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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR EPIDEMIC MODELS WITH VARYING INFECTIVITY AND WANING IMMUNITY

Arsene Brice Zotsa Ngoufack*

Abstract. We study an individual-based stochastic epidemic model in which infected individuals gradually become susceptible again following each infection (generalized SIS model). The epidemic dynamics is described by the average infectivity and susceptibility processes in the population together with the numbers of infected and susceptible/uninfected individuals. In R. Forien et al., Stochastic epidemic models with varying infectivity and susceptibility. arXiv preprint arXiv:2210.04667 (2022), a functional law of large numbers (FLLN) is proved as the population size goes to infinity, and asymptotic endemic behaviors are also studied. In this paper, we prove a functional central limit theorem (FCLT) for the stochastic fluctuations of the epidemic dynamics around the FLLN limit. The FCLT limit for the aggregate infectivity and susceptibility processes is given by a system of stochastic non-linear integral equation driven by a two-dimensional Gaussian process.

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1. Introduction

Many infectious diseases become endemic over a long time horizon, for which waning of immunity plays a critical role in addition to the infection process. The classical compartment model, SIRS (susceptible-infectious-recovered-susceptible), assumes that immunity at the individual level is binary, that is, each individual is either fully immune or fully susceptible. However, that is largely unrealistic since it does not allow for partial immunity or gradual waning of immunity. Various models have been developed to study the effects of the waning of immunity and the associated vaccination policies [1–12]. All the models except [6, 11] start from an ODE model with the additional waning immunity characteristic. In particular, El Khalifi and Britton [10] recently studied an extension of the ODE for the classical SIRS model with a linear or exponential waning function. They considered a model with a fixed number of immunity levels and then discussed the corresponding ODE-PDE limiting model (similar to [3]). See also [12] for an analysis of the stability of the model with an arbitrarily large number of discrete compartments with varying levels of disease immunity. Carlsson *et al.* [6] study an age-structured PDE model that takes into account waning immunity. Despite the interesting findings, there has been lack of individual-based stochastic epidemic models that take into account waning immunity.

Keywords and phrases: Epidemic model, varying infectivity, waning immunity, Gaussian-driven stochastic Volterra integral equations, Poisson random measure, stochastic integral with respect to Poisson random measure, quarantine model.

Aix Marseille Univ, CNRS, I2M, Marseille, France.

* Corresponding author: arsene-brice.zotsa-ngoufack@univ-amu.fr

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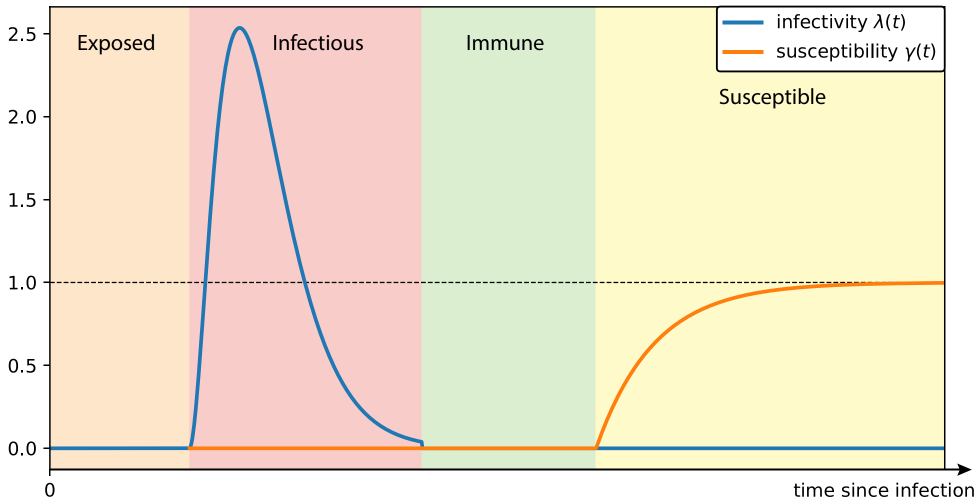

FIGURE 1. Illustration of a typical realization of the random infectivity and susceptibility functions of an individual from the time of infection to the time of recovery, and then to the time of losing immunity and becoming fully susceptible (or in general, partially susceptible).

In [13] the authors Forien, Pang, Pardoux and Zotsa first introduced an individual-based stochastic epidemic model that captures waning immunity as well as varying infectivity [14]. More precisely, they proposed a general stochastic epidemic model which takes into account a random infectivity and a random and gradual loss of immunity (also referred to as waning immunity or varying susceptibility). See Figure 1 for a realization of the infectivity and susceptibility of an individual after an infection. Individuals experience the susceptible-infected-immune-susceptible cycle. When an individual becomes infected, the infected period may include a latent exposed period and then an infectious period. Once an individual recovers from the infection, after some potential immune period (whose duration can be zero), the immunity is gradually lost, and the individual progressively becomes susceptible again. Then the individual may be infected again, and repeat the process at each new infection with a different realization of the random infectivity and susceptibility functions. This model can be regarded as a generalized SIS model. When "I" is interpreted as "infected" including exposed and infectious periods, and "S" is interpreted as including immune and susceptible periods. It can of course also be regarded as a generalized SEIRS model. We also mention the recent work [11], where a similar stochastic model of varying infectivity and waning immunity with vaccination is studied. We notice the difference from our modeling approach besides the vaccination aspect: the random susceptibility function and varying infectivity function are taken independently in each infection. However, we do not impose the independence between the random infectivity and susceptibility functions in each infection.

In [13], the authors have proved a functional law of large numbers (FLLN) in which both the average susceptibility and the total force of infection converge, when the size of the population goes to infinity, to a deterministic limiting model given by a system of integral equations depending on the law of the susceptibility and on the mean of the infectivity function (see Thm. 2.2 below). Note that, the average susceptibility is the sum of the susceptibility of all individuals in the population divided by the size of the population and the total force of infection is the sum of the infectivity of all individuals in the population divided by the population size. Under a particular set of random infectivity and susceptibility functions and initial conditions, they also show that a PDE model with infection-age can be derived from the limiting model, which reduces to the model introduced by Kermack and McKendrick in [15, 16] (see the reformulation in [17]). They also characterize the threshold of endemicity which depends on the law of susceptibility and not only on the mean, and prove the global asymptotical stability of the disease-free steady state when the basic reproduction number is lower than the above-mentioned threshold. When the basic reproduction number is larger than this threshold, they authors prove existence and uniqueness of the endemic equilibrium and under additional assumptions, they prove that the disease-free-steady state is unstable.

The goal of this work is to study the stochastic fluctuations of the dynamics around the deterministic limits for the stochastic epidemic models with random varying infectivity and a random and gradual loss of immunity, see the result in Theorem 2.10. More precisely we study jointly the fluctuation of the average total force of infection and average of susceptibility and then deduce the fluctuations of the proportions of the compartment counting processes. The fluctuation limit of the average total force of infection and average susceptibility is given by a system of stochastic non-linear integral equation driven by a two-dimensional Gaussian process. Given these, the limits of the compartment counting processes are expressed in terms of the solutions of the above non-linear stochastic integral equation driven by another two-dimensional Gaussian process. This result extend the functional central limit theorem (FCLT) results of Pang-Pardoux in [18, 19] for the non-Markovian models without gradual loss of immunity and in [20, 21] for the Markovian case.

To prove the FCLT (Thm. 2.10), we first obtain a decomposition for the scaled infectivity and susceptibility processes, each of which has two component processes. We employ the central limit theorems for $\mathbf{D}$-valued random variables [22] to prove the convergence of one component since it can be regarded as a sum of i.i.d. $\mathbf{D}$-valued random variables. We refer to the Notation section below for the notation $\mathbf{D}$. The convergence of the other component is much more challenging, and we must develop novel methods to prove tightness and convergence. We need more assumptions on the pair of random functions (λ, γ) than the ones used to establish the FLLN, and these assumptions are sufficient to establish tightness. For that purpose, we need to establish moment estimates and maximal inequalities for the increments of the processes. This is extremely difficult. In general the common technique use classical result for moment calculations of stochastic integrals (see for example [18]). However, this result does not work here because of the complicated interactions among the individuals, as well as the randomness in the infectivity and susceptibility. Some of the expressions involve stochastic integrals with respect to Poisson random measures, with integrands which are not predictable but depend on the future. The classical result for moment calculations of stochastic integrals cannot be used in our setting, for example, [23], Theorem 6.2. Thus, we establish a new theorem to calculate the moments for such stochastic integrals (see Thm. 5.1). In fact, in general the common technique use classical result for moment calculations of stochastic integrals see for example [18].

In addition, we develop an approximation technique by introducing a quarantine model, in which one infected individual is quarantined so that the number of infected descendants of that individual can be bounded conveniently (this scheme can be extended to more than one quarantined individual). Using this approximation, we compare the processes counting the number of infections of each individual for the original process to the number of infections of each individual for the quarantine model, and as a consequence, we obtain the moment estimates and maximal inequalities to prove tightness (see Lem. 8.6).

Finally it is worth noting that our individual-based stochastic model resembles the recent studies of models with interactions, for instance, interacting age-dependent Hawkes process in [24, 25], age-structured population model in [26] and stochastic excitable membrane models in [27]. In the proof of the FLLN in [13], the authors adapted the tools to the theory of propagation of chaos (see Sznitman [28]) by constructing a family of i.i.d. processes with a well-chosen coupling. A similar approach was taken in [24, 26]. However, for the FCLT, we derive from the approach of studying fluctuations from the mean limit that were taken in [24, 26], since it is more challenging for our non-Markovian model. In that approach one has to work with processes taking values in a Hilbert space (dual of some Sobolev space of test functions) and the limit is characterized by an SDE in infinite dimension driven by a Gaussian noise. On the contrast, we work directly with the real-valued processes and prove their convergence with the conventional tightness criteria, which leads to a finite-dimensional stochastic integral equation driven by Gaussian processes.

Organization of the paper

The rest of the paper is organized as follows. In Section 2, we describe the model in Subsection 2.1 and we recall the FLLN results from [13] in Subsection 2.2. Next, in Subsection 2.3 we present the main results of this article. The proof of our main result is presented in Section 3, in particular we present an alternative expressions for the limiting system of our main result in Subsection 3.4 and we present a specific cases in Section 4. In Section 5, we present some preliminary results that will be used in the proofs. In Section 6 we characterize the limit of the convergent subsequences. In Section 7 we approximate the limit, and we prove tightness in Section 8. Appendix in Section A.

Notation

Throughout the paper, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\xrightarrow[N\to+\infty]{\mathbb{P}}$ to denote convergence in probability as the parameter $N \to \infty$. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}^k(\mathbb{R}^k_+)$ the space of k-dimensional vectors with real (nonnegative) coordinates, with $\mathbb{R}(\mathbb{R}_+)$ for $k = 1$. We use $\mathbb{1}\{\cdot\}$ for the indicator function. Let $\mathbf{D} = \mathbf{D}(\mathbb{R}_+; \mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in $\mathbf{D}$ meaning convergence in the Skorohod J_1 topology (see, *e.g.*, [29], Chap. 3). Also, we use $\mathbf{D}^k$ to denote the k-fold product with the product J_1 topology. Let $\mathbf{C}$ be the subset of $\mathbf{D}$ consisting of continuous functions and $\mathbf{D}_+$ the subset of $\mathbf{D}$ of càdlàg functions with values in $\mathbb{R}_+$. We use $\Rightarrow$ to denote the weak convergence in $\mathbf{D}$.

2. Model and results

2.1. Model description

We consider a population of fixed size N. Let (λ_0, γ_0) and (λ, γ) be two random variables taking values in $D(\mathbb{R}_+, \mathbb{R}_+)^2$, and let $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \le k \le N\}$ be a family of i.i.d. copies of (λ_0, γ_0) and $\{(\lambda_{k,i}, \gamma_{k,i}), i \ge 1, 1 \le k \le N\}$ be an family of i.i.d. copies of (λ, γ), independent of the previous family. The function $\lambda_{k,0}$ (resp. $\gamma_{k,0}$) is the infectivity (resp. susceptibility) of the k-th individual between time 0 and the time of his/her first (re)-infection, and $\lambda_{k,i}(t)$ denotes the infectivity of the k-th individual, at time t after his/her i-th infection (where we count here only the infections after time 0), and $\gamma_{k,i}(t)$ denotes the susceptibility of the k-th individual, at time t after his/her i-th infection, given that this is his/her most recent infection.

We can think of the law of (λ_0, γ_0) as being a mixture of the law for the initially infected individuals, who have been infected before time 0 and for which $\lambda_0(0) \ge 0$ and $\gamma_0(0) = 0$, and the law for the initially susceptible individuals, for which $\lambda_0(0) = 0$ and $\gamma_0(0) > 0$ (possibly $\gamma_0(0) = 1$).

As in [13] we assume that (λ_0, γ_0) and (λ, γ) satisfy the following assumption.

Assumption 2.1. We assume that:

(i) $0 \le \gamma_0(t), \gamma(t) \le 1$ almost surely and there exists a deterministic constant $\lambda_* < \infty$ such that for all $t \ge 0$, $0 \le \lambda_0(t)$, $\lambda(t) \le \lambda_*$ almost surely.

(ii) Almost surely,

$$\sup\{t \ge 0,\ \lambda_0(t) > 0\} \le \inf\{t \ge 0,\ \gamma_0(t) > 0\} \text{ and } \sup\{t \ge 0,\ \lambda(t) > 0\} \le \inf\{t \ge 0,\ \gamma(t) > 0\}. \tag{2.1}$$

We refer below Assumption 2.6 for example of pair (λ, γ).

Assumption 2.1-(ii) implies that, as long as an individual has not recovered, he/she cannot be reinfected. Hence in each infected-immune-susceptible cycle, the infected and susceptible periods do not overlap. Note that only (i) is necessary for the process to be well defined, and we discuss the consequences of removing part (ii) of the above assumption in Remark 2.13 below.

For $i \ge 0$ and $1 \le k \le N$, we define

$$\eta_{k,0} := \sup\{t \ge 0 : \lambda_{k,0}(t) > 0\}, \quad \text{and} \quad \eta_{k,i} := \sup\{t \ge 0 : \lambda_{k,i}(t) > 0\}.$$

We will also use the notations

$$\eta_0 = \sup\{t > 0,\ \lambda_0(t) > 0\} \quad \text{and} \quad \eta = \sup\{t > 0,\ \lambda(t) > 0\}.$$

Let $A_k^N(t)$ be the number of times that the k-th individual has been (re)-infected between time 0 and t. The time elapsed since this individual's last infection (or since time 0 if no such infection has occurred), is given by

$$\varsigma_k^N(t) := t - \left(\sup\{s \in [0,t] : A_k^N(s) = A_k^N(s^-) + 1\} \vee 0\right), \tag{2.2}$$

where we use the convention $\sup \emptyset = -\infty$. With this notation, the current infectivity and susceptibility of the k-th individual are given by

$$\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)), \qquad \text{and} \qquad \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)).$$

Let us now

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^N \lambda_{k,A_k^N(t)}(\varsigma_k^N(t)), \tag{2.3}$$

the average infectivity at time t and

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^N \gamma_{k,A_k^N(t)}(\varsigma_k^N(t)), \tag{2.4}$$

the average susceptibility of the population at time t.

The instantaneous rate at which the ℓ-th individual infects the k-th individual is

$$\frac{1}{N}\lambda_{\ell,A_\ell^N(t)}(\varsigma_\ell^N(t))\gamma_{k,A_k^N(t)}(\varsigma_k^N(t))$$

where the $\frac{1}{N}$ factor comes from the probability that the k-th individual is chosen as the target of the infectious contact. Summing over the index ℓ, the instantaneous rate at which the k-th individual is (re)infected is given by

$$\Upsilon_k^N(t) := \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\,\overline{\mathfrak{F}}^N(t).$$

Let now $(Q_k, 1 \le k \le N)$ be an i.i.d. family of standard Poisson random measures (PRMs) on $\mathbb{R}_+^2$, independent of the sequence $(\lambda_{k,i}, \gamma_{k,i})_{1\le k\le N, i\ge 0}$. The family of counting processes $(A_k^N(t), t \ge 0, 1 \le k \le N)$ is then defined as the solution of

$$A_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{u \le \Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}u). \tag{2.5}$$

We set

$$\Upsilon^N(t) = \sum_{k=1}^N \Upsilon_k^N(t) = N\overline{\mathfrak{S}}^N(t)\overline{\mathfrak{F}}^N(t).$$

We define the number of infected individuals at time t by

$$I^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) < \eta_{k, A_k^N(t)}}, \tag{2.6}$$

and the number of uninfected individuals at time t by

$$U^N(t) = \sum_{k=1}^{N} \mathbb{1}_{\varsigma_k^N(t) \geq \eta_{k, A_k^N(t)}} = N - I^N(t). \tag{2.7}$$

We then define

$$\overline{I}(0) := \mathbb{P}(\eta_0 > 0), \qquad \overline{U}(0) := \mathbb{P}(\eta_0 = 0) = 1 - \overline{I}(0).$$

Recall that $\{(\lambda_{k,0}, \gamma_{k,0}), 1 \leq k \leq N\}$, and hence $(\eta_{k,0}, 1 \leq k \leq N)$, are independent and identically distributed. Thus by the law of large numbers,

$$\left(\frac{1}{N} I^N(0), \frac{1}{N} U^N(0)\right) = \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0} > 0}, \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\eta_{k,0} = 0}\right) \to \left(\overline{I}(0), \overline{U}(0)\right)$$

2.2. Already known results

Under Assumption 2.1 from [13], Lemma 6.1, there exists a unique $\overline{\mathfrak{F}} \in \mathbf{D}(\mathbb{R}_+)$ such that

$$\overline{\mathfrak{F}}(t) = \mathbb{E}\left[\lambda_{1, A_1(t)}(\varsigma_1(t))\right] \text{ and } \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_{1, A_1(t)}(\varsigma_1(t))\right],$$

where for $k \geq 1$ the process A_k is defined as:

$$A_k(t) = \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{u \leq \Upsilon_k(r^-)} Q_k(\mathrm{d}r, \mathrm{d}u),$$

with

$$\Upsilon_k(t) = \gamma_{k, A_k(t)}(\varsigma_k(t)) \overline{\mathfrak{F}}(t),$$

and ς_k is defined in the same manner as ς_1^N with A_k instead of A_1^N, see (2.2). In this definition we use the same $(\lambda_{k,i}, \gamma_{k,i}, Q_k)$ as in the definition of the model in Subsection 2.1. Moreover, note that, as the $((\lambda_{k,i})_i, (\gamma_{k,i})_i, Q_k)_{k \geq 1}$ are i.i.d, the $(A_k)_k$ are also i.i.d.

Then we recall the following FLLN result from [13]. Let $\left(\overline{U}^N, \overline{I}^N\right) = N^{-1}(U^N, I^N)$. Let

$$\begin{aligned}
&\overline{\lambda}_0(t) := \mathbb{E}\big[\lambda_{1,0}(t) \big| \eta_{1,0} > 0\big], \quad \overline{\lambda}(t) := \mathbb{E}\left[\lambda_{1,1}(t)\right], \\
&F_0^c(t) := \mathbb{P}\left(\eta_{1,0} > t \middle| \eta_{1,0} > 0\right), \quad F^c(t) := \mathbb{P}\left(\eta > t\right),
\end{aligned}$$

and recall that $\overline{I}(0) = \mathbb{P}\left(\eta_{1,0} > 0\right)$. We shall denote below by μ the law of γ.

Theorem 2.2. *Under Assumption 2.1,*

$$\left(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}^N\right) \xrightarrow[N \to +\infty]{\mathbb{P}} \left(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}\right) \quad \textit{in} \quad \mathbf{D}^2 \tag{2.8}$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$ *satisfies the following system of equations,*

$$\begin{cases} \overline{\mathfrak{S}}(t) = \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \\ \qquad\qquad + \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s, & (2.9) \\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s. & (2.10) \end{cases}$$

Given the solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}})$,

$$(\overline{U}^N, \overline{I}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{U}, \overline{I}) \quad \textit{in} \quad \mathbf{D}^2$$

where $(\overline{U}, \overline{I})$ *is given by*

$$\overline{U}(t) = 1 - \overline{I}(t), \tag{2.11}$$

$$\overline{I}(t) = \overline{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s. \tag{2.12}$$

Note that Theorem 2.2 states that in large population, the dynamic of the epidemic becomes deterministic.

Remark 2.3. Note that for each $t \geq 0$, $\overline{\mathfrak{S}}(t) \leq 1$, $\overline{\mathfrak{F}}(t) \leq \lambda_*$.

2.3. Main Results

The purpose of this section is to establish an FCLT for the fluctuations of the stochastic sequence around its deterministic limit. More precisely, we define the following fluctuation process: for all $t \geq 0$,

$$\hat{\mathfrak{F}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\big), \text{ and } \hat{\mathfrak{S}}^N(t) := \sqrt{N}\big(\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\big), \tag{2.13}$$

and we want to find the limiting law of the pair $(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N)$.

We introduce the following Assumptions.

Assumption 2.4. The random functions (λ, γ), of which $(\lambda_{k,i}, \gamma_{k,i})_{k\geq 1, i\geq 1}$ are i.i.d. copies, satisfy the following properties: There exist a number $\ell \in \mathbb{N}^*$, a two random sequences $0 = \xi^0 < \xi^1 < \cdots < \xi^\ell = +\infty$ and $0 = \zeta^0 < \zeta^1 < \cdots < \zeta^\ell = +\infty$ and random functions $\lambda^j \in \mathbf{C}$, $\gamma^j \in \mathbf{C}$, $1 \leq j \leq \ell$ such that

$$\lambda(t) = \sum_{j=1}^{\ell} \lambda^j(t)\mathbf{1}_{[\xi^{j-1},\xi^j)}(t) \quad \text{and} \quad \gamma(t) = \sum_{j=1}^{\ell} \gamma^j(t)\mathbf{1}_{[\zeta^{j-1},\zeta^j)}(t). \tag{2.14}$$

In addition, for any $T > 0$, there exists deterministic nondecreasing function $\varphi_T \in \mathbf{C}$ with $\varphi_T(0) = 0$ such that $|\lambda^j(t) - \lambda^j(s)| \leq \varphi_T(t-s)$ and $|\gamma^j(t) - \gamma^j(s)| \leq \varphi_T(t-s)$ almost surely, for all $0 \leq t, s \leq T$, $1 \leq j \leq k$.

Assumption 2.5. There exists $\alpha > 1/2$ such that for all $0 \leq t \leq T$, the function φ_T from Assumption 2.4 satisfy

$$\varphi_T(t) \leq Ct^\alpha, \tag{2.15}$$

for some constant $C > 0$. Also, if F_j denotes the c.d.f. of the r.v. ξ^j, and G_j denotes the c.d.f. of the r.v. ζ^j, there exist $C' > 0$, and $\rho > 1/2$ such that, for any $1 \leq j \leq \ell - 1$, $0 \leq s < t \leq T$,

$$F_j(t) - F_j(s) \leq C'(t-s)^\rho \quad \text{and} \quad G_j(t) - G_j(s) \leq C'(t-s)^\rho. \tag{2.16}$$

Assumption 2.6. There exist non-decreasing continuous functions ϕ_1, ϕ_2, ψ_1, ψ_2 and constants $\alpha_1 > \frac{1}{2}$, $\alpha_2 > \frac{1}{2}$, $\beta_1 > 1$, $\beta_2 > 1$ such that for all $0 \leq s < u < t$,

i) $\mathbb{E}\left[(\lambda(t) - \lambda(s))^2\right] \leq (\phi_1(t) - \phi_1(s))^{\alpha_1}$;
ii) $\mathbb{E}\left[(\gamma(t) - \gamma(s))^2\right] \leq (\phi_2(t) - \phi_2(s))^{\alpha_2}$;
iii) $\mathbb{E}\left[(\lambda(t) - \lambda(u))^2 (\lambda(u) - \lambda(s))^2\right] \leq (\psi_1(t) - \psi_1(s))^{\beta_1}$;
iv) $\mathbb{E}\left[(\gamma(t) - \gamma(u))^2 (\gamma(u) - \gamma(s))^2\right] \leq (\psi_2(t) - \psi_2(s))^{\beta_2}$.

We note that, Assumptions 2.4, 2.5 and 2.6 are not required to establish the FLLN in [13]. These additional Assumptions are used to establish the tightness, of $\hat{\mathfrak{F}}^N$ and $\hat{\mathfrak{S}}^N$, see the proof of Lemma 3.7. There are many examples of the pair (λ, γ) that satisfy them. A typical example of pair (λ, γ) can be given by:

$$\lambda(t) = \lambda \mathbb{1}_{0 \leq t < \eta}, \text{ and } \gamma(t) = \mathbb{1}_{t \geq \eta} \text{ or } \gamma(t) = \left(1 - \mathrm{e}^{-(t-\eta)}\right) \mathbb{1}_{t \geq \eta},$$

where η is a random variable. For more examples and discussions on $\lambda(\cdot)$ we refer to Section 2.3 in [18], and on $\gamma(\cdot)$ in [10].

Let us now state on main results.

