STOCHASTIC APPROXIMATIONS OF THE SOLUTION OF A FULL BOLTZMANN EQUATION WITH SMALL INITIAL DATA.

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Abstract. This paper gives an approximation of the solution of the Boltzmann equation by stochastic interacting particle systems in a case of cut-off collision operator and small initial data. In this case, following the ideas of Mischler and Perthame, we prove the existence and uniqueness of the solution of this equation and also the existence and uniqueness of the solution of the associated nonlinear martingale problem.

Then, we first delocalize the interaction by considering a mollified Boltzmann equation in which the interaction is averaged on cells of fixed size which cover the space. In this situation, Graham and Méléard have obtained an approximation of the mollified solution by some stochastic interacting particle systems. Then we consider systems in which the size of the cells depends on the size of the system. We show that the associated empirical measures converge in law to a deterministic probability measure whose density flow is the solution of the full Boltzmann equation. That suggests an algorithm based on the Poisson interpretation of the integral term for the simulation of this solution.

1. Introduction

In the upper atmosphere, the gas is rarefied and is described by the non-negative density $f(t,x,v)$ of particles which at time t and point x move with velocity v. Then $f(t,x,v)$ is positive and normalized so that $\int f(t,x,v)dxdv$ is equal to one and satisfies the Boltzmann equation

$$\begin{aligned} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) &= Q(f,f)(t,x,v) \text{ on } [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3 \\ &= Q^+(f,f)(t,x,v) - Q^-(f,f)(t,x,v) \\ f(0,x,v) &= f_0(x,v) \text{ is a density of probability} \end{aligned} \quad (1.1)$$

where

$$Q^+(f,f)(t,x,v) = \int_{S^2} dn \int_{\mathbb{R}^3} dv_* B(v-v_*,n) f(t,x,v') f(t,x,v'_*) \quad (1.2)$$

and

$$Q^-(f,f)(t,x,v) = f(t,x,v) L f(t,x,v)$$

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with

$$\begin{aligned} Lf(t,x,v) &= A * f(t,x,.)(v) = \int_{S^2}\int_{\mathbb{R}^3} B(v-v_*,n) f(t,x,v_*) dv_* dn; \\ A(z) &= \int_{S^2} B(z,n) dn. \end{aligned} \tag{1.3}$$

The nonnegative cross-section $B(z,n)$ depends only on $|z|$ and on $|\langle z.n\rangle|$. The velocities v' and v'_* represent the post-collisional velocities of two particles of velocities v and v_* having collided in a position in which their centers are on a line of direction given by the unit vector n belonging to the unit sphere S^2. Conservation of kinetic energy and momentum for binary collisions implies that

$$v' = v + ((v_* - v)\cdot n)n, \quad v'_* = v_* + ((v - v_*)\cdot n)n.$$

We refer to Cercignani *et al.* (1994) for physical comments on this model.

The Boltzmann equation presents many important difficulties, due to the unboundedness of B and to the localization in space in the quadratic collision term (the interaction is not mean-field). In the general case, uniqueness in not proved and existence of renormalized solutions is showed in the famous paper of DiPerna and Lions (1989). On the other hand, existence and uniqueness have been studied by many authors under restrictive assumptions on the cross-section B (principally a cut-off assumption) and results have been obtained in particular in small time or under small initial data, as it can be found in Kaniel and Shinbrot (1978), Bellomo and Toscani (1985), Hamdache (1985), Toscani (1986), Bellomo *et al.* (1988) and more recently in Mischler and Perthame (1997). One follows the ideas of Mischler and Perthame (1997), obtained in a more general situation of infinite energy, in order to prove by a fixed point argument the existence and uniqueness of the solution in a well chosen functional space $\mathcal{B}_\alpha$, in a case of cut-off collision operator and small initial data. This existence and uniqueness result (Theorem 2.1) is very close to "a priori" assumptions in the paper of Babovsky and Illner (1989).

The Boltzmann equation is an integro-differential equation, in which the integral term comes from the randomness in the geometry of collisions. It is natural to study its probabilistic interpretation. One associates with the equation a nonlinear martingale problem and one obtains the existence and uniqueness of the solution of this martingale problem in the space of probability measures having a measurable version of densities in $\mathcal{B}_\alpha$.

Our aim is then to give a stochastic approximation of the solution of the Boltzmann equation, obtaining thus a theoretical justification of the Nanbu and Bird algorithms in this case (cf. Babovsky and Illner (1989)).

The interaction appearing in the collision term is localized in space and is not mean-field. So we do not know how to construct directly approximating particle systems. One first delocalizes the interaction by considering a mollified Boltzmann equation in which the mollifying kernel is issued from a grid method. The space is shared in disjoint cells of size δ in which the interaction is averaged. In this mean-field case, Graham and Méléard (1997) prove some stochastic approximations of the solution of the mollified Boltzmann equation by interacting particle systems and obtain a precise rate of convergence in $\mathcal{O}(\exp(\frac{K}{\delta^3})/n)$, where n is the size of the particle system. Moreover, a unified approach for systems with simple or binary mean-field interactions is given.

In this paper, one considers such systems in which the size of the cells of the grid depends on the size of the system. More precisely, we assume that δ depends on n in the asymptotic $\exp(\frac{K}{\delta(n)^3})/n \to 0$ (when n tends to infinity). Then one proves that the empirical measures of the associated interacting particle systems converge in law to a deterministic probability measure whose density flow is the solution of the full Boltzmann equation. The convergence is obtained for probability measures on the path space and convergence results for functionals of the paths can be deduced.

At our knowledge, this result (Theorem 5.4) seems to be the first pathwise approximation result in a non mollified case and in dimension 3. Let us quote Caprino and Pulvirenti (1995) and Rezakhzanlou (1996), who obtain the convergence of stochastic particle systems to a one-dimensional Boltzmann equation at fixed times. Our approach is unified for simple or binary mean-field systems, and allows to understand the similarity between Bird's and Nanbu's algorithms. Moreover one gives a precise asymptotic between δ and n; that was an open question in Babovsky and Illner (1989).

One finally suggests an algorithm based on the Poisson interpretation of the integral term to simulate the solution of the Boltzmann equation, which avoids to discretize in time and exactly follows the pathwise history of the particles.

2. The existence and uniqueness result

Let us now prove the existence and uniqueness result obtained for the Boltzmann equation in a case of bounded collision operator and small initial data with finite energy.

Theorem 2.1. *Let $\alpha > 0$ and T be a positive time. Let us assume that*

(H_1): $A \in L^\infty(\mathbb{R}^3)$

(H_2): f_0 is a density function satisfying

$$0 \leq f_0(x,v) \leq \frac{C_0}{6}\exp(-\alpha|v|^2), \tag{2.1}$$

where C_0 is a real number such that $C_0 < \frac{(\sqrt{\alpha})^3}{\|A\|_\infty(\sqrt{\pi})^3 T} = \frac{1}{C_\alpha T}$.

