} - T. \end{aligned} \tag{8.12}$$

According to Corollary 5.3 and Lemma 5.1,

$$|S^{(s)} - T^{(s)}| \leq |U_{N,0} - V_{N,0}| + \sum_{n=s+1}^N |\tilde{U}^{(n)} - \tilde{V}^{(n)}|, \tag{8.13}$$

where $\tilde{U}^{(n)} = \tilde{U}_{n, l_{n,j}}$, $\tilde{V}^{(n)} = \tilde{V}_{n, l_{n,j}}$, $l_{n,j} \cdot 2^n < j \leq (l_{n,j} + 1) \cdot 2^n$ (see (5.13)). As before, $l_{n,j}$ is the integer such that $U_{n, l_{n,j}}$, the sum over

a block of 2^n summands, contains the summand X_j and, consequently, the summand $X_{k_s \cdot 2^s}$, for $s < n \leq N$. Denote as well $U^{(n)} = U_{n, l_{n,j}}$, $V^{(n)} = V_{n, l_{n,j}}$. It is evident that $U^{(N)} = U_{N,0}$, $V^{(N)} = V_{N,0}$. Therefore, from (8.11), (8.13) it follows that

$$|S - T| \leq |U^{(N)} - V^{(N)}| + \sum_{n=s+1}^N |\tilde{U}^{(n)} - \tilde{V}^{(n)}| + |S^{\{s\}}| + |T^{\{s\}}|. \tag{8.14}$$

Define the random integer $\mu = \mu_j$ by $\mu = N + 1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-p}} > a_2, \quad |S^{\{N\}}| > \frac{2a_1 \cdot 2^N}{d^{3/2} \tau}, \quad |U^{(N)}| > \frac{2a_1 \cdot 2^N}{d^{3/2} \tau} \tag{8.15}$$

and by

$$\mu = \min \left\{ m : p \leq m \leq N, \frac{\tau^2 d^3 p}{2^{m-p}} \leq a_2; \right. \\ \left. |S^{\{n\}}| \leq \frac{2a_1 \cdot 2^n}{d^{3/2} \tau}, \quad |U^{(n)}| \leq \frac{2a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}, \quad n = m, \dots, N \right\} \tag{8.16}$$

otherwise. Here the numbers u, v are defined by (8.1). Note that the first of inequalities (8.15) is not satisfied in our conditions (see (8.6)). The same can be said about inequalities (8.31), (8.35).

Denote

$$Y_{*,k} = Y_k^{[N-p]} = Z_{(k-1) \cdot 2^p + 1}^{[N]}, \quad k = 1, \dots, 2^{N-p}. \tag{8.17}$$

By construction and by virtue of (6.14), (8.9), we have

$$\mathbb{P} \{ |Y_{*,k}| \leq 4\lambda \} = 1, \quad k = 1, \dots, 2^{N-p}. \tag{8.18}$$

Let $i = i(j)$ be the largest integer such that $(i - 1) \cdot 2^p + 1 \leq j$. We shall denote by $V_*^{(n)}$, $n = 0, \dots, N - p$, the sum of a block of 2^n summands $Y_{*,k}$ which contains the summand $Y_{*,i}$. Let

$$T_* = T_{*,i} = Y_{*,1} + \dots + Y_{*,i}. \tag{8.19}$$

Recall that, by construction,

$$\bigcup_{k=1}^{2^{N-p}} \left\{ \left\{ \tilde{X}_r = Z_r^{[N]}, \quad r = (k - 1) \cdot 2^p + 2, \dots, k \cdot 2^p \right\} \cup \left\{ Y_{*,k} = Z_{(k-1) \cdot 2^p + 1}^{[N]} \right\} \right\} \tag{8.20}$$

is a collection of random vectors, which are mutually independent. Set

$$U_0^{(n)} = U^{(n)} - V_*^{(n-p)}, \quad n = p, \dots, N. \tag{8.21}$$

It is easy to see that the sum $U_0^{(n)}$ consists of the summands of the sum $U^{(n)}$ with indices which can not be represented in the form $r = (k - 1) \cdot 2^p + 1$

with integer k . In other words, the sum $U_0^{(n)}$ is the sum of contained in $U^{(n)}$ summands $Z_r^{[N]}$ which are chosen as \tilde{X}_r . For the proof of this fact one should note that, by construction, the numbers j and $(i-1) \cdot 2^p + 1$ are contained in the same block of indices $\{m : m = (i-1) \cdot 2^p + 1, \dots, i \cdot 2^p\}$. Recall that, according to our construction, after the first step, when the independent random vectors (8.20) are constructed, we define the rest of the random vectors \tilde{X}_r as the (d -dimensional) functions of the random vectors $Y_{*,k}$, $k = 1, \dots, 2^{N-p}$. Denote

$$\begin{aligned} X_{*,k} &= \tilde{X}_{(k-1) \cdot 2^p + 1}, & k &= 1, \dots, 2^{N-p}; \\ S_* &= S_{*,i} = X_{*,1} + \dots + X_{*,i}, & S_0 &= S - T_*. \end{aligned} \tag{8.22}$$

By analogy with (8.11), (8.12), for $s = p, \dots, N$ we can represent the sums S_*, S_0 and T_* in the form

$$S_0 = S_0^{(s)} - S_0^{\{s\}}, \quad T_* = T_*^{(s-p)} - T_*^{\{s-p\}}, \tag{8.23}$$

where

$$\begin{aligned} T_*^{(s-p)} &= T_{*,k_s} \cdot 2^{s-p}, & T_*^{\{s-p\}} &= T_*^{(s-p)} - T_*, \\ S_0^{(s)} &= S^{(s)} - T_*^{(s-p)}, & S_0^{\{s\}} &= S_0^{(s)} - S_0 = S^{\{s\}} - T_*^{\{s-p\}}, \end{aligned} \tag{8.24}$$

and the numbers k_s are defined above, before the formulas (8.11), (8.12).

From the definition of numbers $i, j, k_s, s = 0, \dots, N$, it follows that

$$\begin{aligned} 0 &\leq k_s \cdot 2^s - j < 2^s, & s &= 0, \dots, N; \\ 0 &\leq k_s \cdot 2^{s-p} - i < 2^{s-p}, & s &= p, \dots, N. \end{aligned} \tag{8.25}$$

It is clear that from the relations (5.5), (8.9), (8.11), (8.12), (8.18), (8.19), (8.24) and (8.25) one can derive that, with probability one,

$$\begin{aligned} |T^{\{s\}}| &\leq 4\lambda \cdot 2^s, & s &= 0, \dots, N; \\ |T_*^{\{s-p\}}| &\leq 4\lambda \cdot 2^{s-p}, & s &= p, \dots, N. \end{aligned} \tag{8.26}$$

Analogously, using (8.18) and the definition of $V_*^{(n)}$, we see that

$$|V_*^{(n)}| \leq 4\lambda \cdot 2^n, \quad n = 0, \dots, N - p. \tag{8.27}$$

Define as well

$$\begin{aligned} \tilde{S} &= \sum_{k=1}^j \tilde{X}_k, & \delta_j(N) &= \left| \sum_{k=1}^j \tilde{Y}_k - \sum_{k=1}^j \tilde{X}_k \right| = |T - \tilde{S}|, \\ \delta_i(N-p) &= \left| \sum_{k=1}^i Y_{*,k} - \sum_{k=1}^i X_{*,k} \right| = |T_* - S_*|, & (8.28) \\ \psi_j(N) &= \left| \sum_{k=1}^j \tilde{Y}_k - \sum_{k=1}^j Z_k^{[N]} \right| = |T - S| \end{aligned}$$

(see (5.5), (5.11), (8.19), (8.22)). It is clear that from (8.17), (8.19), (8.20), (8.22), (8.28) and from the definition of i and S it follows that

$$S = \sum_{k=1}^j Z_k^{[N]} = S_0 + T_*, \quad \tilde{S} = S_0 + S_*, \quad S - \tilde{S} = T_* - S_*. \quad (8.29)$$

It is easy to see that the relations (8.28), (8.29) imply

$$\delta_j(N) \leq \psi_j(N) + \delta_i(N - p). \quad (8.30)$$

Define the random integer $\beta_j(N)$ by $\beta_j(N) = N + 1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-p}} > a_2, \quad |V^{(N)}| > \frac{4a_1 \cdot 2^N}{d^{3/2} \tau} \quad (8.31)$$

and by

$$\beta_j(N) = \min \left\{ m : p \leq m \leq N, \frac{\tau^2 d^3 p}{2^{m-p}} \leq a_2, \right. \\ \left. |V^{(n)}| \leq \frac{4a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}, \quad n = m, \dots, N \right\} \quad (8.32)$$

otherwise.