Definition 2.7. Let $(\hat{\mathfrak{J}}, \hat{\mathfrak{M}})$ be a two-dimensional centered continuous Gaussian process, with covariance functions: for $t, t' \geq 0$,

$$\begin{aligned} \mathrm{Cov}\big(\hat{\mathfrak{J}}(t), \hat{\mathfrak{J}}(t')\big) &= \mathrm{Cov}\left(\gamma_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\right), \\ \mathrm{Cov}\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{M}}(t')\big) &= \mathrm{Cov}\left(\lambda_{1,A_1(t)}(\varsigma_1(t)), \lambda_{1,A_1(t')}(\varsigma_1(t'))\right), \\ \mathrm{Cov}\big(\hat{\mathfrak{M}}(t), \hat{\mathfrak{J}}(t')\big) &= \mathrm{Cov}\left(\lambda_{1,A_1(t)}(\varsigma_1(t)), \gamma_{1,A_1(t')}(\varsigma_1(t'))\right). \end{aligned}$$

Remark 2.8. In Subsection 3.4 the system of equations (3.19) give another expression for $\big(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\big)$.

We consider the following system of stochastic integral equations for which we have $(x, y) \in \mathbf{C}^2$:

$$\begin{cases} x(t) = -\displaystyle\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s) \exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] y(s)\mathrm{d}s \\ \qquad -\displaystyle\int_0^t \int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s) \exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right] y(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ \qquad +\displaystyle\int_0^t \mathbb{E}\left[\gamma(t-s) \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \left(x(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)y(s)\right) \mathrm{d}s \\ \qquad +\hat{\mathfrak{J}}(t), & (2.17) \\ y(t) = \displaystyle\int_0^t \overline{\lambda}(t-s) \left(x(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)y(s)\right) \mathrm{d}s + \hat{\mathfrak{M}}(t). & (2.18) \end{cases}$$

Lemma 2.9. *The set of equations (2.17)–(2.18) has a unique solution $(x, y) \in \mathbf{C}^2$.*

We denote its solution by $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}) \in \mathbf{C}^2$.

Proof. If we denote by (x^1, y^1) and (x^2, y^2) two solutions of (2.17)–(2.18), as $\overline{\mathfrak{S}} \le 1$ and $\overline{\mathfrak{F}} \le \lambda_*$, we obtain by easy computations for all $t \in [0, T]$,

$$|x^1(t) - x^2(t)| + |y^1(t) - y^2(t)| \le (\lambda_*^2 + 2\lambda_* + \lambda_* T + 2) \int_0^t |x^1(s) - x^2(s)| + |y^1(s) - y^2(s)| \mathrm{d}s. \tag{2.19}$$

Uniqueness then follows from Gronwall's Lemma.

Now local existence follows by an approximation procedure, which exploits inequality (2.19). Global existence then follows from the estimates (2.19), which forbid explosion. Lemma 2.9 is established. □

The following Theorem is our main result.

Theorem 2.10. *Under Assumptions 2.1–2.6,*

$$\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N\big) \Rightarrow \big(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\big) \quad \text{in} \quad \mathbf{D}^2,$$

where $\big(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\big)$ *is the unique continuous solution of the system of stochastic integral equations* (2.17)–(2.18)*, that is,*

$$\begin{cases} \hat{\mathfrak{S}}(t) = -\displaystyle\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s \\ \qquad -\displaystyle\int_0^t\int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ \qquad +\displaystyle\int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{I}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s \\ \qquad +\hat{\mathfrak{I}}(t), & (2.20) \\ \\ \hat{\mathfrak{F}}(t) = \displaystyle\int_0^t \overline{\lambda}(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{I}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}(t), & (2.21) \end{cases}$$

where we recall that $\left(\hat{\mathfrak{I}}, \hat{\mathfrak{M}}\right)$ *is specified by Definition 2.7.*

Remark 2.11. Note that in Subsection 3.4, Proposition 3.10 give another expression for $\big(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\big)$.

We define

$$\hat{I}^N(t) = \sqrt{N}\big(\overline{I}^N(t) - \overline{I}(t)\big) \quad \text{and} \quad \hat{U}^N(t) = \sqrt{N}\big(\overline{U}^N(t) - \overline{U}(t)\big), \quad t \ge 0.$$

We first note that as $\overline{I}^N(t) + \overline{U}^N(t) = 1$ and $\overline{I}(t) + \overline{U}(t) = 1$, it follows that $\hat{I}^N(t) + \hat{U}^N(t) = 0$.

As the random function $t \mapsto \mathbb{1}_{\eta \ge t}$ satisfies Assumptions 2.1, 2.4, 2.5 and 2.6, replacing $\lambda_{k,i}(t)$ by $\mathbb{1}_{t<\eta_{k,i}}$ in (2.3), the convergence of $\hat{I}^N(t)$ follows an analogous argument as that used for $\hat{\mathfrak{F}}^N(t)$ in Theorem 2.10. Using the fact that $\hat{I}^N(t) + \hat{U}^N(t) = 0$, we obtain the convergence of $\hat{U}^N$. Therefore, we obtain the following Corollary from Theorem 2.10 and the system of equations (2.20)–(2.21).

Corollary 2.12. *Under Assumptions 2.1–2.6,*

$$\big(\hat{U}^N, \hat{I}^N\big) \Rightarrow \big(-\hat{I}, \hat{I}\big) \quad \text{in} \quad \mathbf{D}^2,$$

where $\hat{I}$ is the unique continuous solution of equation:

$$\hat{I}(t) = \int_0^t F^c(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_1(t), \tag{2.22}$$

where $\hat{\mathfrak{M}}_1$ is a centered continuous Gaussian process as given in Definition 2.7 where we replace $\lambda_{1,A_1(t)}$ by $\mathbb{1}_{t<\eta_{1,A_1(t)}}$ in the expressions of $\hat{\mathfrak{M}}$.

Remark 2.13. If we remove condition (2.1) of Assumption 2.1, this means that an infected individual can be reinfected before the end of his/her infections period. As a result from [13], the limit obtained in the FLLN satisfies a different set of equations. More precisely, equation (2.10) is replaced by (2.23) and (2.12) by (2.24), where (2.23) and (2.24) are given below:

$$\begin{aligned}\overline{\mathfrak{F}}(t) = \mathbb{E}&\left[\lambda_0(t)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\\ &+\int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\overline{I}(t) = \mathbb{E}&\left[\mathbb{1}_{\eta_0>t}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(s)\mathrm{d}r\right)\right]\\ &+\int_0^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.\end{aligned} \tag{2.24}$$

In the same way without condition (2.1) of Assumption 2.1, the limit obtained in the FCLT (Thm. 2.10 and Cor. 2.12) satisfies a different set of equations. More precisely, equation (2.21) is replaced by (2.25) and (2.22) by (2.26), where (2.25) and (2.26) are given below.

$$\begin{aligned}\hat{\mathfrak{F}}(t) = &-\int_0^t \mathbb{E}\left[\lambda_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s\\ &-\int_0^t\int_s^t \mathbb{E}\left[\lambda(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds\\ &+\int_0^t \mathbb{E}\left[\lambda(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s\\ &+\hat{\mathfrak{M}}(t),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\hat{I}(t) = &-\int_0^t \mathbb{E}\left[\mathbb{1}_{\eta_0>t}\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s\\ &-\int_0^t\int_s^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds\\ &+\int_0^t \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s) - \hat{\mathfrak{J}}(s)\overline{\mathfrak{F}}(s) + \overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s\\ &+\hat{\mathfrak{M}}_1(t).\end{aligned} \tag{2.26}$$

In that case, the convergence of $\hat{\mathfrak{F}}^N(t)$ follows an analogous argument as that used for $\hat{\mathfrak{S}}^N(t)$ in Theorem 2.10. More precisely, without condition (2.1) of Assumption 2.1, equation (3.4) is replaced by a similar equation as (3.5). Consequently instead of the expression in (3.9), one gets a different expression, which resembles the expression in (3.10), so that the proof follows from similar arguments.

This model is very versatile. We can construct more types of compartmental epidemic model. From Theorem 2.10 we can deduce the fluctuation of the proportion of individuals in each compartment; for example taking $\mathbb{1}_{\eta_0\le t<\eta_0+\varsigma_0}$ instead of $\lambda_0(t)$ and $\mathbb{1}_{\eta\le t<\eta+\varsigma}$ instead of $\lambda(t)$, in (2.23) we obtain the proportion of recovered individuals at time t denoted $\overline{R}(t)$, where we recall that η_0 and η are infection periods and ς_0 and ς are immune periods, and doing the same in (2.25) we obtain the expression of fluctuation of the proportion of recovered individuals at time t denoted $\hat{R}(t)$. In Section 4 we will deal with a special cases of Theorem 2.10.

3. Proof of Theorem 2.10

We recall the definitions of $\hat{\mathfrak{F}}^N$ and $\hat{\mathfrak{S}}^N$, from (2.13) below:

$$\hat{\mathfrak{F}}^N(t) := \sqrt{N}\left(\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right), \text{ and } \hat{\mathfrak{S}}^N(t) := \sqrt{N}\left(\overline{\mathfrak{S}}^N(t) - \overline{\mathfrak{S}}(t)\right).$$

Let us write,

$$\begin{aligned}\hat{\mathfrak{F}}^N(t) = \sqrt{N}&\left(\frac{1}{N}\sum_{k=1}^N\left(\lambda_{k,A_k^N(t)}(\varsigma_k^N(t)) - \lambda_{k,A_k(t)}(\varsigma_k(t))\right)\right)\\ &+\sqrt{N}\left(\frac{1}{N}\sum_{k=1}^N\lambda_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\lambda_{1,A_1(t)}(\varsigma_1(t))\right]\right)\\ =:&\ \hat{\mathfrak{F}}_1^N(t) + \hat{\mathfrak{F}}_2^N(t).\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\hat{\mathfrak{S}}^N(t) = \sqrt{N}&\left(\frac{1}{N}\sum_{k=1}^N\left(\gamma_{k,A_k^N(t)}(\varsigma_k^N(t)) - \gamma_{k,A_k(t)}(\varsigma_k(t))\right)\right)\\ &+\sqrt{N}\left(\frac{1}{N}\sum_{k=1}^N\gamma_{k,A_k(t)}(\varsigma_k(t)) - \mathbb{E}\left[\gamma_{1,A_1(t)}(\varsigma_1(t))\right]\right)\\ =:&\ \hat{\mathfrak{S}}_1^N(t) + \hat{\mathfrak{S}}_2^N(t).\end{aligned} \tag{3.2}$$

Since $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)), \lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)_k$ are i.i.d $\mathbf{D}^2$-valued random variables with each component satisfying Assumption 2.6, by applying the central limit theorem in $\mathbf{D}$, each component (see Thm. 2 in [22]) it follows that $\hat{\mathfrak{S}}_2^N \Rightarrow \hat{\mathfrak{J}}$ and $\hat{\mathfrak{F}}_2^N \Rightarrow \hat{\mathfrak{M}}$ in $\mathbf{D}$ as $N\to\infty$ respectively. Then, as $\mathbf{D}$ is separable from [30], Lemma 5.2 the pair $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$ is $\mathbf{C}-$tight in $\mathbf{D}^2$ and using the uniqueness of the limit of $\hat{\mathfrak{S}}_2^N$ and $\hat{\mathfrak{F}}_2^N$, the convergence in $\mathbf{D}^2$ of the pair $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$ follows. Moreover, given the convergence of $\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right)$, by the continuous mapping theorem $\hat{\mathfrak{S}}_2^N\hat{\mathfrak{F}}_2^N$ converges in $\mathbf{D}$. It follows that for each $t,t'\ge 0$, the covariance of $\hat{\mathfrak{S}}_2^N(t')$ and $\hat{\mathfrak{F}}_2^N(t)$, which is given by $\mathrm{Cov}\left(\gamma_{1,A_1(t')}(\varsigma_1(t'), \lambda_{1,A_1(t)}(\varsigma_1(t)))\right)$, converges to the covariance of the limit process of $\hat{\mathfrak{S}}_2^N(t')$ and $\hat{\mathfrak{F}}_2^N(t)$. Hence we have the following Lemma:

Lemma 3.1. *Under Assumption 2.6, as $N\to+\infty$,*

$$\left(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N\right) \Rightarrow \left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right) \quad in \quad \mathbf{D}^2.$$

where $\left(\hat{\mathfrak{J}}, \hat{\mathfrak{M}}\right)$ is a centered continuous 2-dimensional Gaussian process given in Definition 2.7.

To conclude the proof of Theorem 2.10, it remains to prove the convergence of the pair $\left(\hat{\mathfrak{S}}_1^N, \hat{\mathfrak{F}}_1^N\right)$, which is very technical. To achieve this, in Subsection 3.1, we provide a decomposition of the pair $\left(\hat{\mathfrak{S}}_1^N, \hat{\mathfrak{F}}_1^N\right)$. In Subsection 3.2 we establish two continuous mapping theorems. Hence, in Subsection 3.3 we are able to prove Theorem 2.10 and in Subsection 3.4 we establish an alternative expressions for the limiting system $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ of Theorem 2.10.

3.1. Decomposition of the fluctuations $\left(\hat{\mathfrak{S}}_1^N, \hat{\mathfrak{F}}_1^N\right)$

In [13] we had made a coupling between A_k^N and A_k and if instead we give ourselves a Poisson random measure on $\mathbb{R}_+ \times \mathbf{D}^2 \times \mathbb{R}_+$, we can define a new coupling. However if we look at the law of the first time for which $A_k^N \neq A_k$, then the law of this time is the same for both couplings.

We then introduce a Poisson random measure Q_k on $\mathbb{R}_+ \times \mathbf{D}^2 \times \mathbb{R}_+$, so that the mean measure of the PRM is

$$\mathrm{d}s \times \mathbb{P}(\mathrm{d}\lambda, \mathrm{d}\gamma) \times \mathrm{d}u,$$

where we consider $\mathbb{P}$ as the law of the pair (λ, γ).

We denote by $\overline{Q}_k$ its compensated measure.

For $0 \le s < t$, and $\gamma \in \mathbf{D}$, $\phi \in \mathbf{D}$ we set

$$P_k(s,t,\gamma,\phi) = \int_{[s,t]\times\mathbf{D}^2\times\mathbb{R}_+} \mathbb{1}_{u<\gamma(r-s)\phi(r)} Q_k(\mathrm{d}r, \mathrm{d}\lambda', \mathrm{d}\gamma', \mathrm{d}u).$$

Note that if individual k-th is infected at time s and his/her susceptibility is γ, $P_k(s,t,\gamma,\overline{\mathfrak{F}}^N) = 0$ if and only if there has not been yet reinfected at time t.

Note that to define A_k^N below we use the fact that,

$$\gamma_{k,A_k^N}(\varsigma_k^N(t)) = \gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \sum_{i=1}^{A_k^N(t)} \gamma_{k,i}(t-\tau_{k,i}^N)\mathbb{1}_{P_k(\tau_{k,i}^N,t,\gamma_{k,i},\overline{\mathfrak{F}}^N)=0},$$

where $(\tau_{k,i}^N)_i$ is the family of infections time of individual k with increasing order and where $(\gamma_{k,i})_i$ is the family of susceptible function which corresponds to this infection. We also have a similarly expression for $\lambda_{k,A_k^N}(\varsigma_k^N(t))$.

So we define A_k^N, as follows

$$A_k^N(t) = \int_{[0,t]\times\mathbf{D}^2\times\mathbb{R}_+} \mathbb{1}_{u\le\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)$$

where

$$\Upsilon_k^N(t) = \left(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s-)} \gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0} Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right) \overline{\mathfrak{F}}^N(t)$$

and

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^N \left\{ \lambda_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s-)} \lambda(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0} Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \right\}. \quad (3.3)$$

Note that the process A_k^N define here has the same law as the process A_k^N given by expression (2.5). Under condition (2.1), (3.3) becomes

$$\overline{\mathfrak{F}}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\left\{\lambda_{k,0}(t) + \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}\lambda(t-s)Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right\}. \tag{3.4}$$

On the other hand, we have

$$\overline{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\left\{\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} + \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right\}. \tag{3.5}$$

Similarly,

$$A_k(t) = \int_{[0,t]\times\mathbf{D}^2\times\mathbb{R}_+}\mathbb{1}_{u\le\Upsilon_k(r^-)}Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)$$

with

$$\Upsilon_k(t) = \left(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} + \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)\overline{\mathfrak{F}}(t)$$

where $\overline{\mathfrak{F}}$ is given by (2.10).

Consequently we have

$$\sum_{k=1}^{N}\lambda_{k,A_k(t)}(\varsigma_k(t)) := \sum_{k=1}^{N}\left\{\lambda_{k,0}(t) + \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s^-)}\lambda(t-s)Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right\}, \tag{3.6}$$

and

$$\sum_{k=1}^{N}\gamma_{k,A_k(t)}(\varsigma_k(t)) := \sum_{k=1}^{N}\left\{\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} + \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right\}. \tag{3.7}$$

Using the fact that

$$\Upsilon_k^N(s) - \Upsilon_k(s) = \gamma_{k,A_k^N(s)}(\varsigma_k^N(s))\overline{\mathfrak{F}}^N(s) - \gamma_{k,A_k(s)}(\varsigma_k(s))\overline{\mathfrak{F}}(s),$$

from expression (3.2), it follows that,

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\left(\Upsilon_k^N(s) - \Upsilon_k(s)\right) = \hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s) - \hat{\mathfrak{S}}_2^N(s)\overline{\mathfrak{F}}^N(s) + \tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s), \tag{3.8}$$

where

$$\tilde{\mathfrak{S}}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\gamma_{k,A_k(t)}(\varsigma_k(t)),$$

and from expressions (3.1) and (3.2), we obtain the following Proposition.

Proposition 3.2. *Under the condition* (2.1) *in Assumption 2.1, for every* $t \geq 0$,

$$\hat{\mathfrak{F}}_1^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(s-)\wedge\Upsilon_k^N(s-)}^{\Upsilon_k(s-)\vee\Upsilon_k^N(s-)} \lambda(t-s)\mathrm{sign}(\Upsilon_k^N(s^-) - \Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)$$
$$+ \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s)\mathrm{d}s - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}_2^N(s)\overline{\mathfrak{F}}^N(s)\mathrm{d}s + \int_0^t \overline{\lambda}(t-s)\tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s)\mathrm{d}s, \tag{3.9}$$

and

$$\hat{\mathfrak{S}}_1^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \gamma_{k,0}(t) \left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \right)$$
$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s)} \gamma(t-s) \left(\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}^N\right)=0} - \mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0} \right) Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)$$
$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(s)\wedge\Upsilon_k^N(s)}^{\Upsilon_k(s)\vee\Upsilon_k^N(s)} \gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}\mathrm{sign}(\Upsilon_k^N(s^-) - \Upsilon_k(s^-))Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)$$
$$=: \hat{\mathfrak{S}}_{1,0}^N(t) + \hat{\mathfrak{S}}_{1,1}^N(t) + \hat{\mathfrak{S}}_{1,2}^N(t). \tag{3.10}$$

Remark 3.3. Note that, in the rest $Q_k(\mathrm{d}r, \mathrm{d}u)$ can be seen as the projection of $Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)$ on $\mathbb{R}_+ \times \mathbb{R}_+$.

3.2. Two continuous integral mappings

Let $\phi_1 : \mathbb{R}_+^3 \to \mathbb{R}_+$ and $\phi_2 : \mathbb{R}_+^2 \to \mathbb{R}$, be given by

$$\phi_1(t,s,r) = \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u \right) \right],$$

and

$$\phi_2(t,s) = \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u \right) \right].$$

Let $\Psi_1 : \mathbf{D}_+^7 \to \mathbf{D}$, be given by

$$\Psi_1(f)(t) = f_2(t) - f_3(t) - f_4(t) + \int_0^t \int_s^t \phi_1(t,s,r)f_1(r)f_5(s)f_6(s)drds$$
$$- \int_0^t \phi_2(t,s)f_5(s)f_2(s)\mathrm{d}s + \int_0^t \phi_2(t,s)f_5(s)f_4(s)\mathrm{d}s - \int_0^t \phi_2(t,s)f_1(s)f_7(s)\mathrm{d}s, \tag{3.11}$$

for all t, where $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathbf{D}_+^7$.

Lemma 3.4. *If* $f^n \to f$ *in* $\mathbf{D}_+^7$*, as* $n \to \infty$ *and* f *is continuous on* $\mathbb{R}_+$*, then* $\Psi_1(f^n) \to \Psi_1(f)$ *in* $\mathbf{D}$ *as* $n \to \infty$.

Proof. Since $f^n \to f$ in $\mathbf{D}_+^7$, as $n \to \infty$ and f is continuous, $\|f^n - f\|_T \to 0$ as $n \to \infty$. Consequently $(\|f^n\|_T)_n$ is bounded and it follows easily that there exists $C_T(\sup_n \|f^n\|_T) > 0$ such that,

$$\|\Psi_1(f^n) - \Psi_1(f)\|_T \leq C_T(\sup_n \|f^n\|_T)\|f^n - f\|_T.$$

Where for $f, g \in \mathbf{D}_+^7$, we define,

$$\|f-g\|_T = \sup_{0\le t\le T} |f(t)-g(t)|,$$

with $|\cdot|$ the Euclidian norm with respect to the dimension of the space. □

On the other hand, let $\Psi_2 : \mathbf{D}_+^6 \to \mathbf{D}$, be given for all t by

$$\Psi_2(f)(t) \;=\; f_1(t) - f_2(t) - \int_0^t \overline{\lambda}(t-s) f_3(s) f_4(s) \mathrm{d}s - \int_0^t \overline{\lambda}(t-s) f_5(s) f_4(s) \mathrm{d}s + \int_0^t \overline{\lambda}(t-s) f_6(s) f_1(s) \mathrm{d}s, \tag{3.12}$$

for any $f = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathbf{D}_+^6$.

Lemma 3.5. *If $f^n \to f$ in $\mathbf{D}_+^6$, as $n\to\infty$ and f is continuous on $\mathbb{R}_+$, then $\Psi_2(f^n) \to \Psi_2(f)$ in $\mathbf{D}$ as $n\to\infty$.*

The proof is similar to Lemma 3.4.

Remark 3.6. If $\Psi_1\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0$ and $\Psi_2\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0$, then $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ is a solution of (2.20)–(2.21).

3.3. Proof of Theorem 2.10

To prove Theorem 2.10, we will show that sequence

$$\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_{1,1}^N\big)_{N\ge 1}$$

is tight and that any limit $\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big)$ and $\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big)$ of a converging subsequences $\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$ and $\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$ respectively satisfies $\Psi_1\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0, \Psi_2\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0$, and

$$\hat{\mathfrak{S}}_{1,0}(t) = -\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s.$$

This will imply that convergence in distribution of $\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N\big)$. This will follows from the following Lemmas.

Lemma 3.7. *The sequence $\big(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_{1,1}^N\big)_{N\ge 1}$ is tight in $\mathbf{D}^6$.*

We refer to Section 8 for the proof.

We have the following characterisation for the limit of any converging subsequence, of $\hat{\mathfrak{S}}_{1,0}^N$ in $\mathbf{D}$ and we refer to Section 6 for the proof.

Lemma 3.8. *Let $\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}_{1,0}\big)$ be a limit of a converging subsequence of $\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_{1,0}^N\big)$. Then, almost surely,*

$$\hat{\mathfrak{S}}_{1,0}(t) = -\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s.$$

Lemma 3.9. *Under Assumptions 2.1–2.6, as $N\to\infty$,*

$$\Psi_1\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big) \Rightarrow 0 \quad \textit{in} \quad \mathbf{D},$$

and

$$\Psi_2\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big) \Rightarrow 0 \quad \textit{in} \quad \mathbf{D}.$$

We refer to Section 7 for the proof.

Proof of Theorem 2.10. As the space $\mathbf{D}$ is separable, and $\overline{\mathfrak{F}}^N$, $\overline{\mathfrak{S}}^N$, and $\tilde{\mathfrak{S}}^N$ are tight in $\mathbf{D}$, from Lemma 3.7.

$$\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$$

is tight in $\mathbf{D}^7$. We can extract a subsequence denoted again $\big(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{S}}_{1,0}^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N\big)$ that converges to $\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big)$ in law in $\mathbf{D}^7$. By Lemma 3.9, and the continuous mapping theorem, it follows that:

$$\begin{cases} \Psi_1\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0, \\ \Psi_2\big(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \overline{\mathfrak{S}}\big) = 0, \end{cases} \tag{3.13}$$

where we recall that the pair $(\hat{\mathfrak{I}}, \hat{\mathfrak{M}})$ is given by Definition 2.7.

Note that

$$\begin{aligned} \Psi_1(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{S}}_{1,0}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \tilde{\mathfrak{S}})(t) &= \hat{\mathfrak{S}}(t) - \hat{\mathfrak{S}}_{1,0}(t) - \hat{\mathfrak{I}}(t) \\ &+ \int_0^t \int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right]\hat{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\ &+ \int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\left(\overline{\mathfrak{F}}(s)\hat{\mathfrak{I}}(s) - \overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}(s) - \hat{\mathfrak{F}}(s)\tilde{\mathfrak{S}}(s)\right)\mathrm{d}s, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\Psi_2(\hat{\mathfrak{F}}, \hat{\mathfrak{S}}, \hat{\mathfrak{M}}, \hat{\mathfrak{I}}, \overline{\mathfrak{F}}, \overline{\mathfrak{S}}, \tilde{\mathfrak{S}})(t) \\ &\quad = \hat{\mathfrak{F}}(t) - \hat{\mathfrak{M}}(t) - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{I}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s + \int_0^t \overline{\lambda}(t-s)\tilde{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\mathrm{d}s. \end{aligned} \tag{3.15}$$

Combining (3.13) with Lemma 3.8, it follows that the pair $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ satisfies the set of equations (2.17)–(2.18) and from Lemma 2.9 we conclude the proof of Theorem 2.10. □

It remains to prove Lemma 3.7, Lemma 3.8 and Lemma 3.9. We refer to Section 6, 7 and 8.