Then there exists a unique function $f \in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ solution of the Boltzmann equation (1.1) satisfying

$$0 \leq f(t,x,v) \leq \frac{C(t)}{6}\exp(-\alpha|v|^2) \tag{2.2}$$

where $C(t)$ is a positive and bounded function on $[0,T]$ defined by $\frac{1}{C(t)} = \frac{1}{C_0} - C_\alpha t$.

Proof. The proof is completely inspired of the proof of Theorem 2 in Mischler and Perthame (1997) given in a case of infinite energy. It consists first in introducing an upper solution related to the Boltzmann equation, and second in obtaining a fixed point theorem in a functional space related to this upper solution.

Let us consider $g(t,v) = C(t)h(v)$, where $h(v) = \exp(-\alpha|v|^2)$. One would like

$$\begin{aligned}\partial_t g(t,v) = \dot{C}(t)h(v) \geq Q^+(g,g)(t,v) &= C^2(t)Q^+(h,h)(v)\\ &= C^2(t)h(v)L(h)(v),\end{aligned}$$

since $Q(h,h) = 0 = Q^+(h,h) - Q^-(h,h) = Q^+(h,h) - hL(h)$.

Then one is looking for C such that $\dot{C} \geq C^2 \sup_v L(h)(v)$. Therefore let us consider $C \in \mathcal{C}^1([0,T],\mathbb{R})$ such that $C(0) = C_0$ and solving $\dot{C}(t) = \|A\|_\infty(\pi/\alpha)^{3/2}C^2(t)$. By denoting $C_\alpha = \|A\|_\infty(\pi/\alpha)^{3/2}$, one finally obtains that $(C(t))^{-1} = (C_0)^{-1} - C_\alpha t$ and the function

$$\hat{h}(t,v) = \frac{C(t)}{6}h(v) = \frac{C(t)}{6}e^{-\alpha|v|^2} \tag{2.3}$$

satisfies

$$\partial_t \hat{h}(t,v) \geq C_\alpha C(t)\hat{h}(t,v) \geq 6Q^+(\hat{h},\hat{h})(t,v). \tag{2.4}$$

Now one considers the set

$$\mathcal{B}_\alpha = \{\varphi \in L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3); 0 \leq \varphi(t,x,v) \leq \hat{h}(t,v)\}$$

with the norm $\|\varphi\|_\alpha = \text{ess}\sup_{t,x,v}\{|\varphi(t,x,v)|/|\frac{C(t)}{6}\exp(-\alpha|v|^2)|\}$ for which it is complete. The global existence and uniqueness of the solution of the Boltzmann equation is deduced from a fixed point theorem in this space. As in Mischler and Perthame (1997), let us define the mapping $\Lambda : \varphi \in \mathcal{B}_\alpha \to \psi = \Lambda(\varphi)$, where ψ is the solution of

$$\begin{aligned}\partial_t\psi + v\cdot\nabla_x\psi + C_\alpha C(t)\psi &= Q^+(\varphi,\varphi) + (C_\alpha C(t) - L(\varphi))\varphi,\\ \psi(0,x,v) &= f_0(x,v).\end{aligned} \tag{2.5}$$

By a maximum principle one observes that Λ sends $\mathcal{B}_\alpha$ into $\mathcal{B}_\alpha$ and by (2.4) that moreover

$$\forall\varphi_1,\varphi_2 \in \mathcal{B}_\alpha, \|\Lambda\varphi_1 - \Lambda\varphi_2\|_\alpha \leq \frac{5}{6}\|\varphi_1 - \varphi_2\|_\alpha.$$

Hence Λ is a contraction and admits a unique fixed point in $\mathcal{B}_\alpha$, which is solution of the Boltzmann equation. $\square$

3. The nonlinear martingale problem associated with the Boltzmann equation

The weak form of the equation (1.1) is given for a function φ in $C_b^1(\mathbb{R}^6)$ by

$$\begin{aligned}&\partial_t\langle f_t,\varphi\rangle - \langle f_t, v\cdot\nabla_x\varphi\rangle\\ &= \langle f_t(x,v)dxdv, \int(\varphi(x,v+((v_*-v).n)n) - \varphi(x,v))\\ &\qquad B(v-v_*,n)f(t,x,v_*)dndv_*\rangle. \end{aligned} \tag{3.1}$$

(Remark that the mapping $(v,v_*) \to (v',v'_*)$ has a determinant equal to 1).

We associate with this evolution equation a nonlinear martingale problem for which every solution is a probability measure on the path space whose marginals are solutions of the equation (1.1).

Let us denote by $\tilde{\mathcal{P}}(\mathbb{D}([0,T],\mathbb{R}^6))$ the space of probability measures on $\mathbb{D}([0,T],\mathbb{R}^6)$ having for every $t\in[0,T]$ a density with respect to the Lebesgue measure. Let us remark that following Meyer (1966) p. 193-194, there exists for P in $\tilde{\mathcal{P}}(\mathbb{D}([0,T],\mathbb{R}^6))$ a measurable function $p(t,x,v)$ on $[0,T]\times\mathbb{R}^6$ such that for any $t\in[0,T]$, $p(t,.)$ is a density of P_t. We call such a function a measurable version of the densities of P.

Definition 3.1. The probability measure $P\in\tilde{\mathcal{P}}(\mathbb{D}([0,T],\mathbb{R}^6))$ is solution of the nonlinear martingale problem $(\mathcal{M})$ if for every function $\varphi\in C_b^1(\mathbb{R}^6)$, for (X,V) the canonical process on $\mathbb{D}([0,T],\mathbb{R}^3\times\mathbb{R}^3)$,

$$\begin{aligned}\varphi(X_t,V_t)-\varphi(X_0,V_0)&-\int_0^t V_s.\nabla_x\varphi(X_s,V_s)ds\\ -\int_0^t\int_{S^2}\int_{\mathbb{R}^3}&(\varphi(X_s,V_s+((v_*-V_s).n)n)-\varphi(X_s,V_s))\\ &\qquad B(V_s-v_*,n)p(s,X_s,v_*)dv_*dnds\end{aligned}$$

is a P-martingale, where $p(t,.)$ is a measurable version of the densities of the flow of marginals $(P_t)_{t\geq 0}$, $P_0(dxdv)=f_0(x,v)dxdv$.

Clearly this definition does not depend on the choice of the measurable version of the densities of P.

Let us denote by $\tilde{\mathcal{P}}_\alpha(\mathbb{D}([0,T],\mathbb{R}^6))$ the subspace of $\tilde{\mathcal{P}}(\mathbb{D}([0,T],\mathbb{R}^6))$ such that a measurable version of the densities p satisfies $0\leq p(t,x,v)\leq\hat{h}(t,v)$ for almost every $(t,x,v)\in[0,T]\times\mathbb{R}^6$, $\hat{h}$ being defined in (2.3). Then it is true for every measurable version of the densities.