Similarly, taking into account the relations (8.7), we define the random integer $\beta_i(N - p)$ by the equality $\beta_i(N - p) = N - p + 1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-2p}} > a_2, \quad |V_*^{(N-p)}| > \frac{4a_1 \cdot 2^{N-p}}{d^{3/2} \tau} \quad (8.33)$$

and by

$$\beta_i(N - p) = \min \left\{ m : p \leq m \leq N - p, \frac{\tau^2 d^3 p}{2^{m-p}} \leq a_2, \right. \\ \left. |V_*^{(n)}| \leq \frac{4a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}, \quad n = m, \dots, N - p \right\} \quad (8.34)$$

otherwise.

Finally, define the random integer $\nu = \nu_j$ by $\nu = N + 1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-p}} > a_2, \quad |U_0^{(N)}| > \frac{a_1 \cdot 2^N}{d^{3/2} \tau}, \quad |S_0^{\{N\}}| > \frac{a_1 \cdot 2^N}{d^{3/2} \tau} \quad (8.35)$$

and by

$$\nu = \min \left\{ m : p \leq m \leq N, \frac{\tau^2 d^3 p}{2^{m-p}} \leq a_2; \right. \\ \left. |S_0^{\{n\}}| \leq \frac{a_1 \cdot 2^n}{d^{3/2} \tau}, \quad |U_0^{(n)}| \leq \frac{a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}, \quad n = m, \dots, N \right\} \quad (8.36)$$

otherwise.

Put now

$$\rho_j(N) = 5 \lambda \cdot 2^{\beta_j(N)}, \quad \rho_i(N - p) = 5 \lambda \cdot 2^{\beta_i(N-p)}, \quad (8.37)$$

$$\gamma_j(N) = \begin{cases} 10 \lambda \cdot 2^\nu, & \text{if } \nu \leq N, \\ U + 22 \lambda \cdot 2^N, & \text{if } \nu = N + 1, \end{cases} \quad (8.38)$$

where

$$U = |U_0^{(N)}| + |S_0^{\{N\}}|. \quad (8.39)$$

LEMMA 8.1. *The following inequalities are valid:*

$$\delta_j(N) + \rho_j(N) \leq \delta_i(N - p) + \rho_i(N - p) + \gamma_j(N) \quad (8.40)$$

and, for each $\varepsilon > 0$,

$$\mathbb{E} \exp(\varepsilon(\delta_j(N) + \rho_j(N))) \leq \mathbb{E} \exp(\varepsilon(\delta_i(N - p) + \rho_i(N - p))) \mathbb{E} \exp(\varepsilon \gamma_j(N)). \quad (8.41)$$

Proof. Assume that $\nu \leq N$ and, taking into account (8.7), define the integer valued random variables

$$\beta = \beta_i(N - p) - 1 \leq N - p, \quad \phi = \max\{\beta, \nu\} \leq N. \quad (8.42)$$

The relations (8.1), (8.4), (8.26) imply

$$|T_*^{\{s-p\}}| \leq \frac{a_1 \cdot 2^s}{d^{3/2} \tau}, \quad s = p, \dots, N. \quad (8.43)$$

Using (8.24), (8.36), (8.42), (8.43), we obtain

$$|S^{\{n\}}| \leq |S_0^{\{n\}}| + |T_*^{\{n-p\}}| \leq \frac{2 a_1 \cdot 2^n}{d^{3/2} \tau}, \quad n = \phi, \dots, N, \quad (8.44)$$

(here and below we use that (8.36), (8.42) imply the inequality $\phi \geq p$). By virtue of (8.1), (8.4), (8.27), (8.42), for $n = \beta, \dots, \beta + p, n \geq p$ we have

$$|V_*^{(n-p)}| \leq 4 \lambda \cdot 2^{n-p} \leq \frac{a_1 \cdot 2^{u\beta+vn}}{d^{3/2} \tau} \leq \frac{a_1 \cdot 2^{u\phi+vn}}{d^{3/2} \tau}. \quad (8.45)$$

If $n = \beta + p + 1, \dots, N$, then, applying (8.1), (8.2), (8.4), (8.34), (8.42), we obtain

$$|V_*^{(n-p)}| \leq \frac{4 a_1 \cdot 2^{u(\beta+1)+v(n-p)}}{d^{3/2} \tau} \leq \frac{a_1 \cdot 2^{u\phi+vn}}{d^{3/2} \tau} \quad (8.46)$$

(we use that $p \geq 4$, (8.1), (8.2) imply $2^{vp} \geq 2^{5p/8} > 4 \cdot 2^{3/8} \geq 4 \cdot 2^u$). From (8.21), (8.36), (8.42), (8.45), (8.46) it follows that

$$\begin{aligned} |U^{(n)}| &\leq |U_0^{(n)}| + |V_*^{(n-p)}| \leq \frac{a_1 \cdot 2^{u\phi+vn}}{d^{3/2} \tau} + \frac{a_1 \cdot 2^{u\phi+vn}}{d^{3/2} \tau} \\ &= \frac{2 a_1 \cdot 2^{u\phi+vn}}{d^{3/2} \tau}, \quad n = \phi, \dots, N. \end{aligned} \quad (8.47)$$

Taking into account (8.16), (8.36), (8.42), (8.44), (8.47), we see that

$$\mu \leq \phi \leq N. \tag{8.48}$$

It is clear that

$$U^{(m)} = \frac{1}{2} U^{(m+1)} + \frac{b_{m+1}}{2} \tilde{U}^{(m+1)}, \quad m = 0, \dots, N - 1, \tag{8.49}$$

where the factors $b_{m+1} = b_{m+1}(j)$ can be equal either to 1 or to -1 (see (5.10)). From (8.49) one can easily derive by induction that the following representation is valid:

$$U^{(m)} = \frac{U^{(N)}}{2^{N-m}} + \sum_{k=m+1}^N \frac{b_k \tilde{U}^{(k)}}{2^{k-m}}, \quad m = 0, \dots, N - 1, \tag{8.50}$$

$$|b_k| = 1, \quad k = 1, \dots, N.$$

It is evident that we can write the analogous equality with change of U by V . Suppose now that an integer n satisfies the inequalities

$$\mu \leq n \leq N. \tag{8.51}$$

Then, taking into account (8.1)–(8.4), (8.16), (8.51), we see that

$$\tau^2 d^3 \leq a_2 \cdot 2^\mu \leq a_1^2 \cdot 2^{2u\mu+(2v-1)n} \leq a_1^2 \cdot 2^{2u\mu+(2v-1)N}. \tag{8.52}$$

Using (8.1), (8.3), (8.16), (8.51), we obtain the inequalities

$$|U^{(n)}| \leq \frac{2a_1 \cdot 2^{u\mu+vn}}{d^{3/2} \tau} \leq \frac{c_{67} \cdot 2^n}{d^{3/2} \tau}. \tag{8.53}$$

Taking here $n = N$ and using the relations (8.1)–(8.3), (8.16), (8.51)–(8.53) and Lemma 7.2(b), we obtain

$$\begin{aligned} |U^{(N)} - V^{(N)}| &\leq c_{47} d^2 \tau^2 \cdot 2^{-N/2} (1 + 2^{-3N/2} |U^{(N)}|^3) \\ &\leq c_{66} d^2 \tau^2 \left(2^{-N/2} + \frac{8a_1^3 \cdot 2^{3u\mu-wN}}{d^{9/2} \tau^3} \right) \\ &\leq \frac{9a_1^3 c_{66} \cdot 2^{3u\mu-wN}}{d^{5/2} \tau} \leq \frac{a_1 \cdot 2^{3u\mu-wN}}{14d^{5/2} \tau}. \end{aligned} \tag{8.54}$$