3.4. Alternative expressions for the limiting system

In this section we establish an alternative expression for the pair $(\hat{\mathfrak{S}}, \hat{\mathfrak{F}})$ solution of (2.20)–(2.21).

Proposition 3.10. *Let* $\big(\hat{\mathfrak{I}}_{0,1}, \hat{\mathfrak{M}}_{0,1}\big)$ *and* $\big(W_1^\gamma, W_2^\gamma, W^\lambda\big)$ *be two independent centered continuous Gaussian process with covariance functions: for* t, t',

$$\begin{aligned}
\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t),\hat{\mathfrak{J}}_{0,1}(t')) &= \mathbb{E}\left[\gamma_0(t)\gamma_0(t')\exp\left(-\int_0^{t\vee t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \\
&\quad - \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^{t}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\mathbb{E}\left[\gamma_0(t')\exp\left(-\int_0^{t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(W_1^\gamma(t),W_1^\gamma(t')) &= \int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\gamma(t'-s)\exp\left(-\int_s^{t\vee t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s \\
&\quad - \int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^{t}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \\
&\qquad \times \mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(W_2^\gamma(t),W_2^\gamma(t')) &= \int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^{t}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \\
&\qquad \times \mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,
\end{aligned}$$

$$\mathrm{Cov}(\hat{\mathfrak{M}}_{0,1}(t),\hat{\mathfrak{M}}_{0,1}(t')) = \mathrm{Cov}\left(\lambda_{1,0}(t),\lambda_{1,0}(t')\right),$$

$$\mathrm{Cov}(\hat{\mathfrak{M}}_{0,1}(t),\hat{\mathfrak{J}}_{0,1}(t')) = -\overline{I}(0)\overline{\lambda}_0(t)\mathbb{E}\left[\gamma_0(t')\exp\left(-\int_0^{t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right],$$

$$\mathrm{Cov}(W^\lambda(t),W^\lambda(t')) = \int_0^{t\wedge t'}\mathbb{E}\left[\lambda(t-s)\lambda(t'-s)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,$$

$$\mathrm{Cov}(W^\lambda(t),W_1^\gamma(t')) = -\int_0^{t\wedge t'}\overline{\lambda}(t-s)\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,$$

$$\mathrm{Cov}(W^\lambda(t),W_2^\gamma(t')) = \int_0^{t\wedge t'}\overline{\lambda}(t-s)\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,$$

$$\begin{aligned}
\mathrm{Cov}(W_1^\gamma(t),W_2^\gamma(t')) &= -\int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{F}}(s) \\
&\qquad \int_0^{s}\mathbb{E}\left[\gamma(s-a)\gamma(t-a)\exp\left(-\int_a^{t}\gamma(r-a)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(a)\overline{\mathfrak{F}}(a)da\mathrm{d}s.
\end{aligned}$$

Let $\left(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{F}}\right)$ *be a solution of* (3.16) *and* (3.17) *below. Then* $\left(\widetilde{\mathfrak{S}}, \widetilde{\mathfrak{F}}\right) \stackrel{d}{=} \left(\hat{\mathfrak{S}}, \hat{\mathfrak{F}}\right)$ *solution of* (2.20)–(2.21).

$$
\begin{cases}
\widetilde{\mathfrak{S}}(t) = -\int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\widetilde{\mathfrak{F}}(s)\mathrm{d}s \\
\qquad -\int_0^t\int_s^t \mathbb{E}\left[\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\right]\widetilde{\mathfrak{F}}(r)\overline{\mathfrak{F}}(s)\overline{\mathfrak{S}}(s)drds \\
\qquad +\int_0^t \mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\left(\widetilde{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\widetilde{\mathfrak{F}}(s)\right)\mathrm{d}s \\
\qquad +\hat{\mathfrak{I}}_{0,1}(t) + W_1^{\gamma}(t) + W_2^{\gamma}(t), \qquad (3.16) \\
\widetilde{\mathfrak{F}}(t) = \int_0^t \overline{\lambda}(t-s)\left(\widetilde{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\widetilde{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_{0,1}(t) + W^{\lambda}(t). \qquad (3.17)
\end{cases}
$$

Moreover

$$
\hat{I}(t) = \int_0^t F^c(t-s)\left(\hat{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)+\overline{\mathfrak{S}}(s)\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_0^F(t) + W^F(t), \tag{3.18}
$$

where $\left(\hat{\mathfrak{M}}_0^F, W^F\right)$ *is a centered continuous* $2-$*dimensional Gaussian process where we replace* $\lambda_0(t)$ *by* $\mathbb{1}_{t<\eta_0}$ *and* $\lambda(t)$ *by* $\mathbb{1}_{t<\eta}$ *in the expressions of* $\left(\hat{\mathfrak{M}}_{0,1}, W^{\lambda}\right)$.

Proof. Note that the following expression follows from (3.6) and (3.5)

$$
\begin{aligned}
\hat{\mathfrak{S}}_2^N(t) &= \hat{\mathfrak{S}}_{2,0}^N(t) + \hat{\mathfrak{S}}_{2,1}^N(t) + \hat{\mathfrak{S}}_{2,2}^N(t) \\
&\qquad + \int_0^t \mathbb{E}\left[\gamma_{1,1}(t-s)\exp\left(-\int_s^t \gamma_{1,1}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}_2^N(s)\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\mathfrak{S}}_{2,0}^N(t) &:= \frac{1}{\sqrt{N}}\sum_{k=1}^N\left(\gamma_{k,0}(t)\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} - \mathbb{E}\left[\gamma_{1,0}(t)\exp\left(-\int_0^t \gamma_{1,0}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s\right)\right]\right), \\
\hat{\mathfrak{S}}_{2,1}^N(t) &:= \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\left[\gamma(t-s)\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right. \\
&\qquad\qquad \left. -\int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\right]Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u), \\
\hat{\mathfrak{S}}_{2,2}^N(t) &:= \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)}\int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\overline{Q}_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).
\end{aligned}
$$

Similarly, we have

$$
\hat{\mathfrak{F}}_2^N(t) = \hat{\mathfrak{F}}_{2,0}^N(t) + \hat{\mathfrak{F}}_{2,1}^N(t) + \int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{F}}(s)\hat{\mathfrak{S}}_2^N(s)\mathrm{d}s,
$$

where

$$\hat{\mathfrak{F}}^N_{2,0}(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N \left(\lambda_{k,0}(t) - \mathbb{E}\left[\lambda_{1,0}(t)\right]\right),$$

$$\hat{\mathfrak{F}}^N_{2,1}(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2}\int_0^{\Upsilon_k(s-)} \lambda(t-s)\overline{Q}_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

We set

$$\hat{\mathfrak{S}}_{2,1}(t) := \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)} \Big[\gamma(t-s)\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0} - \int_{\mathbf{D}} \tilde{\gamma}(t-s)\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\Big] Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),$$

$$\hat{\mathfrak{S}}_{2,2}(t) := \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)} \int_{\mathbf{D}} \tilde{\gamma}(t-s)\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\overline{Q}_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),$$

$$\hat{\mathfrak{F}}_{2,1}(t) := \int_0^t\int_D\int_0^{\Upsilon_1(s-)} \lambda(t-s)\overline{Q}_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}u).$$

Recall that $\left(\gamma_{k,A_k(\cdot)}(\varsigma_k(\cdot)),\lambda_{k,A_k(\cdot)}(\varsigma_k(\cdot))\right)_k$ are i.i.d. $\mathbf{D}^2$-valued random variables and that each component satisfies Assumption 2.6 and $(\overline{Q}_k)_k$ are also i.i.d. (Note that the processes P_k depends only on Q_k and Υ_k depends only on γ_{k,A_k}.) As a result, the processes $\hat{\mathfrak{S}}^N_{2,0}(t)$, $\hat{\mathfrak{S}}^N_{2,1}(t)$, $\hat{\mathfrak{S}}^N_{2,2}(t)$, $\hat{\mathfrak{F}}^N_{2,0}(t)$ and $\hat{\mathfrak{F}}^N_{2,1}(t)$ can all be regarded as sums of i.i.d. random processes in $\mathbf{D}$. Thus, applying the central limit theorem in $\mathbf{D}$ to each term (see Thm. 2 in [22]), we obtain the convergence of these processes: $\left(\hat{\mathfrak{S}}^N_{2,0},\hat{\mathfrak{S}}^N_{2,1},\hat{\mathfrak{S}}^N_{2,2},\hat{\mathfrak{F}}^N_{2,0},\hat{\mathfrak{F}}^N_{2,1}\right)\Rightarrow\left(\hat{\mathfrak{I}}_{0,1},W^\gamma_1,W^\gamma_2,\hat{\mathfrak{M}}_{0,1},W^\lambda\right)$ in $\mathbf{D}^5$, where the limits are Gaussian processes defined as follows:

$\hat{\mathfrak{I}}_{0,1}$ has covariance function, for $t,t'\geq 0$,

$$\begin{aligned}\mathrm{Cov}(\hat{\mathfrak{I}}_{0,1}(t),\hat{\mathfrak{I}}_{0,1}(t')) &= \mathrm{Cov}\left(\gamma_{1,0}(t)\mathbb{1}_{P_1(0,t,\gamma_{1,0},\overline{\mathfrak{F}})=0},\gamma_{1,0}(t')\mathbb{1}_{P_1(0,t',\gamma_{1,0},\overline{\mathfrak{F}})=0}\right)\\ &= \mathbb{E}\left[\gamma_0(t)\gamma_0(t')\exp\left(-\int_0^{t\vee t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\\ &\quad - \mathbb{E}\left[\gamma_0(t)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\mathbb{E}\left[\gamma_0(t')\exp\left(-\int_0^{t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right],\end{aligned}$$

and $\hat{\mathfrak{M}}_{0,1}$ has covariance function, for $t,t'\geq 0$,

$$\mathrm{Cov}(\hat{\mathfrak{M}}_{0,1}(t),\hat{\mathfrak{M}}_{0,1}(t')) = \mathrm{Cov}\left(\lambda_{1,0}(t),\lambda_{1,0}(t')\right),$$

and the covariances between any two processes can be obtained,

$$\mathrm{Cov}(\hat{\mathfrak{I}}_{0,1}(t),W^\gamma_1(t')) = \mathrm{Cov}\left(\gamma_{1,0}(t)\mathbb{1}_{P_1(0,t,\gamma_{1,0},\overline{\mathfrak{F}})=0},\hat{\mathfrak{S}}_{2,1}(t')\right)$$

and so on.

Using Itô formula for Poisson process, Theorem 5.1 and [23], Theorem $VI.2.9$, it is easy to check that these covariance functions are explicitly given by the following:

$$\begin{aligned}
&\mathrm{Cov}(W_1^\gamma(t), W_1^\gamma(t')) = \mathbb{E}\left[\hat{\mathfrak{S}}_{2,1}(t)\hat{\mathfrak{S}}_{2,1}(t')\right]\\
&= \mathbb{E}\Bigg[\int_0^{t\wedge t'}\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)}\left(\gamma(t-s)\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0} - \int_{\mathbf{D}}\tilde{\gamma}(t-s)\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\right)\\
&\qquad \times\left(\gamma(t'-s)\mathbb{1}_{P_1(s,t',\gamma,\overline{\mathfrak{F}})=0} - \int_{\mathbf{D}}\tilde{\gamma}(t'-s)\exp\left(-\int_s^{t'}\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\tilde{\gamma})\right)Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\Bigg]\\
&= \int_0^{t\wedge t'}\mathrm{Cov}\left(\gamma(t-s)\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0}, \gamma(t'-s)\mathbb{1}_{P_1(s,t',\gamma,\overline{\mathfrak{F}})=0}\right)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s\\
&= \int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\gamma(t'-s)\exp\left(-\int_s^{t\vee t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s\\
&\qquad -\int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(W_2^\gamma(t), W_2^\gamma(t')) &= \mathbb{E}\left[\hat{\mathfrak{S}}_{2,2}(t), \hat{\mathfrak{S}}_{2,2}(t')\right]\\
&= \mathbb{E}\Bigg[\int_0^{t\wedge t'}\int_{\mathbf{D}^2}\int_0^{\Upsilon_1(s-)}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\\
&\qquad \times\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\Bigg]\\
&= \int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\\
&\qquad \times\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.
\end{aligned}$$

The remaining covariance are obtained with the same way.

$$\begin{aligned}
\mathrm{Cov}(W^\lambda(t), W^\lambda(t')) &= \int_0^{t\wedge t'}\mathbb{E}\left[\lambda(t-s)\lambda(t'-s)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,\\
\mathrm{Cov}(W^\lambda(t), W_1^\gamma(t')) &= \int_0^{t\wedge t'}\mathrm{Cov}\left(\lambda(t-s), \gamma(t'-s)\mathbb{1}_{P_1(s,t',\gamma,\overline{\mathfrak{F}})=0}\right)\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s\\
&= -\int_0^{t\wedge t'}\overline{\lambda}(t-s)\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,\\
\mathrm{Cov}(W^\lambda(t), W_2^\gamma(t')) &= \int_0^{t\wedge t'}\overline{\lambda}(t-s)\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,\\
\mathrm{Cov}(W_1^\gamma(t), W_2^\gamma(t')) &= -\int_0^{t\wedge t'}\mathbb{E}\left[\gamma(t'-s)\exp\left(-\int_s^{t'}\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{F}}(s)\\
&\qquad \int_0^s\mathbb{E}\left[\gamma(s-a)\gamma(t-a)\exp\left(-\int_a^t\gamma(r-a)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(a)\overline{\mathfrak{F}}(a)da\mathrm{d}s.
\end{aligned}$$

We also have

$$\mathrm{Cov}(\hat{\mathfrak{M}}_{0,1}(t),\hat{\mathfrak{I}}_{0,1}(t')) = -\overline{I}(0)\overline{\lambda}_0(t)\mathbb{E}\left[\gamma_0(t')\exp\left(-\int_0^{t'}\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right].$$

$$\begin{aligned}
&\mathrm{Cov}(W^\lambda(t),\hat{\mathfrak{I}}_{0,1}(t')) = \mathrm{Cov}(\hat{\mathfrak{I}}_{0,1}(t),W_1^\gamma(t')) = \mathrm{Cov}(\hat{\mathfrak{I}}_{0,1}(t),W_2^\gamma(t')) = 0,\\
&\mathrm{Cov}(W^\lambda(t),\hat{\mathfrak{M}}_{0,1}(t')) = \mathrm{Cov}(W_1^\gamma(t),\hat{\mathfrak{M}}_{0,1}(t')) = \mathrm{Cov}(W_2^\gamma(t),\hat{\mathfrak{M}}_{0,1}(t')) = 0.
\end{aligned}$$

Then by the continuous mapping theorem, we obtain the convergence of $\hat{\mathfrak{S}}_2^N$, and then given its convergence and the convergence of $\left(\hat{\mathfrak{F}}_{2,0}^N,\hat{\mathfrak{F}}_{2,1}^N\right)$, we obtain the convergence of $\hat{\mathfrak{F}}_2^N$. Using the uniqueness of the limit of the process $\left(\hat{\mathfrak{S}}_2^N,\hat{\mathfrak{F}}_2^N\right)$, from Lemma 3.1, it follows that,

$$\begin{cases}
\hat{\mathfrak{I}}(t) \overset{d}{=} \hat{\mathfrak{I}}_{0,1}(t) + W_1^\gamma(t) + W_2^\gamma(t) + \displaystyle\int_0^t \mathbb{E}\left[\gamma_{1,1}(t-s)\exp\left(-\int_s^t \gamma_{1,1}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{F}}(s)\hat{\mathfrak{I}}(s)\mathrm{d}s,\\
\hat{\mathfrak{M}}(t) \overset{d}{=} \hat{\mathfrak{M}}_{0,1}(t) + W^\lambda(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)\overline{\mathfrak{F}}(s)\hat{\mathfrak{I}}(s)\mathrm{d}s.
\end{cases} \tag{3.19}$$

where we recall that $\left(\hat{\mathfrak{I}},\hat{\mathfrak{M}}\right)$ is a centered continuous 2-dimensional Gaussian process given in Definition 2.7. This concludes the proof. □

We also refer to the ongoing work [31] for a pde expression equivalent to the system of equations (3.16) and (3.17).

4. Particular cases of Theorem 2.10 or Proposition 3.10

In [13], Remark 3.4, it was proved that Theorem 2.2 coincides to the non-Markovian SIR, SIS and SIRS model in the literature when we remove the gradual loss of immunity. Therefore, in this section, we show that Theorem 2.10 coincides with the corresponding central limit theorems in the literature.

In the sequel we use $\overset{d}{=}$ to denote equality in distribution.

We start to recall that,

$$\overline{U}^N(0) = \frac{1}{N}\sum_{k=0}^N \mathbb{1}_{\eta_{k,0}=0},\ \text{and}\ \overline{I}^N(0) = \frac{1}{N}\sum_{k=0}^N \mathbb{1}_{\eta_{k,0}>0},$$

We also recall that $\hat{I}^N(0) = \sqrt{N}\left(\overline{I}^N(0)-\overline{I}(0)\right)$ and $\hat{U}^N(0) = \sqrt{N}\left(\overline{U}^N(0)-\overline{U}(0)\right)$.

As the family $(\eta_{k,0})_k$ is i.i.d, from classical central limit theorem, it follows that, the pair $(\hat{I}^N(0),\hat{U}^N(0)) \to (\hat{I}(0),\hat{U}(0))$ as $N\to\infty$ and where $(\hat{I}(0),\hat{U}(0))$ is a Gaussian process with covariance given by

$$Var(\hat{U}(0)) = Var(\hat{I}(0)) = \overline{I}(0)(1-\overline{I}(0))\ \text{and}\ \mathrm{Cov}\left(\hat{I}(0),\hat{U}(0)\right) = -\overline{I}(0)\overline{U}(0) = -\overline{I}(0)(1-\overline{I}(0)).$$

4.1. Generalised SIR model

In this case

$$\forall t \geq 0, \quad \gamma(t) = 0, \qquad \text{and} \qquad \gamma_0(t) = \begin{cases} 1 & \text{if } \eta_0 = 0, \\ 0 & \text{if } \eta_0 > 0. \end{cases}$$

From Theorem 2.2 the functional law of large numbers associated is given by the following system

$$\begin{cases} \overline{S}(t) = \overline{S}(0) \exp\left(-\int_0^t \overline{\mathfrak{F}}(r) \mathrm{d}r \right) \\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \int_0^t \overline{\lambda}(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s \end{cases} \tag{4.1}$$

Note that in this case $\overline{\mathfrak{S}}(t) = \overline{S}(t)$.

From Proposition 3.10 the central limit theorem associated is given by

$$\begin{cases} \hat{S}(t) = -\overline{S}(t) \int_0^t \hat{\mathfrak{F}}(s)\mathrm{d}s + \hat{\mathfrak{J}}_{0,1}(t), \\ \\ \hat{\mathfrak{F}}(t) = \int_0^t \overline{\lambda}(t-s) \left(\hat{S}(s)\overline{\mathfrak{F}}(s) + \overline{S}(s)\hat{\mathfrak{F}}(s) \right) \mathrm{d}s + \hat{\mathfrak{M}}_{0,1}(t) + W^{\lambda}(t). \end{cases} \tag{4.2}$$

We prove the following Proposition in Section A.1

Proposition 4.1. *The system of equations (4.2) is equivalent to the following system of equations.*

$$\begin{cases} \hat{S}(t) = \hat{I}(0) + W_0(t) - \int_0^t \left(\hat{S}(s)\overline{\mathfrak{F}}(s) + \overline{S}(s)\hat{\mathfrak{F}}(s) \right) \mathrm{d}s, \\ \\ \hat{\mathfrak{F}}(t) = \hat{I}(0)\overline{\lambda}_0(t) + W_{\overline{\mathfrak{F}}}(t) + \int_0^t \overline{\lambda}(t-s) \left(\hat{S}(s)\overline{\mathfrak{F}}(s) + \overline{S}(s)\hat{\mathfrak{F}}(s) \right) \mathrm{d}s, \end{cases} \tag{4.3}$$

where W_0 is a centered Gaussian process with covariance function for $t, t' \geq 0$,

$$\mathrm{Cov}(W_0(t), W_0(t')) = \int_0^{t \wedge t'} \overline{\mathfrak{F}}(s)\overline{S}(s)\mathrm{d}s,$$

$W_{\overline{\mathfrak{F}}}$ a centered-Gaussian process with covariance functions for $t, t' \geq 0$,

$$\mathrm{Cov}(W_{\overline{\mathfrak{F}}}(t), W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\mathbb{E}\left[\left(\lambda_0(t) - \overline{\lambda}_0(t) \right) \left(\lambda_0(t') - \overline{\lambda}_0(t') \right) | \eta_0 > 0 \right] \\ + \int_0^{t \wedge t'} \mathbb{E}\left[\lambda(t-s)\lambda(t'-s) \right] \overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s. \tag{4.4}$$

and $\hat{I}(0)$ is a centered-Gaussian variable with variance $\overline{I}(0)\overline{S}(0)$ independent of W_0 and $W_{\overline{\mathfrak{F}}}$. Moreover,

$$\mathrm{Cov}(W_0(t), W_{\overline{\mathfrak{F}}}(t')) = \int_0^{t \wedge t'} \overline{\lambda}(t'-s)\overline{\mathfrak{F}}(s)\overline{S}(s)\mathrm{d}s.$$

The set of equations (4.3) describes in Proposition 4.1 corresponds in law to the result described in Pang-Pardoux [18], Theorem 2.2.

4.2. SIS model

We consider the generalised SIS model where

$$\gamma(t) = \mathbb{1}_{t\geq\eta} \text{ and } \gamma_0(t) = \begin{cases} 1 & \text{if } \eta_0 = 0 \\ \mathbb{1}_{\eta_0 > t} & \text{if } \eta_0 > 0 \end{cases}$$

In this case $\overline{\mathfrak{S}}(t) = \overline{U}(t) = \overline{S}(t)$ and as $\overline{I}(t) = 1 - \overline{S}(t)$, the law of large numbers is given by the following system

$$\begin{cases} \overline{I}(t) = \overline{I}(0)F_0^c(t) + \displaystyle\int_0^t F^c(t-s)(1-\overline{I}(s))\overline{\mathfrak{F}}(s)\mathrm{d}s \\ \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s \end{cases}$$

From Proposition 3.10 the central limit theorem associated is given by

$$\begin{cases} \hat{I}(t) = \displaystyle\int_0^t F^c(t-s)\left(-\hat{I}(s)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s))\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_0^F(t) + W^F(t) \\ \hat{\mathfrak{F}}(t) = \displaystyle\int_0^t \overline{\lambda}(t-s)\left(-\hat{I}(s)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s))\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_{0,1}(t) + W^\lambda(t), \end{cases} \tag{4.5}$$

where we use the fact that $\hat{\mathfrak{S}}(t) = -\hat{I}(t)$.

We deduce easily the following Proposition. In fact, it easy to check that

$$\hat{\mathfrak{M}}_0^F(t) + W^F(t) \stackrel{d}{=} \hat{I}(0)F_0^c(t) + W_F(t) \text{ and } \hat{\mathfrak{M}}_{0,1}(t) + W^\lambda(t) \stackrel{d}{=} \hat{I}(0)\overline{\lambda}_0(t) + W_{\overline{\mathfrak{F}}}(t),$$

where the pair $(W_F, W_{\overline{\mathfrak{F}}})$ is a Gaussian process define below in Proposition 4.2.

Proposition 4.2. *The system of equations (4.5) is equivalent to the following system of equations.*

$$\begin{cases} \hat{I}(t) = \hat{I}(0)F_0^c(t) + W_F(t) + \displaystyle\int_0^t F^c(t-s)\left((1-\overline{I}(s))\hat{\mathfrak{F}}(s) - \hat{I}(s)\overline{\mathfrak{F}}(s)\right)\mathrm{d}s, \\ \\ \hat{\mathfrak{F}}(t) = \hat{I}(0)\overline{\lambda}_0(t) + W_{\overline{\mathfrak{F}}}(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)\left((1-\overline{I}(s))\hat{\mathfrak{F}}(s) - \hat{I}(s)\overline{\mathfrak{F}}(s)\right)\mathrm{d}s, \end{cases} \tag{4.6}$$

where W_F is a centered Gaussian process with covariance function for $t, t' \geq 0$,

$$\mathrm{Cov}(W_F(t), W_F(t')) = \overline{I}(0)\left(F_0^c(t\vee t') - F_0^c(t')F_0^c(t)\right) + \int_0^{t\wedge t'} F^c(t\vee t'-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,$$

$W_{\overline{\mathfrak{F}}}$ a centered-Gaussian process with covariance functions for $t, t' \geq 0$,

$$\mathrm{Cov}(W_{\overline{\mathfrak{F}}}(t), W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\mathbb{E}\left[\left(\lambda_0(t) - \overline{\lambda}_0(t)\right)\left(\lambda_0(t') - \overline{\lambda}_0(t')\right)\big|\eta_0 > 0\right] + \int_0^{t\wedge t'} \mathbb{E}\left[\lambda(t-s)\lambda(t'-s)\right]\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.$$

and $\hat{I}(0)$ *is a centered-Gaussian variable with variance* $\overline{I}(0)\overline{S}(0)$ *independent of* W_F *and* $W_{\overline{\mathfrak{F}}}$*. Moreover,*

$$\mathrm{Cov}(W_F(t), W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\left(\mathbb{E}\left[\mathbb{1}_{\eta_0>t}\lambda_0(t')\big|\eta_0 > 0\right] - F_0^c(t)\overline{\lambda}_0(t')\right) + \int_0^{t\wedge t'} \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\lambda(t'-s)\right]\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.$$

The set of equations (4.6) corresponds in law to the result described in Pang-Pardoux [18], Theorem 5.1.