Theorem 3.2. *Under assumptions (H_1) and (H_2), the nonlinear martingale problem $(\mathcal{M})$ has a unique solution P in $\tilde{\mathcal{P}}_\alpha(\mathbb{D}([0,T],\mathbb{R}^6))$. Every measurable version of the densities of P is almost surely equal to the solution f of the Boltzmann equation (1.1) defined in Theorem 2.1.*

Let us first observe that if P is solution of $(\mathcal{M})$, then by taking the expectations in the martingale problem, each measurable version p of its densities is solution of the Boltzmann equation (1.1). Moreover if we assume that $0\leq p(t,x,v)\leq\hat{h}(t,v)$, then p is almost surely equal to f by Theorem 2.1. Therefore, we first study the following classical martingale problem associated with the function f.

Definition 3.3. A probability measure $P\in\mathcal{P}(\mathbb{D}([0,T],\mathbb{R}^6))$ is a solution of the martingale problem $(\mathcal{M}^f)$ if for every function $\varphi\in C_b^1(\mathbb{R}^6)$,

$$\begin{aligned}\varphi(X_t,V_t)-\varphi(X_0,V_0)&-\int_0^t V_s.\nabla_x\varphi(X_s,V_s)ds\\ -\int_0^t\int_{S^2}\int_{\mathbb{R}^3}&(\varphi(X_s,V_s+((v_*-V_s).n)n)-\varphi(X_s,V_s))\\ &\qquad B(V_s-v_*,n)f(s,X_s,v_*)dv_*dnds\end{aligned}\tag{3.2}$$

is a P-martingale and $P_0(dxdv)=f_0(x,v)dxdv$.

Proposition 3.4. *Under assumptions (H_1) and (H_2), the martingale problem $(\mathcal{M}^f)$ has a unique solution P^f absolutely continuous with respect to the Lebesgue measure. Its density q is solution of the evolution equation*

$$q(t,x,v)=f_0(x-tv,v)+\int_0^t(S_{t-s})^*Q(q,f)(s,x,v)ds,\tag{3.3}$$

where S_t is the semigroup associated with the flow solution of $\partial_t q + v.\nabla_x q = 0$, and $(S_t)^*$ is the dual operator.

Proof. 1) (H_1) and (2.2) imply the jump kernel $B(V_s - v_*, n) f(s, X_s, v_*) dv_* dn$ has a finite total mass uniformly in s, X_s, V_s. Moreover, the drift part in (3.2) has a Lipschitz continuous coefficient. In this case, the existence and uniqueness of a solution P^f of $(\mathcal{M}^f)$ is well known.

2) Let us now prove that the solution P^f has a density q. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of random jumps of the process Z under P^f; there is a finite number of random jumps on the time interval $[0, T]$. Following Jacod and Shiryeav (1987) p. 136 and, since the jump measure $B(V_s - v_*, n) f(s, X_s, v_*)$ $dv_* dn ds$ is absolutely continuous with respect to time, the law of the first jump T_0 conditionally to $X_0 = x, V_0 = v$ has a density with respect to the Lebesgue measure. Since the law of (X_0, V_0) has the density f_0, then the triplet (X_0, V_0, T_0) has a density with respect to the Lebesgue measure. Of course, it is the same for $(X_{T_0-}, V_{T_0-}, T_0) = (X_0 + T_0 V_0, V_0, T_0)$. Moreover, conditionally to $(X_{T_0-}, V_{T_0-}, T_0)$, the law of the jump ΔV_{T_0} has clearly a density and we deduce that the law of (X_{T_0}, V_{T_0}, T_0) has a density. By the Markov property, we then obtain that for every T_n, the law of (X_{T_n}, V_{T_n}, T_n) has a density, and so that P^f has a density q with respect to the Lebesgue measure. By taking expectation in (3.2), we obtain moreover that the flow (q_t) satisfies for φ in $C_b^1(\mathbb{R}^6)$

$$\begin{aligned} &\partial_t \langle q_t, \varphi\rangle - \langle q_t, v\cdot\nabla_x \varphi\rangle \\ &= \langle q_t(x,v) dx dv, \int (\varphi(x, v + ((v_* - v).n)n) - \varphi(x,v)) \\ &\qquad B(v - v_*, n) f(t, x, v_*) dn dv_*\rangle. \end{aligned} \tag{3.4}$$

We can extend this formula to functions $\psi(t,x)$ which are in $C_b^1([0,T]\times\mathbb{R}^6)$ by Itô's formula. Let S_t be the semigroup associated with the flow $\partial_t q + v.\nabla_x q = 0$ and S_t^* the dual operator, $S_t^* = S_{-t}$. Of course, $S_t\varphi(x,v) = \varphi(x + tv, v)$. For a fixed t in $[0,T]$ and φ in $C_b^1(\mathbb{R}^6)$, we choose $\psi(s,x,v) = S_{t-s}\varphi(x,v) = \varphi(x + (t-s)v, v)$. Then $\partial_s\psi + v.\nabla_x\psi = 0$ and $\psi(t,.) = \varphi$. The equation (3.4) extended to ψ implies that for every function φ in $C_b^1(\mathbb{R}^6)$,

$$\begin{aligned} &\int \varphi(x,v) q(t,x,v) dx dv \\ &= \int S_t\varphi(x,v) f_0(x,v) dx dv \\ &\quad + \int_0^t \int S_{t-s}\varphi(x,v)[f(s,x,v'_*)q(s,x,v') - f(s,x,v_*)q(s,x,v)] \\ &\qquad B(v - v_*, n) dn dv_* dx dv ds \\ &= \int \varphi(x,v) S_t^* f_0(x,v) dx dv \\ &\quad + \int \varphi(x,v) \int_0^t S_{t-s}^*([f(s,x,v'_*)q(s,x,v') - f(s,x,v_*)q(s,x,v)] \\ &\qquad B(v - v_*, n) dn dv_*) dx dv ds \end{aligned}$$

and then we deduce that for every $t \in [0,T]$, $dxdv$ almost surely,

$$q(t,x,v) = f_0(x-vt,v) + \int_0^t (S_{t-s})^* Q(q,f)(s,x,v)ds. \tag{3.5}$$

□

Proposition 3.5. *Under assumptions (H_1) and (H_2), the evolution equation (3.5) has a unique solution in $L^\infty([0,T], L^1(dxdv))$.*

Proof. Let q' be another solution of the evolution equation. Then

$$\begin{aligned}
&\|q(t)-q'(t)\|_{L^1(dxdv)} \\
&= \|\int_0^t (S_{t-s})^*[Q(q,f)-Q(q',f)](s,x,v)ds\|_{L^1(dxdv)} \\
&\leq \int_0^t \|(S_{t-s})^*[Q(q,f)-Q(q',f)](s,x,v)\|_{L^1(dxdv)}ds \\
&= \int_0^t \|Q(q,f)-Q(q',f)](s,x,v)\|_{L^1(dxdv)}ds \\
&\leq \int_0^t \int \{|f(s,x,v'_*)||q(s,x,v')-q'(s,x,v')| \\
&\qquad + |f(s,x,v_*)||q(s,x,v)-q'(s,x,v)|\}B(v-v_*,n)dndv_*dxdvds \\
&\leq C_\alpha \frac{C(T)}{6}\int_0^t \|q(s)-q'(s)\|_{L^1(dxdv)}ds,
\end{aligned}$$

by (H_1) and (2.2). We deduce by usual Gronwall's Lemma that the solution of the evolution equation is unique in $L^\infty([0,T], L^1(dxdv))$. □

We now prove Theorem 3.2.