From (8.1), (8.53), (8.54) it follows that

$$|V^{(N)}| \leq |U^{(N)}| + |V^{(N)} - U^{(N)}| \leq \frac{4a_1 \cdot 2^{u\mu+vN}}{d^{3/2} \tau}. \tag{8.55}$$

Analogously, if we suppose that

$$\mu \leq n < N, \tag{8.56}$$

then from (8.1)–(8.3), (8.16), (8.49), (8.56) one can easily derive that

$$|\tilde{U}^{(n+1)}| = |U^{(n+1)} - 2U^{(n)}| \leq \frac{2(1 + 2^{3/8})a_1 \cdot 2^{u\mu+v(n+1)}}{d^{3/2}\tau} \leq \frac{c_{67} \cdot 2^{n+1}}{d^{3/2}\tau}. \tag{8.57}$$

Using the induction with respect to n , we shall prove that, if $\mu < N$, then, for $n = N - 1, N - 2, \dots, \mu$,

$$|\tilde{U}^{(n+1)} - \tilde{V}^{(n+1)}| \leq \frac{2a_1 \cdot 2^{3u\mu-w(n+1)}}{d^{5/2}\tau}, \tag{8.58}$$

$$|U^{(n)} - V^{(n)}| \leq \frac{2a_1 \cdot 2^{3u\mu-wn}}{d^{5/2}\tau}, \tag{8.59}$$

$$|V^{(n)}| \leq \frac{4a_1 \cdot 2^{u\mu+vn}}{d^{3/2}\tau}. \tag{8.60}$$

For $n = N$ the relations (8.59), (8.60) are already proved (see (8.54), (8.55)).

Assume that the inequalities (8.59), (8.60) are valid for $n = s + 1, \dots, N$ and the inequality (8.58) holds for $n = s + 1, \dots, N - 1, \mu \leq s \leq N - 1$. It suffices to prove the relations (8.58)–(8.60) for $n = s$.

Using (8.1)–(8.3), (8.16), (8.52), (8.53), (8.56), (8.57), (8.59), (8.60) and applying Lemma 7.2(a), we obtain that in this case

$$\begin{aligned} & |\tilde{U}^{(s+1)} - \tilde{V}^{(s+1)}| \\ & \leq \frac{c_{44}d^2\tau^2}{2^{(s+1)/2}} \left(1 + \frac{\max\{|U^{(s+1)}|^3, |\tilde{U}^{(s+1)}|^3\}}{2^{3(s+1)/2}} \right) + \frac{|U^{(s+1)} - V^{(s+1)}|}{8} \\ & \leq c_{66}d^2\tau^2 \left(2^{-(s+1)/2} + \frac{125a_1^3 \cdot 2^{3u\mu-w(s+1)}}{d^{9/2}\tau^3} \right) + \frac{a_1 \cdot 2^{3u\mu-w(s+1)}}{4d^{5/2}\tau} \\ & \leq \frac{2a_1 \cdot 2^{3u\mu-w(s+1)}}{d^{5/2}\tau}. \end{aligned}$$

The fulfilment of the conditions of Lemma 7.2 can be easily deduced from (8.1), (8.16), (8.53), (8.57), (8.60) (with $n = s + 1$) and from the definitions of a_1, c_{67} (see (8.3)).

From (8.1), (8.2), (8.50), (8.53), (8.54), (8.58) (with $n = s, \dots, N - 1$) it follows that

$$|U^{(s)} - V^{(s)}| \leq \frac{|U^{(N)} - V^{(N)}|}{2^{N-s}} + \sum_{k=s+1}^N \frac{|\tilde{U}^{(k)} - \tilde{V}^{(k)}|}{2^{k-s}} \leq \frac{2a_1 \cdot 2^{3u\mu-ws}}{d^{5/2}\tau}$$

and, consequently,

$$\begin{aligned} |V^{(s)}| & \leq |U^{(s)}| + |U^{(s)} - V^{(s)}| \\ & \leq \frac{2a_1 \cdot 2^{u\mu+vs}}{d^{3/2}\tau} + \frac{2a_1 \cdot 2^{3u\mu-ws}}{d^{5/2}\tau} \leq \frac{4a_1 \cdot 2^{u\mu+vs}}{d^{3/2}\tau}. \end{aligned}$$

This completes the proof of (8.58)–(8.60) for $n = s$.

It is evident that from (8.16), (8.32), (8.55), (8.60) it follows that

$$\beta_j(N) \leq \mu. \tag{8.61}$$

It is clear that the relations (8.42), (8.48), (8.61) imply

$$2^{\beta_j(N)} + 2^\mu \leq 2 \cdot 2^\mu \leq 2 \cdot 2^\phi \leq 2^{\beta_i(N-p)} + 2 \cdot 2^\nu. \tag{8.62}$$

Using (8.2), (8.3), (8.14), (8.16), (8.26), (8.28), (8.54), (8.58) and taking into account that $\lambda \geq 1, \tau \geq 1$, we see that

$$\begin{aligned} \psi_j(N) &\leq \frac{a_1 \cdot 2^{3u\mu - wN}}{14d^{5/2}\tau} + \sum_{n=\mu+1}^N \frac{2a_1 \cdot 2^{3u\mu - wn}}{d^{5/2}\tau} + \frac{2a_1 \cdot 2^\mu}{d^{3/2}\tau} + 4\lambda \cdot 2^\mu \\ &\leq (2(1 - 2^{-w})^{-1}a_1 + 4\lambda) \cdot 2^\mu \leq 5\lambda \cdot 2^\mu. \end{aligned} \tag{8.63}$$

The relations (8.30), (8.37), (8.62), (8.63) imply

$$\begin{aligned} \delta_j(N) + \rho_j(N) &\leq \delta_i(N - p) + 5\lambda(2^\mu + 2^{\beta_j(N)}) \\ &\leq \delta_i(N - p) + \rho_i(N - p) + 10\lambda \cdot 2^\nu. \end{aligned} \tag{8.64}$$

From (8.38), (8.64) it is easy to derive the inequality (8.40) in the case $\nu \leq N$.

Let now $\nu = N + 1$. From (8.12), (8.21) it follows that

$$S^{(N)} = U^{(N)} = U_0^{(N)} + V_*^{(N-p)}. \tag{8.65}$$

Using (8.11), (8.28), we see that

$$\psi_j(N) \leq |S^{(N)}| + |S^{\{N\}}| + |T|. \tag{8.66}$$

By virtue of (8.24) we have

$$|S^{\{N\}}| \leq |S_0^{\{N\}}| + |T_*^{\{N-p\}}|. \tag{8.67}$$

It is not hard to see that (8.10), (8.26), (8.27), (8.39), (8.65)–(8.67) imply

$$\psi_j(N) \leq U + 12\lambda \cdot 2^N. \tag{8.68}$$

From (8.37) and from definition of $\beta_j(N)$ it follows that $\beta_j(N) \leq N + 1$ and

$$\rho_j(N) \leq 10\lambda \cdot 2^N. \tag{8.69}$$

The relations (8.30), (8.38), (8.68), (8.69) imply

$$\delta_j(N) + \rho_j(N) \leq \delta_i(N - p) + U + 22\lambda \cdot 2^N = \delta_i(N - p) + \gamma_j(N). \tag{8.70}$$

Thus, the inequality (8.40) is proved in the case $\nu = N + 1$.