4.3. SIRS model

We consider the generalised SIRS model where

$$\gamma(t) = \mathbb{1}_{t\geq\eta+\theta} \text{ and } \gamma_0(t) = \begin{cases} 1 & \text{if } \eta_0 = 0 \\ \mathbb{1}_{\eta_0+\theta_0>t} & \text{if } \eta_0 > 0 \end{cases}$$

with θ_0 the immune period at time zero, and θ the immune period after time zero.

Note that at the beginning, if the individual is susceptible $\theta_0 = 0$ and $\eta_0 = 0$, and if the individual is recovered $\theta_0 > 0$ and $\eta_0 = 0$. Thus

$$\overline{S}(0) = \mathbb{P}(\eta_0 = 0, \theta_0 = 0),\ \overline{R}(0) = \mathbb{P}(\eta_0 = 0, \theta_0 > 0),\ \text{ and } \overline{I}(0) = \mathbb{P}(\eta_0 > 0).$$

In this case

$$\overline{S}^N(0) = \frac{1}{N}\sum_{k=0}^N \mathbb{1}_{\eta_{k,0}=0,\theta_{k,0}=0},\ \overline{I}^N(0) = \frac{1}{N}\sum_{k=0}^N \mathbb{1}_{\eta_{k,0}>0} \text{ and } \overline{R}^N(0) = \frac{1}{N}\sum_{k=0}^N \mathbb{1}_{\eta_{k,0}=0,\theta_{k,0}>0}.$$

So from classical central limit theorem, it follows that $(\hat{S}^N(0), \hat{I}^N(0), \hat{R}^N(0)) \to (-\hat{I}(0) - \hat{R}(0), \hat{I}(0), \hat{R}(0))$ as $N \to \infty$, where $(\hat{I}(0), \hat{R}(0))$ is a Gaussian process with covariance given by

$$Var(\hat{R}(0)) = \overline{R}(0)(1 - \overline{R}(0)),\ Var(\hat{I}(0)) = \overline{I}(0)(1 - \overline{I}(0)) \text{ and } \mathrm{Cov}\left(\hat{I}(0), \hat{R}(0)\right) = -\overline{I}(0)\overline{R}(0). \tag{4.7}$$

In this case $\overline{S}(t) = \overline{\mathfrak{S}}(t)$. As we said before, taking $\mathbb{1}_{\eta_0\leq t<\eta_0+\varsigma_0}$ instead of $\lambda_0(t)$ and $\mathbb{1}_{\eta\leq t<\eta+\varsigma}$ instead of $\lambda(t)$, in (2.23) we obtain the proportion of recovered individuals at time t denoted $\overline{R}(t)$ and in (2.25) the expression of associated central limit theorem's denoted $\hat{R}(t)$. As a result,

$$\begin{aligned}\overline{R}(t) &= \mathbb{E}\left[\mathbb{1}_{\eta_0\leq t<\eta_0+\theta_0}\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \\ &\qquad + \int_0^t \mathbb{E}\left[\mathbb{1}_{\eta\leq t-s<\eta+\theta}\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\overline{\mathfrak{S}}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s, \\ &= \overline{R}(0)G_0^c(t) + \overline{I}(0)\Psi_0(t) + \int_0^t \Psi(t-s)\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}&G_0^c(t) = \mathbb{P}\left(\theta_0 \geq t\big|\eta_0 = 0, \theta_0 > 0\right),\ G^c(t|u) = \mathbb{P}\left(\theta \geq t\big|\eta = u\right), \\ &\Psi_0(t) = \mathbb{P}\left(\eta_0 \leq t < \eta_0 + \theta_0\big|\eta_0 > 0\right) \text{ and } \Psi(t) = \mathbb{P}\left(\eta \leq t < \eta + \theta\right).\end{aligned}$$

Consequently the functional law of large numbers associated is given by the following system

$$\begin{cases} \overline{\mathfrak{F}}(t) = \overline{I}(0)\overline{\lambda}_0(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)(1-\overline{I}(s)-\overline{R}(s))\overline{\mathfrak{F}}(s)\mathrm{d}s \\ \overline{I}(t) = \overline{I}(0)F_0^c(t) + \displaystyle\int_0^t F^c(t-s)(1-\overline{I}(s)-\overline{R}(s))\overline{\mathfrak{F}}(s)\mathrm{d}s, \\ \overline{R}(t) = \overline{R}(0)G_0^c(t) + \overline{I}(0)\Psi_0(t) + \displaystyle\int_0^t \Psi(t-s)(1-\overline{I}(s)-\overline{R}(s))\overline{\mathfrak{F}}(s)\mathrm{d}s \end{cases}$$

From Proposition 3.10 the central limit theorem associated is given by

$$\begin{cases} \hat{\mathfrak{F}}(t) = \displaystyle\int_0^t \overline{\lambda}(t-s)\left(-\hat{I}(s)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s))\hat{\mathfrak{F}}(s)\right)\mathrm{d}s + \hat{\mathfrak{M}}_{0,1}(t) + W^{\lambda}(t), \\ \hat{I}(t) = \displaystyle\int_0^t F^c(t-s)\left[-\left(\hat{I}(s)+\hat{R}(s)\right)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s)-\overline{R}(s))\hat{\mathfrak{F}}(s)\right]\mathrm{d}s + \hat{\mathfrak{M}}_0^F(t) + W^F(t) \\ \hat{R}(t) = \displaystyle\int_0^t \Psi(t-s)\left[-\left(\hat{I}(s)+\hat{R}(s)\right)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s)-\overline{R}(s))\hat{\mathfrak{F}}(s)\right]\mathrm{d}s + \hat{\mathfrak{M}}_{0,1}^R(t) + W^R(t), \end{cases} \tag{4.8}$$

where $\left(\hat{\mathfrak{M}}_0^R, W^R\right)$ is a centered continuous 2−dimensional Gaussian process as given in Proposition 3.10 where we replace $\gamma_0(t)$by $\mathbb{1}_{\eta_0\le t<\eta_0+\theta_0}$ and $\gamma(t)$ by $\mathbb{1}_{\eta\le t<\eta+\theta}$, in the expressions of $\left(\hat{\mathfrak{I}}_{0,1}, W_1^{\gamma}, W_2^{\gamma}\right)$.

We deduce easily the following Proposition.

Proposition 4.3. *The system of equations (4.8) is equivalent to the following system of equations.*

$$\begin{cases} \hat{\mathfrak{F}}(t) = \hat{I}(0)\overline{\lambda}_0(t) + W_{\overline{\mathfrak{F}}}(t) + \displaystyle\int_0^t \overline{\lambda}(t-s)\left[-\left(\hat{I}(s)+\hat{R}(s)\right)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s)-\overline{R}(s))\hat{\mathfrak{F}}(s)\right]\mathrm{d}s, \\ \hat{I}(t) = \hat{I}(0)F_0^c(t) + W_F(t) + \displaystyle\int_0^t F^c(t-s)\left[-\left(\hat{I}(s)+\hat{R}(s)\right)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s)-\overline{R}(s))\hat{\mathfrak{F}}(s)\right]\mathrm{d}s, \\ \hat{R}(t) = \hat{R}(0)G_0^c(t) + \hat{I}(0)\Psi_0(t) + W_R(t) + \displaystyle\int_0^t \Psi(t-s)\left[-\left(\hat{I}(s)+\hat{R}(s)\right)\overline{\mathfrak{F}}(s) + (1-\overline{I}(s)-\overline{R}(s))\hat{\mathfrak{F}}(s)\right]\mathrm{d}s, \end{cases} \tag{4.9}$$

where W_R is a centered Gaussian process with covariance function for $t,t'\ge 0$,

$$\begin{aligned} \mathrm{Cov}(W_R(t),W_R(t')) = \overline{I}(0)\left(\int_0^{t\wedge t'}\Psi_0(t\vee t'-r|r)F_0(\mathrm{d}r) - \Psi_0(t)\Psi_0(t')\right) + \overline{R}(0)\left(G_0^c(t\vee t') - G_0^c(t)G_0^c(t')\right) \\ + \int_0^{t\wedge t'}\int_0^{t\wedge t'-s} G^c(t\vee t'-s-r|r)F(\mathrm{d}r)\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s, \end{aligned}$$

$W_{\overline{\mathfrak{F}}}$ a centered-Gaussian process with covariance functions for $t,t'\ge 0$,

$$\begin{aligned} \mathrm{Cov}(W_{\overline{\mathfrak{F}}}(t),W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\mathbb{E}\left[\left(\lambda_0(t)-\overline{\lambda}_0(t)\right)\left(\lambda_0(t')-\overline{\lambda}_0(t')\right)\big|\eta_0>0\right] \\ + \int_0^{t\wedge t'}\mathbb{E}\left[\lambda(t-s)\lambda(t'-s)\right]\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s. \end{aligned}$$

and $(\hat{I}(0), \hat{R}(0))$ *is a centered-Gaussian variable independent of* W_F *and* $W_{\overline{\mathfrak{F}}}$, *with covariance given by* (4.7). *Moreover,*

$$\mathrm{Cov}(W_F(t), W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\left(\mathbb{E}\left[\mathbb{1}_{\eta_0>t}\lambda_0(t')\middle|\eta_0>0\right] - F_0^c(t)\overline{\lambda}_0(t')\right) + \int_0^{t\wedge t'} \mathbb{E}\left[\mathbb{1}_{\eta>t-s}\lambda(t'-s)\right]\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.$$

The set of equations (4.9) corresponds in law to the result described in Pang-Pardoux [18], Theorem 5.2.

5. Some preliminary results

To establish the tightness of $\mathfrak{F}^N$, $\mathfrak{S}^N$ we need a **C**-tightness criterion (see App. Sect. A.2), a new result on stochastic integrals with respect to Poisson random measures, the moment estimates, an approximation of the original model by a quarantine model, and an estimate on the pair (λ, γ), which are given in the next subsections.

5.1. A property of stochastic integrals with respect to Poisson random measures

Theorem 5.1. *Let* $(E, \mathcal{B}(E), \nu)$ *be a* $\sigma-$*finite measured space and let* Q *be a Poisson random measure on* $\mathbb{R}_+ \times E$ *of intensity* $\mathrm{d}s\nu(\mathrm{d}u)$ *and* $(\mathcal{F}_t)_t$ *a filtration which such that for all* $t \geq 0$, $Q|_{[0,t]\times E}$ *is* $\mathcal{F}_t-$*measurable, and for* $0 \leq s < t$, $\mathcal{F}_s$ *and* $Q|_{]s,t]\times E}$ *are independent. Let* $h : \mathbb{R}_+ \times E \to \mathbb{R}$, *be a predictable process such that for all* $t \in \mathbb{R}_+$,

$$\mathbb{E}\left[\int_0^t \int_E |h(s,u)|\nu(\mathrm{d}u)\mathrm{d}s\right] < \infty.$$

Let $f : \mathbb{R}_+ \times E \times \mathcal{M}_F(\mathbb{R}_+ \times E) \to \mathbb{R}$, *be a bounded and measurable deterministic function. Then*

$$\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(\mathrm{d}s,\mathrm{d}u)\right] = \mathbb{E}\left[\int_0^t \int_E h(s,u)\overline{f}(s,u,t)\nu(\mathrm{d}u)\mathrm{d}s\right],$$

where

$$\overline{f}(s,u,t) = \mathbb{E}\left[f(s,u,Q|_{]s,t]\times E})\right].$$

Proof. Let $(s_i, u_i)_i$ be an arbitrary ordering be the atoms of the measure Q. We note that

$$\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(\mathrm{d}s,\mathrm{d}u) = \sum_{i\geq 1} h(s_i,u_i) f(s_i,u_i,Q|_{]s_i,t]\times E})\mathbb{1}_{s_i\leq t}.$$

As

$$\mathbb{E}\left[\int_0^t \int_E |h(s,u)|\nu(\mathrm{d}u)\mathrm{d}s\right] < \infty,$$

by Fubini's theorem

$$\mathbb{E}\left[\int_{[0,t]\times E} h(s,u) f(s,u,Q|_{]s,t]\times E}) Q(\mathrm{d}s,\mathrm{d}u)\right] = \sum_{i\geq 1} \mathbb{E}\left[h(s_i,u_i)\mathbb{E}\left[f(s_i,u_i,Q|_{]s_i,t]\times E})\middle|\mathcal{F}_{s_i}\right]\mathbb{1}_{s_i\leq t}\right],$$

and as $Q|_{]s_i,t]\times E}$ and $\mathcal{F}_{s_i}$ are independent,

$$\mathbb{E}\left[f(s_i,u_i,Q|_{]s_i,t]\times E})\middle|\mathcal{F}_{s_i}\right]=\overline{f}(s_i,u_i,t),$$

it follows that,

$$\begin{aligned}\mathbb{E}\left[\int_{[0,t]\times E}h(s,u)f(s,u,Q|_{]s,t]\times E})Q(\mathrm{d}s,\mathrm{d}u)\right]&=\sum_{i\geq 1}\mathbb{E}\left[h(s_i,u_i)\overline{f}(s_i,u_i,t)\mathbb{1}_{s_i\leq t}\right]\\&=\mathbb{E}\left[\int_{[0,t]\times E}h(s,u)\overline{f}(s,u,t)Q(\mathrm{d}s,\mathrm{d}u)\right].\end{aligned}$$

Consequently, as h is a predictable process, by a classical result on Poisson random measures [23], Theorem 6.2

$$\mathbb{E}\left[\int_{[0,t]\times E}h(s,u)f(s,u,Q|_{]s,t]\times E})Q(\mathrm{d}s,\mathrm{d}u)\right]=\mathbb{E}\left[\int_0^t\int_E h(s,u)\overline{f}(s,u,t)\nu(\mathrm{d}u)\mathrm{d}s\right].$$

□

5.2. Moment Inequalities

We recall the following Lemma from [13].

Lemma 5.2. *Assumption 2.1 for $k\in\mathbb{N}$ and $T\geq 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_k^N(t)-A_k(t)\right|\right]\leq\int_0^T\mathbb{E}\Big[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\Big]dt=:\delta^N(T) \tag{5.1}$$

and

$$\mathbb{P}\Big((\varsigma_k^N(s))_{t\in[0,T]}\neq(\varsigma_k(t))_{t\in[0,T]}\Big)\leq T\delta^N(T).$$

Moreover,

$$\delta^N(T)\leq\frac{\lambda_*}{\sqrt{N}}T\exp(2\lambda_* T). \tag{5.2}$$

From Lemma 5.2 or [13], Lemma 6.3 in [13], we also have the following Corollary.

Corollary 5.3. *Assumption 2.1, for $k\in\mathbb{N}$ and $t\geq 0$,*

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t)-\overline{\mathfrak{F}}(t)\right|\right]\leq\frac{\lambda_*}{\sqrt{N}}\left(1+\lambda_* t\exp(2\lambda_* t)\right),\qquad\mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}(t)\right|\right]\leq\frac{1}{\sqrt{N}}\left(1+\lambda_* t\exp(2\lambda_* t)\right)$$
$$\text{and}\quad\mathbb{E}\left[\left|\Upsilon_k^N(t)-\Upsilon_k(t)\right|\right]\leq\frac{\lambda_*}{\sqrt{N}}\left(1+2\lambda_* t\exp(2\lambda_* t)\right).$$

By exchangeability from Corollary 5.3, it easy to check that for $t\geq 0$,

$$\mathbb{E}\left[\left|\overline{\Upsilon}^N(t)-\tilde{\Upsilon}^N(t)\right|\right]\leq\frac{\lambda_*}{\sqrt{N}}\left(1+2\lambda_* t\exp(2\lambda_* t)\right), \tag{5.3}$$

where

$$\tilde{\Upsilon}^N(t) = \frac{1}{N}\sum_{k=1}^{N}\Upsilon_k(t).$$

Now we establish similar inequalities as in Corollary 5.3 for higher moments. Let

$$\chi_N^{(k)}(t) := \mathbb{P}\Big((\varsigma_{k'}^N(s))_{s\in[0,t]} \neq (\varsigma_{k'}(s))_{s\in[0,t]},\, \forall k' = 1,\cdots,k\Big).$$

We establish the following Proposition.

Proposition 5.4. *Assumption 2.1, for all $N \geq k$, and $t \in [0,T]$, there are positives constants $C_{k,T}$ and $C'_{k,T}$ depending on k and T, such that*

$$\chi_N^{(k)}(t) \leq C_{k,T}N^{-k/2}, \quad \textit{and} \quad \xi_N^{(k)}(t) := \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t)\right|^k\right] \leq C'_{k,T}(t)N^{-k/2}.$$

Proof. We adapt the proof of [25], Proposition 3.1. Let

$$\Delta_k^N(t) := \int_0^t \int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)} Q_k(\mathrm{d}s,\mathrm{d}u).$$

Observe that for all k', $(\varsigma_{k'}^N(s))_{s\in[0,t]} = (\varsigma_{k'}(s))_{s\in[0,t]}$ if and only if $\Delta_{k'}^N(t) = 0$. As $\Delta_{k'}^N$ takes integer values,

$$\chi_N^{(k)}(t) \leq \mathbb{E}\left[\prod_{k'=1}^{k}\Delta_{k'}^N(t)\right].$$

Let us set, for all $k, p \in \mathbb{N}$ such that $N \geq k$,

$$\varepsilon_N^{(k,p)}(t) = \mathbb{E}\left[\prod_{k'=1}^{k}\left(\Delta_{k'}^N(t)\right)^p\right].$$

We next show by induction on k that

$$\varepsilon_N^{(k,p)}(t) \leq C_{T,k,p}N^{-k/2}.$$

From Lemma 5.2,

$$\varepsilon_N^{(1,1)}(t) \leq C_T N^{-1/2}. \tag{5.4}$$

Note that for all $p \leq q$,

$$\varepsilon_N^{(k,p)}(t) \leq \varepsilon_N^{(k,q)}(t),$$

because the counting process Δ_k^N takes values in $\mathbb{N}$.

As in [25], Proposition 3.1, we have for all $p \in \mathbb{N}$,

$$\varepsilon_N^{(1,p)}(t) = \mathbb{E}\left[\left(\Delta_1^N(t)\right)^p\right] \leq C_{T,p}N^{-1/2}. \tag{5.5}$$

Indeed, noting that, as the process $(\Delta_1^N(t))_t$ jumps at each time from $\Delta_1^N(t^-)$ to $\Delta_1^N(t^-)+1$, the process $((\Delta_1^N(t))^p)_t$ jumps from $(\Delta_1^N(t^-))^p$ to $(\Delta_1^N(t^-)+1)^p$. Consequently, from the fact that

$$(\Delta_1^N(t^-)+1)^p - (\Delta_1^N(t^-))^p = \sum_{p'=0}^{p-1} \binom{p}{p'} \left(\Delta_1^N(t^-)\right)^{p'},$$

it follows that

$$\left(\Delta_1^N(t)\right)^p = \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \left(\Delta_1^N(s^-)\right)^{p'} \Delta_1^N(\mathrm{d}s).$$

Moreover, as $\left(\Delta_1^N(s^-)\right)^{p'} \le \left(\Delta_1^N(s^-)\right)^p$ for $p' \le p$, we deduce that

$$\begin{aligned}
\varepsilon_N^{(1,p)}(t) = \mathbb{E}\left[\left(\Delta_1^N(t)\right)^p\right] &\le \mathbb{E}\left[\int_0^t \Delta_1^N(\mathrm{d}s)\right] + 2^p \mathbb{E}\left[\int_0^t \left(\Delta_1^N(s^-)\right)^p \Delta_1^N(\mathrm{d}s)\right] \\
&= \varepsilon_N^{(1,1)}(t) + 2^p \int_0^t \mathbb{E}\left[\left(\Delta_1^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] \mathrm{d}s \\
&\le C_{T,p} N^{-1/2} + 2^p \lambda_* \int_0^t \varepsilon_N^{(1,p)}(s)\mathrm{d}s,
\end{aligned}$$

where the last inequality comes from (5.4) and the fact that $|\Upsilon_1(s) - \Upsilon_1^N(s)| \le \lambda^*$. The conclusion (5.5) follows by Gronwall's Lemma.

When $k \ge 2$ and $p \ge 1$, as above, noting that

$$\prod_{i=1}^k \left(\Delta_i^N(t^-)\right)^p = \sum_{j=1}^k \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \left(\prod_{i\neq j, i=1}^k \left(\Delta_i^N(s^-)\right)^p\right) \left(\Delta_j^N(s^-)\right)^{p'} \Delta_j^N(\mathrm{d}s),$$

almost surely, and using exchangeability of the processes $(\Delta_i^N)_i$ and the fact that the integrand is predictable, it follows that

$$\begin{aligned}
\varepsilon_N^{(k,p)}(t) &= \sum_{j=1}^k \sum_{p'=0}^{p-1} \binom{p}{p'} \mathbb{E}\left[\int_0^t \prod_{i\neq j, i=1}^k \left(\Delta_i^N(s^-)\right)^p \left(\Delta_j^N(s^-)\right)^{p'} \Delta_j^N(\mathrm{d}s)\right] \\
&= k \sum_{p'=0}^{p-1} \binom{p}{p'} \int_0^t \mathbb{E}\left[\left(\Delta_1^N(s)\right)^{p'} \prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] \mathrm{d}s \\
&\le k \int_0^t \left(\mathbb{E}\left[\prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\Upsilon_1(s) - \Upsilon_1^N(s)|\right] + 2^p \lambda_* \varepsilon_N^{(k,p)}(s)\right) \mathrm{d}s.
\end{aligned}$$

However, as

$$\begin{aligned}
|\Upsilon_1(s) - \Upsilon_1^N(s)| &\le |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* |\gamma_{1,A_1^N(s)}(\varsigma_1^N(s)) - \gamma_{1,A_1(s)}(\varsigma_1(s))| \\
&\le |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* \mathbb{1}_{\varsigma_1^N(s)\neq\varsigma_1(s) \text{ or } A_1^N(s)\neq A_1(s)} \\
&\le |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| + \lambda_* \Delta_1^N(s),
\end{aligned}$$

it follows that,

$$\varepsilon_N^{(k,p)}(t) \leq k \int_0^t \left(M^N(s) + 2^{p+1} \lambda_* \varepsilon_N^{(k,p)}(s) \right) \mathrm{d}s,$$

with $M^N(s) = \mathbb{E}\left[\prod_{i=2}^k \left(\Delta_i^N(s)\right)^p |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)|\right]$. Using exchangeability we can replace each term $\left(\Delta_i^N(s)\right)^p$ in the expression of M^N by the following sum

$$\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor + 1}^{i \lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p$$

without changing the value of M^N, since the sums are taken on disjoint indices. Then,

$$\begin{aligned} M^N(s) &= \mathbb{E}\left[\prod_{i=2}^k \left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor + 1}^{i \lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p \right) |\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)| \right] \\ &\leq \left(\prod_{i=2}^k \mathbb{E}\left[\left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=(i-1)\lfloor \frac{N}{k} \rfloor + 1}^{i \lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p \right)^k \right] \right)^{1/k} \left(\mathbb{E}\left[|\overline{\mathfrak{F}}^N(s) - \overline{\mathfrak{F}}(s)|^k \right] \right)^{1/k}. \end{aligned}$$

Consequently, using Young's inequality, it follows that

$$\varepsilon_N^{(k,p)}(t) \lesssim \int_0^t \left(\frac{k-1}{k} E_{N,k,p}(s) + \frac{1}{k} \xi_N^{(k)}(s) + \varepsilon_N^{(k,p)}(s) \right) \mathrm{d}s,$$

where

$$\begin{aligned} E_{N,k,p}(s) &:= \mathbb{E}\left[\left(\frac{1}{\lfloor \frac{N}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{N}{k} \rfloor} (\Delta_j^N(s))^p \right)^k \right] \\ &\leq C_k \left(\sum_{k'=1}^{k-1} N^{k'-k} \varepsilon_N^{(k',kp)}(s) + \varepsilon_N^{(k,p)}(s) \right), \end{aligned}$$

and

$$\begin{aligned} \xi_N^{(k)}(t) &:= \mathbb{E}\left[\left| \overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}(t) \right|^k \right] \\ &\leq 2^{k-1} \left(\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N \lambda_{i,A_i^N(t)}(\varsigma_i^N(t)) - \lambda_{i,A_i(t)}(\varsigma_i(t)) \right)^2 \right] + \mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N \lambda_{i,A_i(t)}(\varsigma_i(t)) - \overline{\mathfrak{F}}(t) \right)^2 \right] \right) \\ &\leq 2^{k-1} \left(\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N \Delta_i^N(t) \right)^2 \right] + \mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N \lambda_{i,A_i(t)}(\varsigma_i(t)) - \overline{\mathfrak{F}}(t) \right)^2 \right] \right) \\ &\leq \sum_{k'=1}^{k-1} N^{k'-k} \varepsilon_N^{(k',k)}(t) + \varepsilon_N^{(k,p)}(t) + C_k N^{-k/2}, \end{aligned} \tag{5.6}$$

We refer to [25], Proposition 3.1 for more details. The conclusion of the proof of this Proposition follows by induction and Gronwall's Lemma. □

The proof of the following Corollary is similar to (5.6) replacing $\lambda_{i,j}$ by $\gamma_{i,j}$.