Proof. We first consider $\psi \in C_b^1([0,T]\times\mathbb{R}^6)$ with compact support. We can then prove by using Fubini's theorem and the integration by part formula that

$$\begin{aligned}
&\partial_t\langle f_t,\psi\rangle - \langle f_t, v\cdot\nabla_x\psi + \partial_t\psi\rangle \\
&\quad = \langle f_t(x,v)dxdv, \int (\psi(t,x,v+((v_*-v).n)n) - \psi(t,x,v)) \\
&\qquad\qquad B(v-v_*,n)f(t,x,v_*)dndv_*\rangle.
\end{aligned}$$

We obtain by approximation the same formula for every function $\psi \in C_b^1([0,T]\times\mathbb{R}^6)$, and considering $\psi(s,x,v) = S_{t-s}\varphi(x,v)$, $\varphi \in C_b^1(\mathbb{R}^6)$, we obtain as before that the solution solution f of the full Boltzmann equation (1.1) is solution of the evolution equation (3.5). The uniqueness proved in Proposition 3.5 implies that $q = f$. Then the solution of $(\mathcal{M}^f)$ is in fact a solution of $(\mathcal{M})$.

Let us now consider two solutions P and Q of $(\mathcal{M})$ with measurable versions of the densities bounded by $\hat{h}$. Thus these densities are solutions of (1.1) bounded by $\hat{h}$ and so are almost surely equal, and equal to f, by Theorem 2.1. Since the nonlinearity in the martingale problem just depends on this common flow f, we now get a (classical) martingale problem in which the jump measure is given and bounded. The uniqueness in this martingale problem implies that $P = Q$, hence we get Theorem 3.2.

The flow of densities is equal to f and satisfies moreover for each $t \in [0,T]$

$$f(t,x,v) = f_0(x-vt,v) + \int_0^t (S_{t-s})^* Q(f,f)(s,x,v)ds, \quad \text{a.s. in } x, v. \tag{3.6}$$

□

Let us now give a regularity result for the function f, useful later in the proof of Proposition 5.2. The property stated below, as well as Theorem 2.1, are very close to properties presented as conjectures in Babovsky and Illner (1989).

PROPOSITION 3.6. *Let us assume that*
(H_3)*: There exists* $K > 0$ *such that for every* $h \in \mathbb{R}^3$*,*

$$\text{ess sup}_{x\in\mathbb{R}^3, v\in\mathbb{R}^3} \frac{|f_0(x+h,v) - f_0(x,v)|}{e^{-\alpha|v|^2}} \leq K|h|, \tag{3.7}$$

then the same property holds for f*: there exists* $K_T > 0$ *such that*

$$\text{ess sup}_{t\in[0,T], x\in\mathbb{R}^3, v\in\mathbb{R}^3} \frac{|f(t,x+h,v) - f(t,x,v)|}{e^{-\alpha|v|^2}} \leq K_T|h|, \quad \forall h \in \mathbb{R}^3. \tag{3.8}$$

Proof. f is solution of the evolution equation (3.6) and

$$\begin{aligned}
&f(t,x+h,v) - f(t,x,v) \\
&= f_0(x+h-tv,v) - f_0(x-tv,v) \\
&+ \int_0^t \int \Bigg[\Bigg(f(s,x+h-(t-s)v,v'_*) f(s,x+h-(t-s)v,v') \\
&\quad - f(s,x-(t-s)v,v'_*) f(s,x-(t-s)v,v') \Bigg) \\
&\quad - \Bigg(f(s,x+h-(t-s)v,v_*) f(s,x+h-(t-s)v,v) \\
&\quad - f(s,x-(t-s)v,v_*) f(s,x-(t-s)v,v) \Bigg) \Bigg] B(v-v_*,n) dndv_*ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{|f(t,x+h,v) - f(t,x,v)|}{e^{-\alpha|v|^2}} \\
&\leq \frac{|f_0(x+h-tv,v) - f_0(t,x-tv,v)|}{e^{-\alpha|v|^2}} \\
&+ \int_0^t \int \Bigg(\frac{f(s,x+h-(t-s)v,v'_*)}{e^{-\alpha|v|^2}} \\
&\qquad \times |f(s,x+h-(t-s)v,v') - f(s,x-(t-s)v,v')| \\
&\qquad + \frac{f(s,x-(t-s)v,v')}{e^{-\alpha|v|^2}} \\
&\qquad \times |f(s,x+h-(t-s)v,v'_*) - f(s,x-(t-s)v,v'_*)| \Bigg) \\
&\quad \times B(v-v_*,n) dndv_*ds
\end{aligned}$$

$$
\begin{aligned}
+ \quad \int_0^t \int \Bigg(& \frac{f(s, x+h-(t-s)v, v_*)}{e^{-\alpha|v|^2}} \\
& \quad \times |f(s, x+h-(t-s)v, v) - f(s, x-(t-s)v, v)| \\
& + \frac{f(s, x-(t-s)v, v)}{e^{-\alpha|v|^2}} \\
& \quad \times |f(s, x+h-(t-s)v, v_*) - f(s, x-(t-s)v, v_*)| \Bigg) \\
& \times B(v-v_*, n) dn dv_* ds.
\end{aligned}
$$

Let $\Delta^\alpha_{h,t}(f) = \text{ess sup}_{x,v} \frac{|f(t,x+h,v)-f(t,x,v)|}{e^{-\alpha|v|^2}}$. We have

$$
\begin{aligned}
& \frac{|f(t, x+h, v) - f(t, x, v)|}{e^{-\alpha|v|^2}} \\
\leq \quad & K|h| + \frac{C(T)}{6} \int_0^t \int \Bigg(\frac{e^{-\alpha|v'_*|^2} \Delta^\alpha_{h,s}(f) e^{-\alpha|v'|^2}}{e^{-\alpha|v|^2}} + \frac{e^{-\alpha|v'|^2} \Delta^\alpha_{h,s}(f) e^{-\alpha|v'_*|^2}}{e^{-\alpha|v|^2}} \\
& + e^{-\alpha|v_*|^2} \Delta^\alpha_{h,s}(f) + e^{-\alpha|v_*|^2} \Delta^\alpha_{h,s}(f) \Bigg) B(v-v_*, n) dn dv_* ds.
\end{aligned}
$$