By construction (see (8.19)–(8.24), (8.28), (8.33)–(8.38)), $\gamma_j(N)$ is a function of

$$\bigcup_{k=1}^{2^{N-p}} \{ \tilde{X}_r = Z_r^{[N]}, r = (k - 1) \cdot 2^p + 2, \dots, k \cdot 2^p \} \tag{8.71}$$

and the random variables $\delta_i(N - p), \rho_i(N - p)$ are functions of

$$\{Y_{*,k} = Z_{(k-1) \cdot 2^p + 1}^{[N]}, k = 1, \dots, 2^{N-p}\}. \tag{8.72}$$

The random variables $\gamma_j(N)$ and $\delta_i(N - p) + \rho_i(N - p)$ are independent, since the sets (8.71), (8.72) are disjoint sets consisting of mutually independent vectors (see (8.20)), Therefore, the inequality (8.41) follows from (8.40) and the proof of Lemma 8.1 is completed. \square

In this section we shall denote by ε the quantity (see (8.3))

$$\varepsilon = \frac{a_1^2}{512 \lambda d^3 \tau^2} \leq \frac{a_1}{16 d^{3/2} \tau}. \tag{8.73}$$

LEMMA 8.2. *There exists a positive quantity a_3 depending only on v and such that the following inequality is valid:*

$$\mathbb{E} \exp(\varepsilon \gamma_j(N)) \leq \exp(a_3 p (\lambda d^{3/2} \tau)^{1/v}). \tag{8.74}$$

Proof. Denote by

$$\zeta = \mathbf{1}_{\{\nu=N+1\}} \tag{8.75}$$

the indicator function of the event $\{\nu = N + 1\}$ (see (8.35)). Using (8.38) and the Hölder inequality, we obtain

$$\mathbb{E} \zeta \exp(\varepsilon \gamma_j(N)) \leq (\mathbb{E} \zeta^2)^{1/2} \exp(22\varepsilon \lambda \cdot 2^N) (\mathbb{E} e^{2\varepsilon U})^{1/2}. \tag{8.76}$$

From the definitions of $U_0^{(n)}, S_0^{\{n\}}$ it follows that $U_0^{(N)}, S_0^{\{N\}}$ are sums of not more than 2^N independent random vectors \tilde{X}_k with $\mathcal{L}(\tilde{X}_k) \in \mathcal{A}_d(\tau)$, $\text{cov } \tilde{X}_k = I$ (see (8.21), (8.24)). Therefore, the relations (8.3), (8.6), (8.35), (8.75) and Lemma 2.11 imply

$$\begin{aligned} \mathbb{E} \zeta^2 &\leq \mathbb{P} \left\{ |U_0^{(N)}| > \frac{a_1 \cdot 2^N}{d^{3/2} \tau} \right\} + \mathbb{P} \left\{ |S_0^{\{N\}}| > \frac{a_1 \cdot 2^N}{d^{3/2} \tau} \right\} \\ &\leq 4d \exp\left(-\min\left\{\frac{a_1^2 \cdot 2^N}{4d^3 \tau^2}, \frac{a_1 \cdot 2^N}{4d^{3/2} \tau^2}\right\}\right) \\ &= 4d \exp\left(-\frac{a_1^2 \cdot 2^N}{4d^3 \tau^2}\right). \end{aligned} \tag{8.77}$$

It is clear that (see (8.39))

$$\mathbb{E} e^{2\varepsilon U} \leq \left(\mathbb{E} \exp(4\varepsilon |U_0^{(N)}|) \mathbb{E} \exp(4\varepsilon |S_0^{\{N\}}|)\right)^{1/2}. \tag{8.78}$$

By virtue of (8.3), (8.73) we have

$$4\varepsilon \tau < 1. \tag{8.79}$$

Taking into account (8.79) and applying Lemma 2.11 with $B^2 \leq 2^N$, $h = 4\varepsilon$, we see that

$$\begin{aligned} \mathbb{E} \exp(4\varepsilon |U_0^{(N)}|) &\leq 2d \exp(16\varepsilon^2 \cdot 2^N), \\ \mathbb{E} \exp(4\varepsilon |S_0^{\{N\}}|) &\leq 2d \exp(16\varepsilon^2 \cdot 2^N). \end{aligned} \tag{8.80}$$

From (8.3), (8.4), (8.6), (8.73), (8.76)–(8.78), (8.80) it follows that

$$\begin{aligned} \mathbb{E} \zeta \exp(\varepsilon \gamma_j(N)) &\leq 2\sqrt{2}d \exp\left(-\frac{a_1^2 \cdot 2^N}{8d^3 \tau^2} + 22\varepsilon \lambda \cdot 2^N + 8\varepsilon^2 \cdot 2^N\right) \\ &\leq 2\sqrt{2}d \exp\left(-\frac{a_1^2 \cdot 2^N}{32d^3 \tau^2}\right) \leq 2\sqrt{2}d \exp\left(-\frac{p \cdot 2^p}{32}\right) \\ &\leq 2\sqrt{2}d \exp\left(-\frac{d^{3/2} \tau \lambda}{8a_1}\right) \leq c. \end{aligned} \tag{8.81}$$

Denote now by m_0 the minimal integer such that

$$\frac{\tau^2 d^3 p}{2^{m_0-p}} < a_2. \tag{8.82}$$

It is evident that (8.3), (8.5)–(8.7), (8.82) imply $p \leq \tilde{m}_0 = M + 1 \leq N$. Using the definition of the random variable ν , we see that $\mathbb{P}\{\nu = m + 1\}$, $m = m_0, \dots, N - 1$, can be estimated from above by

$$\sum_{n=m}^N \left(\mathbb{P}\left\{|U_0^{(n)}| \geq \frac{a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}\right\} + \mathbb{P}\left\{|S_0^{\{n\}}| \geq \frac{a_1 \cdot 2^n}{d^{3/2} \tau}\right\} \right). \tag{8.83}$$

It is clear that the vectors $U_0^{(n)}, S_0^{\{n\}}$ are sums of not more than 2^n independent random vectors \tilde{X}_k with $\mathcal{L}(\tilde{X}_k) \in \mathcal{A}_d(\tau)$, $\text{cov} \tilde{X}_k = I$ (see (8.21) and (8.24)). Therefore, for $m \geq m_0$, the relations (8.1), (8.3), (8.82) and Lemma 2.11 imply

$$\begin{aligned} \mathbb{P}\left\{|U_0^{(n)}| \geq \frac{a_1 \cdot 2^{um+vn}}{d^{3/2} \tau}\right\} &\leq 2d \exp\left(-\min\left\{\frac{a_1^2 \cdot 2^{2um+2vn}}{4d^3 \tau^2 \cdot 2^n}, \frac{a_1 \cdot 2^{um+vn}}{4d^{3/2} \tau^2}\right\}\right) \\ &= 2d \exp\left(-\frac{a_1^2 \cdot 2^{2um+(2v-1)n}}{4d^3 \tau^2}\right), \quad n = m, \dots, N, \end{aligned} \tag{8.84}$$

and, analogously,

$$\mathbb{P}\left\{|S_0^{\{n\}}| \geq \frac{a_1 \cdot 2^n}{d^{3/2} \tau}\right\} \leq 2d \exp\left(-\frac{a_1^2 \cdot 2^n}{4d^3 \tau^2}\right), \quad n = m, \dots, N. \tag{8.85}$$

Taking into account (8.38), (8.75), we see that

$$\mathbb{E}(1-\zeta) \exp(\varepsilon \gamma_j(N)) \leq \exp(10 \lambda \varepsilon \cdot 2^{m_0}) + \sum_{m=m_0+1}^N e^{10 \lambda \varepsilon \cdot 2^m} \mathbb{P}\{\nu = m\}. \tag{8.86}$$

From (8.1), (8.3), (8.73), (8.82) and from the definition of p (see (8.4)) it follows that

$$10 \lambda \varepsilon \cdot 2^{m_0} \leq \frac{10 a_1^2}{512 d^3 \tau^2} \cdot \frac{d^3 \tau^2 p \cdot 2^{p+1}}{a_2} \leq a p \cdot 2^p \leq a p (\lambda d^{3/2} \tau)^{1/v}. \tag{8.87}$$