Corollary 5.5. *For all ℓ, k, $T>0$, There exists $C_{T,k}>0$, such that for all $t\in[0,T]$, $\forall N\geq k$,*

$$\mathbb{E}\left[\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}(t)\right|^k\right]\leq C_{T,k}N^{-k/2},\ \textit{and}\ \mathbb{E}\left[\left|\Upsilon_\ell^N(t)-\Upsilon_\ell(t)\right|^k\right]\leq C_{T,k}N^{-k/2}. \tag{5.7}$$

5.3. Quarantine model

Fix $k\in\mathbb{N}$, let us consider the model above where the k-th individual is quarantined. Then we define

$$\overline{\mathfrak{F}}^N_{(k)}(t)=\frac{1}{N}\sum_{\ell=1,\ell\neq k}^N\lambda_{\ell,A^N_{\ell,(k)}(t)}(\varsigma^N_{\ell,(k)}(t)),\ \text{and}\ \overline{\mathfrak{S}}^N_{(k)}(t)=\frac{1}{N}\sum_{\ell=1,\ell\neq k}^N\gamma_{\ell,(k)}(t)(\varsigma^N_{\ell,(k)}(t)),$$

where

$$A^N_{\ell,(k)}(t)=\int_0^t\int_0^{+\infty}\mathbb{1}_{\gamma_{\ell,A^N_{\ell,(k)}(s^-)}(\varsigma^N_{\ell,(k)}(s^-))\overline{\mathfrak{F}}^N_{(k)}(s^-)>u}Q_\ell(\mathrm{d}s,\mathrm{d}u),\ \ell\neq k,$$

and we recall that Q_k is a standard Poisson random measure on $\mathbb{R}^2_+$. In the original model, the number of individuals infected by the k-th individual can be described by

$$\sum_{\ell=1,\ell\neq k}^N\int_0^t\int_0^{+\infty}\mathbb{1}_{\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k-1}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))<u\leq\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\frac{1}{N}\sum_{i=1}^{k}\lambda_{i,A^N_i(s^-)}(\varsigma^N_i(s^-))}Q_\ell(\mathrm{d}s,\mathrm{d}u).$$

Since $\lambda_{k,A^N_k(s^-)}(\varsigma^N_k(s^-))\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))\leq\lambda_*$, we have

$$\sum_{\ell=1,\ell\neq k}^N\int_0^t\int_0^{+\infty}\mathbb{1}_{\frac{1}{N}\lambda_{k,A^N_k(s^-)}(\varsigma^N_k(s^-))\gamma_{\ell,A^N_\ell(s^-)}(\varsigma^N_\ell(s^-))>u}Q_\ell(\mathrm{d}s,\mathrm{d}u)\leq\int_0^t\int_0^{\lambda_*}Q'_k(\mathrm{d}s,\mathrm{d}u),$$

where Q'_k is a standard Poisson random measure on $\mathbb{R}^2_+$ independent of Q_k.

So we can bound the number of infected descendants of the individual k, by a pure birth process with birth rate λ_*, denoted by $Y(t-\tau^N_{k,1})$ (where $\tau^N_{k,1}$ denotes the first time of infection of the individual k in the full model). Note that $Y(t-\tau^N_{k,1}))\leq Y(t)$, and that $Y(t)$ follows a geometric distribution with parameter $\exp(-\lambda_*t)$ and is independent of Q_k. As the random variable Y bounds the number of individuals who do not have the same state between the two models, as a result, we have

$$\left|\overline{\mathfrak{F}}^N(t)-\overline{\mathfrak{F}}^N_{(k)}(t)\right|\leq\frac{\lambda_*}{N}Y(t)\quad\text{and}\quad\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}^N_{(k)}(t)\right|\leq\frac{Y(t)}{N}.$$

Similarly we can define a quarantine model $(\overline{\mathfrak{F}}^N_{(k,\ell)},\overline{\mathfrak{S}}^N_{(k,\ell)})$ where the k-th and ℓ-th individuals are quarantined such that

$$\left|\overline{\mathfrak{F}}^N(t)-\overline{\mathfrak{F}}^N_{(k,\ell)}(t)\right|\leq\frac{\lambda_*}{N}\tilde{Y}(t)\quad\text{and}\quad\left|\overline{\mathfrak{S}}^N(t)-\overline{\mathfrak{S}}^N_{(k,\ell)}(t)\right|\leq\frac{\tilde{Y}(t)}{N}, \tag{5.8}$$

where $\tilde{Y}(t)$ follows a geometric distribution with parameter $\exp(-2\lambda_*t)$ and is independent of Q_k and Q_ℓ.

5.4. Useful inequalities for $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$

We establish the following Lemma where λ and γ satisfy Assumption 2.4–2.5.

Lemma 5.6. *For $t \geq s \geq 0$, with $\alpha > 1/2$ as in Assumption 2.4–2.5, there exists $C > 0$ independent of s and t such that,*

$$\big|\lambda(t) - \lambda(s)\big| \leq C(t-s)^{\alpha} + \lambda_* \sum_{j=1}^{\ell-1} \mathbf{1}_{s<\xi^j\leq t},$$

$$\big|\gamma(t) - \gamma(s)\big| \leq C(t-s)^{\alpha} + \sum_{j=1}^{\ell-1} \mathbf{1}_{s<\zeta^j\leq t},$$

and

$$|\bar{\lambda}(t) - \bar{\lambda}(s)| \leq C(t-s)^{\alpha} + \lambda^* \sum_{j=1}^{\ell-1} (F_j(t) - F_j(s)).$$

Proof. We have

$$\lambda(t) - \lambda(s) = \sum_{j=1}^{\ell} \big(\lambda^j(t) - \lambda^j(s)\big)\mathbf{1}_{\xi^{j-1}\leq s,\, t<\xi^j} + \big(\lambda(t)-\lambda(s)\big)\sum_{j=1}^{\ell-1} \mathbf{1}_{s<\xi^j\leq t}.$$

Thus the statement follows from Assumption 2.5. □

6. Characterization of the limit of converging subsequences of $(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N_{1,0})$

The aim of this section is to prove Lemma 3.8. We recall that

$$\hat{\mathfrak{S}}^N_{1,0}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \gamma_{k,0}(t) \left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \right).$$

Let

$$\Xi^N_1(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \chi^N_k(t), \tag{6.1}$$

where

$$\chi^N_k(t) = \gamma_{k,0}(t)\Bigg(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} - \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right) + \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\Bigg).$$

Lemma 6.1. *Under Assumption 2.6, for all $t \geq 0$, as $N \to +\infty$,*

$$\Xi_1^N(t) \to 0 \quad \textit{in probability.}$$

Proof. It suffices to show that

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\Xi_1^N(t)\right)^2\right] = 0. \tag{6.2}$$

By exchangeability, we have

$$\mathbb{E}\left[\left(\Xi_1^N(t)\right)^2\right] = \mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]. \tag{6.3}$$

From Theorem 2.2, as $N \to \infty$, $\overline{\mathfrak{F}}^N \Rightarrow \overline{\mathfrak{F}}$, in $\mathbf{D}$, consequently as $N \to \infty$, $\chi_1^N(t) \to 0$ in probability and as $|\chi_1^N(t)| \leq 2$, it follows that, as $N \to \infty$,

$$\mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right] \to 0. \tag{6.4}$$

To obtain (6.2) it remains to show that

$$\left|\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]\right| = o\left(\frac{1}{N}\right). \tag{6.5}$$

We consider a quarantine model where the first and second individuals are in quarantine (see Sect. 5.3). We denote by $\overline{\mathfrak{F}}^N_{(1,2)}(t)$ the force of infection in the population at time t in this model. As in (5.8), for any $T \geq 0$, there exists a geometric random variable Y independent from Q_1 and Q_2 such that almost surely for all $t \in [0,T]$,

$$\overline{\mathfrak{F}}^N_{(1,2)}(t) - \frac{Y}{N} \leq \overline{\mathfrak{F}}^N(t) \leq \overline{\mathfrak{F}}^N_{(1,2)}(t) + \frac{Y}{N}.$$

Therefore,

$$P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} - \frac{Y}{N}) \leq P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N) \leq P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} + \frac{Y}{N}).$$

Let

$$E_k^N = \left\{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} - \frac{Y}{N}) = P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)} + \frac{Y}{N})\right\}.$$

Then, for $k \in \{1,2\}$ as Q_k is independent of $(Q_i, \gamma_{k,0}, Y, \overline{\mathfrak{F}}^N_{(1,2)})$ for $k \neq i$, it follows by Markov's inequality that

$$\mathbb{P}\left(\left(E_k^N\right)^c \big| \gamma_{k,0}, \overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_i\right) \leq \frac{2tY}{N}. \tag{6.6}$$

Let

$$\begin{aligned}\tilde{\chi}_k^N(t) = \gamma_{k,0}(t)\Big(&\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N_{(1,2)})=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0} \\ &- \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}^N_{(1,2)}(r)\mathrm{d}r\right) + \exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\Big).\end{aligned}$$

For $k \in \{1,2\}$ as Q_k, $\gamma_{k,0}$, and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}^N_k(t)|\overline{\mathfrak{F}}^N_{(1,2)}, \gamma_{k,0}\right] = 0.$$

Moreover, as Q_1, $\gamma_{1,0}$, Q_2, $\gamma_{2,0}$ and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] = \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mathbb{E}\left[\tilde{\chi}^N_2(t)|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right] = 0. \tag{6.7}$$

We have

$$\begin{aligned}\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) - \tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] &= \mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t) - \tilde{\chi}^N_2(t)\right)\right] + \mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right] \\ &\quad + \mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t)\right)\left(\chi^N_2(t) - \tilde{\chi}^N_2(t)\right)\right]. \end{aligned} \tag{6.8}$$

However,

$$\left|\chi^N_k(t) - \tilde{\chi}^N_k(t)\right| \le \mathbb{1}_{(E^N_k)^c} + \frac{Yt}{N}. \tag{6.9}$$

Hence from (6.6) and (6.9),

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t) - \tilde{\chi}^N_2(t)\right)\right]\right| &\le \mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|\mathbb{1}_{(E^N_2)^c}\right] + \frac{t}{N}\mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|Y\right] \\ &= \mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|\mathbb{P}\left(\left(E^N_2\right)^c|\gamma_{1,0}, \overline{\mathfrak{F}}^N_{(1,2)}, Y, Q_1\right)\right] + \frac{t}{N}\mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|Y\right] \\ &\le \frac{3t}{N}\mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|Y\right] \\ &= \frac{3t}{N}\mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|\right]\mathbb{E}\left[Y\right],\end{aligned}$$

where we use the fact that Y and $\tilde{\chi}^N_1$ are independent.

From (5.3) as $\overline{\mathfrak{F}}^N \to \overline{\mathfrak{F}}$, and $|\overline{\mathfrak{F}}^N - \overline{\mathfrak{F}}^N_{(1,2)}| \le \frac{Y}{N}$, it follows that as $N \to \infty$,

$$\begin{aligned}\mathbb{E}\left[\left|\tilde{\chi}^N_1(t)\right|\right] &\le \mathbb{E}\left[\int_0^t \int_{\gamma_{1,0}(r^-)\left(\overline{\mathfrak{F}}^N_{(1,2)}(r^-)\wedge\overline{\mathfrak{F}}(r^-)\right)}^{\gamma_{1,0}(r^-)\left(\overline{\mathfrak{F}}^N_{(1,2)}(r^-)\vee\overline{\mathfrak{F}}(r^-)\right)} Q_1(\mathrm{d}r, \mathrm{d}u)\right] + \int_0^t \mathbb{E}\left[\gamma_{1,0}(r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(r) - \overline{\mathfrak{F}}(r)\right|\right]\mathrm{d}r \\ &= 2\int_0^t \mathbb{E}\left[\gamma_{1,0}(r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(r) - \overline{\mathfrak{F}}(r)\right|\right]\mathrm{d}r \to 0.\end{aligned}$$

Thus

$$\left|\mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t) - \tilde{\chi}^N_2(t)\right)\right]\right| = o\left(\frac{1}{N}\right). \tag{6.10}$$

Similarly, we show that,

$$\left|\mathbb{E}\left[\left(\chi^N_1(t) - \tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right]\right| = o\left(\frac{1}{N}\right). \tag{6.11}$$

Moreover, from (6.9), by the fact that Q_1 and Q_2 are independent, and exchangeability,

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]\right| \\
&\qquad\le \mathbb{E}\left[\left(\mathbb{1}_{(E_1^N)^c}+\frac{Yt}{N}\right)\left(\mathbb{1}_{(E_2^N)^c}+\frac{Yt}{N}\right)\right] \\
&\qquad= \mathbb{E}\left[\mathbb{P}\left(\left(E_1^N\right)^c|\overline{\mathfrak{F}}^N_{(1,2)},Y\right)\mathbb{P}\left(\left(E_2^N\right)^c|\overline{\mathfrak{F}}^N_{(1,2)},Y\right)\right]+\frac{2t}{N}\mathbb{E}\left[Y\mathbb{P}\left(\left(E_1^N\right)^c|\overline{\mathfrak{F}}^N_{(1,2)},Y\right)\right]+\frac{t^2}{N^2}\mathbb{E}\left[Y^2\right] \\
&\qquad\le \frac{10t^2}{N^2}\mathbb{E}\left[Y^2\right],
\end{aligned}$$

where we use (6.6).

Consequently,

$$\left|\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]\right| = o\left(\frac{1}{N}\right). \tag{6.12}$$

Thus from (6.12), (6.11), (6.10), (6.8) and (6.7), (6.5) holds. □

We set,

$$\mathbb{V}_0^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\gamma_{k,0}(t)\left(\exp\left(-\int_0^t\gamma_{k,0}(r)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)-\exp\left(-\int_0^t\gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right).$$

Consequently, from (6.1) for any $t\ge 0$, $\hat{\mathfrak{S}}^N_{1,0}(t)=\Xi^N_1(t)+\mathbb{V}^N_0(t)$. Therefore from Corollary 6.1 as $\Xi^N_1(t)\to 0$ in probability, to conclude the proof of Lemma 3.8, it suffices to establish the following Lemma.

Lemma 6.2. *Let $\hat{\mathfrak{F}}$ be a limit of a converging subsequence of $\hat{\mathfrak{F}}^N$. Then, for any $t\ge 0$, almost surely, as $N\to\infty$,*

$$\left(\mathbb{V}_0^N(t),\hat{\mathfrak{F}}^N\right)\to\left(Z(t),\hat{\mathfrak{F}}\right),$$

where

$$Z(t)=-\int_0^t\mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t\gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s.$$

Proof. By Taylor's formula we have

$$\mathbb{V}_0^N(t)=-Z^N(t)+\frac{1}{N}\sum_{k=1}^N r_k^N(t), \tag{6.13}$$

where

$$|r_k^N(t)|\le\frac{t}{2}\sqrt{N}\int_0^t\left(\overline{\mathfrak{F}}^N(s)-\overline{\mathfrak{F}}(s)\right)^2\mathrm{d}s,$$

and

$$Z^N(t)=\int_0^t\left(\frac{1}{N}\sum_{k=1}^N\gamma_{k,0}(t)\gamma_{k,0}(s)\exp\left(-\int_0^t\gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\hat{\mathfrak{F}}^N(s)\mathrm{d}s.$$

From Proposition 5.4, it follows that,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{1}{N}\sum_{k=1}^{N} r_k^N(t)\right|\right] \le T\sqrt{N}\int_0^T \mathbb{E}\left[\left(\overline{\mathfrak{F}}^N(s)-\overline{\mathfrak{F}}(s)\right)^2\right]\mathrm{d}s \le \frac{C_T}{\sqrt{N}}. \tag{6.14}$$

Hence $\frac{1}{N}\sum_{k=1}^{N} r_k^N(t) \to 0$ in probability.

On the other hand, for any fixed t, as $(\gamma_{k,0})_k$ are i.i.d by the law of large numbers in $\mathbf{D}$ it follows that, as $N\to\infty$,

$$\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,0}(t)\gamma_{k,0}(\cdot)\exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right) \to \mathbb{E}\left[\gamma_0(t)\gamma_0(\cdot)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right] \quad a.s.$$

in $\mathbf{D}$ and as $N\to\infty, \hat{\mathfrak{F}}^N \Rightarrow \hat{\mathfrak{F}}$ in $\mathbf{D}$ and $\hat{\mathfrak{F}}$ is continuous, it follows that,

$$\left(\frac{1}{N}\sum_{k=1}^{N}\gamma_{k,0}(t)\gamma_{k,0}(\cdot)\exp\left(-\int_0^t \gamma_{k,0}(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\hat{\mathfrak{F}}^N(\cdot) \to \mathbb{E}\left[\gamma_0(t)\gamma_0(\cdot)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(\cdot),$$

in law in $\mathbf{D}$. For any fixed t, since the mapping $f \to \int_0^t f(s)\mathrm{d}s$ is continuous from $\mathbf{D}$ into $\mathbb{R}$, as $N\to\infty$,

$$Z^N(t) \to \int_0^t \mathbb{E}\left[\gamma_0(t)\gamma_0(s)\exp\left(-\int_0^t \gamma_0(r)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right]\hat{\mathfrak{F}}(s)\mathrm{d}s.$$

This concludes the proof.

□

7. Approximation of the limit

In this section we establish the proof of Lemma 3.9. We organise this section as follows: In subsection 7.1 we give the proof of Lemma 3.9 with the proofs of the supporting lemmas in subsection 7.2.

7.1. Proof of Lemma 3.9

We first make the following observation related to the process $\hat{\mathfrak{S}}^N_{1,1}$.

Lemma 7.1. *For $t\ge 0$,*

$$\begin{aligned}\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\left(\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)-\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)\mathrm{d}s\\ = -\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\hat{\mathfrak{F}}^N(r)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)drds + r^N(t),\end{aligned}$$

where as $N\to\infty$, $r^N\to 0$ in probability in $\mathbf{D}$.

Proof. By Taylor's formula we have

$$\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\left(\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)-\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)\mathrm{d}s$$
$$=-\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\hat{\mathfrak{F}}^N(r)\exp\left(-\int_s^t\gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)drds+r^N(t),$$

where

$$|r^N(t)|\le\frac{\lambda_*\sqrt{N}}{2}\int_0^t\int_{\mathbf{D}}\left(\int_s^t\gamma(r-s)\left(\overline{\mathfrak{F}}^N(r)-\overline{\mathfrak{F}}(r)\right)\mathrm{d}r\right)^2\mu(\mathrm{d}\gamma)\mathrm{d}s. \tag{7.1}$$

By Hölder's inequality and Proposition 5.4 it follows that,

$$\mathbb{E}\left[\sup_{0\le t\le T}|r^N(t)|\right]\le\frac{T\lambda_*\sqrt{N}}{2}\int_0^T\int_s^T\mathbb{E}\left[\left(\overline{\mathfrak{F}}^N(r)-\overline{\mathfrak{F}}(r)\right)^2\right]drds\le\frac{T^3\lambda_*}{2\sqrt{N}}.$$

□

Hence, from (3.2), (3.10), (3.11) and Lemma 7.1, it follows that

$$\begin{aligned}
&\Psi_1(\hat{\mathfrak{F}}^N,\hat{\mathfrak{S}}^N,\hat{\mathfrak{S}}^N_{1,0},\hat{\mathfrak{S}}^N_2,\overline{\mathfrak{F}}^N,\overline{\mathfrak{S}}^N,\tilde{\mathfrak{S}}^N)(t)\\
&=\hat{\mathfrak{S}}^N(t)-\hat{\mathfrak{S}}^N_{1,0}(t)-\hat{\mathfrak{S}}^N_2(t)\\
&+\int_0^t\int_s^t\int_{\mathbf{D}}\gamma(t-s)\gamma(r-s)\exp\left(-\int_s^t\gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\mu(\mathrm{d}\gamma)\hat{\mathfrak{F}}^N(r)\overline{\mathfrak{F}}^N(s)\overline{\mathfrak{S}}^N(s)drds\\
&-\int_0^t\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\left(\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N_2(s)-\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N(s)-\hat{\mathfrak{F}}^N(s)\tilde{\mathfrak{S}}^N(s)\right)\mathrm{d}s\\
&:=\Xi^N_2(t)+r^N(t)+\Xi^N_3(t),
\end{aligned}$$

where r^N is given by (7.1),

$$\begin{aligned}\Xi^N_2(t)=\hat{\mathfrak{S}}^N_{1,1}(t)-\sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)&\left(\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)\right.\\
&\left.-\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)\mathrm{d}s,\end{aligned} \tag{7.2}$$

$$\Xi^N_3(t)=\hat{\mathfrak{S}}^N_{1,2}(t)-\int_0^t\int_{\mathbf{D}}\gamma(t-s)\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\sqrt{N}\left(\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s)\right)\mathrm{d}s, \tag{7.3}$$

and where we use the fact that,

$$\sqrt{N}(\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s))=\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N_2(s)-\overline{\mathfrak{F}}^N(s)\hat{\mathfrak{S}}^N(s)-\hat{\mathfrak{F}}^N(s)\tilde{\mathfrak{S}}^N(s).$$

On the other hand, from (3.1) and (3.9),

$$\begin{aligned}
&\Psi_2(\hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_2^N, \overline{\mathfrak{F}}^N, \overline{\mathfrak{S}}^N, \tilde{\mathfrak{S}}^N)(t)\\
&= \hat{\mathfrak{F}}^N(t) - \hat{\mathfrak{F}}_2^N(t) - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}^N(s)\overline{\mathfrak{F}}^N(s)\mathrm{d}s - \int_0^t \overline{\lambda}(t-s)\hat{\mathfrak{S}}_2^N(s)\overline{\mathfrak{F}}^N(s)\mathrm{d}s\\
&\qquad + \int_0^t \overline{\lambda}(t-s)\tilde{\mathfrak{S}}^N(s)\hat{\mathfrak{F}}^N(s)\mathrm{d}s\\
&= \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2}\int_{\Upsilon_k^N(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon_k^N(r^-)\vee\Upsilon_k(r^-)} \lambda(t-s)\mathrm{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&=: \Xi_4^N(t). \qquad (7.4)
\end{aligned}$$

To prove Lemma 3.9 it suffices to establish the following three lemmas, whose proofs are given in the next subsection.

Lemma 7.2. *Under Assumption 2.6, as $N \to +\infty$,*

$$\Xi_4^N \Rightarrow 0 \quad \text{in} \quad \mathbf{D}.$$

Lemma 7.3. *As $N \to +\infty$,*

$$\Xi_2^N \to 0 \quad \text{in} \quad \mathbf{D},$$

in probability.

Lemma 7.4. *As $N \to \infty$,*

$$\Xi_3^N \to 0 \quad \text{in} \quad \mathbf{D},$$

in probability.

7.2. Proofs of Lemmas 7.2–7.4

7.2.1. Proof of Lemma 7.2

We recall that

$$\Xi_4^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2}\int_{\Upsilon_k^N(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon_k^N(r^-)\vee\Upsilon_k(r^-)} \lambda(t-s)\mathrm{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u). \qquad (7.5)$$

As $(\overline{Q}_k)_k$ are i.i.d, by exchangeability and from Corollary 5.3, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left(\Xi_4^N(t)\right)^2\right] &= \mathbb{E}\left[\int_0^t \lambda(t-s)^2|\Upsilon_1^N(s)-\Upsilon_1(s)|\mathrm{d}s\right]\\
&\le (\lambda_*)^2\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\right]\mathrm{d}s \to 0 \text{ as } N\to+\infty.
\end{aligned}$$

To conclude, it suffices to prove tightness of $(\Xi_4^N)_N$. By the expression in (7.5), tightness of the sequence processes $(\Xi_4^N)_N$ can be deduced from the tightness of the following sequence processes since $\mathrm{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))$ does not modify the setting:

$$\mathbb{V}_1^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k^N(r^-)\wedge\Upsilon_k(r^-)}^{\Upsilon_k^N(r^-)\vee\Upsilon_k(r^-)} \lambda(t-s) Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u),$$
$$\mathbb{V}_2^N(t) = \sqrt{N} \int_0^t \overline{\lambda}(t-s) \left| \tilde{\Upsilon}^N(s) - \overline{\Upsilon}^N(s) \right| \mathrm{d}s,$$

where we recall that

$$\overline{\Upsilon}^N(t) = \frac{1}{N} \sum_{k=1}^N \Upsilon_k^N(t) \text{ and } \tilde{\Upsilon}^N(t) = \frac{1}{N} \sum_{k=1}^N \Upsilon_k(t).$$

The tightness of $(\mathbb{V}_1^N)_N$ will be established in Lemma 8.1 below. Hence it remains to prove the tightness of $(\mathbb{V}_2^N)_N$.