We use the conservation of energy: $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. So we get

$$
\begin{aligned}
& \frac{|f(t, x+h, v) - f(t, x, v)|}{e^{-\alpha|v|^2}} \\
\leq \quad & K|h| + 4\frac{C(T)}{6} \int_0^t \int \Delta^\alpha_{h,s}(f) e^{-\alpha|v_*|^2} B(v-v_*, n) dn dv_* \\
\leq \quad & K|h| + \frac{2C(T)}{3} C_\alpha \int_0^t \Delta^\alpha_{h,s}(f) ds,
\end{aligned}
$$

and finally

$$
\Delta^\alpha_{h,t}(f) \leq K|h| + \frac{2}{3} C(T) C_\alpha \int_0^t \Delta^\alpha_{h,s}(f) ds. \tag{3.9}
$$

Gronwall's Lemma allows to conclude. □

4. The mollified problem

4.1. The mollified nonlinear martingale problem

Mollifying consists in delocalizing in space the interaction appearing in the Boltzmann equation in order to obtain a mean-field model. We cover $\mathbb{R}^3$ by a grid of cubic, uniform disjoint cells Δ of volume $|\Delta| = \delta^3$, and we introduce the regularizing kernel

$$
I^\delta(x, y) = \frac{1}{\delta^3} \sum_\Delta \mathbb{1}_{x \in \Delta} \mathbb{1}_{y \in \Delta}. \tag{4.1}
$$

The kernel Q is replaced by Q^δ, defined by

$$
\begin{aligned}
Q^\delta(f, f)(t, x, v) \quad = \quad & \int [f(t, x, v') f(t, y, v'_*) - f(t, x, v) f(t, y, v_*)] \\
& \times B(v-v_*, n) I^\delta(x, y) dy dn dv_*
\end{aligned}
$$

which leads to the mollified equation

$$\partial_t f + v \cdot \nabla_x f = Q^\delta(f, f). \tag{4.2}$$

By using the fact that

$$\|Q^\delta(f,f) - Q^\delta(g,g)\|_{L^1(dxdv)} \le C(\delta)\|f+g\|_{L^1(dxdv)}\|f-g\|_{L^1(dxdv)},$$

one prove that this equation has a unique solution in $L^\infty([0,T], L^1(dxdv))$. Indeed, the above inequality implies that there exists T_0 such that there is existence and uniqueness in $L^\infty([0,T_0], L^1(dxdv))$ and one constructs piecewise the solution on $[0,T]$. But under small initial data, one can prove moreover that

PROPOSITION 4.1. *Under assumptions* (H_1) *and* (H_2)*, the unique solution* f^δ *of (4.2) with initial condition* f_0 *satisfies for almost* $(t,x,v) \in [0,T] \times \mathbb{R}^6$*,*

$$0 \le f^\delta(t,x,v) \le \hat{h}(t,v).$$

Proof. Since $Q^\delta(\hat{h},\hat{h}) = Q(\hat{h},\hat{h})$, the function $\hat{h}$ is also an upper solution for the mollified problem. The same arguments as in the proof of Theorem 2.1 imply that a solution of the mollified equation is also obtained by a fixed point theorem in $\mathcal{B}_\alpha$ associated with the contraction Λ^δ defined as in (2.5) by replacing Q^+ by $Q^{\delta+}$. □

One deduces from Proposition 4.1 the existence and uniqueness of the solution for the mollified nonlinear martingale problem associated with Q^δ, whose proof is completely similar to that of Theorem 3.2.

THEOREM 4.2. *Under assumptions* (H_1) *and* (H_2)*, there exists a unique probability measure* $P^\delta \in \tilde{\mathcal{P}}_\alpha(\mathbb{D}([0,T],\mathbb{R}^6))$*, solution of the mollified nonlinear martingale problem defined as follows. For every function* $\varphi(x,v)$ *on* $\mathbb{R}^3 \times \mathbb{R}^3$*,* C_b^1 *in* x *and bounded in* v*,*

$$\begin{aligned} &\varphi(X_t,V_t) - \varphi(X_0,V_0) - \int_0^t V_s.\nabla_x\varphi(X_s,V_s)ds \\ &- \int_0^t \int (\varphi(X_s, V_s + ((v_* - V_s).n)n) - \varphi(X_s,V_s)) \\ &\qquad \times B(V_s - v_*, n) I^\delta(X_s, y) P_s^\delta(dydv_*)dnds \end{aligned} \tag{4.3}$$

is a P^δ *martingale, where* P_s^δ *is the marginal at time* s *of* P^δ*,* $P_0^\delta(dxdv) = f_0(x,v)dxdv$*.*

4.2. THE APPROXIMATING INTERACTING PARTICLE SYSTEMS

As in Graham and Méléard (1997), one defines mean-field interacting particle systems which approximate the solution of the nonlinear martingale problem (4.3) associated with the mollified equation. These systems correspond to different physical models: first a simple mean-field model (Nanbu's system), second a binary mean-field model (Bird's system).

Let $(x^n, v^n) = ((x_1,v_1),(x_2,v_2),...,(x_n,v_n))$ be the generic point in $(\mathbb{R}^6)^n$. We introduce the mapping $\mathbf{e_i} : h \to \mathbf{e_i}.h = (0,...,0,h,0,...,0) \in (\mathbb{R}^3)^n$ with h at the i-th place. We consider $\phi \in C_b^1((\mathbb{R}^6)^n)$ and define two systems of

particles:
The Nanbu system is a Markov process in $\mathbb{D}(\mathbb{R}_+, (\mathbb{R}^6)^n)$ with generator

$$\begin{aligned}&\sum_{i=1}^{n} v_i.\nabla_{x_i}\phi(x^n, v^n)\\ &\quad + \frac{1}{n-1}\sum_{1\le i\ne j\le n}\int (\phi(x^n, v^n + \mathbf{e_i}.(((v_j - v_i).n)n)) - \phi(x^n, v^n))\\ &\qquad\qquad\qquad \times B(v_i - v_j, n) I^\delta(x_i, x_j)dn. \qquad (4.4)\end{aligned}$$

The Bird system is a Markov process on the same space with generator

$$\begin{aligned}&\sum_{i=1}^{n} v_i.\nabla_{x_i}\phi(x^n, v^n)\\ &\quad + \frac{1}{n-1}\sum_{1\le i\ne j\le n}\int \frac{1}{2}(\phi(x^n, v^n + \mathbf{e_i}.(((v_j - v_i).n)n\\ &\quad + \mathbf{e_j}.(((v_i - v_j).n)n)) - \phi(x^n, v^n))B(v_i - v_j, n)I^\delta(x_i, x_j)dn. \qquad (4.5)\end{aligned}$$

In both cases, $Z^{\delta,n} = (X^{\delta,n}, V^{\delta,n}) = (Z^{\delta,1n}, Z^{\delta,2n}, ..., Z^{\delta,nn})$ denotes the Markov process and $|.|_T$ the variation norm in the space of signed measures on the Skorohod space $\mathbb{D}([0,T], (\mathbb{R}^6)^k)$, for any $k \in \mathbb{N}$.