Using (8.1)–(8.4), (8.73), (8.82)–(8.85), we obtain

$$\begin{aligned}
 & \sum_{m=m_0+1}^N e^{10\lambda\varepsilon\cdot 2^m} \mathbb{P}\{\nu = m\} \\
 & \leq \sum_{m=m_0+1}^N \sum_{n=m-1}^N 4d \exp\left(\frac{10a_1^2 \cdot 2^m}{512d^3\tau^2} - \frac{a_1^2 \cdot 2^{2um+(2v-1)n}}{8d^3\tau^2}\right) \\
 & \leq 4d \sum_{m=m_0+1}^N \sum_{n=m-1}^N \exp\left(-\frac{a_1^2 \cdot 2^{2um+(2v-1)n}}{16d^3\tau^2}\right) \\
 & \leq 4d \sum_{m=m_0+1}^{\infty} cL(a_1^{-1}d\tau) \exp\left(-\frac{a_1^2 \cdot 2^m}{32d^3\tau^2}\right) \\
 & \leq cdL^2(a_1^{-1}d\tau) \exp\left(-\frac{a_1^2 \cdot 2^{m_0}}{16d^3\tau^2}\right) \\
 & \leq cdL^2(a_1^{-1}d\tau) \exp\left(-\frac{p \cdot 2^p}{16}\right) \\
 & \leq cdL^2(a_1^{-1}d\tau) \exp\left(-\frac{d^{3/2}\tau\lambda}{4a_1}\right) \leq c. \tag{8.88}
 \end{aligned}$$

The inequality (8.74) follows from (8.81), (8.86)–(8.88). □

LEMMA 8.3. *Assume now (instead of (8.6)) that $N = M$. Then there exists a positive quantity a_4 depending only on v and such that the following inequality is valid:*

$$\mathbb{E} \exp(\varepsilon(\rho_j(M) + \delta_j(M))) \leq \exp(a_4 p(\lambda d^{3/2} \tau)^{1/v}), \quad j = 1, \dots, 2^M. \tag{8.89}$$

Proof. Similarly to (8.69), one can verify that $\rho_j(M) \leq 10\lambda \cdot 2^M$. Therefore, using (8.28), we obtain

$$\mathbb{E} \exp(\varepsilon(\rho_j(M) + \delta_j(M))) \leq \exp(10\varepsilon\lambda \cdot 2^M) \mathbb{E} e^{\varepsilon W}, \tag{8.90}$$

where $W = |\tilde{S}| + |T|$. From the definitions of \tilde{S} , $T = T_j$ (see (5.5) and (8.28)) it follows that they are sums of not more than 2^M independent random vectors \tilde{X}_k and $Y_k^{[M]}$ respectively. By analogy with (8.10), we have $\mathbb{P}\{|T| \leq 4\lambda \cdot 2^M\} = 1$. Hence,

$$\mathbb{E} e^{\varepsilon W} \leq \mathbb{E} e^{\varepsilon|\tilde{S}|} \exp(4\varepsilon\lambda \cdot 2^M). \tag{8.91}$$

Recall that $\text{cov} \tilde{X}_k = I$, $\mathcal{L}(\tilde{X}_k) \in \mathcal{A}_d(\tau)$. Taking into account (8.79) and applying Lemma 2.11 with $B^2 \leq 2^M$, $h = \varepsilon$, we see that

$$\mathbb{E} e^{\varepsilon|\tilde{S}|} \leq 2d \exp(\varepsilon^2 \cdot 2^M). \tag{8.92}$$

From (8.1), (8.3), (8.5), (8.73), (8.90)–(8.92) and from the definition of p (see (8.4)) it follows that

$$\begin{aligned}
 \mathbb{E} \exp(\varepsilon(\rho_j(M) + \delta_j(M))) & \leq 2d \exp\left(14\varepsilon\lambda \cdot 2^M + \varepsilon^2 \cdot 2^M\right) \\
 & \leq 2d \exp\left(\frac{15a_1^2 \cdot 2^M}{512d^3\tau^2}\right) \leq 2d \exp(ap \cdot 2^p) \\
 & \leq 2d \exp(ap(\lambda d^{3/2} \tau)^{1/v}). \tag{8.93}
 \end{aligned}$$

It is evident that (8.1), (8.4), (8.93) implies (8.89). □

Denote

$$\begin{aligned}
 b(N) &= \max_{1 \leq m \leq 2^N} \mathbb{E} \exp(\varepsilon(\delta_m(N) + \rho_m(N))), \\
 b(N - p) &= \max_{1 \leq m \leq 2^{N-p}} \mathbb{E} \exp(\varepsilon(\delta_m(N - p) + \rho_m(N - p))) \\
 r(N) &= \max_{1 \leq m \leq 2^N} \mathbb{E} \exp(\varepsilon\gamma_m(N)).
 \end{aligned} \tag{8.94}$$

One can easily verify that the definitions of $b(N)$, $b(N - p)$ are good coordinated: if we begin with 2^{N_0} vectors, where $N_0 = N - p$, then we have $b(N_0) = b(N - p)$ (see (8.17), (8.28)). Since j , $1 \leq j \leq 2^N$, is arbitrary, we can easily derive from (8.41), (8.94) and Lemma 8.2 that

$$b(N) \leq b(N - p)r(N) \leq b(N - p) \exp(a_3 p(\lambda d^{3/2} \tau)^{1/v}). \tag{8.95}$$

Continuing this procedure and taking into account (8.5)–(8.7), we obtain

$$b(N) \leq b(M) \exp(a_3(N - M)(\lambda d^{3/2} \tau)^{1/v}). \tag{8.96}$$

From (8.7), (8.89), (8.94), (8.96) it follows that

$$b(N) \leq \exp(a_5 N(\lambda d^{3/2} \tau)^{1/v}) \tag{8.97}$$

with $a_5 = \max\{a_3, a_4\}$. Using (8.28), (8.94), (8.97), we obtain

$$\begin{aligned}
 \mathbb{E} \exp(\varepsilon\Delta(\tilde{X}, \tilde{Y})) &\leq \sum_{m=1}^{2^N} \mathbb{E} \exp(\varepsilon\delta_m(N)) \\
 &\leq \sum_{m=1}^{2^N} \mathbb{E} \exp(\varepsilon(\delta_m(N) + \rho_m(N))) \\
 &\leq 2^N \exp(a_5 N(\lambda d^{3/2} \tau)^{1/v}) \\
 &\leq \exp(a N(\lambda d^{3/2} \tau)^{1/v}).
 \end{aligned} \tag{8.98}$$

Assume now, instead of (8.6), that $N \leq M$ and suppose that the vectors \tilde{X}_k , $k = 1, \dots, 2^M$, are constructed as functions of the vectors \tilde{Y}_k , $k = 1, \dots, 2^M$, in the same way as $X_k^* = \tilde{X}_{(k-1) \cdot 2^{N-M} + 1}$, $k = 1, \dots, 2^M$, were constructed as functions of $Y_k^* = Z_{(k-1) \cdot 2^{N-M} + 1}^{[M+p]}$, $k = 1, \dots, 2^M$. For $k = 2^N + 1, \dots, 2^M$ we can take $\mathcal{L}(\tilde{X}_k) = \mathcal{L}(\tilde{X}_1)$, $\mathcal{L}(\tilde{Y}_k) = \mathcal{L}(\tilde{Y}_1)$. Now, if we denote

$$\Delta_N(\tilde{X}, \tilde{Y}) = \max_{1 \leq j \leq 2^N} \left| \sum_{k=1}^j \tilde{Y}_k - \sum_{k=1}^j \tilde{X}_k \right|, \tag{8.99}$$

then, using (8.28), (8.89), (8.94), (8.99), we obtain

$$\begin{aligned}
 \mathbb{E} \exp(\varepsilon\Delta_N(\tilde{X}, \tilde{Y})) &\leq \sum_{m=1}^{2^N} \mathbb{E} \exp(\varepsilon\delta_m(M)) \leq 2^N \exp(a_4 p(\lambda d^{3/2} \tau)^{1/v}) \\
 &\leq \exp(a(N + p)(\lambda d^{3/2} \tau)^{1/v}).
 \end{aligned} \tag{8.100}$$

Proof of Theorem 1.5. Without loss of generality we shall suppose that $0 < \alpha \leq 1/5$ and $n = 2^N$, with $N \geq 1$. Assume that independent Gaussian vectors Y_1, \dots, Y_{2^N} are already constructed. According to Theorem 6.4, we can construct the independent random vectors $\tilde{Y}_1, \dots, \tilde{Y}_{2^N}$ so that $\mathcal{L}(\tilde{Y}_k) = \Psi(\tilde{\eta}_k)$, $k = 1, \dots, n$, and

$$\mathbb{E} \exp\left(\frac{c_{16} \Delta(\tilde{Y}, Y)}{d^{3/2} \tau}\right) \leq \exp(c_{17} N L(d)). \tag{8.101}$$

Then we construct the vectors $\tilde{X}_1, \dots, \tilde{X}_{2^N}$ as functions of $\tilde{Y}_1, \dots, \tilde{Y}_{2^N}$ as it was described above in this section.