From (5.3) we have

$$\mathbb{E}\left[\sup_{0\le t\le T} \mathbb{V}_2^N(t)\right] \le \lambda_* \int_0^T \mathbb{E}\left[\left| \sqrt{N}\left(\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right)\right|\right] \mathrm{d}r \le \lambda_* C_T. \tag{7.6}$$

From Lemma 5.6, for any $0 \le s \le t \le T$,

$$\begin{aligned}
\left|\mathbb{V}_2^N(t) - \mathbb{V}_2^N(s)\right| &\le \lambda_*\sqrt{N} \int_s^t \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r + \sqrt{N} \int_0^s \left|\overline{\lambda}(t-r) - \overline{\lambda}(s-r)\right| \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r \\
&\le \lambda_*\sqrt{N} \int_s^t \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r + \sqrt{N}(t-s)^\alpha \int_0^s \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r \\
&\quad + \lambda_*\sqrt{N} \sum_{j=1}^{\ell-1} \int_0^s \left(F_j(t-r) - F_j(s-r)\right) \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r.
\end{aligned} \tag{7.7}$$

By Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le s<t<T, |t-s|\le\delta} \sqrt{N} \int_s^t \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right| \mathrm{d}r \ge \theta\right) \\
&\le \frac{N}{\theta^2} \mathbb{E}\left[\sup_{0\le s<t<T, |t-s|\le\delta} (t-s) \int_s^t \left|\tilde{\Upsilon}^N(r) - \overline{\Upsilon}^N(r)\right|^2 dr\right] \\
&\le \frac{\delta}{\theta^2} N \int_0^T \mathbb{E}\left[\left|\overline{\Upsilon}^N(r) - \tilde{\Upsilon}^N(r)\right|^2\right] \mathrm{d}r \\
&\le \frac{\delta C_T}{\theta^2},
\end{aligned} \tag{7.8}$$

where the last line follows from applying (5.7) with $k = 2$.

On the other hand,

$$\mathbb{P}\left(\sup_{0\le s<t<T,|t-s|\le\delta}(t-s)^{\alpha}\sqrt{N}\int_0^s\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\mathrm{d}r\ge\theta\right)\le\frac{\delta^{\alpha}}{\theta}\sqrt{N}\int_0^T\mathbb{E}\left[\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\right]\mathrm{d}r$$
$$\le\frac{C_T}{\theta}\delta^{\alpha},\tag{7.9}$$

where the last line follows from applying (5.7) with $k=1$.

Applying again (5.7) with $k=1$, by Markov's inequality and Assumption 2.5,

$$\mathbb{P}\left(\sup_{0\le s<t<T,|t-s|\le\delta}\sqrt{N}\sum_{j=1}^{\ell-1}\int_0^s\left(F_j(t-r)-F_j(s-r)\right)\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\mathrm{d}r\ge\theta\right)$$
$$\begin{aligned}&\le\frac{\ell\delta^{\rho}}{\theta}\sqrt{N}\int_0^T\mathbb{E}\left[\left|\tilde{\Upsilon}^N(r)-\overline{\Upsilon}^N(r)\right|\right]\mathrm{d}r\\&\le\frac{\ell C_T}{\theta}\delta^{\rho}.\end{aligned}\tag{7.10}$$

Thus from (7.7), (7.8), (7.9) and (7.10),

$$\lim_{\delta\to0}\limsup_N\mathbb{P}\left(w_T(\mathbb{V}_2^N,\delta)\ge\theta\right)=0,$$

combined with (7.6), it follows from Theorem A.1 that $(\mathbb{V}_2^N)_N$ is **C**-tight.

This concludes the proof of Lemma 7.2.

7.2.2. Proof of Lemma 7.3

We recall that

$$\hat{\mathfrak{S}}_{1,1}^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}-\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right)Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

Define

$$\Xi_{2,1}^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\chi_k^N(t).$$

where

$$\begin{aligned}\chi_k^N(t)=\int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(s^-)}&\Big\{\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)=0}-\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right)\\&-\int_{\mathbf{D}}\tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)-\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\mu(\mathrm{d}\tilde{\gamma})\Big\}Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).\end{aligned}$$

and

$$\Xi_{2,2}^N(t)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\chi_k^N(t),$$

where

$$\chi_k^N(t) = \int_0^t \int_0^{\Upsilon_k^N(s^-)} \int_{\mathbf{D}} \tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\mu(\mathrm{d}\tilde{\gamma}) \overline{Q}_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u).$$

From the expression (7.2), we rewrite Ξ_2^N as follows: $\Xi_2^N(t) = \Xi_{2,1}^N(t) + \Xi_{2,2}^N(t)$.

Hence to prove Lemma 7.3 we first prove that, for fixed t, as $N \to \infty, \Xi_{2,1}^N(t) \to 0$ and $\Xi_{2,2}^N(t) \to 0$ in probability. Therefore, we require the following two Lemmas.

Lemma 7.5. *For any $t \geq 0$, as $N \to +\infty$,*

$$\mathbb{E}\left[\left(\Xi_{2,1}^N(t)\right)^2\right] \to 0.$$

Proof. By exchangeability we have

$$\mathbb{E}\left[\left(\Xi_{2,1}^N(t)\right)^2\right] = \mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right]. \tag{7.11}$$

Set

$$\begin{aligned}\tilde{\chi}_k^N(t) = \int_0^t \int_{\mathbf{D}^2} \int_0^{\Upsilon_k^N(s^-)} &\Big\{\gamma(t-s)\left(\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N)=0} - \mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right) \\ &- \int_{\mathbf{D}} \tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}_{(1,2)}^N(r)\mathrm{d}r\right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\mu(\mathrm{d}\tilde{\gamma})\Big\} Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u),\end{aligned}$$

and

$$\begin{aligned}\hat{\chi}_k^N(t) := \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} &\Big\{\gamma(t-s)\left|\mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N)=0} - \mathbb{1}_{P_k(s,t,\gamma,\overline{\mathfrak{F}})=0}\right| \\ &+ \int_{\mathbf{D}} \tilde{\gamma}(t-s)\left|\exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}_{(1,2)}^N(r)\mathrm{d}r\right) - \exp\left(-\int_s^t \tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right|\mu(\mathrm{d}\tilde{\gamma})\Big\} Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u).\end{aligned}$$

For $0 \leq s \leq t$, and $\gamma \in D$, let $E_k^N(s,\gamma)$ denote the event

$$E_k^N(s,\gamma) = \left\{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N - \frac{Y}{N}\right) = P_k\left(s,t,\gamma,\overline{\mathfrak{F}}_{(1,2)}^N + \frac{Y}{N}\right)\right\},$$

where Y is a geometric random variable independent from Q_1 and Q_2 chosen as in (5.8), such that almost surely for all $t \in [0,T]$,

$$\overline{\mathfrak{F}}_{(1,2)}^N(t) - \frac{Y}{N} \leq \overline{\mathfrak{F}}^N(t) \leq \overline{\mathfrak{F}}_{(1,2)}^N(t) + \frac{Y}{N}.$$

We denote

$$\mathcal{F}_t^N = \sigma\{(\lambda_{k,i}(\varsigma_k^N(\cdot)))_{1\leq k\leq N, i\leq A_k^N(t)}, (\gamma_{k,i}(\varsigma_k^N(\cdot)))_{1\leq k\leq N, i\leq A_k^N(t)}, (Q_k|_{[0,t]\times E})_{k\leq N}\}.$$

As

$$P_k(s,t,\gamma,\overline{\mathfrak{F}}^N)\mathbb{1}_{E_k^N(s,\gamma)} = P_k\left(s,t,\gamma,\overline{\mathfrak{F}}^N_{(1,2)}\right)\mathbb{1}_{E_k^N(s,\gamma)},$$

and $\Upsilon_k^N(s^-) \leq \lambda_*$, it follows that

$$\left|\chi_k^N(t) - \tilde{\chi}_k^N(t)\right| \leq \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} \mathbb{1}_{(E_k^N(s,\gamma))^c} Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u) + \frac{Yt}{N}B_k(t), \tag{7.12}$$

where

$$B_k(t) = \int_0^t \int_{\mathbf{D}^2} \int_0^{\lambda_*} Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

Moreover, as $Q_k|_{]s,t]}$ is independent of $(\mathcal{F}_s^N,\, \overline{\mathfrak{F}}^N_{(1,2)},\, Y,\, Q_\ell)$, for $\ell \neq k$ and $k \in \{1,2\}$,

$$\mathbb{P}\left((E_k^N(s,\gamma))^c \middle| \mathcal{F}_s^N,\, \overline{\mathfrak{F}}^N_{(1,2)},\, Y, Q_\ell\right) \leq \frac{2tY}{N}. \tag{7.13}$$

We have

$$\begin{aligned}\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t) - \tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right] &= \mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right] + \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right] \\ &\quad + \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]. \end{aligned} \tag{7.14}$$

Note that

$$|\chi_k^N| \leq \hat{\chi}_k^N \text{ and } |\tilde{\chi}_k^N| \leq \hat{\chi}_k^N.$$

As Q_2 is independent of $(\overline{\mathfrak{F}}^N_{(1,2)},\, Y,\, Q_1)$, from (7.12), (7.13) and Theorem 5.1,

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| &\leq \lambda_*\mathbb{E}\left[\hat{\chi}_1^N(t)\int_0^t \int_{\mathbf{D}} \mathbb{P}\left((E_2^N(s,\gamma))^c \middle| \mathcal{F}_s^N,\, \overline{\mathfrak{F}}^N_{(1,2)},\, Y, Q_1\right)\mu(\mathrm{d}\gamma)\mathrm{d}s\right] \\ &\quad + \frac{t}{N}\mathbb{E}\left[\hat{\chi}_1^N(t)YB_2(t)\right] \\ &\leq \frac{3t^2\lambda_*}{N}\mathbb{E}\left[Y\hat{\chi}_1^N(t)\right],\end{aligned}$$

where as $N \to \infty$, $\mathbb{E}\left[Y\hat{\chi}_1^N(t)\right] \to 0$, because $\hat{\chi}_1^N(t) \to 0$ in probability as $N \to \infty$ and $\hat{\chi}_1^N$ is bounded.

Thus it follows that

$$\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| = o\left(\frac{1}{N}\right).$$

Similarly we show that

$$\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\right| = o\left(\frac{1}{N}\right).$$

From (7.12),

$$
\begin{aligned}
&\mathbb{E}\left[\left|\chi_1^N(t)-\tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t)-\tilde{\chi}_2^N(t)\right|\right]\\
&\qquad\le \mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_1^N(s,\gamma))^c}Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_2^N(s,\gamma))^c}Q_2(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right]\\
&\qquad+\frac{t}{N}\mathbb{E}\left[YB_2(t)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_1^N(s,\gamma))^c}Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right]\\
&\qquad+\frac{t}{N}\mathbb{E}\left[YB_1(t)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_2^N(s,\gamma))^c}Q_2(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right]+\frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right].
\end{aligned}
$$

Conditionally on $\overline{\mathfrak{F}}^N_{(1,2)}$ and Y, the terms inside the first expectation above are independent because Q_1 and Q_2 are independent, and from (7.13) and Theorem 5.1, it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_1^N(s,\gamma))^c}Q_1(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{(E_2^N(s,\gamma))^c}Q_2(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right]\\
&\qquad=\lambda_*^2\mathbb{E}\left[\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_1^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)},\,Y\right)\mu(\mathrm{d}\gamma)\mathrm{d}s\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_2^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)},\,Y\right)\mu(\mathrm{d}\gamma)\mathrm{d}s\right]\\
&\qquad\le\frac{4\lambda_*^2t^4}{N^2}\mathbb{E}\left[Y^2\right].
\end{aligned}
$$

Hence using again (7.12) and the fact that Q_k is independent of $(\overline{\mathfrak{F}}^N_{(1,2)},\,Y,\,Q_\ell)$, with $k\neq\ell$ and $k,\ell\in\{1,2\}$, we deduce that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\chi_1^N(t)-\tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t)-\tilde{\chi}_2^N(t)\right|\right]\\
&\qquad\le\frac{4\lambda_*^2t^4}{N^2}\mathbb{E}\left[Y^2\right]+\frac{\lambda_*t}{N}\mathbb{E}\left[YB_2(t)\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_1^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)},\,Y,Q_2\right)\mu(\mathrm{d}\gamma)\mathrm{d}s\right]\\
&\qquad\quad+\frac{\lambda_*t}{N}\mathbb{E}\left[YB_1(t)\int_0^t\int_{\mathbf{D}}\mathbb{P}\left((E_2^N(s,\gamma))^c\middle|\overline{\mathfrak{F}}^N_{(1,2)},\,Y,Q_1\right)\mu(\mathrm{d}\gamma)\mathrm{d}s\right]+\frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right]\\
&\qquad\le\frac{4\lambda_*^2t^4}{N^2}\mathbb{E}\left[Y^2\right]+\frac{2\lambda_*t^2}{N^2}\mathbb{E}\left[YB_2(t)\right]+\frac{2\lambda_*t^2}{N^2}\mathbb{E}\left[YB_1(t)\right]+\frac{t^2}{N^2}\mathbb{E}\left[Y^2B_1(t)B_2(t)\right].
\end{aligned}
$$

Hence,

$$
\left|\mathbb{E}\left[\left(\chi_1^N(t)-\tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t)-\tilde{\chi}_2^N(t)\right)\right]\right|=o\left(\frac{1}{N}\right).
$$

In conclusion, coming back to (7.14)

$$
\left|\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)-\tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right]\right|=o\left(\frac{1}{N}\right).\tag{7.15}
$$

On the other hand, since Q_1, Q_2 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent,

$$\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right] = 0.$$

Hence from (7.11) and (7.15), it follows that

$$\begin{aligned}\mathbb{E}\left[\left(\Xi^N_{2,1}(t)\right)^2\right] &\le \mathbb{E}\left[\left(\hat{\chi}^N_1(t)\right)^2\right] + (N-1)\left|\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t) - \tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right]\right| \\ &= \mathbb{E}\left[\left(\hat{\chi}^N_1(t)\right)^2\right] + o\left(1\right).\end{aligned}$$

However, as Q_1 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, from Theorem 5.1

$$\begin{aligned}\mathbb{E}\left[\hat{\chi}^N_1(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right] &= \lambda_*\int_0^t\int_{\mathbf{D}}\gamma(t-s)\mathbb{E}\left[\left|\mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}}^N_{(1,2)})=0} - \mathbb{1}_{P_1(s,t,\gamma,\overline{\mathfrak{F}})=0}\right|\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mu(\mathrm{d}\gamma)\mathrm{d}s \\ &+ \lambda_*\int_0^t\int_{\mathbf{D}}\tilde{\gamma}(t-s)\mathbb{E}\left[\left|\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N_{(1,2)}(r)\mathrm{d}r\right) - \exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right|\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mu(\mathrm{d}\tilde{\gamma})\mathrm{d}s.\end{aligned}$$

Hence since $\overline{\mathfrak{F}}^N_{(1,2)} \to \overline{\mathfrak{F}}$, as $N\to\infty$, and $\hat{\chi}^N_1$ is bounded, it follows that as $N\to\infty$, $\mathbb{E}\left[\hat{\chi}^N_1(t)\right]\to 0$ and $\mathbb{E}\left[(\hat{\chi}^N_1(t))^2\right]\to 0$. ☐

Lemma 7.6. *For any $t\ge 0$, as $N\to\infty$,*

$$\mathbb{E}\left[|\Xi^N_{2,2}(t)|\right]\to 0.$$

Proof. Let

$$\tilde{\chi}^N_k(t) = \int_0^t\int_0^{\Upsilon^N_k(s^-)}\int_{\mathbf{D}}\tilde{\gamma}(t-s)\left(\exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}^N_{(1,2)}(r)\mathrm{d}r\right) - \exp\left(-\int_s^t\tilde{\gamma}(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\mu(\mathrm{d}\tilde{\gamma}) \\ \overline{Q}_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

As in Lemma 7.5 we have

$$\left|\chi^N_k(t) - \tilde{\chi}^N_k(t)\right| \le \frac{tY}{N}\left(B_k(t)+\lambda_*\right).$$

On the other hand, since Q_1, Q_2 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, it follows that

$$\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right] = 0.$$

The Lemma follows by exchangeability and the fact that as $N\to\infty$, $\tilde{\chi}^N_k(t)\to 0$, proceeding as in (7.11) and (7.14) ☐

We can now conclude the proof of Lemma 7.3.

Proof of Lemma 7.3. We recall that

$$\Xi^N_2(t) = \hat{\mathfrak{G}}^N_{1,1}(t) - \sqrt{N}\int_0^t\int_{\mathbf{D}}\gamma(t-s)\left(\exp\left(-\int_s^t\gamma(r-s)\overline{\mathfrak{F}}^N(r)\mathrm{d}r\right)\right.$$

$$
- \exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)\mathrm{d}s
$$

$$
:= \Xi_{2,1}^N(t) + \Xi_{2,2}^N(t).
$$

From Lemmas 7.5 and 7.6, for each $t \geq 0$, as $N \to \infty$, $\Xi_2^N(t) \to 0$ in probability. As a consequence, for all $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_n$,

$$
\left(\Xi_2^N(t_0), \Xi_2^N(t_1), \cdots, \Xi_2^N(t_n)\right) \to (0, 0, \cdots, 0)\,.
$$

Note that, from Lemma 7.1, it follows that, $\Xi_2^N(t) = \hat{\mathfrak{S}}_{1,1}^N(t) - r^N(t)$, where

$$
\mathbb{V}_3^N(t) = \int_0^t \int_s^t \int_{\mathbf{D}} \gamma(t-s)\gamma(r-s)\hat{\mathfrak{F}}^N(r)\exp\left(-\int_s^t \gamma(u-s)\overline{\mathfrak{F}}(u)\mathrm{d}u\right)\overline{\Upsilon}^N(s)\mu(\mathrm{d}\gamma)drds. \tag{7.16}
$$

To conclude it remains to show the tightness of $(\Xi_2^N)_N$, but from Lemma 3.7 $(\hat{\mathfrak{S}}_{1,1}^N)_N$ is **C**-tight and as r^N is also **C**-tight (see Lem. 7.1), it remains to show the tightness of $\mathbb{V}_3^N$. Since the pair $(\hat{\mathfrak{F}}^N, \overline{\Upsilon}^N)$ is tight in $\mathbf{D}^2$, the result follows using the continuous mapping theorem.

This concludes the proof of Lemma 7.3. □

7.2.3. Proof of Lemma 7.4

We recall that

$$
\hat{\mathfrak{S}}_{1,2}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2}\int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)} \gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}\mathrm{sign}(\Upsilon_k^N(s^-) - \Upsilon_k(s^-))Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),
$$

and from (7.3) we also recall

$$
\Xi_3^N(t) = \hat{\mathfrak{S}}_{1,2}^N(t) - \int_0^t \int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\sqrt{N}\left(\overline{\Upsilon}^N(s) - \tilde{\Upsilon}^N(s)\right)\mathrm{d}s. \tag{7.17}
$$

We define for any $k \in \mathbb{N}$,

$$
\Delta_k^N(t) = \int_0^t \int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)} Q_k(\mathrm{d}s,\mathrm{d}u)\,. \tag{7.18}
$$

and let $(\vartheta_{k,i}^N)_{k,i}$ be such that

$$
\Delta_k^N(t) = \sum_{i\geq 1}\mathbb{1}_{\vartheta_{k,i}^N \leq t}.
$$

We note that, for some i.i.d $(\gamma_{k,i},\, i \geq 1)$,

$$
\hat{\mathfrak{S}}_{1,2}^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N\sum_{i\geq 1}\gamma_{k,i}(t-\vartheta_{k,i}^N)\mathbb{1}_{P_k(\vartheta_{k,i}^N,t,\gamma_{k,i},\overline{\mathfrak{F}})=0}\mathrm{sign}(\Upsilon_k^N(\vartheta_{k,i}^N) - \Upsilon_k(\vartheta_{k,i}^N))\mathbb{1}_{\vartheta_{k,i}^N\leq t}\,. \tag{7.19}
$$

We recall that

$$\overline{\Upsilon}^N(t) = \overline{\mathfrak{S}}^N(t)\overline{\mathfrak{F}}^N(t) \text{ and } \tilde{\Upsilon}^N(t) = \frac{1}{N}\sum_{k=1}^N \gamma_{k,A_k(t)}(\varsigma_k(t))\overline{\mathfrak{F}}(t).$$

We set

$$\mathcal{G}_t^N = \sigma\{(\lambda_{k,i})_{1\le k\le N, i<A_k^N(t)},\, (\gamma_{k,i})_{1\le k\le N, i<A_k^N(t)},\, (Q_k|_{[0,t]\times E})_{k\le N}\}. \tag{7.20}$$

Note that from Lermma 3.7 Ξ_3^N is tight.

We want to show that, as N tends to ∞, $\Xi_3^N \to 0$ in $\mathbf{D}$ in probability. To do this, we define

$$\Xi_{3,1}^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N\sum_{i\ge 1}\chi_{k,i}^N(t), \tag{7.21}$$

with

$$\begin{aligned}\chi_{k,i}^N(t) = \bigg(\gamma_{k,i}(t-\vartheta_{k,i}^N)\mathbb{1}_{P_k(\vartheta_{k,i}^N, t, \gamma_{k,i}, \overline{\mathfrak{F}})=0} - \int_{\mathbf{D}}\gamma(t-\vartheta_{k,i}^N)\exp\left(-\int_{\vartheta_{k,i}^N}^t \gamma(r-\vartheta_{k,i}^N)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\bigg)& \\ \times \operatorname{sign}(\Upsilon_k^N(\vartheta_{k,i}^N) - \Upsilon_k(\vartheta_{k,i}^N))\mathbb{1}_{\vartheta_{k,i}^N\le t}\,.& \end{aligned} \tag{7.22}$$

We establish the following Lemma.

Lemma 7.7. *For any $t\ge 0$, as $N\to\infty$,*

$$\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right] \to 0.$$

Proof. For all $k,i\in\mathbb{N}$, $\mathbb{E}\left[\chi_{k,i}^N(t)\middle|\mathcal{G}_{\vartheta_{k,i}^N}^N\right] = 0$. Moreover, for any $k\ne\ell$,

$$\begin{aligned}\mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\right] &= \mathbb{E}\left[\mathbb{1}_{\vartheta_{k,i}^N<\vartheta_{\ell,j}^N}\mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\middle|\mathcal{G}_{\vartheta_{\ell,j}^N}^N, Q_k\right]\right] + \mathbb{E}\left[\mathbb{1}_{\vartheta_{\ell,j}^N<\vartheta_{k,i}^N}\mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\middle|\mathcal{G}_{\vartheta_{k,i}^N}^N, Q_\ell\right]\right] \\ &= \mathbb{E}\left[\chi_{k,i}^N(t)\mathbb{1}_{\vartheta_{k,i}^N<\vartheta_{\ell,j}^N}\mathbb{E}\left[\chi_{\ell,j}^N(t)\middle|\mathcal{G}_{\vartheta_{\ell,j}^N}^N, Q_k\right]\right] + \mathbb{E}\left[\chi_{\ell,j}^N(t)\mathbb{1}_{\vartheta_{\ell,j}^N<\vartheta_{k,i}^N}\mathbb{E}\left[\chi_{k,i}^N(t)\middle|\mathcal{G}_{\vartheta_{k,i}^N}^N, Q_\ell\right]\right] \\ &= 0\end{aligned} \tag{7.23}$$

because $\chi_{\ell,j}^N$ and Q_k are independent and similarly for $\chi_{k,i}^N$ and Q_ℓ. Hence, as $\left|\chi_{k,i}^N(t)\right| \le \mathbb{1}_{\vartheta_{k,i}^N\le t}$, from (7.23) and exchangeability,

$$
\begin{aligned}
\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right] &= \frac{1}{N}\sum_{k=1}^N \mathbb{E}\left[\left(\sum_{i\geq 1}\chi_{k,i}^N(t)\right)^2\right] + \frac{N-1}{2}\sum_{1\leq k<\ell\leq N}\sum_{i,j\geq 1}\mathbb{E}\left[\chi_{k,i}^N(t)\chi_{\ell,j}^N(t)\right] \\
&\leq \frac{1}{N}\sum_{k=1}^N \mathbb{E}\left[\left(\Delta_k^N(t)\right)^2\right] \\
&= \mathbb{E}\left[\left(\Delta_1^N(t)\right)^2\right].
\end{aligned}
$$

where Δ_k^N is given by (7.18).