THEOREM 4.3. *Under hypothesis (H_1), let us denote by Λ_δ the real number $\|A\|_\infty/\delta^3$ and assume that $(Z_0^{\delta,in})_{1\le i\le n}$ are i.i.d. with law P_0.*

(i) There is propagation of chaos in a strong sense: for given T and k,

$$|\mathcal{L}(Z^{\delta,1n}, ..., Z^{\delta,kn}) - \mathcal{L}(Z^{\delta,1n})^{\otimes k}|_T \le 2k(k-1)\frac{\Lambda_\delta T + \Lambda_\delta^2 T^2}{n-1} \qquad (4.6)$$

(ii) If moreover $P_0(dxdv) = f_0(x,v)dxdv$ and f_0 satisfies the assumption (H_2),

$$|\mathcal{L}(Z^{\delta,in}) - P^\delta|_T \le 6\frac{e^{\Lambda_\delta T} - 1}{n-1}, \quad \forall i \in \{1, ..., n\}, \qquad (4.7)$$

where P^δ is the solution of the nonlinear martingale problem (4.3).

(iii) The empirical measures $\mu^{\delta,n} = n^{-1}\sum_{i=1}^n \delta_{Z^{\delta,in}}$ converge in probability to P^δ in $\mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^6))$ for the weak convergence for the Skorohod metric on $\mathbb{D}([0,T], \mathbb{R}^6)$ with the convergence rate $\sqrt{K + e^{\Lambda_\delta T}}/\sqrt{n-1}$.

Proof. In Graham and Méléard (1997), (4.6) is proved, and also the convergence of $Z^{\delta,in}$ to a probability measure Q in the sense of (4.7), where Q is uniquely defined using a Boltzmann tree and is solution of the mollified nonlinear martingale problem (4.3).

It is easy to show, as in the proof of Proposition 3.4, that for each t, Q_t is absolutely continuous with respect to Lebesgue's measure. The only difference between the two proofs concerns the conditional law of ΔV_{T_0}. In the present case, this law is absolutely continuous with respect to $Q_{T_0}(dydv_*)$ which is the law of (X_{T_0-}, V_{T_0-}) and is then absolutely continuous with respect to Lebesgue's measure. So the conditional law of ΔV_{T_0} under Q has a density. Let us denote by q the measurable version of densities of Q belonging to $L^\infty([0,T], L^1(dxdv))$. By taking expectations in (4.3), we prove that q is solution of (4.2). The uniqueness of the solution of this equation in $L^\infty([0,T], L^1(dxdv))$ implies that for every t, $q(t) = f^\delta(t), dxdv$ a.s.. By Proposition 4.1 and Theorem 4.2, we conclude that $Q = P^\delta$. So we have (4.7) and (iii). □

5. Approximation of the non mollified Boltzmann equation

We consider interacting systems in which the size of the cells depends on the size of the system. We will prove that if we assume $(H_1), (H_2)$ and take an asymptotic $(n, \delta(n))$ which tends to $(+\infty, 0)$ such that $\exp(\frac{T\|A\|_\infty}{\delta(n)^3})/n$ tends to zero, then the empirical measures of the system $(Z^{\delta(n),n})$ converge when n tends to infinity to the solution of $(\mathcal{M})$.

As seen in Theorem 4.3, the empirical measures of $(Z^{\delta,n})$ converge to P^δ. Let us now study the convergence of P^δ to P when δ tends to zero. Consider a sequence $(\delta(l))_{l\in\mathbb{N}}$ which tends to zero when l tends to infinity. Denote $P^l = P^{\delta(l)}$ and f^l a measurable version of the densities of P^l solution of (4.2), and in the same way $Q^l = Q^{\delta(l)}$ and $I^l = I^{\delta(l)}$.

Proposition 5.1. *Let us assume (H_1), (H_2), that P_0 has a second moment and that for each l, $P_0^l(dxdv) = P_0(dxdv) = f_0(x,v)dxdv$.*

Then the family (P^l) is tight when l tends to infinity.

Proof. The canonical process (X^l, V^l) under P^l belongs to $\mathbb{D}([0,T], \mathbb{R}^6)$ and is a sequence of $\mathcal{D}$-semimartingales in the sense of Joffe and Métivier (1986). Moreover, since

$$\begin{aligned}
& |\int ((v-v_*).n)nB(v-v_*,n)I^l(x,y)dnP_t^l(dydv_*)| \\
\le\ & \|A\|_\infty \int |v-v_*| f^l(t,y,v_*)I^l(x,y)dydv_* \\
\le\ & \|A\|_\infty \frac{C(T)}{6}\int (|v|+|v_*|)e^{-\alpha|v_*|^2}dv_* \\
\le\ & K(|v|+1).
\end{aligned}$$

Hypotheses 3.2.1 in Joffe and Métivier (1986) are satisfied and Proposition 3.2.3 in Joffe and Métivier (1986) implies in our case that the family (P^l) is tight in $\mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^6))$. □

Proposition 5.2. *Assume $(H_1), (H_2), (H_3)$ (cf. Proposition 3.6) and if f denotes the unique solution of the Boltzmann equation in $\mathcal{B}_\alpha$, then*

$$\sup_{t\le T} \|f^l(t,.) - f(t,.)\|_{L^1} \le K_T \delta(l) \tag{5.1}$$

and thus tends to zero when l tends to infinity.

Proof. Let us first remark that there exists a constant K_T such that for each (x,v) in $\mathbb{R}^6$ and t in $[0,T]$,

$$|\int f(t,y,v)I^l(x,y)dy - f(t,x,v)| \le K_T\delta(l)e^{-\alpha|v|^2}. \tag{5.2}$$

Indeed, we have

$$
\begin{aligned}
& \left|\int f(t,y,v) I^l(x,y)dy - f(t,x,v)\right| \\
&= \left|\int (f(t,y,v) - f(t,x,v)) \frac{1}{\delta(l)^3} \mathbb{1}_{\{y\in\Delta_x^l\}} dy\right| \\
&\leq K_T \int |y-x| e^{-\alpha|v|^2} \frac{1}{\delta(l)^3} \mathbb{1}_{\{y\in\Delta_x^l\}} dy \quad \text{by Proposition 3.6} \\
&\leq K_T \delta(l) e^{-\alpha|v|^2}.
\end{aligned}
$$