Taking $v = \frac{2}{3+\alpha}$, we see that v satisfies (8.1) and

$$v^{-1} = \frac{3 + \alpha}{2}. \tag{8.102}$$

The relations (8.1), (8.3), (8.102) show that a, a_1, \dots, a_5 can be considered as quantities depending only on α .

Let M be the maximal integer satisfying the inequality (8.5) with a_2, p defined by (8.3), (8.4). Define N as the minimal integer such that $n \leq 2^N$, if $n \leq 2^M$, and as the minimal integer satisfying the relations $n \leq 2^N$ and (8.6), if $n > 2^M$. In both cases we have

$$N \leq c \log n + p. \tag{8.103}$$

It is evident that from (8.1), (8.4), (8.102) it follows that

$$p \leq a_6 L(\lambda d \tau) \leq a_7 (\lambda d^{3/2} \tau)^{\alpha/2}, \tag{8.104}$$

where a_6, a_7 depend only on α . The statement of Theorem 1.5 follows now from (8.73), (8.98)–(8.104). \square

9. PROOFS OF THEOREMS 1.3 AND 1.6

A complication of the formulation of Theorem 9.1 below in comparison with that of Theorem 1.5 is necessary to make more precise the dependence of the denominators of the fractions in the left-hand sides of inequalities (1.4), (1.7) on τ .

THEOREM 9.1. *Suppose that $\tau_1 \geq \tau_2 \geq 1$, $\tau_3 \geq \tau_2$, $\lambda \geq 1$, $\alpha > 0$ and ξ_1, \dots, ξ_n are independent random vectors with $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau_1)$, $\mathbb{E} \xi_k = 0$, $\text{cov} \xi_k = I$, $k = 1, \dots, n$. Let $n = 2^N \geq \tau_1^2 \tau_2^{-2}$ for some integer N and let the integer $M \leq N$ be defined by the relation*

$$\tau_1^2 \leq 2^M \tau_2^2 < 2 \tau_1^2. \tag{9.1}$$

Denote

$$\tilde{\xi}_m = 2^{-M/2} (\xi_{(m-1) \cdot 2^M + 1} + \dots + \xi_{m \cdot 2^M}), \quad m = 1, \dots, 2^{N-M}. \tag{9.2}$$

Assume that there exist random vectors $\tilde{\eta}_m$, $m = 1, \dots, 2^{N-M}$, which have the same moments of the first three orders as $\tilde{\xi}_m$, $\mathcal{L}(\tilde{\eta}_m) \in \mathcal{A}_d(\tau_3)$ and $\mathbb{P}\{|\tilde{\eta}_m| \leq \lambda\} = 1$. Then one can construct on a probability space a sequence of independent random vectors X_1, \dots, X_n and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, and

$$\mathbb{E} \exp\left(\frac{c_7(\alpha) \tau_2 \Delta(X, Y)}{\lambda d^3 \tau_1 \tau_3^2}\right) \leq \exp(c_8(\alpha)(\lambda d^{3/2} \tau_3)^{3/2+\alpha} L(n \tau_2^2 / \tau_1^2)),$$

where $c_7(\alpha)$, $c_8(\alpha)$ are positive quantities depending only on α .

Proof. It is clear that $0 \leq M \leq N$ (see (9.1)) and the random vectors $\tilde{\xi}_1, \dots, \tilde{\xi}_{2^{N-M}}$ satisfy the conditions of Theorem 1.5 with the change of n by 2^{N-M} and τ by τ_3 (one should use Lemma 2.5, the completeness of $\mathcal{A}_d(\tau)$ with respect to convolution and the monotonicity of $\mathcal{A}_d(\tau)$ with respect to τ). Therefore, one can construct on a probability space a sequence of independent random vectors $\tilde{X}_1, \dots, \tilde{X}_{2^{N-M}}$ and a sequence of independent Gaussian random vectors $\tilde{Y}_1, \dots, \tilde{Y}_{2^{N-M}}$ so that $\mathcal{L}(\tilde{X}_m) = \mathcal{L}(\tilde{\xi}_m)$, $\mathcal{L}(\tilde{Y}_m) = \mathcal{N}(0, I)$, $m = 1, \dots, 2^{N-M}$, and

$$\mathbb{E} \exp(2^{M/2} b \Delta(\tilde{X}, \tilde{Y})) \leq \exp(c_4(\alpha)(\lambda d^{3/2} \tau_3)^{3/2+\alpha} L(2^{N-M})), \tag{9.3}$$

where

$$b = \frac{c_3(\alpha)}{\lambda d^3 \tau_3^2 \cdot 2^{M/2}} \tag{9.4}$$

and $c_3(\alpha)$, $c_4(\alpha)$ are quantities depending only on α from Theorem 1.5. Using the Berkes–Philipp lemma (1979, Lemma A1, p. 53) and (9.2), we can suppose without loss of generality that

$$\begin{aligned} \tilde{X}_m &= 2^{-M/2} (X_{(m-1) \cdot 2^M + 1} + \dots + X_{m \cdot 2^M}), \\ \tilde{Y}_m &= 2^{-M/2} (Y_{(m-1) \cdot 2^M + 1} + \dots + Y_{m \cdot 2^M}), \end{aligned} \quad m = 1, \dots, 2^{N-M}, \tag{9.5}$$

where $\{X_1, \dots, X_{2^N}\}$ and $\{Y_1, \dots, Y_{2^N}\}$ are two collections of mutually independent random vectors with $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $\mathcal{L}(Y_k) = \mathcal{N}(0, I)$, for $k = 1, \dots, 2^N$.

Denote

$$\begin{aligned} S_j &= X_1 + \dots + X_j, & j = 1, \dots, 2^N; & & S_0 &= T_0 = 0; \\ T_j &= Y_1 + \dots + Y_j, & & & & \end{aligned} \tag{9.6}$$

and, by analogy with the proof of Theorem 6.4,

$$\Delta_1 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^M} |S_{k \cdot 2^M + s} - S_{k \cdot 2^M}|, \tag{9.7}$$

$$\Delta_2 = \max_{0 \leq k < 2^{N-M}} \max_{1 \leq s \leq 2^M} |T_{k \cdot 2^M + s} - T_{k \cdot 2^M}|, \tag{9.8}$$

$$\Delta_3 = \max_{1 \leq k \leq 2^{N-M}} |S_{k \cdot 2^M} - T_{k \cdot 2^M}|, \tag{9.9}$$

where the maximums are taken over integer values of k and s . It is evident that from (1.1), (9.6)–(9.9) it follows that

$$\Delta(X, Y) \leq \Delta_1 + \Delta_2 + \Delta_3. \tag{9.10}$$

Moreover, the relations (9.5), (9.6), (9.9), imply

$$\Delta_3 = 2^{M/2} \Delta(\tilde{X}, \tilde{Y}). \tag{9.11}$$

It is clear that we can take $c_3(\alpha) < 1/2$. From (9.4) and from the conditions of Theorem 9.1 it follows that $0 < b\tau_1 < 1/2$. Recall that Gaussian distributions from \mathcal{G}_d belong to all classes $\mathcal{A}_d(\tau)$. Using Lemma 2.11 with $t = b$, $\tau = \tau_1$, $B^2 = 2^M$, we have

$$\mathbb{E} e^{b\Delta_l} \leq 2^{N-M} \cdot 3de, \quad l = 1, 2. \tag{9.12}$$

Now the statement of Theorem 9.1 can be easily derived from (9.1), (9.3), (9.4) and (9.10)–(9.12) by an application of the Hölder inequality to the exponential moment $\mathbb{E} \exp(b\Delta(X, Y)/3)$. \square

For the proof of Theorem 1.3 we need the following Lemma 9.2.