However from Corollary 5.3,

$$
\begin{aligned}
\mathbb{E}\left[\left(\Delta_1^N(t)\right)^2\right] &\leq 2\left(\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\right]\mathrm{d}s + \mathbb{E}\left[\left(\int_0^t \left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\mathrm{d}s\right)^2\right]\right) \\
&\leq 2\left(\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|\right]\mathrm{d}s + t\int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(s)-\Upsilon_1(s)\right|^2\right]\mathrm{d}s\right) \\
&\leq \frac{C_t}{\sqrt{N}}.
\end{aligned}
$$

Consequently

$$
\mathbb{E}\left[\Xi_{3,1}^N(t)^2\right] \leq \frac{C_t}{\sqrt{N}}.
$$

□

Note that from (7.17),

$$
\begin{aligned}
&\Xi_3^N(t) - \Xi_{3,1}^N(t) \\
&= \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)} \int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\mathrm{sign}(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s,\mathrm{d}u).
\end{aligned}
$$

Hence, as $(Q_k)_k$ are i.i.d.,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)} \int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\right.\right. \\
&\qquad\qquad\left.\left.\times sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s,\mathrm{d}u)\right)^2\right] \\
&= \frac{1}{N}\sum_{k=1}^N \mathbb{E}\left[\left(\int_0^t \int_{\Upsilon_k^N(s^-)\wedge\Upsilon_k(s^-)}^{\Upsilon_k^N(s^-)\vee\Upsilon_k(s^-)} \int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\right.\right. \\
&\qquad\qquad\left.\left.\times sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))\overline{Q}_k(\mathrm{d}s,\mathrm{d}u)\right)^2\right] \\
&= \int_0^t \left(\int_{\mathbf{D}} \gamma(t-s)\exp\left(-\int_s^t \gamma(r-s)\overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mu(\mathrm{d}\gamma)\right)^2 \mathbb{E}\left[\left|\overline{\Upsilon}^N(s)-\tilde{\Upsilon}^N(s)\right|\right]\mathrm{d}s.
\end{aligned}
$$

From Corollary 5.5 and Lemma 7.7 it follows that, as N tends to ∞, $\Xi_3^N(t) \to 0$ in probability for any $t \geq 0$. This implies the finite dimensional convergence and as $(\Xi_3^N)_N$ is tight in $\mathbf{D}$, it follows that as $N \to \infty, \Xi_3^N \to 0$ in law in $\mathbf{D}$ and hence in probability.

8. Proof of Tightness

In this section we prove Lemma 3.7.

To prove Lemma 3.7, since $\mathbf{D}$ is separable, it suffices to establish that each component of $\left(\hat{\mathfrak{S}}^N, \hat{\mathfrak{F}}^N, \hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N, \hat{\mathfrak{S}}_{1,0}^N\right)$ is tight in $\mathbf{D}$. Moreover, from Lemma 3.1 the pair $(\hat{\mathfrak{S}}_2^N, \hat{\mathfrak{F}}_2^N)$ is C-tight in $\mathbf{D}^2$, since it converges in $\mathbf{D}^2$ to a continuous limit. The tightness of the rest follows from the following Lemmas.

Lemma 8.1. *$\hat{\mathfrak{F}}^N$ is* $\mathbf{C}$*-tight.*

Lemma 8.2. *$\hat{\mathfrak{S}}_{1,2}^N$ is* $\mathbf{C}$*-tight.*

Lemma 8.3. *$\hat{\mathfrak{S}}_{1,1}^N$ is* $\mathbf{C}$*-tight.*

Lemma 8.4. *$\hat{\mathfrak{S}}_{1,0}^N$ is* $\mathbf{C}$*-tight.*

8.1. Proof of Lemma 8.1

Since $(\hat{\mathfrak{F}}_2^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$, it suffices to prove that $(\hat{\mathfrak{F}}_1^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$. By the expression in (3.9), that claim can be deduced from the tightness of the following sequence of processes

$$\mathbb{V}_4^N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(s-)\wedge\Upsilon_k^N(s-)}^{\Upsilon_k(s-)\vee\Upsilon_k^N(s-)} \lambda(t-s) Q_k(\mathrm{d}s, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u).$$

From Assumption 2.4 and Lemma 5.6, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned}
|\mathbb{V}_4^N(t) - \mathbb{V}_4^N(s)| \leq & \frac{\lambda_*}{\sqrt{N}} \sum_{k=1}^N \int_s^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \\
& + \frac{\ell(t-s)^\alpha}{\sqrt{N}} \sum_{k=1}^N \int_0^s \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \\
& + \frac{\lambda_*}{\sqrt{N}} \sum_{j=1}^{\ell-1} \sum_{k=1}^N \int_0^s \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{s-r<\xi^j\leq t-r} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u).
\end{aligned} \tag{8.1}$$

By Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\ell\delta^\alpha}{\sqrt{N}} \sum_{k=1}^N \int_0^T \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \geq \theta\right) \\
&\qquad \leq \frac{1}{\theta^2} \mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}} \sum_{k=1}^N \int_0^T \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right)^2\right].
\end{aligned}$$

However, by exchangeability and Hölder's inequality, it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^T\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\qquad\le 2\mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^T\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \overline{Q}_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\qquad\quad + 2\mathbb{E}\left[\left(\frac{\ell\delta^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^T\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|\mathrm{d}r\right)^2\right]\\
&\qquad\le 2\ell^2\delta^{2\alpha}\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]\mathrm{d}r + 2\ell^2\delta^{2\alpha}N\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|^2\right]\mathrm{d}r\\
&\qquad\le 2\ell^2\delta^{2\alpha}\int_0^T\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]\mathrm{d}r + 2C_T\ell^2\delta^{2\alpha},
\end{aligned}
$$

where the second term in the last line follows from applying (5.7) with $k=2$. Note that from Corollary 5.3 the first term in the last line tends to 0 as $N\to\infty$, while the second term is independent of N.

Then, thanks to $\alpha>\frac{1}{2}$ (Asm. 2.5),

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\le v\le\delta}\frac{\ell v^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^T\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\theta\right)=0. \tag{8.2}
$$

On the other hand, for all $0<t\le T$, since $(Q_k)_k$ are i.i.d, by Hölder's inequality and exchangeability, we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\frac{\lambda_*}{\sqrt{N}}\sum_{k=1}^N\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)>\theta\right)\\
&\le\frac{1}{\theta^2}\mathbb{E}\left[\left(\frac{\lambda_*}{\sqrt{N}}\sum_{k=1}^N\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\le\frac{2\lambda_*^2}{\theta^2}\left\{\mathbb{E}\left[\frac{1}{N}\sum_{k=1}^N\left(\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \overline{Q}_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\right.\\
&\qquad\qquad\left.+\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_t^{t+\delta}\left|\Upsilon_k^N(r)-\Upsilon_k(r)\right|\mathrm{d}r\right)^2\right]\right\}\\
&\le\frac{2\lambda_*^2}{\theta^2}\left\{\mathbb{E}\left[\int_t^{t+\delta}\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\mathrm{d}r\right]+N\mathbb{E}\left[\left(\int_t^{t+\delta}\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\mathrm{d}r\right)^2\right]\right\}\\
&\le\frac{2\lambda_*^2}{\theta^2}\left\{\int_0^{T+\delta}\mathbb{E}\left[\left|\Upsilon_1^N(r)-\Upsilon_1(r)\right|\right]\mathrm{d}r+C_T\delta^2\right\},
\end{aligned}
$$

where the last line follows from Hölder's inequality and applying (5.7) with $k = 2$. Note that the second term is independent of N and from Corollary 5.3 the first term tends to 0 as $N \to \infty$. Consequently,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{0\le t\le T} \frac{1}{\delta}\mathbb{P}\left(\sup_{0\le v\le\delta} \frac{\lambda_*}{\sqrt{N}} \sum_{k=1}^{N} \int_t^{t+v} \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) > \epsilon\right) = 0. \tag{8.3}$$

Moreover, by Markov's inequality,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N}\sum_{j=1}^{\ell-1} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{t-r<\xi^j\le t+\delta-r} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \ge \theta\right) \\
&\qquad\le \frac{1}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N}\sum_{j=1}^{\ell-1} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{t-r<\xi^j\le t+\delta-r} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right)^2\right] \\
&\qquad\le \frac{2}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N}\sum_{j=1}^{\ell-1} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{t-r<\xi^j\le t+\delta-r} \overline{Q}_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right)^2\right] \\
&\qquad\quad + \frac{2}{\theta^2}\mathbb{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{j=1}^{\ell-1}\sum_{k=1}^{N} \int_0^t (F_j(s+\delta-r) - F_j(t-r)) \left|\Upsilon_k^N(r) - \Upsilon_k(r)\right| \mathrm{d}r\right)^2\right].
\end{aligned}$$

Consequently, since $(Q_k)_k$ are i.i.d., from (2.16) and Hölder's inequality and from exchangeability, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N}\sum_{j=1}^{\ell-1} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{t-r<\xi^j\le t+\delta-r} Q_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u) \ge \theta\right) \\
&\le \frac{2}{N\theta^2} \sum_{k=1}^{N} \mathbb{E}\left[\left(\sum_{j=1}^{\ell-1} \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)} \mathbb{1}_{t-r<\xi^j\le t+\delta-r} \overline{Q}_k(\mathrm{d}r, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right)^2\right] \\
&\quad + \frac{2\ell^2\delta^{2\rho}}{\theta^2} \sum_{k=1}^{N} \mathbb{E}\left[\left(\int_0^t \left|\Upsilon_k^N(r) - \Upsilon_k(r)\right| \mathrm{d}r\right)^2\right] \\
&\le \frac{2}{\theta^2} \sum_{j=1}^{\ell-1} \int_0^t (F_j(s+\delta-r) - F_j(t-r))\, \mathbb{E}\left[\left|\Upsilon_1^N(r) - \Upsilon_1(r)\right|\right] \mathrm{d}r + \frac{2\ell^2\delta^{2\rho}}{\theta^2} N \int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(r) - \Upsilon_1(r)\right|^2\right] \mathrm{d}r \\
&\le \frac{2\ell\delta^\rho}{\theta^2} \int_0^t \mathbb{E}\left[\left|\Upsilon_1^N(r) - \Upsilon_1(r)\right|\right] \mathrm{d}r + \frac{2C_T\ell^2\delta^{2\rho}}{\theta^2}.
\end{aligned}$$

where the last line follows from Hölder's inequality and applying (5.7) with $k = 2$. Note that the second term is independent of N and from Corollary 5.3 the first term tend to 0 as $N \to \infty$.

Therefore, as $\rho > \frac{1}{2}$ (Asm. 2.5),

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\le v\le\delta}\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{t-r<\xi^j\le t+v-r}Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\theta\right)=0. \tag{8.4}$$

Thus, from (8.1), (8.2), (8.3) and (8.4), we deduce that

$$\lim_{\delta\to 0}\limsup_{N}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\le v\le\delta}|\mathbb{V}_4^N(t+v)-\mathbb{V}_4^N(t)|\ge\theta\right)=0,$$

and thanks to Lemme A.2, $(\mathbb{V}_4^N)_N$ is $\mathbf{C}-$tight in $\mathbf{D}$.

8.2. Proof of Lemma 8.2

We recall that

$$\hat{\mathfrak{S}}_{1,2}^N(t) := \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)}\gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}sign(\Upsilon_k^N(s^-)-\Upsilon_k(s^-))Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u). \tag{8.5}$$

By the expression in (8.5), tightness of the processes $(\hat{\mathfrak{S}}_{1,2}^N)_N$ can be deduced from the tightness of the following processes

$$\mathbb{V}_5^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(s^-)\wedge\Upsilon_k^N(s^-)}^{\Upsilon_k(s^-)\vee\Upsilon_k^N(s^-)}\gamma(t-s)\mathbb{1}_{P_k\left(s,t,\gamma,\overline{\mathfrak{F}}\right)=0}Q_k(\mathrm{d}s,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u). \tag{8.6}$$

For all $0\le s\le t\le T$,

$$\begin{aligned}
&\left|\mathbb{V}_5^N(t)-\mathbb{V}_5^N(s)\right|\\
&\le\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_s^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&+\frac{\ell(t-s)^\alpha}{\sqrt{N}}\sum_{k=1}^{N}\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&+\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\mathbb{1}_{s-r<\zeta^j\le t-r}Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&+\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^s\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\left(\mathbb{1}_{P_k\left(r,s,\gamma,\overline{\mathfrak{F}}\right)=0}-\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),
\end{aligned}$$

where we use Lemma 5.6.

From (8.2), (8.3) and (8.4) it remains to prove that, as $\delta\to 0$,

$$\limsup_{N}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(r^-)\wedge\Upsilon_k^N(r^-)}^{\Upsilon_k(r^-)\vee\Upsilon_k^N(r^-)}\left(\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}-\mathbb{1}_{P_k\left(r,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0}\right)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0. \tag{8.7}$$

We recall that for $0 \leq t \leq T$,

$$\mathcal{F}_t^N = \sigma\left\{ (\lambda_{k,i}, \gamma_{k,i})_{\substack{1\leq k\leq N \\ 1\leq i\leq A_k^N(t)}}, \ (Q_k|_{[0,t]})_{1\leq k\leq N} \right\}.$$

We set

$$D_k^N(t) = \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(v^-)\wedge\Upsilon_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Upsilon_k^N(v^-)} Q_k(, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u),$$

and $\overline{D}_k^N$ is given by the same expression as D_k^N but where we replace Q_k by its compensated measure.

Since

$$0 \leq \mathbb{1}_{P_k\left(v,t,\gamma,\overline{\mathfrak{F}}\right)=0} - \mathbb{1}_{P_k\left(v,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0} \leq \int_t^{t+\delta}\int_0^{\lambda_*} Q_k(\mathrm{d}u_1, \mathrm{d}u_2) = C_k(t, t+\delta),$$

by exchangeability and the fact that $(Q_k)_k$ are independent and $C_k(t,t+\delta)$ and $Q_k|_{[0,t]}$ are independent, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t \int_{\mathbf{D}^2} \int_{\Upsilon_k(v^-)\wedge\Upsilon_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Upsilon_k^N(v^-)} \left(\mathbb{1}_{P_k\left(v,t,\gamma,\overline{\mathfrak{F}}\right)=0} - \mathbb{1}_{P_k\left(v,t+\delta,\gamma,\overline{\mathfrak{F}}\right)=0}\right) Q_k(\mathrm{d}v, \mathrm{d}\lambda, \mathrm{d}\gamma, \mathrm{d}u)\right)^2\right] \\
&\quad \leq \mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N C_k(t,t+\delta) D_k^N(t)\right)^2\right] \\
&\quad = \mathbb{E}\left[C_1^2(t,t+\delta)\left(D_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[C_1(t,t+\delta)C_2(t,t+\delta)D_1^N(t)D_2^N(t)\right] \\
&\quad = \mathbb{E}\left[C_1^2(t,t+\delta)\right]\mathbb{E}\left[\left(D_1^N(t)\right)^2\right] + (N-1)\mathbb{E}\left[C_1(t,t+\delta)\right]\mathbb{E}\left[C_2(t,t+\delta)\right]\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \\
&\quad \leq 2(\lambda_*\delta + \lambda_*^2\delta^2)\left(\mathbb{E}\left[\left(\overline{D}_1^N(t)\right)^2\right] + \mathbb{E}\left[\left(\int_0^t |\Upsilon_1^N(v) - \Upsilon_1(v)|\mathrm{d}v\right)^2\right]\right) + \lambda_*^2\delta^2 N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \\
&\quad \leq 2(\lambda_*\delta + \lambda_*^2\delta^2)\left(\int_0^t \mathbb{E}\left[|\Upsilon_1(v) - \Upsilon_1^N(v)|\right]\mathrm{d}v + T\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v) - \Upsilon_1(v)|^2\right]\mathrm{d}v\right) \\
&\qquad + \lambda_*^2\delta^2 N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right],
\end{aligned} \tag{8.8}$$

where we use Hölder's inequality in the last inequality.

At this stage, we admit the following Lemma holds (and we will show this immediately below):

Lemma 8.5. *There exists $C > 0$ such that for all N.*

$$N\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] \leq C, \tag{8.9}$$

for some $C > 0$ independent of N.

Consequently from Corollary 5.5, from (8.9) and (8.8), (8.7) follows and so does Lemma 8.2.

Now we establish Lemma 8.5. To do this, we define the following process for each $k \in \mathbb{N}$,

$$B_k^N(t) = \int_0^t \int_0^{\Theta_k^N(r^-)} Q_k(\mathrm{d}r, \mathrm{d}u),$$

where

$$\Theta_k^N(t) = \gamma_{k,B_k^N(t)}(\vartheta_k^N(t))\overline{\mathfrak{F}}_{(1,2)}^N(t),$$

and ϑ_k^N is defined in the same manner as ς_1^N with B_k^N instead of A_1^N in (2.2).

Lemma 8.6. *For $k \in \mathbb{N}$ and $T \geq 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right|\right] \leq \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big]dt =: \delta^N(T) \tag{8.10}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\varsigma_k^N(t) - \vartheta_k^N(t)\right|\right] \leq T\delta^N(T).$$

Moreover,

$$\delta^N(T) \leq \frac{\mathbb{E}\left[Y\right]}{N} T \exp(2\lambda^* T). \tag{8.11}$$

Proof. Since

$$\left|A_k^N(t) - B_k^N(t)\right| \leq \int_0^t \int_0^{+\infty} \mathbb{1}_{\min\left(\Upsilon_k^N(r^-),\Theta_k^N(r^-)\right)<u\leq\max\left(\Upsilon_k^N(r^-),\Theta_k^N(r^-)\right)} Q_k(\mathrm{d}u, \mathrm{d}r),$$

we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|A_k^N(t) - B_k^N(t)\right|\right] \leq \int_0^T \mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big]dt = \delta^N(T).$$

We recall that

$$\Upsilon_k^N(t) = \gamma_{k,A_k^N(t)}(\varsigma_k^N(t))\overline{\mathfrak{F}}^N(t) \quad \text{ and } \quad \Theta_k^N(t) = \gamma_{k,B_k^N(t)}(\vartheta_k^N(t))\overline{\mathfrak{F}}_{(1,2)}^N(t).$$

However, since $\gamma_{k,i} \leq 1$ and $0 \leq \overline{\mathfrak{F}}^N(t)$, $\overline{\mathfrak{F}}_{(1,2)}^N(t) \leq \lambda^*$, we obtain

$$\begin{aligned}\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|\Big] \leq \mathbb{E}\left[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right| \mathbb{1}_{A_k^N(t)=B_k^N(t),\varsigma_k(t)=\vartheta_k^N(t)}\right] \\ + \lambda^*\mathbb{P}\left(A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\right).\end{aligned} \tag{8.12}$$

On the other hand, as $\gamma_{k,i} \leq 1$, from (5.8), we have

$$\begin{aligned}\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right| \mathbb{1}_{A_k^N(t)=B_k^N(t),\varsigma_k^N(t)=\vartheta_k^N(t)}\right] &\leq \mathbb{E}\left[\left|\overline{\mathfrak{F}}^N(t) - \overline{\mathfrak{F}}_{(1,2)}^N(t)\right|\right] \\ &\leq \frac{1}{N}\mathbb{E}\left[Y\right].\end{aligned} \tag{8.13}$$

Moreover, since

$$\left\{A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\right\} \subset \left\{\sup_{r\in[0,t]} \left|A_k^N(r) - B_k^N(r)\right| \geq 1\right\},$$

we have

$$\mathbb{P}\left(A_k^N(t) \neq B_k^N(t) \text{ or } \varsigma_k^N(t) \neq \vartheta_k^N(t)\right) \leq \mathbb{E}\left[\sup_{r\in[0,t]} |A_k^N(r) - B_k^N(r)|\right] \leq \delta^N(t).$$

Thus, from (8.12) and (8.13), we have

$$\mathbb{E}\Big[\left|\Upsilon_k^N(t) - \Theta_k(t)\right|\Big] \leq \frac{\mathbb{E}\left[Y\right]}{N} + \lambda^*\delta^N(t).$$

Hence, from (8.10), we deduce that

$$\delta^N(T) \leq \frac{\mathbb{E}\left[Y\right]}{N}T + \lambda^* \int_0^T \delta^N(t)dt,$$

and by Gronwall's lemma, it follows that

$$\delta^N(T) \leq \frac{\mathbb{E}\left[Y\right]}{N}T\exp(\lambda^* T).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\varsigma_k^N(t) - \vartheta_k^N(t)\right|\right] &= \mathbb{E}\left[\mathbb{1}_{\{\exists t\in[0,T], \varsigma_k^N(t)\neq\vartheta_k^N(t)\}} \sup_{t\in[0,T]}\left|\varsigma_k^N(t) - \vartheta_k^N(t)\right|\right] \\
&\leq T\mathbb{P}\left(\exists t\in[0,T], \varsigma_k^N(t) \neq \vartheta_k^N(t)\right) \\
&= T\mathbb{P}\left(\sup_{t\in[0,T]}\left|A_k^N(t) - B_k^N(t)\right| \neq 0\right) \\
&\leq T\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_k^N(t) - B_k^N(t)\right|\right] \\
&\leq T\delta^N(T).
\end{aligned}$$

This concludes the proof of the lemma. $\square$

We can now establish Lemma 8.5.

Proof of Lemma 8.5. We first note that, from the proof of Proposition 5.4, where we replace A_k by B_k^N and Υ_k by Θ_k^N and using Lemma 8.6 instead of Lemma 5.2, we deduce that, for all $p \in \mathbb{N}$, $t \in [0, T]$,

$$\mathbb{P}\left((\varsigma_{k'}^N(s))_{s\in[0,t]} \neq (\vartheta_{k'}^N(s))_{s\in[0,t]}, \forall k' = 1, \cdots, p\right) \leq \frac{C_T}{N^p},$$

and

$$\mathbb{E}\left[\left|\Upsilon_k^N(t) - \Theta_k^N(t)\right|^p\right] \leq \frac{C_T}{N^p}. \tag{8.14}$$

By exchangeability and the fact that Q_1 and Q_2 are independent,

$$\begin{aligned}
\mathbb{E}\left[D_1^N(t)D_2^N(t)\right] &= \mathbb{E}\left[\overline{D}_1^N(t)\overline{D}_2^N(t)\right] + 2\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)|\overline{D}_2^N(t)\right]\mathrm{d}v \\
&\qquad + \int_0^t\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dvdr \\
&= 2\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)|\overline{D}_2^N(v)\right]\mathrm{d}v \\
&\quad + \int_0^t\int_0^t \mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]dvdr, \qquad (8.15)
\end{aligned}$$

where the first term in the last equality follows from the fact that

$$\mathbb{E}\left[\overline{D}_2^N(t)-\overline{D}_2^N(v)\Big|\mathcal{F}_v^N\right] = \mathbb{E}\left[\int_v^t\int_{\mathbf{D}^2}\int_{\Upsilon_2(r^-)\wedge\Upsilon_2^N(r^-)}^{\Upsilon_2(r^-)\vee\Upsilon_2^N(r^-)} \overline{Q}_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\Big|\mathcal{F}_v^N\right] = 0.$$

As

$$|\Upsilon_1^N(v)-\Upsilon_1(v)| \le |\Upsilon_1^N(v)-\Theta_1^N(v)| + |\Theta_1^N(v)-\Upsilon_1(v)|,$$

by Hölder's inequality, it follows that

$$\begin{aligned}
\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\le \mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)||\overline{D}_2^N(v)|\right] + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] \\
&\le \left(\mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\overline{D}_2^N(v))^2\right]\right)^{1/2} + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right].
\end{aligned}$$

Moreover, as

$$\begin{aligned}
|\overline{D}_2^N(v)| &\le |D_2^N(v)| + \int_0^v |\Upsilon_2^N(r)-\Upsilon_2(r)|\mathrm{d}r \\
&\le |D_2^N(v)-\tilde{D}_2^N(v)| + \tilde{D}_2^N(v) + \int_0^v |\Upsilon_2^N(r)-\Upsilon_2(r)|\mathrm{d}r,
\end{aligned}$$

where

$$\tilde{D}_k^N(t) = \int_0^t\int_{\mathbf{D}^2}\int_{\Upsilon_k(v^-)\wedge\Theta_k^N(v^-)}^{\Upsilon_k(v^-)\vee\Theta_k^N(v^-)} Q_k(\mathrm{d}v,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),$$

it follows that,

$$\begin{aligned}
\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\le \left(\mathbb{E}\left[|\Upsilon_1^N(v)-\Theta_1^N(v)|^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\overline{D}_2^N(v))^2\right]\right)^{1/2} + \lambda_*\mathbb{E}\left[|D_2^N(v)-\tilde{D}_2^N(v)|\right] \\
&\quad + \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)|\mathbb{E}\left[\tilde{D}_2^N(v)\Big|\overline{\mathfrak{F}}_{(1,2)}^N, Q_1\right]\right] \\
&\quad + \int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]\mathrm{d}r,
\end{aligned}$$

where we use the fact that $|\Theta_1^N(v)-\Upsilon_1(v)| \le \lambda_*$.

Consequently from (8.14) and the fact that Q_2 and $\left(Q_1, \overline{\mathfrak{F}}^N_{(1,2)}\right)$ are independent, it follows that

$$\begin{aligned}
\mathbb{E}\left[|\Upsilon_1^N(v)-\Upsilon_1(v)||\overline{D}_2^N(v)|\right] &\le \frac{C_T}{N^{5/4}} + \lambda_*\mathbb{E}\left[\int_0^v \int_{\mathbf{D}^2}\int_{\Upsilon_2^N(r^-)\wedge\Theta_2^N(r^-)}^{\Upsilon_2^N(r^-)\vee\Theta_2^N(r^-)} Q_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right] \\
&\quad + \int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Theta_2^N(r)-\Upsilon_2(r)|\right]\mathrm{d}r + \int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]\mathrm{d}r \\
&= \frac{C_T}{N^{5/4}} + \lambda_*\int_0^v \mathbb{E}\left[|\Theta_2^N(r)-\Upsilon_2^N(r)|\right]\mathrm{d}r + \int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Theta_2^N(r)-\Upsilon_2(r)|\right]\mathrm{d}r \\
&\quad + \int_0^v \mathbb{E}\left[|\Theta_1^N(v)-\Upsilon_1(v)||\Upsilon_2^N(r)-\Upsilon_2(r)|\right]\mathrm{d}r \\
&\le \frac{2C_T}{N^{5/4}} + \frac{C_T(\lambda_*+1)}{N},
\end{aligned}$$

where the last line follows from (8.14) and Corollary 5.5.