We now use the evolution equations satisfied by f^l and f:

$$
\begin{aligned}
f(t,x,v) &= f_0(x-vt,v) + \int_0^t (S_{t-s})^* Q(f,f)(s,x,v)ds, \quad \text{a.s. in } x,v, \\
f^l(t,x,v) &= f_0(x-vt,v) + \int_0^t (S_{t-s})^* Q^l(f^l,f^l)(s,x,v)ds, \quad \text{a.s. in } x,v.
\end{aligned}
$$

So we have

$$
f^l(t,x,v) - f(t,x,v) = \int_0^t (S_{t-s})^* \Big(Q^l(f^l,f^l) - Q(f,f) \Big)(s,x,v)ds,
$$

and if we denote $\|f\|_{L^1,t} = \|f(t,.)\|_{L^1(dxdv)}$, we obtain

$$
\|f^l - f\|_{L^1,t} \leq \int_0^t \|Q^l(f^l,f^l) - Q(f,f)\|_{L^1,s} ds. \tag{5.3}
$$

But

$$
\begin{aligned}
&(Q^l(f^l,f^l) - Q(f,f))(s,x,v) \\
&\quad = \int B(v-v_*,n) \\
&\qquad \times \Big(\int I^l(x,y)dy (f^l(s,x,v')f^l(s,y,v'_*) - f^l(s,x,v)f^l(s,y,v_*)) \\
&\qquad\qquad - [f(s,x,v')f(s,x,v'_*) - f(s,x,v)f(s,x,v_*)] \Big) dndv_* \\
&\quad = \int B(v-v_*,n) \\
&\qquad \times \Big(\int I^l(x,y)dy (f^l(s,x,v')f^l(s,y,v'_*) - f^l(s,x,v)f^l(s,y,v_*)) \\
&\qquad\qquad - \int I^l(x,y)dy (f(s,x,v')f(s,y,v'_*) - f(s,x,v)f(s,y,v_*)) \\
&\qquad\qquad + \int I^l(x,y)dy (f(s,x,v')f(s,y,v'_*) - f(s,x,v)f(s,y,v_*)) \\
&\qquad\qquad - [f(s,x,v')f(s,x,v'_*) - f(s,x,v)f(s,x,v_*)] \Big) dndv_*
\end{aligned}
$$

$$\begin{aligned}
&= \int B(v-v_*,n)\Big(\int I^l(x,y)dy\Big[f^l(s,x,v')(f^l(s,y,v'_*)-f(s,y,v'_*))\\
&\qquad\qquad + f(s,y,v'_*)(f^l(s,x,v')-f(s,x,v'))\\
&\qquad\qquad + (f(s,x,v')(f(s,y,v'_*)-f(s,x,v'_*))\Big]\\
&\qquad - \int I^l(x,y)dy\Big[f^l(s,x,v)(f^l(s,y,v_*)-f(s,y,v_*))\\
&\qquad\qquad + f(s,y,v_*)(f^l(s,x,v)-f(s,x,v))\\
&\qquad\qquad + (f(s,x,v)(f(s,y,v_*)-f(s,x,v_*))\Big]\Big)dndv_*\\
&\leq \|A\|_\infty\Big[\int\Big(\int I^l(x,y)dy\Big(f^l(s,x,v')|f^l(s,y,v'_*)-f(s,y,v'_*)|\\
&\qquad\qquad + f(s,y,v'_*)|f^l(s,x,v')-f(s,x,v')|\\
&\qquad\qquad + f^l(s,x,v)|f^l(s,y,v_*)-f(s,y,v_*)|\\
&\qquad\qquad + f(s,y,v_*)|f^l(s,x,v)-f(s,x,v)|\Big)\\
&\qquad + f(s,x,v')|\int f(s,y,v'_*)I^l(x,y)dy - f(s,x,v'_*)|\\
&\qquad + f(s,x,v)|\int f(s,y,v_*)I^l(x,y)dy - f(s,x,v_*)|\Big)dv_*\Big]\\
&\leq \|A\|_\infty(T_1+T_2+T_3+T_4+T_5+T_6).
\end{aligned}$$

The equation (5.2) implies that

$$\begin{aligned}
T_5 &\leq K_T\delta(l)\int f(s,x,v')e^{-\alpha|v'_*|^2}dv_*\\
T_6 &\leq K_T\delta(l)\int f(s,x,v)e^{-\alpha|v_*|^2}dv_*.
\end{aligned}$$

Otherwise,

$$T_1 \leq \frac{C(T)}{6}\int e^{-\alpha|v'|^2}\int I^l(x,y)dy|f^l(s,y,v'_*)-f(s,y,v'_*)|dv_*$$

and in the same way,

$$\begin{aligned}
T_2 &\leq \frac{C(T)}{6}\int e^{-\alpha|v'_*|^2}|f^l(s,x,v')-f(s,x,v')|dv_*,\\
T_3 &\leq \frac{C(T)}{6}\int e^{-\alpha|v|^2}\int I^l(x,y)dy|f^l(s,y,v_*)-f(s,y,v_*)|dv_*,\\
T_4 &\leq \frac{C(T)}{6}\int e^{-\alpha|v_*|^2}|f^l(s,x,v)-f(s,x,v)|dv_*.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
&\int |(Q^l(f^l,f^l) - Q(f,f))(s,x,v)|dxdv \\
&\leq \quad K_T\|A\|_\infty \delta(l) \\
&\qquad\qquad \times \int \left(\int f(s,x,v')e^{-\alpha|v'_*|^2}dv_* + \int f(s,x,v)e^{-\alpha|v_*|^2}dv_* \right) dxdv \\
&+ \quad \frac{C(T)}{6}\|A\|_\infty \int \left(\int e^{-\alpha|v'|^2} \int I^l(x,y)dy|f^l(s,y,v'_*) - f(s,y,v'_*)|dv_* \right.\\
&+ \quad \int e^{-\alpha|v'_*|^2}|f^l(s,x,v') - f(s,x,v')|dv_* \\
&+ \quad \int e^{-\alpha|v|^2} \int I^l(x,y)dy|f^l(s,y,v_*) - f(s,y,v_*)|dv_* \\
&+ \quad \left. \int e^{-\alpha|v_*|^2}|f^l(s,x,v) - f(s,x,v)|dv_* \right) dxdv.
\end{aligned}$$

We apply Fubini's theorem, the change of variable $(v,v_*) \to (v',v'_*)$ with determinant 1 and observe that the terms containing (v',v'_*) are equal to the correspondent terms with (v,v_*). Hence we obtain

$$\begin{aligned}
\int |(Q^l(f^l,f^l) - Q(f,f))(s,x,v)|dxdv \quad &\leq \quad 2K_TC_\alpha\delta(l) \\
&+ \quad 4C_\alpha \frac{C(T)}{6}\|f^l - f\|_{L^1,s}.
\end{aligned}$$