LEMMA 9.2. *Let $F = \mathcal{L}(\xi) \in \mathcal{A}_d(1)$, $\mathbb{E}\xi = 0$ and $\text{cov}\xi = I$. Then there exists a random vector η and absolute positive constants $c_{69}, c_{70} \geq 1$ such that*

$$\begin{aligned} \mathbb{E}\eta = \mathbb{E}\xi = 0, \quad \text{cov}\eta = \text{cov}\xi = I, \\ \mathbb{P}\{|\eta| \leq c_{69}(1 + \log d)\} = 1, \quad \mathcal{L}(\eta) \in \mathcal{A}_d(c_{70}), \end{aligned} \tag{9.13}$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbb{E}\langle \eta, u \rangle \langle \eta, v \rangle \langle \eta, w \rangle = \mathbb{E}\langle \xi, u \rangle \langle \xi, v \rangle \langle \xi, w \rangle. \tag{9.14}$$

The following Lemma 9.3 (due to Yurinskii (1978), see Sazonov (1981, Lemma 3, p. 84)) will be used in the proof of Lemma 9.2.

LEMMA 9.3. *Let $0 < \gamma < 1$ and let the random variable μ take values $\frac{1}{2\gamma} (1 \pm (1 + 4\gamma^3)^{1/2})$ with probabilities $\frac{1}{2} \mp \frac{1}{2(1+4\gamma^3)^{1/2}}$. Then $\mathbb{E}\mu = 0$, $\mathbb{E}\mu^2 = \gamma$, $\mathbb{E}\mu^3 = 1$.*

Proof of Lemma 9.2. Denote

$$R = 4c_{71}(1 + \log d), \tag{9.15}$$

where an absolute positive constant c_{71} is so large as it will be necessary for the arguments below. Define the "cubic layers" A_j by

$$A_j = \{x \in \mathbb{R}^d : R(j - 1) \leq |x| < Rj\}, \quad j = 1, 2, \dots, \tag{9.16}$$

and represent the distribution F as the mixture

$$F = \sum_{j=1}^{\infty} q_j G_j, \tag{9.17}$$

where

$$q_j = F\{A_j\}, \quad G_j \in \mathcal{F}_d, \quad G_j\{A_j\} = 1, \quad j = 1, 2, \dots \tag{9.18}$$

Below we shall denote by ξ_j the random vectors with distributions

$$\mathcal{L}(\xi_j) = G_j, \quad j = 1, 2, \dots \tag{9.19}$$

Lemma 2.11 implies that, for $x \geq 1$,

$$\mathbb{P} \{ |\xi| \geq x \} \leq 2d e^{-x/4}. \tag{9.20}$$

Using (9.15), (9.16), (9.18), (9.20), we obtain

$$\begin{aligned} q_j &\leq \mathbb{P} \{ |\xi| \geq R(j-1) \} \leq 2d \exp\left(-\frac{R(j-1)}{4}\right) = \frac{2d}{(e^{c_{71}}d^{c_{71}})^{j-1}} \\ &< \frac{1}{d^3 j^3 R^2 (4d)^j}, \quad j = 2, 3, \dots, \end{aligned} \tag{9.21}$$

if c_{71} is sufficiently large.

Below we define a transform of d -dimensional distributions which will be denoted for $H = \mathcal{L}(\zeta) \in \mathcal{F}_d$, $r > 0$ by $H^{(r)}$. Suppose that μ is a random variable which is independent of ζ and has the distribution described in Lemma 9.2 with $\gamma = 1 - \frac{1}{8d}$. According to Lemma 9.2, we have

$$\mathbb{E} \mu = 0, \quad \mathbb{E} \mu^2 = 1 - \frac{1}{8d}, \quad \mathbb{E} \mu^3 = 1, \quad \mathbb{P} \{ |\mu| \leq 2 \} = 1. \tag{9.22}$$

Define

$$s = q_1 + \sum_{j=2}^{\infty} d^3 j^3 q_j. \tag{9.23}$$

The quantity s can be estimated with the help of (9.15), (9.17), (9.18), (9.21), (9.23) as follows:

$$1 = \sum_{j=1}^{\infty} q_j \leq s \leq 1 + \sum_{j=2}^{\infty} d^3 j^3 q_j < 1 + \sum_{j=2}^{\infty} \frac{1}{(4d)^j} < 1 + \frac{1}{8d}. \tag{9.24}$$

Below we shall use the notation $H^{(r)}$ for

$$H^{(r)} = \mathcal{L}(s^{1/3} \mu r^{-1} \zeta). \tag{9.25}$$

Consider the probability measure

$$F_1 = s^{-1} \left(q_1 G_1^{(1)} + \sum_{j=1}^{\infty} d^3 j^3 q_j G_j^{(dj)} \right). \tag{9.26}$$

Let the random vector η_1 have distribution $\mathcal{L}(\eta_1) = F_1$. The relations (9.17), (9.19), (9.22), (9.25), (9.26) imply

$$\mathbb{E} \eta_1 = \int_{\mathbb{R}^d} x F_1 \{dx\} = s^{-1} \left(q_1 s^{1/3} \mathbb{E} \mu \mathbb{E} \xi_1 + \sum_{j=1}^{\infty} d^2 j^2 q_j s^{1/3} \mathbb{E} \mu \mathbb{E} \xi_j \right) = 0 \tag{9.27}$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E} \langle \eta_1, u \rangle \langle \eta_1, v \rangle \langle \eta_1, w \rangle &= s^{-1} \sum_{j=1}^{\infty} d^3 j^3 q_j \left(\frac{s^{1/3}}{dj} \right)^3 \mathbb{E} \mu^3 \mathbb{E} \langle \xi_j, u \rangle \langle \xi_j, v \rangle \langle \xi_j, w \rangle \\ &= \sum_{j=1}^{\infty} q_j \mathbb{E} \langle \xi_j, u \rangle \langle \xi_j, v \rangle \langle \xi_j, w \rangle \\ &= \int_{\mathbb{R}^d} \langle x, u \rangle \langle x, v \rangle \langle x, w \rangle F\{dx\}. \end{aligned} \tag{9.28}$$

Furthermore, from (9.15), (9.16), (9.18), (9.22), (9.24)–(9.26) it follows that

$$\mathbb{P} \{ |\eta_1| > 3R \} = F_1 \{ \{x : |x| > 3R\} \} = 0. \tag{9.29}$$

Below we shall denote by t an arbitrary vector $t \in \mathbb{R}^d$ with $\|t\| = 1$. Using (9.17)–(9.19) and taking into account that $\text{cov } F = \text{cov } \xi = I$, we see that

$$q_1 \mathbb{E} \langle \xi_1, t \rangle^2 = \int_{\mathbb{R}^d} \langle x, t \rangle^2 q_1 G_1\{dx\} \leq \int_{\mathbb{R}^d} \langle x, t \rangle^2 F\{dx\} = \mathbb{E} \langle \xi, t \rangle^2 = 1. \tag{9.30}$$