Hence, from (8.15) and Corollary 5.5 it follows that the inequality (8.9) holds. □

8.3. Proof of Lemma 8.3

We recall that

$$\hat{\mathfrak{S}}_{1,1}^N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \int_0^t\int_{\mathbf{D}^2}\int_0^{\Upsilon_k^N(r^-)} \gamma(t-r)\left(\mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}^N\right)=0} - \mathbb{1}_{P_k\left(r,t,\gamma,\overline{\mathfrak{F}}\right)=0}\right) Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u). \tag{8.16}$$

We set

$$H_k(s,t,\gamma,\phi_1,\phi_2) = \int_s^t \int_{\gamma(v-s)(\phi_1\wedge\phi_2(v-))}^{\gamma(v-s)(\phi_1\vee\phi_2(v-))} Q_k(\mathrm{d}v, dw).$$

Using the fact that for all A, $B\in\mathbb{N}$,

$$|\mathbb{1}_{A=0} - \mathbb{1}_{B=0}| \le |A-B|.$$

From Assumptions 2.4 and 2.5 and Lemma 5.6, it follows that

$$\begin{aligned}
&\left|\hat{\mathfrak{S}}_{1,1}^N(t) - \hat{\mathfrak{S}}_{1,1}^N(s)\right| \\
&\le +\frac{2}{\sqrt{N}}\sum_{k=1}^N\int_s^t\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u) \\
&\quad + \frac{2(t-s)^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u) \\
&\quad + \frac{2}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_0^{\lambda_*} \mathbb{1}_{s-r<\zeta^j\le t-r} H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u) \\
&\quad + \frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^s\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) + H_k(r,s,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})\right)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).
\end{aligned}$$

Therefore, tightness of $\hat{\mathfrak{S}}_{1,1}^N$ follows from the following Lemmas.

Lemma 8.7. *As $\delta \to 0$,*

$$\limsup_{N\to\infty} \sup_{0\le t\le T} \frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0. \tag{8.17}$$

Proof. By exchangeability, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\quad=\mathbb{E}\left[\left(\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\qquad+(N-1)\mathbb{E}\left[\left(\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)\right.\\
&\qquad\qquad\left.\left(\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_2(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)\right]
\end{aligned}\tag{8.18}$$

Let

$$\chi_k^N(t)=\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),$$

and

$$\tilde{\chi}_k^N(t)=\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

Note that, as $\overline{\mathfrak{F}}^N_{(1,2)}$ and Q_k for $k\in\{1,2\}$ are independent, from Theorem 5.1 we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{\chi}_k^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]&=\mathbb{E}\left[\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&=\lambda_*\mathbb{E}\left[\int_t^{t+\delta}\int_{\mathbf{D}}\int_r^{t+\delta}\gamma(v-r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v\mu(\mathrm{d}\gamma)\mathrm{d}r\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&\le\lambda_*\int_t^{t+\delta}\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|dvdr\\
&=\lambda_*\delta\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v.
\end{aligned}\tag{8.19}$$

Consequently, conditioning on $\overline{\mathfrak{F}}^N_{(1,2)}$, and using the fact that Q_1, Q_2 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mathbb{E}\left[\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right] \\
&\le \lambda_*^2\delta^2\mathbb{E}\left[\left(\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v\right)^2\right] \\
&\le \lambda_*^2\delta^3\int_t^{t+\delta}\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|^2\right]\mathrm{d}v \\
&\le 2\lambda_*^2\delta^4\left(\frac{1}{N^2}\mathbb{E}\left[Y^2\right]+\frac{C_T}{N}\right),
\end{aligned} \tag{8.20}$$

where we use Hölder's inequality and the fact that $|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)| \le \frac{Y}{N}+|\overline{\mathfrak{F}}^N(v)-\overline{\mathfrak{F}}(v)|$, $(a+b)^2\le 2(a^2+b^2)$ and Proposition 5.4.

In addition for $k\in\{1,2\}$, using Subsection 5.3

$$\begin{aligned}
\left|\chi^N_k(t)-\tilde{\chi}^N_k(t)\right| &= \int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*} H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}}^N)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u) \\
&\le \int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_k(\mathrm{d}v,dw)\right)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).
\end{aligned} \tag{8.21}$$

Consequently, using Theorem 5.1

$$\mathbb{E}\left[\left|\chi^N_1(t)-\tilde{\chi}^N_1(t)\right|\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right] \le \frac{2Y\lambda_*\delta}{N}, \tag{8.22}$$

and from (8.21) we deduce that as $N\to\infty$, $\chi^N_1(t)\to 0$ in probability and as

$$\chi^N_1(t) \le \left(\int_t^{t+\delta}\int_0^{\lambda_*} Q_1(\mathrm{d}v,dw)\right)^2,$$

it follows that, as $N\to\infty$,

$$\mathbb{E}\left[\left(\chi^N_1(t)\right)^2\right]\to 0. \tag{8.23}$$

We have

$$\begin{aligned}
\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t)-\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right] &= \mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t)-\tilde{\chi}^N_2(t)\right)\right]+\mathbb{E}\left[\left(\chi^N_1(t)-\tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right] \\
&\quad+\mathbb{E}\left[\left(\chi^N_1(t)-\tilde{\chi}^N_1(t)\right)\left(\chi^N_2(t)-\tilde{\chi}^N_2(t)\right)\right].
\end{aligned} \tag{8.24}$$

As $\left(Q_1, Q_2\right)$ and $\left(\overline{\mathfrak{F}}^N_{(1,2)}, Y\right)$ are independent, from Theorem 5.1, (8.21), (8.22) and (8.19) it follows that

$$\begin{aligned}
&\left|\mathbb{E}\left[\tilde{\chi}^N_1(t)\left(\chi^N_2(t)-\tilde{\chi}^N_2(t)\right)\right]\right| \\
&\le \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_2(\mathrm{d}v, dw)\right) Q_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\Big|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\Big|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right. \\
&\qquad\qquad \left.\times\mathbb{E}\left[\int_t^{t+\delta}\int_{\mathbf{D}^2}\int_0^{\lambda_*}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)} Q_2(\mathrm{d}v, dw)\right) Q_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\Big|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right] \\
&\le \frac{2\lambda_*^2\delta^2}{N}\mathbb{E}\left[Y\int_t^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v\right] \\
&\le \frac{2\lambda_*^3\delta^3}{N}\mathbb{E}\left[Y\right],
\end{aligned} \tag{8.25}$$

where the last line follows from the fact that $\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right| \le \lambda_*$.

Similarly, we show that

$$\left|\mathbb{E}\left[\left(\chi^N_1(t)-\tilde{\chi}^N_1(t)\right)\tilde{\chi}^N_2(t)\right]\right| \le \frac{2\lambda_*^3\delta^3}{N}\mathbb{E}\left[Y\right]. \tag{8.26}$$

From (8.21), and as in the setting of (8.25), it follows that

$$\mathbb{E}\left[\left|\chi^N_1(t)-\tilde{\chi}^N_1(t)\right|\left|\chi^N_2(t)-\tilde{\chi}^N_2(t)\right|\right] \le \frac{4\lambda_*^4\delta^4}{N^2}\mathbb{E}\left[Y^2\right]. \tag{8.27}$$

Consequently from (8.27),(8.26), (8.25), (8.24), and (8.20), it follows that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}N\mathbb{E}\left[\chi^N_1(t)\chi^N_2(t)\right] = 0.$$

Therefore, from (8.23) and (8.18), it follows that (8.17) holds. □

Applying the same method as in the previous lemma, we establish the following Lemma,

Lemma 8.8. *As $\delta\to 0$,*

$$\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\frac{\delta^\alpha}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0,$$

$$\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^N\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0,$$

and

$$\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0,$$

To conclude the tightness of $\hat{\mathfrak{S}}^N_{1,1}$, it remains to establish the following Lemma.

Lemma 8.9. *As $\delta\to 0$,*

$$\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\ge\epsilon\right)\to 0. \tag{8.28}$$

Proof. By exchangeability it follows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{\ell-1}\sum_{k=1}^{N}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\quad=\mathbb{E}\left[\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)^2\right]\\
&\quad+(N-1)\mathbb{E}\left[\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_1(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_1(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)\right.\\
&\qquad\left.\left(\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_2(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_2(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\right)\right].
\end{aligned}\tag{8.29}$$

Let

$$\chi^N_k(t)=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u),$$

and

$$\tilde{\chi}^N_k(t)=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u).$$

Since $\overline{\mathfrak{F}}^N_{(1,2)}$ and Q_k for $k\in\{1,2\}$ are independent, from Theorem 5.1, we have

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\chi}^N_k(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&\qquad=\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&\qquad=\lambda_*\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\int_{\mathbf{D}}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}\int_r^{t+\delta}\gamma(v-r)\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v\mu(\mathrm{d}\gamma)\mathrm{d}r\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&\qquad\le\lambda_*\sum_{j=1}^{\ell-1}\mathbb{E}\left[\int_0^t\left(G_j(t+\delta-r)-G_j(t-r)\right)\int_r^{t+\delta}\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|dvdr\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\\
&\qquad\le\lambda_*\ell T\delta^\rho\int_0^T\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v,
\end{aligned}
\tag{8.30}
$$

where we use $\gamma\le 1$, to get the third line.

Consequently, conditioning by $\overline{\mathfrak{F}}^N_{(1,2)}$, and using the fact that Q_1, Q_2 and $\overline{\mathfrak{F}}^N_{(1,2)}$ are independent, we have

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\right]&=\mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}^N_1(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\mathbb{E}\left[\tilde{\chi}^N_2(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}\right]\right]\\
&\le\lambda_*^2\ell^2T^2\delta^{2\rho}\mathbb{E}\left[\left(\int_0^T\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|\mathrm{d}v\right)^2\right]\\
&\le\lambda_*^2\ell^2T^3\delta^{2\rho}\int_0^T\mathbb{E}\left[\left|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)\right|^2\right]\mathrm{d}v\\
&\le 2\lambda_*^2\ell^2T^3\delta^{2\rho}\left(\frac{1}{N^2}\mathbb{E}\left[Y^2\right]+\frac{C_T}{N}\right),
\end{aligned}
\tag{8.31}
$$

where we use Hölder's inequality and the fact that $|\overline{\mathfrak{F}}^N_{(1,2)}(v)-\overline{\mathfrak{F}}(v)|\le\frac{Y}{N}+|\overline{\mathfrak{F}}^N(v)-\overline{\mathfrak{F}}(v)|$, $(a+b)^2\le 2(a^2+b^2)$ and Proposition 5.4.

In addition for $k\in\{1,2\}$, using Subsection 5.3

$$
\begin{aligned}
&\left|\chi^N_k(t)-\tilde{\chi}^N_k(t)\right|\\
&=\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}H_k(r,t+\delta,\gamma,\overline{\mathfrak{F}}^N_{(1,2)},\overline{\mathfrak{F}})Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&\le\sum_{j=1}^{\ell-1}\int_0^t\int_{\mathbf{D}^2}\int_0^{\lambda_*}\mathbb{1}_{t-r<\zeta^j\le t+\delta-r}\left(\int_r^{t+\delta}\int_{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)-Y/N)}^{\gamma(v-r)(\overline{\mathfrak{F}}^N_{(1,2)}(v^-)+Y/N)}Q_k(\mathrm{d}v,dw)\right)Q_k(\mathrm{d}r,\mathrm{d}\lambda,\mathrm{d}\gamma,\mathrm{d}u)\\
&=:\hat{\chi}^N_k(t).
\end{aligned}
\tag{8.32}
$$

Consequently, using Theorem 5.1, it follows that,

$$
\mathbb{E}\left[\left|\chi^N_1(t)-\tilde{\chi}^N_1(t)\right|\right]\le\frac{2T^2\lambda_*\ell\delta^\rho}{N}\mathbb{E}\left[Y\right].
\tag{8.33}
$$

and from (8.32) we deduce that as $N \to \infty$, $\chi_1^N \to 0$ in probability, and as it is bounded by a square-integrable process, it follows that, as $N \to \infty$,

$$\mathbb{E}\left[\left(\chi_1^N(t)\right)^2\right] \to 0. \tag{8.34}$$

We have

$$\begin{aligned}\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t) - \tilde{\chi}_1^N(t)\tilde{\chi}_2^N(t)\right] = \mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right] + \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\\ + \mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right],\end{aligned} \tag{8.35}$$

As (Q_1, Q_2) and $(\overline{\mathfrak{F}}^N_{(1,2)}, Y)$ are independent from (8.32) and (8.30),

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{\chi}_1^N(t)\left(\chi_2^N(t) - \tilde{\chi}_2^N(t)\right)\right]\right| &\le \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\hat{\chi}_k^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\tilde{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\mathbb{E}\left[\hat{\chi}_1^N(t)\middle|\overline{\mathfrak{F}}^N_{(1,2)}, Y\right]\right]\\ &\le \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N}\mathbb{E}\left[Y\int_0^T\left|\overline{\mathfrak{F}}^N_{(1,2)}(v) - \overline{\mathfrak{F}}(v)\right|\mathrm{d}v\right]\\ &\le \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N^{3/2}}\mathbb{E}\left[Y\right].\end{aligned} \tag{8.36}$$

Similarly we show that

$$\left|\mathbb{E}\left[\left(\chi_1^N(t) - \tilde{\chi}_1^N(t)\right)\tilde{\chi}_2^N(t)\right]\right| \le \frac{2T^3\lambda_*^2\ell^2\delta^{2\rho}}{N^{3/2}}\mathbb{E}\left[Y\right]. \tag{8.37}$$

From (8.32), and as in the setting in (8.36), it follows that

$$\mathbb{E}\left[\left|\chi_1^N(t) - \tilde{\chi}_1^N(t)\right|\left|\chi_2^N(t) - \tilde{\chi}_2^N(t)\right|\right] \le \frac{4T^4\lambda_*^4\ell^4\delta^{4\rho}}{N^2}\mathbb{E}\left[Y^2\right]. \tag{8.38}$$

Consequently from (8.38),(8.37), (8.36), (8.35), and (8.31), it follows that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}N\mathbb{E}\left[\chi_1^N(t)\chi_2^N(t)\right] = 0.$$

Therefore, (8.28) follows from (8.34) and (8.29). $\square$

8.4. Proof of Lemma 8.4

We recall that

$$\hat{\mathfrak{S}}^N_{1,0}(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \gamma_{k,0}(t)\left(\mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N)=0} - \mathbb{1}_{P_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}})=0}\right).$$

Using the fact that for all A, $B \in \mathbb{N}$,

$$\left|\mathbb{1}_{A=0} - \mathbb{1}_{B=0}\right| \le |A - B|,$$

from Assumption 2.4 and 2.5 and Lemma 5.6, it follows that

$$
\begin{aligned}
\left|\hat{\mathfrak{S}}^N_{1,0}(t)-\hat{\mathfrak{S}}^N_{1,0}(s)\right| \le \frac{(t-s)^\alpha}{\sqrt{N}}\sum_{k=1}^N H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \\
+\frac{1}{\sqrt{N}}\sum_{k=1}^N\sum_{j=1}^{\ell-1}\mathbb{1}_{s-r<\zeta_k^j\le t-r}H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}}) \\
+\frac{1}{\sqrt{N}}\sum_{k=1}^N\Big(H_k(0,t,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})+H_k(0,s,\gamma_{k,0},\overline{\mathfrak{F}}^N,\overline{\mathfrak{F}})\Big)\,.
\end{aligned}
\tag{8.39}
$$

Hence as in the setting of (8.2), (8.3) and (8.4) respectively, we establish the tightness of $\hat{\mathfrak{S}}^N_{1,0}$.

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Appendix A.

A.1 Proof of Proposition 4.1

It is easy to check that

$$\hat{\mathfrak{M}}_{0,1}(t) + W^{\lambda}(t) \overset{d}{=} \hat{I}(0)\overline{\lambda}_0(t) + \hat{W}_{\overline{\mathfrak{F}}}(t),$$

where $\hat{I}(0)$ is a Gaussian variable with variance $\overline{I}(0)\overline{S}(0)$ and $W_{\overline{\mathfrak{F}}}$ a centered-Gaussian process independent of $\hat{I}(0)$ with covariance function for $t, t' \geq 0$,

$$\begin{aligned}\mathrm{Cov}(W_{\overline{\mathfrak{F}}}(t), W_{\overline{\mathfrak{F}}}(t')) = \overline{I}(0)\mathbb{E}\left[\left(\lambda_0(t) - \overline{\lambda}_0(t)\right)\left(\lambda_0(t') - \overline{\lambda}_0(t')\right)|\eta_0 > 0\right] \\ + \int_0^{t\wedge t'} \mathbb{E}\left[\lambda(t-s)\lambda(t'-s)\right]\overline{S}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s.\end{aligned}$$

This lead to the second equation in the set of equations (4.3).

Recall that the unique solution of the linear differential equation:

$$x(t) = x(0) - \int_0^t \overline{\mathfrak{F}}(s)x(s)\mathrm{d}s + y(t),$$

with $y(0) = 0$, is given by the formula for $t \geq 0$,

$$x(t) = x(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) - \int_0^t y(s)\overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathrm{d}s + y(t). \tag{A.1}$$

We set,

$$X(t) = -\overline{S}(t)\int_0^t \hat{\mathfrak{F}}(s)\mathrm{d}s.$$

By derivation,

$$\begin{aligned}X'(t) &= -\overline{S}'(t)\int_0^t \hat{\mathfrak{F}}(s)\mathrm{d}s - \overline{S}(t)\hat{\mathfrak{F}}(t) \\ &= \overline{\mathfrak{F}}(t)\overline{S}(t)\int_0^t \hat{\mathfrak{F}}(s)\mathrm{d}s - \overline{S}(t)\hat{\mathfrak{F}}(t) \\ &= -X(t)\overline{\mathfrak{F}}(t) - \overline{S}(t)\hat{\mathfrak{F}}(t),\end{aligned}$$

where the second line follows from the fact that $\overline{S}'(t) = -\overline{S}(t)\overline{\mathfrak{F}}(t)$ and the third line follows from definition of $X(t)$. As a result,

$$X(t) = -\int_0^t X(s)\overline{\mathfrak{F}}(s)\mathrm{d}s - \int_0^t \overline{S}(s)\hat{\mathfrak{F}}(s)\mathrm{d}s. \tag{A.2}$$

We next show that,

$$\hat{\mathfrak{I}}_{0,1}(t) = \hat{I}(0) - \int_0^t \hat{\mathfrak{I}}_{0,1}(s)\overline{\mathfrak{F}}(s)\mathrm{d}s + W_0(t), \tag{A.3}$$

where W_0 is a centered Gaussian process with covariance function for $t, t' \geq 0$,

$$\mathrm{Cov}(W_0(t), W_0(t')) = \int_0^{t\wedge t'} \overline{\mathfrak{F}}(s)\overline{S}(s)\mathrm{d}s,$$

and $\hat{I}(0)$ is a centered-Gaussian variable with variance $\overline{I}(0)\overline{S}(0)$.

Note that combining (A.2) and (A.3) from (4.2), we obtain the first equation in the set of equations (4.3). So to conclude it suffices to compute the covariance function of $\hat{\mathfrak{I}}_{0,1}$.

From (A.1) we have for $t \geq 0$,

$$\hat{\mathfrak{J}}_{0,1}(t) = \hat{I}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) - \int_0^t W_0(s)\overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathrm{d}s + W_0(t).$$

We compute the covariance function $\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t), \hat{\mathfrak{J}}_{0,1}(t'))$ for $t \leq t'$,

$$\begin{aligned}\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t), \hat{\mathfrak{J}}_{0,1}(t')) = &\int_0^t \int_0^{t'} \overline{\mathfrak{F}}(s)\overline{\mathfrak{F}}(u)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_u^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(u)\right]dsdu \\ &- \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(t')\right]\mathrm{d}s \\ &- \int_0^{t'} \overline{\mathfrak{F}}(s)\exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(t)\right]\mathrm{d}s \\ &+ \mathbb{E}\left[W_0(t)W_0(t')\right] + \overline{I}(0)\overline{S}(0)\exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_0^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right).\end{aligned}$$

For the first term by integration by part twin, we have

$$\begin{aligned}&\int_0^t \int_0^{t'} \overline{\mathfrak{F}}(s)\overline{\mathfrak{F}}(u)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_u^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(u)\right]dsdu \\ &\quad= \int_0^t \int_0^s \overline{\mathfrak{F}}(s)\overline{\mathfrak{F}}(u)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_u^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^u \overline{\mathfrak{F}}(r)\overline{S}(r)drduds \\ &\qquad+ \int_0^t \int_s^{t'} \overline{\mathfrak{F}}(s)\overline{\mathfrak{F}}(u)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_u^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drduds \\ &\quad= -2\int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds \\ &\qquad+ \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds + \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds \\ &\quad= -\exp\left(-\int_t^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^t \overline{\mathfrak{F}}(r)\overline{S}(r)\mathrm{d}r + \exp\left(-\int_t^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^t \overline{S}(s)\overline{\mathfrak{F}}(s)\exp\left(-2\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathrm{d}s \\ &\qquad+ \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds + \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds.\end{aligned}$$

For the second term

$$\int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(t')\right]\mathrm{d}s = \int_0^t \overline{\mathfrak{F}}(s)\exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds.$$

For the third term

$$\int_0^{t'} \overline{\mathfrak{F}}(s)\exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\mathbb{E}\left[W_0(s)W_0(t)\right]\mathrm{d}s$$

$$= \int_0^t \overline{\mathfrak{F}}(s) \exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds + \int_t^{t'} \overline{\mathfrak{F}}(s) \exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \mathrm{d}s \int_0^t \overline{\mathfrak{F}}(r)\overline{S}(r)\mathrm{d}r$$
$$= \int_0^t \overline{\mathfrak{F}}(s) \exp\left(-\int_s^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \int_0^s \overline{\mathfrak{F}}(r)\overline{S}(r)drds + \int_0^t \overline{\mathfrak{F}}(r)\overline{S}(r)\mathrm{d}r - \exp\left(-\int_t^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \int_0^t \overline{\mathfrak{F}}(r)\overline{S}(r)\mathrm{d}r,$$

where the last line follows by integration.

Consequently using the fact that

$$\overline{S}(t') = \overline{S}(t) \exp\left(-\int_t^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right),$$

it follows that,

$$\begin{aligned}
\mathrm{Cov}(\hat{\mathfrak{J}}_{0,1}(t), \hat{\mathfrak{J}}_{0,1}(t')) &= \exp\left(-\int_t^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \int_0^t \overline{S}(s)\overline{\mathfrak{F}}(s) \exp\left(-2\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \mathrm{d}s \\
&\qquad + \overline{I}(0)\overline{S}(0) \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r - \int_0^{t'} \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \\
&= \overline{S}(t') \int_0^t \overline{\mathfrak{F}}(s) \exp\left(-\int_s^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \mathrm{d}s + (1-\overline{S}(0))\overline{S}(t') \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \\
&= \overline{S}(t') \left(1 - \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right)\right) + (1-\overline{S}(0))\overline{S}(t') \exp\left(-\int_0^t \overline{\mathfrak{F}}(r)\mathrm{d}r\right) \\
&= \overline{S}(t') - \overline{S}(t')\overline{S}(t).
\end{aligned}$$

This conclude the proof.

A.2 Tightness criterion

We recall the following theorem used to prove of **C**-tightness in **D** (see Thm. 3.21 in [32], p. 350).

Theorem A.1. *Let $(X^N)_N$ be a sequence of r.v. taking values in* **D**. *It is* **C***-tight in* **D**, *if*

(i) for any $T > 0$, $\epsilon > 0$ there exist $N_0 \in \mathbb{N}$ and $C > 0$ such that

$$N \geq N_0 \implies \mathbb{P}\left(\sup_{0\leq s\leq T} |X_s^N| > C\right) \leq \epsilon. \tag{A.4}$$

(ii) for any $T > 0$, $\epsilon > 0$, $\theta > 0$ there exist $N_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$N \geq N_0 \implies \mathbb{P}\left(w_T(X^N, \delta) \geq \theta\right) \leq \epsilon. \tag{A.5}$$

where

$$w_T(\alpha, \delta) = \sup_{0\leq s<t<T, |t-s|\leq\delta} |\alpha(t) - \alpha(s)|.$$

Since we work with processes in **D**, we will simply write **C**-tightness below for brevity. In fact, **C**-tightness means that the limit of subsequences are continuous. We also recall the following result from [18], Lemma 3.1.

Lemma A.2. *Let* $\{X^N\}_{N\geq 1}$ *be a sequence of random elements in* $\mathbf{D}$ *such that* $X^N(0)=0$. *If for all* $T>0$, $\epsilon>0$, *as* $\delta\to 0$,

$$\limsup_{N\to\infty}\sup_{0\leq t\leq T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leq u\leq \delta}|X^N(t+u)-X^N(t)|>\epsilon\right)\to 0,$$

then the sequence X^N *is* $\mathbf{C}$*-tight.*