We deduce that

$$\|f^l - f\|_{L^1,t} \leq 2K_TC_\alpha\delta(l)T + 4C_\alpha \frac{C(T)}{6} \int_0^t \|f^l - f\|_{L^1,s}ds \tag{5.4}$$

and by Gronwall's Lemma there exists a constant $K > 0$ such that

$$\sup_{t\leq T} \|f^l - f\|_{L^1,t} \leq K\delta(l).$$

Finally, $\sup_{t\leq T}\|f^l - f\|_{L^1,t}$ tends to zero when l tends to infinity. □

Theorem 5.3. *Under $(H_1),(H_2),(H_3)$ and if P_0 has a second order moment, then the sequence (P^l) converges to the unique probability measure $P \in \tilde{\mathcal{P}}_\alpha(\mathbb{D}([0,T],\mathbb{R}^6))$ defined in Theorem 3.2.*

Proof. The sequence (P^l) is tight, so a subsequence of (P^l), still denoted by (P^l), converges to a probability measure Q. Then, almost surely in t, P^l_t converge to Q_t. Let us consider the sequence of associated measurable densities f^l. By Proposition 5.2, (f^l) converges to f in $L^\infty([0,T],L^1(dxdv))$. Therefore, for almost every t, Q_t has a density which is equal to $f(t,.)$. Let us now prove that Q is the unique solution of $(\mathcal{M}^f)$ defined in (3.2).

Following Joffe and Métivier (1986), it suffices to prove that

$$\begin{aligned}\lim_{l\to+\infty}\int_0^T ds\int dxdv f^l(s,x,v)\\ \times\Big(\int \Delta\varphi(x,v,v_*,n)B(v-v_*,n)I^l(x,y)f^l(s,y,v_*)dndv_*dy\\ -\int \Delta\varphi(x,v,v_*,n)B(v-v_*,n)f(s,x,v_*)\Big)dndv_* = 0 \qquad (5.5)\end{aligned}$$

where $\Delta\varphi(x,v,v_*,n)=\varphi(x,v+((v_*-v).n)n)-\varphi(x,v)$.

Since all the terms are bounded,

$$\begin{aligned}&\int_0^T ds\int dxdv f^l(s,x,v)\\ &\quad\times\Big(\int \Delta\varphi(x,v,v_*,n)B(v-v_*,n)I^l(x,y)f^l(s,y,v_*)dndv_*dy\\ &\quad-\int \Delta\varphi(x,v,v_*,n)B(v-v_*,n)f(s,x,v_*)\Big)dndv_*\\ =&\int_0^T ds\int f^l(s,x,v)B(v-v_*,n)\Delta\varphi(x,v,v_*,n)I^l(x,y)\\ &\quad\Big(f^l(s,y,v_*)-f(s,y,v_*)\Big)dndydv_*dxdv\\ +&\int_0^T ds\int f^l(s,x,v)B(v-v_*,n)\Delta\varphi(x,v,v_*,n)\\ &\quad\Big(\int I^l(x,y)f(s,y,v_*)dy-f(s,x,v_*)\Big)dndv_*dxdv\\ =&\ T_1+T_2.\end{aligned}$$

We have

$$\begin{aligned}|T_1| &\leq 2\|\varphi\|_\infty\|A\|_\infty\int_0^T ds\int I^l(x,y)f^l(s,x,v)\\ &\qquad\times\Big(\int |f^l(s,y,v_*)-f(s,y,v_*)|dydv_*\Big)dxdv\\ &\leq 2\|\varphi\|_\infty\|A\|_\infty K\delta(l)T\int I^l(x,y)dx\frac{C(T)}{6}\int e^{-\alpha|v|^2}dv\\ &\leq \frac{C(T)}{3}\|\varphi\|_\infty C_\alpha K\delta(l)T.\end{aligned}$$

The second inequality comes from equations (5.5) and (2.5). This term tends to zero when l tends to infinity.

Now, we have using Lemma 5.3

$$\begin{aligned}|T_2| &\leq 2\|\varphi\|_\infty\|A\|_\infty K_T\delta(l)\int_0^T ds\int f^l(s,x,v)e^{-\alpha|v_*|^2}dv_*dxdv\\ &\leq 2\|\varphi\|_\infty C_\alpha K_T\delta(l)T\end{aligned}$$

and T_2 tends to zero when l tends to infinity.

Then Q is the unique probability measure P defined in Theorem 3.2 and so the sequence (P^l) has a unique limit value P and thus converges in law to P. $\square$

We finally conclude by our main result which proves that for a good asymptotic $(n, \delta(n))$, we can construct a stochastic interacting particle system whose law converges when n tends to infinity to the law P associated with the solution of the full Boltzmann equation. More precisely we prove:

THEOREM 5.4. *Let us assume* $(H_1), (H_2), (H_3)$ *and that* P_0 *has a second order moment. Let* $n \in \mathbb{N}^*$ *and consider a sequence of positive real numbers* $\delta(n)$ *which tends to zero in an asymptotic such that* $\exp(\frac{\|A\|_\infty T}{\delta(n)^3})/n$ *tends to zero when* n *tends to infinity, then*
(i) for every $1 \leq i \leq n$, *the sequence of laws of* $Z^{\delta(n),in}$ *converges in* $\mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^6))$ *to the probability measure* P *defined in Theorem 3.2.*
(ii) The empirical measures of the interacting particle system $(Z^{\delta(n),in})_{1\leq i \leq n}$ *converge in law (and in probability) to* P *in* $\mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^6))$.

The proof just consists in associating Theorems 4.3 and 5.4.

6. Two algorithms for the Boltzmann equation

We deduce from the above study two algorithms associated either with the simple mean-field interacting particle system or with the binary mean-field interacting particle system. The description of the algorithms is the same in both cases, since the theoretical justification is unified for the two systems.

As seen previously, the empirical measure $\mu^{\delta(n),n}$ approximates the law of the Boltzmann process whose marginal at time t is equal to the solution $f(t,.)$ of the Boltzmann equation.

We simulate the particle system. For a fixed n, there are n particles and the total jump rate for the $n(n-1)/2$ pairs of possible interactions is $n\Lambda_{\delta(n)}/2$, as seen in (4.4) and (4.5), where $\Lambda_{\delta(n)} = \|A\|_\infty/(\delta(n)^3)$. A Poisson process of rate $n\Lambda_{\delta(n)}/2$ gives the sequence of collision times. At each of these times, we choose uniformly the pair of particles which interact, update the particles under the free transport, compute the mass of the jump measure following (4.4) or (4.5). Let us denote by $\hat{B}$ this jump measure and by $|\hat{B}|$ its mass. Then we follow an acceptance-rejection procedure. We discard the jump with probability $1 - |\hat{B}|/\Lambda_{\delta(n)}$ and with probability $|\hat{B}|/\Lambda_{\delta(n)}$ we choose the joint jump amplitude according to $\hat{B}/|\hat{B}|$. All this is done independently. We only evaluate at each step the cross-section of the interacting pair and not those of the $n(n-1)/2$ pairs. This simulation is exact if we simulate exactly the exponential variables related to the Poisson process, instead of discretizing in time.

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