The relations (9.17), (9.19), (9.22), (9.24)–(9.26), (9.30) imply

$$\begin{aligned} \mathbb{E} \langle \eta_1, t \rangle^2 &= s^{-1} \left(q_1 s^{2/3} \mathbb{E} \mu^2 \mathbb{E} \langle \xi_1, t \rangle^2 + \sum_{j=2}^{\infty} d^3 j^3 q_j \left(\frac{s^{1/3}}{dj} \right)^2 \mathbb{E} \mu^2 \mathbb{E} \langle \xi_j, t \rangle^2 \right) \\ &= s^{-1/3} \mathbb{E} \mu^2 \left(q_1 \mathbb{E} \langle \xi_1, t \rangle^2 + \sum_{j=2}^{\infty} dj q_j \mathbb{E} \langle \xi_j, t \rangle^2 \right) \\ &\geq s^{-1} \mathbb{E} \mu^2 \sum_{j=1}^{\infty} q_j \mathbb{E} \langle \xi_j, t \rangle^2 = s^{-1} \mathbb{E} \mu^2 \mathbb{E} \langle \xi, t \rangle^2 \geq 1 - \frac{1}{4d}. \end{aligned} \tag{9.31}$$

According to (9.16), (9.18), (9.19), (9.21), we have

$$\sum_{j=2}^{\infty} dj q_j \mathbb{E} \langle \xi_j, t \rangle^2 \leq \sum_{j=2}^{\infty} dj q_j \mathbb{E} \|\xi_j\|^2 \leq d^2 R^2 \sum_{j=2}^{\infty} j^3 q_j < \sum_{j=2}^{\infty} \frac{1}{(4d)^j} < \frac{1}{8d}. \tag{9.32}$$

From (9.22), (9.24), (9.27), (9.30)–(9.32) it follows that

$$\mathbb{E} \langle \eta_1, t \rangle^2 \leq 1 - \frac{1}{64d^2} < 1 \tag{9.33}$$

and, hence, the maximal eigenvalue of the covariance operator $D_1 = \text{cov } \eta_1$ does not exceed 1. Let $D_2 = I - D_1$. From (9.27), (9.31), (9.33) one can easily derive that

$$0 < \|D_2^{1/2} t\|^2 = \langle D_2 t, t \rangle = \langle I t, t \rangle - \langle D_1 t, t \rangle = 1 - \mathbb{E} \langle \eta_1, t \rangle^2 \leq \frac{1}{4d}. \tag{9.34}$$

Therefore, D_2 can be considered as covariance operator. Let η_2 be the random vector which is independent of η_1 and

$$\mathcal{L}(\eta_2) = \mathcal{L}(\sqrt{2}D_2^{1/2} K), \tag{9.35}$$

where K is the vector from Definition 6.3. From (9.34), (9.35) it follows that

$$\mathbb{E} \eta_2 = 0, \quad \text{cov } \eta_2 = D_2, \quad \mathbb{P} \{ \|\eta_2\| \leq 1 \} = 1, \tag{9.36}$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbb{E} \langle \eta_2, u \rangle \langle \eta_2, v \rangle \langle \eta_2, w \rangle = 0 \tag{9.37}$$

(we use that the vector K has symmetric distribution).

Now we define the vector η by $\eta = \eta_1 + \eta_2$. The first of the relations (9.13) follows now from (9.27), (9.36). The second one is a simple consequence of (9.34), (9.36). The third can be easily derived from (9.15), (9.29), (9.36). The simplest way to prove the fourth relation in (9.13) is to use (6.14), Lemma 2.5 and the connection between classes $\mathcal{A}_d(\tau)$ and classes of distributions satisfying Bernstein-type inequality conditions (see Zaitsev (1986, p. 205; 1987, 1988)). This proof is elementary and therefore omitted. Finally, it is evident that the relation (9.14) follows from (9.27), (9.28), (9.36), (9.37). \square

Proof of Theorem 1.3. Without loss of generality we assume that $n = 2^N$, where N is some integer. Let the integer M be defined by the relation

$$\tau^2 \leq 2^M < 2\tau^2. \tag{9.38}$$

Let $M \leq N$. Denote

$$\tilde{\xi}_m = 2^{-M/2} (\xi_{(m-1) \cdot 2^M + 1} + \dots + \xi_{m \cdot 2^M}), \quad m = 1, \dots, 2^{N-M}. \tag{9.39}$$

From the conditions of Theorem 1.3 and from the properties of classes $\mathcal{A}_d(\tau)$ it follows that $\mathcal{L}(\tilde{\xi}_m) \in \mathcal{A}_d(1)$, $\mathbb{E} \tilde{\xi}_m = 0$ and $\text{cov } \tilde{\xi}_m = I$. According to Lemma 9.2, there exist random vectors $\tilde{\eta}_m$ satisfying the relation

$$\mathbb{P} \{ |\tilde{\eta}_m| \leq \rho \} = 1, \quad \rho = c_{69}(1 + \log d), \quad \mathcal{L}(\tilde{\eta}_m) \in \mathcal{A}_d(c_{70}), \quad c_{70} \geq 1, \tag{9.40}$$

and having the same moments of the first three orders as $\tilde{\xi}_m$. Thus, we see that the conditions of Theorem 9.1 are fulfilled with $\tau_1 = \tau$, $\tau_2 = 1$, $\tau_3 = c_{70}$, $\lambda = \rho$. Therefore, for any $\beta > 0$ one can construct on a probability space a sequence of independent random vectors X_1, \dots, X_n and a corresponding sequence of independent Gaussian random vectors Y_1, \dots, Y_n so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, and

$$\mathbb{E} \exp\left(\frac{c_5(\beta)\Delta(X, Y)}{c_{70}^2 d^3 \tau \rho}\right) \leq \exp(c_6(\beta)(c_{70} d^{3/2} \rho)^{3/2+\beta} L(n/\tau^2)). \tag{9.41}$$

The relations (9.40) and (9.41) imply the statement of Theorem 1.3 in the case $M \leq N$.

Let now $M > N$. Then we take the vectors $\{X_1, \dots, X_n; Y_1, \dots, Y_n\}$ with needed distributions to be mutually independent. From (9.38) it follows that $n = 2^N < 2\tau^2$. Recall that all d -dimensional Gaussian distributions belong to classes $\mathcal{A}_d(\tau)$ for all $\tau \geq 0$. Using the completeness of $\mathcal{A}_d(\tau)$ with respect to convolution and applying Lemma 2.11 with $t = \frac{1}{3\tau}$, $B^2 = 2^{N+1}$, we obtain the inequality $\mathbb{E} \exp(\Delta(X, Y)/3\tau) \leq 3de^2$, which implies the statement of Theorem 1.3 in the case $M > N$. \square

Proof of Theorem 1.6. It repeats the proof of Theorem 1.3. The only difference is in the construction of the vectors $\tilde{\eta}_m$. Let $n = 2^N$. Assume that the integer $M \leq N$ and the vectors $\tilde{\xi}_m$, $m = 1, \dots, 2^{N-M}$, are defined by (9.38) and (9.39). From the conditions of Theorem 1.6 it follows that, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbb{E} \langle \tilde{\xi}_m, u \rangle \langle \tilde{\xi}_m, v \rangle \langle \tilde{\xi}_m, w \rangle = 0, \quad m = 1, \dots, 2^{N-M}.$$

Consider the random vectors $\tilde{\eta}_m$, $m = 1, \dots, 2^{N-M}$, with distributions $\mathcal{L}(\tilde{\eta}_m) = \mathcal{L}(\sqrt{2}K)$, where K is the vector with symmetric distribution involved in Definition 6.3. It is clear that the random vectors $\tilde{\eta}_m$ have the same moments of the first three orders as $\tilde{\xi}_m$, $\mathbb{P}\{|\tilde{\eta}_m| \leq \sqrt{2}\} = 1$, $\mathcal{L}(\tilde{\eta}_m) \in \mathcal{A}_d(\sqrt{2}c_{14})$. The last inclusion follows from (6.14) and from the properties of classes $\mathcal{A}_d(\tau)$. Thus, we see that the conditions of Theorem 9.1 are fulfilled with $\tau_1 = \tau$, $\tau_2 = 1$, $\tau_3 = \sqrt{2}c_{14}$, $\lambda = \sqrt{2}$. An application of Theorem 9.1 completes the proof of Theorem 1.6 in the case $M \leq N$. The case $M > N$ can be considered as in the proof of Theorem 1.3. \square

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